Hindawi
Journal of Mathematics
Volume 2022, Article ID 8339837, 10 pages
https://doi.org/10.1155/2022/8339837

Research Article

A Class of Symmetric Fractional Differential Operator Formed by Special Functions

Ibtisam Aldawish,[1] **Rabha W. Ibrahim**,[2] **and Suzan J. Obaiys**[3]

[1]*Department of Mathematics and Statistics, College of Science, IMSIU (Imam Mohammad Ibn Saud Islamic University), Riyadh, Saudi Arabia*
[2]*The Institute of Electrical and Electronics Engineers(IEEE): 94086547, Portland 13765, NY, USA*
[3]*Department of Computer Systems and Technology, Faculty of Computer Science and Information Technology, University Malaya, Kuala Lumpur 50603, Malaysia*

Correspondence should be addressed to Rabha W. Ibrahim; rabhaibrahim@yahoo.com

Received 16 May 2022; Accepted 4 July 2022; Published 2 August 2022

Academic Editor: Arzu Akbulut

In light of a certain sort of fractional calculus, a generalized symmetric fractional differential operator based on Raina's function is built. The generalized operator is then used to create a formula for analytic functions of type normalized. We use the ideas of subordination and superordination to show a collection of inequalities using the suggested differential operator. The new Raina's operator is also used to the generalized kinematic solutions (GKS). Using the concepts of subordination and superordination, we provide analytic solutions for GKS. As a consequence, a certain hypergeometric function provides the answer. A fractional coefficient differential operator is also created. The geometric and analytic properties of the object are being addressed. The symmetric differential operator in a complex domain is shown to be a generalized fractional differential operator. Finally, we explore the characteristics of the Raina's symmetric differential operator.

1. Introduction

Symmetry is both an abstract basis of attractiveness and an applied tool for resolving convoluted problems. As a consequence, symmetry is a well-known foundation in numerous fields of physics. Despite a well-developed abstract theory of analytic symmetry, symmetry in real-world complex networks has established little attention [1]. Many scientists in many domains of mathematical sciences have been interested in learning more about the theory of symmetric operators. A special class of symmetric operators is defined by using some special functions, which are satisfying the symmetric behavior. The Mittag–Leffler function and its extensions, including Raina's functions, are solutions for all categories of fractional differential equations (see [2–8]).

We examine how Raina's function may utilize to expand a symmetric fractional differential operator in a complex domain in this research. A range of new normalized analytic functions are explained using the fractional symmetric operator. The idea of differential subordination and superordination is applied to study a collection of differential inequalities. The geometric behavior of the generalized kinematic solution (GKS), a family of analytic solutions, is also studied. A variety of applications employ the new convolution linear operator.

2. Methods and Techniques

We will go through the strategies we used in this part.

2.1. Geometric Concepts. We start by the following definition [9]:

Concept 2.1. The analytic functions ψ_1, ψ_2 in $\mathbb{U}: =\{\xi \in \mathbb{C}: |\xi| < 1\}$ are subordinated $\psi_1 \prec \psi_2$ or

$$\psi_1(\xi) \prec \psi_2(\xi), \quad \xi \in \mathbb{U}. \tag{1}$$

If for an analytic function $\upsilon, |\upsilon| \le |\xi| < 1$ owning

$$\psi_1(\xi) = \psi_2(\upsilon(\xi)), \quad \xi \in \mathbb{U}. \tag{2}$$

Concept 2.2. Consider the subclass of analytic functions Λ by

$$\psi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n, \quad \eta \in \mathbb{U}, \tag{3}$$

satisfying $\psi(0) = 0, \psi'(0) = 1$.

Furthermore, the functions $\psi_1, \psi_2 \in \Lambda$ are called convoluted $(\psi_1 * \psi_2)$ if they admin the operation [10]

$$\begin{aligned}(\psi_1 * \psi_2)(\xi) &= \left(\xi + \sum_{n=2}^{\infty} \phi_n \xi^n\right) * \left(\xi + \sum_{n=2}^{\infty} \varphi_n \xi^n\right) \\ &= \xi + \sum_{n=2}^{\infty} \phi_n \varphi_n \xi^n.\end{aligned} \tag{4}$$

Concept 2.3. The $\mathcal{S}^*$ class of star-like functions and the $\mathcal{C}$ class of convex univalent functions are both related to the class of normalized analytic functions (Λ). In addition, we require the class of analytic functions

$$\mathcal{P} \coloneqq \left\{\varrho\colon \varrho(\xi) = 1 + \varrho_1 \xi + \varrho_2 \xi^2 + \ldots, \xi \in \mathbb{U}\right\}. \tag{5}$$

2.2. Modified Special Function. Special functions include integrals and the outputs of many different types of differential equations. Therefore, most integral sets include special duty descriptions, and these duties include the elementary integrals. Since symmetries are important in real life, the philosophy of special functions is tightly linked to various mathematical physics topics [11]. We will start with a well-known special function, the Mittag–Leffler function.

Concept 2.4. The extended Mittag–Leffler function is formulated by the series [12]

$$\mathcal{T}^{\upsilon}_{\alpha,\beta}(\xi) = \sum_{n=0}^{\infty} \left(\frac{(\upsilon)_n}{\Gamma(\alpha n + \beta)}\right) \frac{\xi^n}{n!}, \tag{6}$$

such that $(\upsilon)_n = \Gamma(\upsilon + n)/\Gamma(\upsilon)$. Clearly, for $\upsilon = 1$, implies that

$$\mathcal{T}^{1}_{\alpha,\beta}(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(\beta + \alpha n)}. \tag{7}$$

After that, we will go through Raina's function.

Concept 2.5. Raina's function is determined by the power series as follows [12]:

$$\mathcal{F}^{\upsilon}_{\alpha,\beta}(\xi) = \sum_{n=0}^{\infty} \frac{\upsilon(n)}{\Gamma(\alpha n + \beta)} \xi^n, \quad \xi \in \mathbb{U}, \tag{8}$$

such that $\alpha \in (0, \infty), \beta \in [1, \infty)$ and $\upsilon\colon = \{\upsilon(0), \upsilon(1), \ldots, \upsilon(n)\}$ is a collection of real or complex numbers.

Notice 2.6. We have the following well-known special cases:

(i) $\upsilon(n) = 1 \Longrightarrow \mathcal{T}^{1}_{\alpha,\beta}(\xi)$
(ii) $\upsilon(n) = ((\upsilon)_n/n!) \Longrightarrow \mathcal{T}^{\upsilon}_{\alpha,\beta}(\xi)$
(iii) $\alpha = 1, \beta = 1, \upsilon(n) = ((x)_n (y)_n/(s)_n) \Longrightarrow {}_2\Theta_1(x, y; s; \xi) = \sum_{n=0}^{\infty} ((x)_n (y)_n/(s)_n)(\xi^n/\Gamma(n+1))$

Employing the functional $\mathcal{F}^{\upsilon}_{\alpha,\beta}(\xi)$, we get the convolution operator, for $\psi \in \Lambda$

$$\begin{aligned}\mathbb{I}^{\upsilon}_{\alpha,\beta}\psi(\xi) &= \left(\frac{\Gamma(\alpha + \beta)}{\upsilon(1)}\right)\left(\mathcal{F}^{\upsilon}_{\alpha,\beta} * \psi\right)(\xi) \\ &= \left(\left(\frac{\Gamma(\alpha + t\beta)}{\upsilon(1)}\right)\upsilon(0) + \xi \right.\\ &\quad \left. + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha + \beta)}{\upsilon(1)}\right)\left(\frac{\upsilon(n)}{\Gamma(\alpha n + \beta)}\right)\xi^n\right) * \left(\xi + \sum_{n=2}^{\infty} a_n \xi^n\right) \\ &= \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha + \beta)}{\Gamma(\beta + \alpha n)}\right)\left(\frac{\upsilon(n)}{\upsilon(1)}\right) a_n \xi^n \\ &\coloneqq \xi + \sum_{n=2}^{\infty} \varsigma_n a_n \xi^n,\end{aligned} \tag{9}$$

such that

$$\varsigma_n \coloneqq \left(\frac{\Gamma(\alpha + \beta)}{\Gamma(\beta + \alpha n)}\right)\left(\frac{\upsilon(n)}{\upsilon(1)}\right)$$
$$(\xi \in \mathbb{U}, \psi \in \Lambda, \alpha \in (0, \infty), \beta \in [1, \infty), \upsilon = \{\upsilon(0), \ldots, \upsilon(n)\}). \tag{10}$$

Clearly, $\mathbb{I}^{\upsilon}_{\alpha,\beta}\psi(\xi) \in \Lambda$. From the above structure, the fractional differential operator can be viewed geometrically.

Note that the operator $\mathbb{I}^{\upsilon}_{\alpha,\beta}\psi(\xi)$ is a new type of the convoluted Carlson–Shaffer operator [13] satisfying $\alpha = \beta = 1$, and $\upsilon(n) = ((1)_n (y)_n/(s)_n), \forall n$, with

$$\mathbb{I}^{\upsilon}_{\alpha,\beta}\psi(\xi) = \sum_{n=0}^{\infty}\left(\frac{(1)_n (y)_n}{\Gamma(n+1)(s)_n}\right) a_n \xi^n. \tag{11}$$

Moreover, when $\upsilon(n) = \Gamma(\beta + \alpha n)$ for all $n \ge 1$, we have the Sàlàgean operator [14]:

$$\mathbb{I}^{\upsilon}_{\alpha,\beta}\psi(\xi) = \xi + \sum_{n=2}^{\infty} n a_n \xi^n. \tag{12}$$

2.3. Arguments. The following precursors are utilized to develop the results of this inquiry into differential subordination theory:

Argument 2.7 (see [9]). Suppose that $f_1(\xi)$ and $f_2(\xi)$ are convex univalent in $\mathbb{U}$ with $f_1(0) = f_2(0)$. Then, for a fixed value $\eta \neq 0, \Re(\eta) \geq 0$, the subordination

$$f_1(\xi) + \left(\frac{1}{\eta}\right)\xi f_1'(\xi) \prec f_2(\xi), \tag{13}$$

gives

$$f_1(\xi) \prec f_2(\xi). \tag{14}$$

Argument 2.8 (see [9]). Consider the class of holomorphic functions as follows:

$$\Pi[b,n] = \left\{\varphi\colon \varphi(\xi) = b + b_n\xi^n + b_{n+1}\xi^{n+1} + \cdots\right\}, \tag{15}$$

where $b \in \mathbb{C}$ and $n \in \mathbb{Z}^+$. The condition $\imath \in \mathbb{R}$ implies

$$\Re\{\varphi(\xi) + \imath\xi\varphi'(\xi)\} > 0 \Rightarrow \Re(\varphi(\xi)) > 0. \tag{16}$$

In addition, if $\imath > 0$ and $\varphi \in \Pi[1,n]$, then for $\eta_1, \eta_2 \in (0,\infty)$ satisfying

$$\varphi(\xi) + \imath\xi\varphi'(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{\eta_1} \Rightarrow \varphi(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{\eta_2}. \tag{17}$$

Argument 2.9 (see [15]). Let $\hbar, \varphi \in \Pi[b,n]$, where $\varphi \in \mathscr{C}$ and for $w_1, w_2 \in \mathbb{C}, w_2 \neq 0$. Then, the subordination

$$w_1\hbar(\xi) + w_2\xi\hbar'(\xi) \prec w_1\varphi(\xi) + w_2\xi\varphi'(\xi), \tag{18}$$

yields

$$\hbar(\xi) \prec \varphi(\xi). \tag{19}$$

Argument 2.10 (see [16]). Let $\varphi, \phi \in \Pi[b,n]$, where $\phi \in \mathscr{C}$ and the functional $\varphi(\xi) + v\xi\varphi'(\xi)$ is univalent for some positive fixed number v. Then the differential inequality

$$\phi(\xi) + v\xi\phi'(\xi) \prec \varphi(\xi) + v\xi\varphi'(\xi), \tag{20}$$

implies

$$\phi(\xi) \prec \varphi(\xi). \tag{21}$$

3. Consequences

The next class of normalized analytic functions is defined in this paper, and its features are investigated employing differential subordination and superordination theory.

Concept 3.1. A function $\psi \in \Lambda$ aims to be in the class $\Omega^{\mu}_{\alpha,\beta}(\lambda,\rho)$ if it fulfills the inequality

$$\left(\frac{1-\lambda}{\xi}\right)\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] + \lambda\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]' \prec \rho(\xi),$$

$$(\xi \in \mathbb{U}, \lambda \in (0,1], \rho(0) = 1, \alpha \in (0,\infty), \beta \in [1,\infty)), \tag{22}$$

whenever $\rho \in \mathscr{C}$.

Eventually, the convexity of the univalent function

$$\rho(\xi) = \frac{A\xi + 1}{B\xi + 1}, \tag{23}$$

implies that

$$\rho \in \mathscr{P} \coloneqq \left\{\rho \in \mathbb{U}\colon \rho(\xi) = 1 + \sum_{n=1}^{\infty} \rho_i \xi^n\right\}. \tag{24}$$

Consider the functional $\Sigma^{\lambda}_{\psi}\colon \mathbb{U} \longrightarrow \mathbb{U}$, as in the following structure:

$$\Sigma^{\lambda}_{\psi}(\xi) \coloneqq \left(\frac{1-\lambda}{\xi}\right)\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] + \lambda\left[\mathbb{I}^{\mu}_{\alpha,\beta}\psi(\xi)\right]'. \tag{25}$$

Consequently, in view of Concept 3.1, we get the next inequality

$$\Sigma^{\lambda}_{\psi}(\xi) \prec \frac{A\xi + 1}{B\xi + 1}, \quad \xi \in \mathbb{U}. \tag{26}$$

We proceed to investigate the geometric possessions of the suggested operators.

3.1. Results of Subordination Formula. We begin with the following outcome.

Proposition 1. *Assume that $\psi \in \Omega^{\mu}_{\alpha,\beta}(\lambda,\rho)$. If*

$$\begin{aligned}\Re\left\{\Sigma^{\lambda}_{\psi}(\xi)\right\} &= \Re \\ &\left\{\left(\frac{1-\lambda}{\xi}\right)\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] + \lambda\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]'\right\} \\ &\coloneqq \Re\left\{1 + \sum_{n=1}^{\infty} \varsigma_n\right\} > 0,\end{aligned} \tag{27}$$

then the upper bound of the coefficients ς_n is determined by the probability measure $d\omega$:

$$\frac{|\varsigma_n|}{2} \leq \int_0^{2\pi} \left|e^{-in\tau}\right| d\omega(\tau). \tag{28}$$

In addition, if

$$\Re\left(e^{i\tau}\Sigma^{\lambda}_{\psi}(\xi)\right) > 0, \quad \xi \in \mathbb{U}, \tau \in \mathbb{R}, \tag{29}$$

then $\psi \in \Omega^{\mu}_{\alpha,\beta}(A\xi + 1/B\xi + 1)$ and

$$\Sigma^{\lambda}_{\psi}(\xi) = \frac{A\xi + 1}{B\xi + 1}, \quad \xi \in \mathbb{U}. \tag{30}$$

Proof. Suppose that

$$\Re\left(\Sigma^{\lambda}_{\psi}(\xi)\right) = \Re\left(1 + \sum_{n=1}^{\infty} \varsigma_n \xi^n\right) > 0. \tag{31}$$

Continuously, the Carathéodory positivist lemma entails

$$\left|\varsigma_n\right| \le 2\int_0^{2\pi}\left|e^{-in\tau}\right| d\omega(\tau), \tag{32}$$

where $d\omega$ is a probability measure. Besides, if

$$\Re\left(e^{i\tau}\Sigma_{\psi}^{\lambda}(\xi)\right) > 0, \quad \xi \in \mathbb{U}, \tau \in \mathbb{R}, \tag{33}$$

then according to Theorem 1.6 in [10] and for a fixed number $\tau \in \mathbb{R}$, we have

$$\Sigma_{\psi}^{\lambda}(\xi) = \frac{A\xi + 1}{B\xi + 1}, \quad \xi \in \mathbb{U}. \tag{34}$$

Hence, $\psi \in \Omega_{\alpha,\beta}^{\mu}(\lambda, (A\xi + 1/B\xi + 1))$.

The following findings reveal the functional sandwich theory's required and adequate methodology. □

Proposition 2. *Suppose that*

$$\lambda\xi\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'' + \left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]' \prec F_2(\xi) + \xi F_2'(\xi), \tag{35}$$

where $F_2(0) = 1$ and convex in $\mathbb{U}$. Moreover, let $\Sigma_{\psi}^{\lambda}(\xi)$ be univalent in $\mathbb{U}$ with $\Sigma_{\psi} \in \Pi[F_1(0), 1] \cap \mathbb{D}$, where $\mathbb{D}$ indicates the class of all univalent analytic functions F having the limit $\lim_{\xi\in\partial\mathbb{D}} F \neq \infty$ and

$$F_1(\xi) + \xi F_1'(\xi) \prec \lambda\xi\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'' + \left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'. \tag{36}$$

Then

$$F_1(\xi) \prec \Sigma_{\psi}^{\lambda}(\xi) \prec F_2(\xi), \tag{37}$$

and $F_1(\xi)$ is the best subdominant and $F_2(\xi)$ is the best dominant.

Proof. Putting

$$\Sigma_{\psi}^{\lambda}(\xi) = \left(\frac{1-\lambda}{\xi}\right)\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right] + \lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]', \tag{38}$$

a calculation yields

$$\begin{aligned}\Sigma_{\psi}^{\lambda}(\xi) + \xi\left(\Sigma_{\psi}^{\lambda}(\xi)\right)' &= \lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]' \\ &\quad + \frac{\xi\left(\lambda\xi\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'' - (\lambda-1)\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'\right) + (\lambda-1)\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]}{\xi} \\ &\quad + \frac{(1-\lambda)\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]}{\xi} \\ &= \lambda\xi\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'' + \left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'.\end{aligned} \tag{39}$$

As a consequence, the double inequality produced is as follows:

$$F_1(\xi) + \xi F_1'(\xi) \prec \Sigma_{\psi}^{\lambda}(\xi) + \xi\left(\Sigma_{\psi}^{\lambda}(\xi)\right)' \prec F_2(\xi) + \xi F_2'(\xi). \tag{40}$$

Finally, Arguments 2.9 and 2.10 provide the required outcome. □ □

Proposition 3. *Assume that*

$$\Sigma_{\psi}^{\lambda}(\xi) = \frac{(1-\lambda)}{\xi}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right] + \lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'. \tag{41}$$

Then this leads to

$$\begin{aligned}&\left(\frac{\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'}{\xi}\right)\varepsilon_1 + \left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]\left[\varepsilon_1 + 3\varepsilon_2\right] + \varepsilon_2\xi\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'' \prec \left(\frac{1+\xi}{1-\xi}\right)^{\gamma_1} \\ &\Rightarrow \Sigma_{\psi}^{\lambda}(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{\gamma_2}, \\ &\left(\gamma_1 > 0, \gamma_2 > 0, \varepsilon_1 = 1-\lambda, \varepsilon_2 = \lambda > 0\right).\end{aligned} \tag{42}$$

Proof. A calculation gives that

$$\begin{aligned}\Sigma_{\psi}^{\lambda}(\xi)+\xi\left(\Sigma_{\psi}^{\lambda}(\xi)\right)' &= \frac{(1-\lambda)}{\xi}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]+\lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]' \\ &\quad+\xi\left(\frac{(1-\lambda)}{\xi}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]+\lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'\right)' \\ &= \left(\frac{\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'}{\xi}\right)\varepsilon_1+\left[\varepsilon_1+3\varepsilon_2\right]\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]+\varepsilon_2\xi\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'' \\ &\prec\left(\frac{1+\xi}{1-\xi}\right)^{\gamma_1}.\end{aligned} \tag{43}$$

In view of Argument 2.8, we obtain

$$\Sigma_{\psi}^{\lambda}(\xi)\prec\left(\frac{1+\xi}{1-\xi}\right)^{\gamma_2}. \tag{44}$$

□

3.2. Fractional Differential Equation with Kinematic Solutions. We will use the generalized differential operator to continue our research in this section. A generalized formula for the kinematic solutions (GKS) is presented using the suggested operator. Kinematic behaviors describe the motion of an item with constant acceleration in a dynamic system.

We aim to utilize the class $\Omega_{\alpha,\beta}^{\mu}(\lambda,(1+\xi/1-\xi))$ to extend GKSs. We deal with the upper bound solution, for the fractional differential equation

$$\begin{aligned}&\left(\frac{1-\lambda}{\xi}\right)\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]+\lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'=\frac{A\xi+1}{B\xi+1}, \\ &\left(\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(0)\right]=0,\varsigma\in[0,1],\xi\in\mathbb{U}\right).\end{aligned} \tag{45}$$

The outcome of (45) is formulated as follows.

Proposition 4. *Let $\xi\in\Omega_{\alpha,\beta}^{\upsilon}(\varsigma,(1+\xi/1-\xi))$. Then (45) has a solution expressed by*

$$\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]=c\xi^{(\lambda-1)/\lambda}+\frac{2\xi^2{}_2\Theta_1(1,1+1/\lambda;2+1/\lambda;\xi)}{\lambda+1}+\frac{\lambda\xi}{\lambda+1}+\frac{\xi}{\lambda+1}, \tag{46}$$

where c is a constant and ${}_2\Theta_1(x,y,s;\xi)$ is the hypergeometric function.

Proof. Let $\xi\in\Omega_{\alpha,\beta}^{\upsilon}(\varsigma,(1+\xi/1-\xi))$. Thus, we obtain

$$\left(\frac{1-\lambda}{\xi}\right)\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]+\lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'=\frac{\varphi(\xi)+1}{1-\varphi(\xi)}, \tag{47}$$

where $|\chi|\le|\xi|<1$ and $\chi(0)=0$. As a result, we get the integral formula

$$\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]=\xi^{(\lambda-1)/\lambda}\int_0^{\xi}-\eta^{1/(\lambda-1)}\left(\frac{\chi(\eta)+1}{\lambda(\chi(\eta)-1)}\right)d\eta. \tag{48}$$

In view of Schwarz lemma, we get $\chi(\xi)=w\xi,|w|=1$ (see Theorem 5.34 in [17]). Therefore, by assuming $\chi(\xi)=\xi$, we obtain the differential equation

$$\frac{(1-\lambda)}{\xi}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]+\lambda\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'=\frac{1+\xi}{1-\xi}. \tag{49}$$

If we reorganize the previous equation, we conclude that

$$\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'+\frac{1-\lambda}{\lambda\xi}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]=\left(\frac{1}{\lambda}\right)\left(\frac{1+\xi}{1-\xi}\right). \tag{50}$$

Then multiplying by the functional

$$\mathbb{T}(\xi)=\exp\left(\int\frac{1-\lambda}{\lambda\xi}\mathrm{d}\xi\right)=\xi^{1/(\lambda-1)}, \tag{51}$$

we obtain

$$\begin{aligned}&\xi^{1/(\lambda-1)}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]'-\frac{\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]\left((1-\lambda)\xi^{1/(\lambda-2)}\right)}{\lambda} \\ &\quad=\left(\frac{\xi^{1/(\lambda-1)}}{\lambda}\right)\left(\frac{1+\xi}{1-\xi}\right).\end{aligned} \tag{52}$$

As a result, we receive the solution

$$\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]=c\xi^{(\lambda-1)/\lambda}+\frac{2\xi^2{}_2\Theta_1(1,1+1/\lambda;2+1/\lambda;\xi)}{\lambda+1}+\frac{\lambda\xi}{\lambda+1}+\frac{\xi}{\lambda+1}. \tag{53}$$

□

Example 1. Let $\psi\in\Omega_{\alpha,\beta}^{\upsilon}(\lambda,(1+\xi/1-\xi))$, where $\lambda=1/2$ and $c=0$. According to Proposition 5, we have

$$\begin{aligned}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]&=\xi\left(\frac{2\xi_2\Theta_1(1,1+(1/\lambda),2+(1/\lambda),\xi)}{\lambda+1}+1\right), \\ c=0&=\xi\left(\frac{4\xi_2\Theta_1(1,3,4,\xi)}{3}+1\right), \\ \lambda=\frac{1}{2}&=\xi+\xi^2+\xi^3+\xi^4+\cdots+O\left(\xi^7\right),\quad|\xi|<1.\end{aligned} \tag{54}$$

Let $\psi(\xi) = (\xi/1-\xi)$. Then

$$\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta+\alpha n)}\right)\left(\frac{v(n)}{v(1)}\right)\xi^{n}. \tag{55}$$

Comparing the right sides of the above equations, we obtain that $v(n) = \Gamma(\beta+\alpha n), \forall n$. But $\psi(\xi) = (\xi/1-\xi)$ is the optimal convex function in the open unit disk; thus, the operator $[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)]$ is convex whenever ψ is convex (see Figure 1).

3.3. Symmetric Differential Operator. The Raina's convoluted operator is assumed to present an extended symmetric differential operator.

$$\begin{aligned}
\mathscr{M}^{0}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] &= \left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]\\
\mathscr{M}^{1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] &= \ell\xi\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]' - (1-\ell)\xi\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(-\xi)\right]'\\
&= \ell\left(\xi + \sum_{n=2}^{\infty} n a_n \varsigma_n \xi^{n}\right) - (1-\ell)\left(-\xi + \sum_{n=2}^{\infty} n(-1)^{n} a_n \varsigma_n \xi^{n}\right)\\
&= \xi + \sum_{n=2}^{\infty}\left[n\left(\ell - (1-\ell)(-1)^{n}\right)\right] a_n \varsigma_n \xi^{n}\\
\mathscr{M}^{2}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] &= \mathscr{M}^{1}_{\ell}\left[\mathscr{M}^{1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]\right]\\
&= \xi + \sum_{n=2}^{\infty}\left[n\left(\ell - (1-\ell)(-1)^{n}\right)\right]^{2} a_n \varsigma_n \xi^{n}\\
&\vdots\\
\mathscr{M}^{k}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] &= \mathscr{M}^{1}_{\ell}\left[\mathscr{M}^{k-1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]\right]\\
&= \xi + \sum_{n=2}^{\infty}\left[n\left(\ell - (1-\ell)(-1)^{n}\right)\right]^{k} \varsigma_n a_n \xi^{n}.
\end{aligned} \tag{56}$$

When $\varsigma_n = 1, \forall n$, we have the symmetric operator in [18]. Moreover, when $\varsigma_n = 1$ and $\ell = 1$, we receive the Sàlàgean integral operator [14].

The following classes will be studied:

Concept. Let $\psi \in \Lambda$. Then, we define the subclass of star-like functions:

(i) $\psi \in S^{*v,k}_{\alpha,\beta,\ell}(\hbar)$ if and only if there occurs a convex function $\hbar \in \mathscr{C}$ satisfying the subordination

$$\frac{\xi\left(\mathscr{M}^{k}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]\right)'}{\mathscr{M}^{k}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]} \prec \hbar(\xi). \tag{57}$$

(ii) $\psi \in \mathbb{J}^{\natural}_{\ell}(A,B,k)$ if and only if

$$1 + \frac{1}{\natural}\left(\frac{2\mathscr{M}^{k+1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]}{\mathscr{M}^{k}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] - \mathscr{M}^{k}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(-\xi)\right]}\right) \prec \frac{1+A\xi}{1+B\xi},$$
$$(\xi \in \mathbb{U}, -1 \le B < A \le 1, k = 1,2,\ldots, \natural \in \mathbb{C}\setminus\{0\}, \ell \in [0,1]). \tag{58}$$

Proposition 5. *Consider $\psi \in S^{*v,k}_{\alpha,\beta,\ell}(\hbar)$. Then*

$$\mathscr{M}^{k+1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] \prec \xi e^{\left(\int_{0}^{\xi}(\hbar(y(z)) - 1/z)dz\right)}, \tag{59}$$

where $y(\xi)$ is analytic in $\mathbb{U}$ with $y(0) = 0$ and $|y(\xi)| < 1$. Additionally, for $|\xi| = \aleph$, $\mathscr{M}^{k+1}_{\ell}[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)]$ satisfies the inequality

$$\exp\left(\int_{0}^{1}\frac{\hbar(y(-\aleph)) - 1}{\aleph}\right)d\aleph \le \left|\frac{\mathscr{M}^{k+1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]}{\xi}\right| \le \exp\left(\int_{0}^{1}\frac{\hbar(y(\aleph)) - 1}{\aleph}\right)d\aleph. \tag{60}$$

Proof. Because $\psi \in S^{*\mu,k}_{\alpha,\beta,\ell}(\hbar)$, then we conclude that

$$\left(\frac{\xi\left(\mathscr{M}^{k+1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]\right)'}{\mathscr{M}^{k+1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]}\right) \prec \hbar(\xi), \quad \xi \in \mathbb{U}. \tag{61}$$

This leads to the existence of a Schwarz function with $y(0) = 0$ and $|y(\xi)| < 1$ such that

$$\frac{\xi\left(\mathscr{M}^{k+1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]\right)'}{\mathscr{M}^{k+1}_{\ell}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]} = \hbar(y(\xi)), \quad \xi \in \mathbb{U}, \tag{62}$$

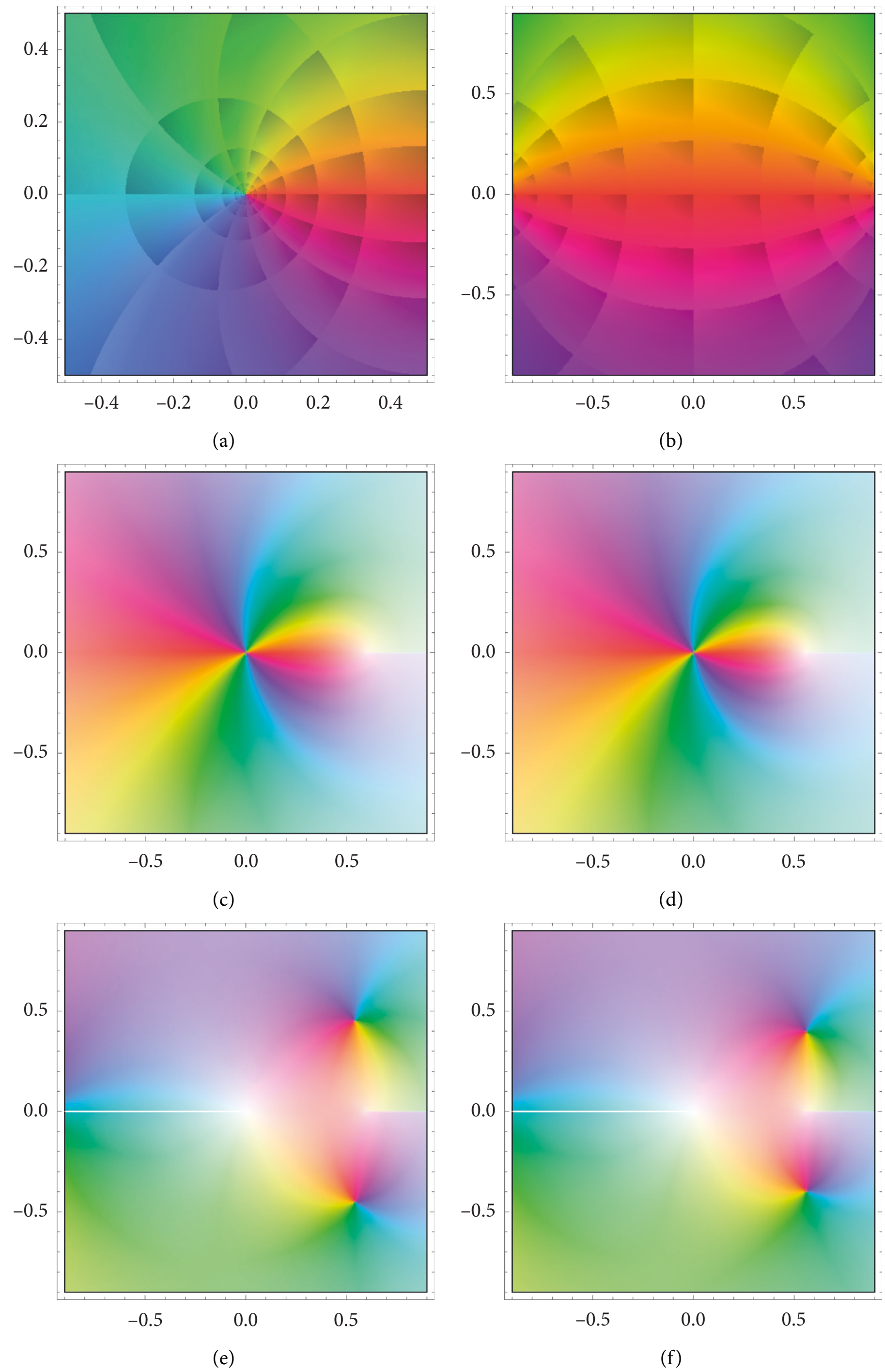

Figure 1: Plots of GKS of equation (45). (a) $\psi(\xi) = \xi/(1-\xi)$. (b) $(1+\xi)/(1-\xi)$. (c) $[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)]$, when $\lambda = 0.5, c = 0$. (d) $[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)]$, when $\lambda = 0.25, c = 0$. (e) $[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)]$, when $\lambda = 0.5, c = 1$. (f) $[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)]$, when $\lambda = 0.25, c = 1$.

which implies that

$$\frac{\left(\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]\right)'}{\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]} - \frac{1}{\xi} = \frac{\hbar(y(\xi)) - 1}{\xi}. \tag{63}$$

Integration implies that

$$\log \mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right] - \log \xi = \int_{0}^{\xi} \frac{\hbar(y(z)) - 1}{z} dz. \tag{64}$$

A computation brings

$$\log\left(\frac{\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}^{v}_{\alpha,\beta}\psi(\xi)\right]}{\xi}\right) = \int_{0}^{\xi} \frac{\hbar(y(z)) - 1}{z} dz. \tag{65}$$

The subordination yields

$$\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right] \prec \xi \exp\left(\int_0^{\xi}\frac{\hbar(y(z))-1}{z}dz\right). \tag{66}$$

Moreover, the disk is mapped by $\hbar(\xi)$. When we apply $0<|\xi|<\aleph<1$ to an area that is convex and symmetric with respect to the real axis, we get

$$\hbar(-\aleph|\xi|) \leq \Re(\hbar(y(\aleph\xi))) \leq \hbar(\aleph|\xi|), \quad \aleph \in (0,1), \tag{67}$$

which brings

$$\begin{gathered}\hbar(-\aleph) \leq \hbar(-\aleph|\xi|), \quad \hbar(\aleph|\xi|) \leq \hbar(\aleph),\\ \int_0^1\frac{\hbar(y(-\aleph|\xi|))-1}{\aleph}d\aleph \leq \Re\left(\int_0^1\frac{\hbar(y(\aleph))-1}{\aleph}d\aleph\right) \leq \int_0^1\frac{\hbar(y(\aleph|\xi|))-1}{\aleph}d\aleph.\end{gathered} \tag{68}$$

Employing equation (65), we obtain

$$\int_0^1\frac{\hbar(y(-\aleph|\xi|))-1}{\aleph}d\aleph \leq \log\left|\frac{\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]}{\xi}\right| \leq \int_0^1\frac{\hbar(y(\aleph|\xi|))-1}{\aleph}d\aleph. \tag{69}$$

As a result, we get the inequality

$$\exp\left(\int_0^1\frac{\hbar(y(-\aleph|\xi|))-1}{\aleph}d\aleph\right) \leq \left|\frac{\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]}{\xi}\right| \leq \exp\left(\int_0^1\frac{\hbar(y(\aleph|\xi|))-1}{\aleph}d\aleph\right). \tag{70}$$

Hence, we receive

$$\exp\left(\int_0^1\frac{\hbar(y(-\aleph))-1}{\aleph}\right)d\aleph \leq \left|\frac{\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\psi(\xi)\right]}{\xi}\right| \leq \exp\left(\int_0^1\frac{\hbar(y(\aleph))-1}{\aleph}\right)d\aleph. \tag{71}$$

□

Proposition 6. *Suppose that $\psi \in \mathbb{J}_{\ell}^{\natural}(A,B,k)$ then the odd function*

$$\mathfrak{L}(\xi) = \frac{1}{2}[\psi(\xi)-\psi(-\xi)], \quad \xi \in \mathbb{U}, \tag{72}$$

fulfills the inequality

$$\begin{gathered}1+\frac{1}{\natural}\left(\frac{\mathcal{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\mathfrak{L}(\xi)\right]}{\mathcal{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{\upsilon}\mathfrak{L}(\xi)\right]}-1\right) \prec \frac{1+A\xi}{1+B\xi},\\ \Re\left(\frac{\xi\mathfrak{L}(\xi)'}{\mathfrak{L}(\xi)}\right) \geq \frac{1-\eta^2}{1+\eta^2}, \quad |\xi|=\eta<1,\\ (\xi \in \mathbb{U}, -1 \leq B < A \leq 1, k=1,2,\ldots, \natural \in \mathbb{C}\setminus\{0\}, \ell \in [0,1]).\end{gathered} \tag{73}$$

Proof. By the condition $\psi \in \mathbb{J}_{\ell}^{\natural}(A,B,k)$, we obtain the existence of a function $G \in \mathbb{J}(A,B)$ such that

$$\natural(G(\xi)-1)=\left(\frac{2\mathscr{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{v}\psi(\xi)\right]}{\mathscr{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{v}\psi(\xi)\right]-\mathscr{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{\mu}\psi(-\xi)\right]}\right),$$
$$\natural(G(-\xi)-1)=\left(\frac{-2\mathscr{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{v}\psi(-\xi)\right]}{\mathscr{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{v}\psi(\xi)\right]-\mathscr{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{v}\psi(-\xi)\right]}\right). \tag{74}$$

This leads to

$$1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}{\mathscr{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}-1\right)=\frac{G(\xi)+G(-\xi)}{2}. \tag{75}$$

Also, because

$$G(\xi)\prec\frac{1+A\xi}{1+B\xi}, \tag{76}$$

where $(1+A\xi/1+B\xi)$ is univalent, then the above subordination yields

$$1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}{\mathscr{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{77}$$

Additionally, the function $\mathfrak{L}(\xi)$ is star-like in $\mathbb{U}$, which gives the inequality

$$\frac{\xi\mathfrak{L}(\xi)'}{\mathfrak{L}(\xi)}\prec\frac{1-\xi^2}{1+\xi^2}. \tag{78}$$

As a consequence, we confirm the existence of Schwarz function $\wp\in\mathbb{U}, |\wp(\xi)|\le|\xi|<1, \wp(0)=0$ such that

$$\Upsilon(\xi)\coloneqq\frac{\xi\mathfrak{L}(\xi)'}{\mathfrak{L}(\xi)}\prec\frac{1-\wp(\xi)^2}{1+\wp(\xi)^2}, \tag{79}$$

which yields that there is $\xi, |\xi|=\eta<1$ such that

$$\wp^2(\xi)=\frac{1-\Upsilon(\xi)}{1+\Upsilon(\xi)},\quad \xi\in\mathbb{U}. \tag{80}$$

By rearranging the above inequality, we receive

$$\left|\frac{1-\Upsilon(\xi)}{1+\Upsilon(\xi)}\right|=|\wp(\xi)|^2\le|\xi|^2. \tag{81}$$

Hence, we have the following conclusion:

$$\left|\Upsilon(\xi)-\frac{1+|\xi|^4}{1-|\xi|^4}\right|^2\le\frac{4|\xi|^4}{\left(1-|\xi|^4\right)^2}, \tag{82}$$

or

$$\left|\Upsilon(\xi)-\frac{1+|\xi|^4}{1-|\xi|^4}\right|\le\frac{2|\xi|^2}{\left(1-|\xi|^4\right)}. \tag{83}$$

This yields

$$\mathfrak{R}(\Upsilon(\xi))\ge\frac{1-\eta^2}{1+\eta^2},\quad |\xi|=\eta<1. \tag{84}$$

As a result, we obtain the next outcomes. □

Corollary 1 (see [18]). *Let $\mu(n)=\Gamma(\alpha n+\beta)$ in Proposition 6. Then*

$$1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{\ell}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}{\mathscr{M}_{\ell}^{k}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}-1\right)=1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{\ell}^{k+1}[\mathfrak{L}(\xi)]}{\mathscr{M}_{\ell}^{k}[\mathfrak{L}(\xi)]}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{85}$$

Corollary 2 (see [19]). *Let $\ell=1$ and $v(n)=\Gamma(\alpha n+\beta)$ in Theorem 3.9. Then*

$$1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{1}^{k+1}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}{\mathscr{M}_{1}^{k}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}-1\right)=1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{1}^{k+1}[\mathfrak{L}(\xi)]}{\mathscr{M}_{1}^{k}[\mathfrak{L}(\xi)]}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{86}$$

Corollary 3 (see [20]). *Let $\ell=1, k=1$ and $v(n)=\Gamma(\alpha n+\beta)$ in Theorem 3.9. Then*

$$1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{1}^{2}\left[\mathbb{I}_{\alpha,\beta}^{\mu}\mathfrak{L}(\xi)\right]}{\mathscr{M}_{1}^{1}\left[\mathbb{I}_{\alpha,\beta}^{\mu}\mathfrak{L}(\xi)\right]}-1\right)=1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{1}^{2}[\mathfrak{L}(\xi)]}{\mathscr{M}_{1}^{1}[\mathfrak{L}(\xi)]}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{87}$$

Corollary 4. *Let $k=0, \ell=1$ and $v(n)=\Gamma(\alpha n+\beta)$ in Theorem 3.9. Then*

$$1+\frac{1}{\natural}\left(\frac{\mathscr{M}_{1}^{1}\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}{\left[\mathbb{I}_{\alpha,\beta}^{v}\mathfrak{L}(\xi)\right]}-1\right)\prec\frac{1+A\xi}{1+B\xi}. \tag{88}$$

4. Conclusion

The preceding study used symmetric derivative and Jackson's calculus to generalize Raina's transformations in $\mathbb{U}$. We used the suggested linear convolution operator on the normalized subclass. The operator is utilized to analyze the outcome of a specific form of GKS, which is utilized as an application. The hypergeometric function was used to determine the behavior of solutions. We further stressed that the answer belongs to the normalized analytic functions category.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

References

[1] H. Weyl, "Symmetry," in *Symmetry*Princeton University Press, Princeton, NY, USA, 2015.
[2] H. J. Seybold and R. Hilfer, "Numerical results for the generalized Mittag-Leffler function," *Fractional Calculus and Applied Analysis*, vol. 8, no. 2, pp. 127–139, 2005.
[3] F. Mainardi, "Why the mittag-leffler function can be considered the queen function of the fractional calculus?" *Entropy*, vol. 22, no. 12, p. 1359, 2020.
[4] R. Gorenflo and F. Mainardi, "The Mittag-Leffler function in the thinning theory for renewal processes," 2018, https://arXiv.org/abs/1808.06563.
[5] R. W. Ibrahim, "Normalized symmetric differential operators in the open unit disk," in *Approximation and Computation in Science and Engineering*, pp. 417–434, Springer, Berlin, Germany, 2022.
[6] R. W. Ibrahim, "Analytic solvability of a class of symmetric nonlinear second order differential equations," *Montes Taurus Journal of Pure and Applied Mathematics*, vol. 4, no. 3, pp. 80–92, 2022.
[7] R. W. Ibrahim and D. Baleanu, "Convoluted fractional differentials of various forms utilizing the generalized Raina's function description with applications," *Journal of Taibah University for Science*, vol. 16, no. 1, pp. 432–441, 2022.
[8] A. Tassaddiq and A. Alruban, "On modifications of the gamma function by using Mittag-Leffler function," *Journal of Mathematics*, vol. 2021, p. 1, 2, 2021.
[9] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, Baco Raton, FL, USA, 2000.
[10] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Les Presses De L'Universite De Montreal, 1982.
[11] A. Gil, J. Segura, and M. Nico, "Numerical methods for special functions," *Society for Industrial and Applied Mathematics*, 2007.
[12] R. K. Raina, "On generalized wright's hypergeometric functions and fractional calculus operators," *East Asian Mathematical Journal*, vol. 21, pp. 191–203, 2015.
[13] B. C. Carlson and D. B. Shaffer, "Starlike and prestarlike hypergeometric functions," *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.
[14] G. S. Sàlàgean, "Subclasses of univalent functions, complex analysis-fifth Romanian-Finnish seminar, part 1 (Bucharest, 1981)," *Lecture Notes in Math*, vol. 1013, pp. 362–372, Springer, Berlin, Germany, 1983.
[15] T. N. Shanmugam, S. Sivasubramanian, and H. M. Srivastava, "Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations," *Integral Transforms and Special Functions*, vol. 17, no. 12, pp. 889–899, 2006.
[16] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables, Theory and Application: An International Journal*, vol. 48, no. 10, pp. 815–826, 2003.
[17] R. E. Rodriguez, I. Kra, and J. P. Gilman, *Complex Analysis: In the Spirit of Lipman Bers*, Springer, Berlin, Germany, 2013.
[18] R. W. Ibrahim and M. Darus, "New symmetric differential and integral operators defined in the complex domain," *Symmetry*, vol. 11, no. 7, p. 906, 2019.
[19] M. Arif, K. Ahmad, J. L. Liu, and J. Sokol, "A new class of analytic functions associated with Salagean operator," *Journal of Function Spaces*, pp. 1–8, 2019.
[20] R. N. Das and P. Singh, "On subclasses of Schlicht mapping," *Indian Journal of Pure and Applied Mathematics*, vol. 8, pp. 864–872, 1977.