Research Article

Some Bond Incident Degree Indices of Cactus Graphs

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A connected graph in which no edge lies on more than one cycle is called a cactus graph (also known as Husimi tree). A bond incident degree (BID) index of a graph G is defined as $\sum_{uv \in E(G)} f(d_G(u), d_G(v))$, where $d_G(w)$ denotes the degree of a vertex w of G, $E(G)$ is the edge set of G, and f is a real-valued symmetric function. This study involves extremal results of cactus graphs concerning the following type of the BID indices: $I_{f_i}(G) = \sum_{uv \in E(G)} [f_i(d_G(u))/d_G(u) + f_i(d_G(v))/d_G(v)]$, where $i \in \{1, 2\}$, f_1 is a strictly convex function, and f_2 is a strictly concave function. More precisely, graphs attaining the minimum and maximum I_{f_i} values are studied in the class of all cactus graphs with a given number of vertices and cycles. The obtained results cover several well-known indices including the general zeroth-order Randić index, multiplicative first and second Zagreb indices, and variable sum exdeg index.

1. Introduction

All the graphs considered in this study are connected. The notation and terminology that are used in this study but not defined here can be found in some standard graph-theoretical books [6, 7].

Graph invariants of the following form are known as the bond incident degree (BID) indices [5]:

$$BID(G) = \sum_{uv \in E(G)} f(d_G(u), d_G(v)), \quad (1)$$

where $d_G(w)$ denotes the degree of a vertex $w \in V(G)$ of the graph G, $E(G)$ is the edge set of G, and f is a real-valued symmetric function. In this study, we are concerned with the following type [2] of the BID indices:

$$I_{f_i}(G) = \sum_{uv \in E(G)} \left[\frac{f_i(d_G(u))}{d_G(u)} + \frac{f_i(d_G(v))}{d_G(v)} \right] = \sum_{v \in V(G)} f_i(d_G(v)) \quad (2)$$

where $i \in \{1, 2\}$, f_1 is a strictly convex function, and f_2 is a strictly concave function.

A connected graph in which no edge lies on more than one cycle is called a cactus graph (also known as Husimi tree [9]). In the present study, we study the graphs attaining the minimum and maximum I_{f_i} values from the class of all cactus graphs with a given number of vertices and cycles. Our main results cover the general zeroth-order Randić index $^0R_\alpha$ [3], variable sum exdeg index SEI_a [13], multiplicative first Zagreb index Π_1 [8], multiplicative second Zagreb index Π_2 [1, 8, 10], and sum lordeg index SL [12, 14], where the aforementioned indices for a graph G are defined as follows:

$$^0R_\alpha(G) = \sum_{v \in V(G)} [d_G(v)]^\alpha,$$

$$SEI_a(G) = \sum_{v \in V(G)} d_G(v) a^{d_G(v)},$$

$$\Pi_1(G) = \prod_{v \in V(G)} [d_G(v)]^2,$$

$$\Pi_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v) = \prod_{v \in V(G)} [d_G(v)]^{d_G(v)},$$

$$SL(G) = \sum_{v \in V(G)} d_G(v)\sqrt{\ln[d_G(v)]}$$

$$= \sum_{v \in V(G); d_G(v) \geq 2} d_G(v)\sqrt{\ln[d_G(v)]}. \tag{3}$$

A graph in which every vertex has degree less than 5 is known as a chemical graph.

Although we cannot apply our main result on the Lanzhou index [15] for finding the extremal graphs from the class of all cactus graphs, we still are able to utilize one of our main results for finding the graphs having the minimum Lanzhou index among all chemical cactus graphs, where the Lanzhou index for a graph G is defined as

$$Lz(G) = \sum_{v \in V(G)} (n - d_G(v) - 1)[d_G(v)]^2. \tag{4}$$

We end this section with the remark that the Lanzhou index is same [11] as the graph invariant $^0R_3$.

2. Main Results

By an n-vertex graph, we mean a graph of order n. In a graph, a set of pairwise nonadjacent edges is called a matching. The elements of a matching are known as independent edges.

Theorem 1. *The graph formed by adding r-independent edges in the n-vertex star S_n (Figure 1) uniquely attains the maximum I_{f_1} value and minimum I_{f_2} value in the class of all n-vertex cactus graphs having r cycles, where n and r are the fixed integers satisfying the inequalities $n \geq 2r + 1$, $n \geq 4$, and $r \geq 0$.*

Proof. We prove the result for the graph invariant I_{f_1}. The result regarding the other invariant can be proved in a fully analogous way. Let G be a graph having the maximum I_{f_1} value in the given class of graphs. It is enough to show that G has the maximum degree $n - 1$. Contrarily, we assume that that the maximum degree of G is at most $n - 2$. Let $v \in V(G)$ be a vertex of maximum degree. Then, there exists at least one neighbor, say w of v which has at least one neighbor not adjacent to v. Let $w_1, w_2, \ldots, w_k$ be those neighbors of w that are not adjacent to v. If G' is the graph formed by adding the edges $w_1v, w_2v, \ldots, w_kv$ in G and removing the edges $w_1w, w_2w, \ldots, w_kw$ from G (Figure 2), then we have

$$I_{f_1}(G) - I_{f_1}(G') = f_1(d_G(v)) - f_1(d_G(v) + k) + f_1(d_G(w)) - f_1(d_G(w) - k). \tag{5}$$

Note that the cactus graphs G and G' have the same number of cycles as well as order. By using Lagrange's mean
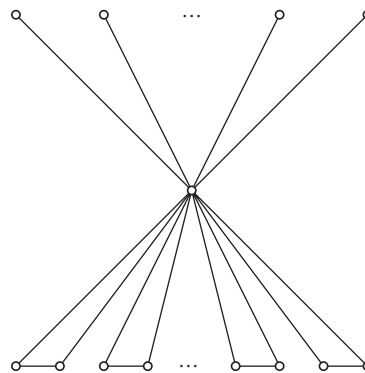

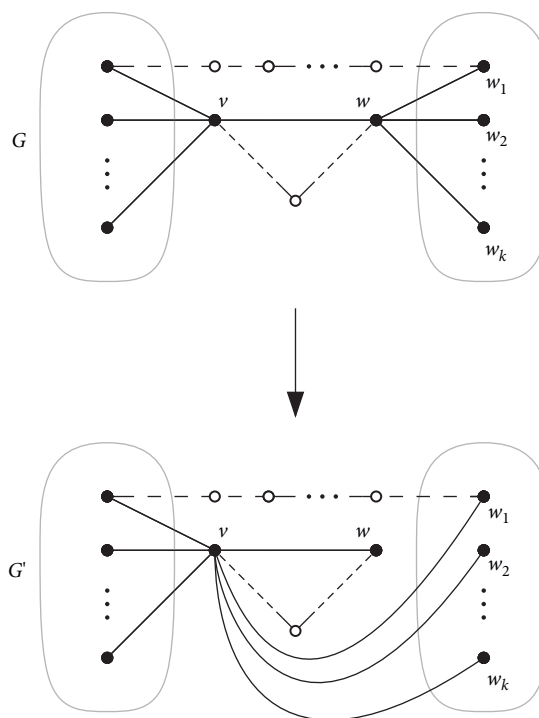Figure 1: The extremal graph mentioned in Theorem 1.

Figure 2: The graph transformation used in the proof of Theorem 1. The white vertices and doted edges may or may not exist provided that every edge lies on at most one cycle.

value theorem, we conclude that there exist real numbers a_1 and a_2, such that

$$a_1 \in (d_G(w) - k, d_G(w)),$$

$$a_2 \in (d_G(v), d_G(v) + k), \tag{6}$$

$$I_{f_1}(G) - I_{f_1}(G') = k[f_1'(a_1) - f_1'(a_2)].$$

The fact $d_G(w) \leq d_G(v)$ implies that $a_1 < a_2$, which further implies that the right hand side of equation (6) is negative because f_1 is a strictly convex function. Thus, one has $I_{f_1}(G) - I_{f_1}(G') < 0$, a contradiction to the maximality of $I_{f_1}(G)$. □

Corollary 1. *In the class of all n-vertex cactus graphs having r cycles, the graph formed by adding r independent edges in the n-vertex star S_n uniquely attains the maximum general zeroth-order Randić index $^0R_\alpha$ for $\alpha > 1$ or $\alpha < 0$, maximum variable sum exdeg index SEI_a for $a > 1$, maximum multiplicative second Zagreb index Π_2, maximum sum lordeg index SL, minimum general zeroth-order Randić index $^0R_\alpha$ for $0 < \alpha < 1$, and minimum multiplicative first Zagreb index Π_1, where n and r are the fixed integers satisfying the inequalities $n \geq 2r + 1$, $n \geq 4$, and $r \geq 0$.*

Proof. We observe that a graph G attains its maximum Π_2 value or minimum Π_1 value in a class of graphs if and only if G attains its maximum $\ln \Pi_2$ value or minimum $\ln \Pi_1$ value, respectively, in the considered class of graphs. We define $\phi_1(x) = xa^x$ with $a > 1$ and $x \geq 1$; $\phi_2(x) = x^\alpha$ with $x \geq 1$ and $\alpha > 1$ or $\alpha < 0$; $\phi_3(x) = x \ln x$ with $x \geq 1$; $\phi_4(x) = x\sqrt{\ln x}$ with $x \geq 2$; $\phi_5(x) = 2 \ln x$ with $x \geq 1$; and $\phi_6(x) = x^\alpha$ with $x \geq 1$ and $0 < \alpha < 1$. It can be easily verified that for each $i \in \{1, 2, 3, 4\}$, ϕ_i is strictly convex and for each $j \in \{5, 6\}$, ϕ_j is strictly concave. Thus, the required conclusion follows from Theorem 1.

A graph of order n and size m is called an (n, m)-graph. □

Lemma 2 (see [4]). *If G attains the minimum I_{f_1} value or maximum I_{f_2} value among all connected (n, m)-graphs and $v \in V(G)$, then the minimum degree of G is at least 2, where n and m are the fixed integers satisfying the conditions $3n \geq 2m$, $n \geq 4$, and $m \geq n$.*

The next result is a direct consequence of Lemma 2.

Corollary 3. *If G is a graph attaining the minimum I_{f_1} value or maximum I_{f_2} value in the class of all n-vertex cactus graphs having r cycles, then the minimum degree of G is 2, where n and r are the fixed integers satisfying the inequalities $n \geq 2r + 1$, $n \geq 6$, and $r \geq 2$.*

Denote by $N_G(v)$ the set of neighbors of a vertex $v \in V(G)$ of a graph G.

Theorem 2. *If G is a graph attaining the minimum I_{f_1} value or maximum I_{f_2} value in the class of all n-vertex cactus graphs having r cycles and $v \in V(G)$, then the minimum degree of G is 2 and*

$$d_G(v) \leq \begin{cases} 4, & \text{if } v \text{ lies on some cycle,} \\ 3, & \text{otherwise,} \end{cases} \tag{7}$$

where n and r are the fixed integers satisfying the inequalities $n \geq 2r + 1$, $n \geq 6$, and $r \geq 2$.

Proof. We prove the result for the graph invariant I_{f_1}. The result regarding the other invariant can be proved in a fully analogous way. Let G be a graph having the minimum I_{f_1} value in the given class of graphs.

From Corollary 3, it follows that the minimum degree of G is 2. Next, we prove that

$$d_G(v) \leq \begin{cases} 4, & \text{if } v \text{ lies on some cycle,} \\ 3, & \text{otherwise,} \end{cases} \tag{8}$$

First, assume that v lies one some cycle C of G. Suppose to the contrary that $d_G(v) \geq 5$. Let $N_G(v)\backslash V(C) := \{v_1, v_2, \ldots, v_r\}$, where $V(C)$ denotes the set of vertices of the cycle C. For $i = 1, 2, \ldots, r$, denote by M_i the component of the graph $G - \{v\}$ containing the vertex v_i. It is claimed that no more than two vertices of $N_G(v)\backslash V(C)$ lie on the same component of the graph $G - \{v\}$; if v_k, v_l, and v_m lie on the same component of the graph $G - \{v\}$, then the vertices v_k, v_l, v_m, and v lie on a cycle whose each edge belongs to more than one cycle of G, which contradicts the definition of G. □

Case 1. There exists at least one i, such that the component M_i contains a unique vertex of $N_G(v)\backslash V(C)$.

Suppose, without loss of generality, that the component M_1 contains none of $v_2, v_3, \ldots, v_r$. We note that there exists at least one component M_j with $j \geq 2$, such that M_j contains at least one vertex $u \in V(G)$ satisfying $d_G(u) = 2$. Certainly, both the graphs G and $G' \cong G - \{vv_1\} + \{uv_1\}$ have the same number of cycles and vertices. On the other hand, we have

$$\begin{aligned} I_{f_1}(G) - I_{f_1}(G') &= f_1(d_G(v)) - f_1(d_G(v) - 1) \\ &\quad + f_1(d_G(u)) - f_1(d_G(u) + 1) \\ &= f_1(d_G(v)) - f_1(d_G(v) - 1) \\ &\quad - [f_1(3) - f_1(2)]. \end{aligned} \tag{9}$$

By using Lagrange's mean value theorem, we conclude that there exist real numbers a_1 and a_2, such that

$$\begin{aligned} a_1 &\in (2, 3), \\ a_2 &\in (d_G(v) - 1, d_G(v)), \\ I_{f_1}(G) - I_{f_1}(G') &= f_1'(a_2) - f_1'(a_1). \end{aligned} \tag{10}$$

The assumption $d_G(v) \geq 5$ implies that $a_1 < a_2$, which further implies that the right hand side of equation (2) is positive because f_1 is a strictly convex function. Thus, one has $I_{f_1}(G) - I_{f_1}(G') > 0$, a contradiction to the minimality of $I_{f_1}(G)$.

Case 2. For each $i \in \{1, 2, \ldots, r\}$, exactly two vertices of the set $N_G(v)\backslash V(C)$ lie on the component M_i.

Suppose, without loss of generality, that $M_1 = M_2$. It is clear that $M_1 \neq M_j$ for each $j \in \{3, \ldots, r\}$, and there exists at least one component M_j with $j \geq 3$, such that M_j contains at least one vertex $w \in V(G)$ satisfying $d_G(w) = 2$. It is obvious that both the graphs G and $G'' \cong G - \{vv_1, vv_2\} + \{wv_1, wv_2\}$ have the same number of cycles and vertices. On the other hand, we have

$$\begin{aligned} I_{f_1}(G) - I_{f_1}(G'') &= f_1(d_G(v)) - f_1(d_G(v) - 2) \\ &\quad + f_1(d_G(w)) - f_1(d_G(w) + 2) \\ &= f_1(d_G(v)) - f_1(d_G(v) - 2) \\ &\quad - [f_1(4) - f_1(2)]. \end{aligned} \tag{11}$$

By using Lagrange's mean value theorem, we conclude that there exist real numbers a_3 and a_4, such that

$$a_3 \in (2, 4),$$
$$a_4 \in (d_G(v) - 2, d_G(v)), \tag{12}$$
$$I_{f_1}(G) - I_{f_1}(G'') = 2[f_1'(a_4) - f_1'(a_3)].$$

We note, for the present case, that the degree of v is at least 6, which implies that $a_3 < a_4$, which further implies that the right hand side of (12) is positive because f_1 is a strictly convex function. Thus, one has $I_{f_1}(G) - I_{f_1}(G'') > 0$, a contradiction to the minimality of $I_{f_1}(G)$.

Thus, $d_G(v) \leq 4$ when v lies on some cycle of G.

It is still left to prove that $d_G(v) \leq 3$ when v does not belong to any cycle of G. Suppose to the contrary that $d_G(v) = r \geq 4$ and that v does not belong to any cycle of G. As before, we take $N_G(v) = \{v_1, v_2, \ldots, v_r\}$, and for $i \in \{1, 2, \ldots, r\}$, we denote by M_i the component of the graph $G - v$ containing the vertex v_i. We observe that $M_i \neq M_j$ whenever $i \neq j$; if the components M_i and M_j are the same for some $i \neq j$, then the path from v_i to v_j in $G - \{v\}$ together with the path $v_i v v_j$ yields a cycle in G containing v, which is a contradiction. We note that there exists at least one component M_j with $j \geq 2$, such that M_j contains at least one vertex $x \in V(G)$ satisfying $d_G(x) = 2$. It is obvious that both the graphs G and $G''' \cong G - \{v_1 v\} + \{v_1 x\}$ have the same number of cycles and vertices. On the other hand, we have

$$\begin{aligned} I_{f_1}(G) - I_{f_1}(G'') &= f_1(d_G(v)) - f_1(d_G(v) - 1) \\ &\quad + f_1(d_G(x)) - f_1(d_G(x) + 1) \\ &= f_1(d_G(v)) - f_1(d_G(v) - 1) \\ &\quad - [f_1(3) - f_1(2)]. \end{aligned} \tag{13}$$

By using Lagrange's mean value theorem, we conclude that there exist real numbers a_5 and a_6, such that

$$a_5 \in (2, 3),$$
$$a_6 \in (d_G(v) - 1, d_G(v)), \tag{14}$$
$$I_{f_1}(G) - I_{f_1}(G'') = f_1'(a_6) - f_1'(a_5).$$

The assumption $d_G(v) \geq 4$ implies that $a_3 < a_4$, which further implies that the right hand side of equation (4) is positive because f_1 is a strictly convex function. Thus, one has $I_{f_1}(G) - I_{f_1}(G'') > 0$, a contradiction to the minimality of $I_{f_1}(G)$. Thus, $d_G(v) \leq 3$ when v does not belong to any cycle of G.

Corollary 4. *If G is a graph attaining the minimum general zeroth-order Randić index $^0R_\alpha$ for $\alpha > 1$ or $\alpha < 0$, minimum variable sum exdeg index SEI_a for $a > 1$, minimum multiplicative second Zagreb index Π_2, minimum sum lordeg index SL, maximum general zeroth-order Randić index $^0R_\alpha$ for $0 < \alpha < 1$, and maximum multiplicative first Zagreb index Π_1, in the class of all n-vertex cactus graphs having r cycles and $v \in V(G)$, then the minimum degree of G is 2 and*

$$d_G(v) \leq \begin{cases} 4, & \text{if } v \text{ lies on some cycle,} \\ 3, & \text{otherwise,} \end{cases} \tag{15}$$

where n and r are the fixed integers satisfying the inequalities $n \geq 2r + 1$, $n \geq 6$, and $r \geq 2$.

Proof. We observe that a graph G attains its minimum Π_2 value or maximum Π_1 value in a class of graphs if and only if G attains its minimum $\ln \Pi_2$ value or maximum $\ln \Pi_1$ value, respectively, in the considered class of graphs. We define $\phi_1(x) = xa^x$ with $a > 1$ and $x \geq 1$; $\phi_2(x) = x^\alpha$ with $x \geq 1$ and $\alpha > 1$ or $\alpha < 0$; $\phi_3(x) = x \ln x$ with $x \geq 1$; $\phi_4(x) = x\sqrt{\ln x}$ with $x \geq 2$; $\phi_5(x) = 2 \ln x$ with $x \geq 1$; and $\phi_6(x) = x^\alpha$ with $x \geq 1$ and $0 < \alpha < 1$. It can be easily verified that for each $i \in \{1, 2, 3, 4\}$, ϕ_i is strictly convex, and for each $j \in \{5, 6\}$, ϕ_j is strictly concave. Thus, the required conclusion follows from Theorem 2.

We observe that the function $\psi(x) = (n - 1 - x)x^2$ is strictly convex for $x < (n - 1)/3$. Thus, we have the next corollary regarding the Lanzhou index. $\square$

Corollary 5. *If G is a graph attaining the minimum Lanzhou index in the class of all n-vertex chemical cactus graphs having r cycles and $v \in V(G)$, then the minimum degree of G is 2 and*

$$d_G(v) \leq \begin{cases} 4, & \text{if } v \text{ lies on some cycle,} \\ 3, & \text{otherwise,} \end{cases} \tag{16}$$

where n and r are the fixed integers satisfying the inequalities $n \geq 2r + 1$, $n \geq 6$, and $r \geq 2$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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