Research Article

Some Notes of the Fractional Integral and Derivatives and Its Applications

Fen Qin, Fuyi Xu, and Huizeng Qin

School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong, China

Correspondence should be addressed to Fuyi Xu; zbxufuyi@163.com and Huizeng Qin; qinhuizeng@163.com

Received 16 March 2022; Accepted 13 May 2022; Published 10 November 2022

Academic Editor: Serkan Araci

Copyright © 2022 Fen Qin et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, the definitions of the fractional derivative and integrals are given by the neutrix limit. Not only is it consistent with the classical results but also the representations of the fractional derivative and integrals are obtained for

\[(x-a)^{-k}, (x-a)^{-k}\ln p (x-a), \text{ and } (x-a)^{-k}\ln p (x-a)f(x), \]

where \(k, p \geq 1, 2, \ldots\), \(f(x)\) is the analytic function.

1. Introduction

Fractional calculus has been used to model physics and engineering processes widely since standard mathematical models of integer-order derivatives, including nonlinear models. In fact, in many fields such as mechanics, electronics, chemistry, biology, economics, notably control theory, and signal and image processing, fractional calculus has been playing more and more important roles in recent years [1–5]. There are several definitions for fractional derivatives and integrals.

The Riemann–Liouville fractional integral and derivatives [6] are defined as

\[ RL_a J^\alpha_x f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \text{ Re} \alpha > 0, \] (1)

and

\[ RL_a D^\alpha_x f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \left[ \int_a^x (x-s)^{m-\alpha-1} f(s) ds \right], & m-1 < \text{Re} \alpha < m, \\ \frac{d^m}{dx^m} f(x), & \alpha = m, \end{cases} \] (2)

for \(\beta > -1\) and \(x > a\), respectively. The Grünwald–Letnikov fractional derivative [7] is defined as

\[ GL_a D^\alpha_x f(x) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[x-a/h]} (-1)^j \binom{\alpha}{j} f(x-jh), \] (3)

where \([x]\) is the integer part of \(x\), \(\binom{\alpha}{j} = (\Gamma(\alpha+1)/j!\Gamma(1+\alpha-j))\). It has been proved that this definition is equivalent to
where \( m - 1 < \Re \alpha < m \), \( f(x) \in C^m[a, x] \). The Caputo fractional derivative, from [8], is defined as

\[
\frac{C^\alpha_aD_x^m f(x)}{x} = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(m - \alpha)} \int_a^x (x - s)^{m-\alpha-1} f(s)ds, & m - 1 < \Re \alpha < m, \\
\frac{d^m}{dx^m} f(x), & \alpha = m,
\end{array} \right.
\]

and

\[
\frac{RL_x^\alpha D_x^m f(x)}{x} = \left\{ \begin{array}{ll}
\frac{(-1)^m}{\Gamma(m - \alpha)} \left[ \frac{d^m}{dx^m} \int_a^x (s - x)^{m-\alpha-1} f(s)ds \right], & m - 1 < \alpha < m, \\
\frac{d^m}{dx^m} f(x), & \alpha = m.
\end{array} \right.
\]

From the definitions (4) and (5), we have \( \frac{C^\alpha_aD_x^m f(x)}{x} = \frac{GL_aD_x^m f(x)}{x} \) when \( f^{(k)}(a) = 0, k = 0, 1, \ldots, m - 1 \).

For the fractional integral and derivative of the power function \( (x - a)^\beta \), there are

\[
\frac{RL_a^{\alpha} J_x^\beta (x - a)^\beta}{x} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 + \alpha)} (x - a)^{\beta + \alpha},
\]

and

\[
\frac{RL_a^{\alpha} D_x^\beta (x - a)^\beta}{x} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 + \alpha)} (x - a)^{\beta + \alpha}.
\]

for \( \Re \alpha > 0, \Re \beta > -1 \) and \( x > a \). When \( \beta = -1, -2, \ldots \), (7) and (8) are clearly not valid. However,

\[
D^\beta (x - a)^{-m} = (-m - n + 1)_n (x - a)^{-m-n},
\]

where the symbol \( (a)_n \) represents the Pochhammer symbol, i.e.,

\[
(a)_n = a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)},
\]

and it requires a proper definition of the fractional derivative of \( (x - a)^{-k} \), \( k = 1, 2, \ldots \). To remedy this problem, the following definition is given by Mauro Bologna [9]:

\[
\frac{RL_0^{\alpha} D_x^\beta (x - a)^{-m}}{x} = \frac{(-1)^m \Gamma(\alpha + m)}{\Gamma(m)} (x - a)^{-m-\alpha}.
\]
It is clear that (8) and (11) there is no essential connection, to be more reasonable definition. In fact, if (11) is true, then we have by (7)

\[ RL \frac{D^a_x f}{f_x} = \frac{(-1)^m \Gamma(m)}{m!} \cdot \frac{RL \int_0^x f_s^{m-a-1} ds}{RL D_x^a f(x)} \]

which is impossible.

Therefore, the traditional definition of fractional derivatives for \( D^k (x-a)^{-k} \) \((k = 1, 2, \ldots)\) is unreasonable. In this paper, our goal is to give a reasonable definition and representations of fractional derivatives for \((x-a)^{-k}\), \((x-a)^{-k} \ln^m (x-a)\), \((x-a)^{-k} f(x)\), \((x-a)^{-k} \ln^m (x-a) f(x)\), where \(k, p = 1, 2, \ldots\) \((x)\) is an analytic function. The properties of the fractional derivatives of the correlation function and the solution of the correlation fractional differential equation are discussed. For simplicity, we take,

\[
m = \begin{cases} \text{Re} + 1, \text{Re} > 0, & a \neq 0, 1, 2, \ldots, \\ \alpha, & a = 0, 1, 2, \ldots, \\ 0, & \text{Re} < 0. \end{cases}
\]

We use neutrix limit \([9–16]\) to define the fractional derivatives for (1)–(6).

**Definition 1.** For complex number \(a\), let

\[
RL \frac{D^a_x f}{f_x} = \frac{1}{\Gamma(a)} \lim_{\epsilon \to 0, \epsilon > 0} \int_0^x (x-s)^{a-1} f(s) ds, \text{Re} > 0,
\]

or

\[
RL \frac{D^a_x f}{f_x} = \frac{1}{\Gamma(m-a)} \int_0^x (x-s)^{m-a-1} f(s) ds, \text{Re} > 0.
\]

\[
D^a_x f(x) = \frac{d^m}{dx^m} \left[ RL \frac{D^a_x f}{f_x} \right],
\]

and

\[
D^a_x f(x) = \frac{1}{\Gamma(m-a)} \int_0^x (x-s)^{m-a-1} f(s) ds.
\]

For convenience of writing, let \( D^a_x f(x) = \frac{RL D^a_x f}{f_x} \).

**Remark 1.** If \( \text{Re} < 0, m = 0 \), then \( D^a_x f(x) = GL \frac{D^a_x f}{f_x} \).

The so-called neutrix limit is defined as follows.

**Definition 2.** If

\[
F(x, \epsilon) = \epsilon^\lambda \sum_{k=0}^{N} A_k(x) \epsilon^k + \epsilon^{-\mu} \sum_{l=1}^{M} B_{k,l}(x) \ln \epsilon^l
\]

for \( N, M, L = 0, 1, 2, 3, \ldots, N < \lambda, M \leq \mu \), then

\[
\lim_{\epsilon \to 0, \epsilon > 0} F(x, \epsilon) = C(x).
\]

For example, by

\[
(x-s)^{a-1} (s-a)^{b} = (x-a)^{a-1} \sum_{l=0}^{\infty} \frac{(1-a)l}{l!} \frac{(s-a)^{l+b}}{(x-a)^{l+b}},
\]

for \( \text{Re} > 0 \), we have

\[
\int_0^x (x-s)^{a-1} (s-a)^{b} ds = (x-a)^{a-1} \int_0^\infty \sum_{l=0}^{\infty} \frac{(1-a)l}{l!} \frac{(s-a)^{l+b}}{(x-a)^{l+b}} ds.
\]
\[
\int_{a\epsilon}^x (x-s)^{\alpha-1} (s-a)^\beta \, ds = \left\{ \begin{array}{ll}
\sum_{l=0}^{\infty} \frac{(1-\alpha)!}{l!(l+\beta+1)} x^{l+\beta+1} - \sum_{l=0}^{[-\beta]-1} \frac{(1-\alpha)!}{l!(l+\beta+1)} x^{l+\beta+1} + o(e), & \beta \neq -1, -2, \ldots, \\
\end{array} \right.
\]

\[
\text{For Beta function}
\]

\[
\text{Lemma 1.} \quad \text{In this section, we shall present some important lemmas that}
\]

\[
\text{present the following two examples to illustrate our}
\]

\[
\text{conclusions.}
\]

\[
2. \text{Lemmas}
\]

\[
\text{In this section, we shall present some important lemmas that}
\]

\[
\text{will be frequently used and their proof may be found in [15, 16].}
\]

\[
\text{Lemma 1. For Beta function } B(x, y), \text{ there is}
\]

\[
B(x, -m) = B(-m, x) = \sum_{k=0, k \neq m}^{\infty} \left( \frac{1-x_k}{k!(k-m)} \right)
\]

\[
= \frac{(-1)^m \Gamma(x)(H_m - \gamma - \psi(x-m))}{m! \Gamma(x-m)},
\]

\[
\text{for } x \neq 0, -1, -2, \ldots, m = 0, 1, 2, \ldots,
\]

\[
\text{Lemma 2. For the complex number } \beta \text{ and } p = 1, 2, \ldots, \text{ there is}
\]

\[
\frac{d^p}{dx^p} [ (x-a)^\beta \ln^p(x-a) ] = (x-a)^\beta - m \sum_{l=0}^{m} \left( \frac{p}{l} \right) a_{m,l} (1+\beta-m) \ln^{p-l}(x-a),
\]

where \( a_{m,l}(x) = (d^l/dx^l)(x)_m \left( \frac{p}{l} \right) = 0, \ l = p + 1, p + 2, \ldots \) In particular,
\[
\frac{d^n}{dx^n} \left[ (x-a)^\beta \ln(x-a) \right] = (x-a)^{\beta-m}(1+\beta-m)_m \left( \ln(x-a) + \sum_{j=1}^{m} \frac{1}{\beta-m+j} \right). \tag{30}
\]

**Lemma 3.** For \(\beta \neq -1, -2, \ldots, n = 1, 2, \ldots\), one has
\[
B_{a,p}^{(p)}(\beta) = H_p^\beta(a,\beta)B_n(\beta),
\]
where \(B_{a,p}^{(p)}(\beta) = (d^{p+1}/dx^p dy^p)B(x,y), B_n(\beta) = (\Gamma(\beta)/\Gamma(\beta+n)), B_{a,p}^{(p)}(\beta) = (d^p/dx^p)B_n(a)|_{x=\beta}.
\]

\[
B_0(\beta) = H_p^\beta(a,\beta), \tag{31}
\]

\[
H_p^\beta(a,\beta) = \sum_{l=0}^{p-1} H_{p,l}^\beta(a,\beta),
\]

\[
H_{1,0}^\beta(a,\beta) = \psi(\beta) - \psi(\beta+a),
\]

\[
H_{n,0}^\beta(a,\beta) = (-1)^n(n-1)!(\zeta(n,\beta) - \zeta(n,\beta+a)),
\]

\[
H_{n,l}^\beta(a,\beta) = (\psi(\beta) - \psi(\beta+a))H_{n-l-1}^\beta(a,\beta) + (n-1)! \sum_{i=1}^{n-2} \frac{(-1)^{n-i}(\zeta(n-i,\beta) - \zeta(n-i,\beta+a))H_{i,l-1}^\beta(a,\beta)}{i!}, \quad l = 1, 2, \ldots, n, n = 1, 2, \ldots, p,
\]

Here, \(\psi(x)\) is the digamma function defined by
\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = -\gamma - \frac{1}{x} + \sum_{l=1}^{\infty} \left( \frac{1}{l} - \frac{1}{l+x} \right), \tag{33}
\]

and \(\zeta(p,z)\) is the Hurwitz zeta function defined by
\[
\zeta(p,z) = \sum_{l=0}^{\infty} \frac{1}{(l+z)^p}, \quad \zeta(p,0) = \zeta(p,1) = \zeta(p), \tag{34}
\]

where \(\zeta(q)\) is the Riemann zeta function.

\[
\int_a^x (s-a)^\beta \ln^p(s-a)ds = p! (x-a)^{\beta+1} \sum_{j=0}^{p} \frac{(-1)^j \ln^{p-j}(x-a)}{(\beta+1)^{j+1} (p-j)!} - p! \epsilon^{\beta+1} \sum_{j=0}^{p} \frac{(-1)^j \ln^{p-j}(x-a)}{(\beta+1)^{j+1} (p-j)!}, \tag{36}
\]

for \(\beta \neq -1\) and
\[
\int_a^x (s-a)^{-1} \ln^p(s-a)ds = \frac{\ln^{p+1}(x-a) - \ln^{p+1}(x-a)}{p+1}. \tag{37}
\]

**Lemma 4.** For complex number \(\alpha, \beta(\beta \neq -1, -2, \ldots, n = 0, 1, 2, \ldots)\), there is
\[
\sum_{l=0}^{\infty} \frac{(1-\alpha)_l}{(l+\beta)_n} = \frac{(-1)^p\alpha_n(a,\beta)}{n!}. \tag{35}
\]

**Lemma 5.** For complex number \(\beta\), there is.

\[
\int_a^x (s-a)^{-\epsilon} \ln^p(s-a)ds = \frac{\ln^{p+1}(x-a) - \ln^{p+1}(x-a)}{p+1}. \tag{37}
\]

**Lemma 6.** For \(n, p, k = 1, 2, \ldots\), there are
\[
\frac{(x-a)^{-\epsilon-1} \ln^{p+1}(x-a)}{p+1} \sum_{l=1}^{n+1} \left( \frac{p+1}{l} \right) a_{n+1}(l) \ln^{p+1-l}(x-a) = \left( \frac{\ln^p(x-a)}{x-a} \right)^{(\alpha)}, \tag{38}
\]

and
\[(x - a)^{-\alpha - k} p! \sum_{j=0}^{p} (-1)^{j} \frac{(p-j)!}{(p-j)!(1-k)^{p-j}} \sum_{l=0}^{n+1} \binom{p-j}{l} a_{n+l} (1-n-k)\ln^{p-j-l} (x - a) = (x - a)^{-k}\ln^{p} (x - a)^{(n)}.
\]

**Lemma 7.** For the complex number \(\alpha, \beta (\beta \neq -1, -2, \ldots, p = 1, 2, \ldots, n)\) there is

\[
1 \Gamma (m - \alpha) \sum_{l=0}^{m} \binom{p}{l} a_{m,l} \Gamma (t + 1 + \beta - m) \sum_{j=0}^{p-l} \binom{p-l}{j} B_{\alpha, \beta}(u, v) |_{u = m - \alpha v = m + 1} \ln^{p-l-j} (x - a) = \frac{1}{\Gamma (\alpha)} \sum_{j=0}^{p} \binom{p}{j} B_{\alpha, \beta}(u, v) \ln^{p-j} (x - a).
\]

which implies that (7) holds.

Employing (16) and (7), we have.

\[
D_{a}^{\alpha} (x - a)^{\beta} = \frac{d^{m}}{dx^{m}} [a_s^{m-a} (x - a)^{\beta}]
\]

\[
= \frac{d^{m}}{dx^{m}} \left[ \frac{\Gamma (\beta + 1)}{\Gamma (m - \alpha + \beta + 1)} (x - a)^{\beta - m - a} \right]
\]

\[
= \frac{(\beta - \alpha + 1)}{\Gamma (m - \alpha + \beta + 1)} (x - a)^{\beta - a}
\]

which concludes that (8) holds.

**3. Fractional Derivatives of \((x - a)^{\beta}\) and \((x - a)^{\beta} \ln^{p} (x - a)\)**

Through the analytical continuation of \(\Gamma (\beta + 1)\), we see that (7) and (8) still hold for \(\alpha > 0, \beta \neq -1, -2, \ldots, \) and \(x > a\). In this section, we shall consider the fractional order derivatives of \((x - a)^{\beta}\) and \((x - a)^{\beta} \ln^{p} (x - a)\) for all complex \(\beta\) and \(p = 1, 2, \ldots,\) and expect to get similar results according to the definition of neutrix limit. Our main results read as follows.

**Theorem 1.** According to definition (14)–(16), (7), and (8) still hold for the complex \(\alpha, \beta (\beta \neq -1, -2, -3, \ldots)\).

**Proof.** For \(Re > 0, \beta \neq -1, -2, -3, \ldots, x - a > 0\), using (21), Definitions 2 and (35) that

\[
\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{x} (x - s)^{\alpha - 1} (s - a)^{\beta} ds
\]

\[
= (x - a)^{\alpha + \beta} \sum_{l=0}^{\infty} \frac{1}{\Gamma (l + \beta + 1)} = B(\alpha, \beta + 1) (x - a)^{\alpha + \beta}.
\]

By (16) and (41), we have

\[
D_{a}^{\alpha} (x - a)^{\beta} = \frac{B(\alpha, \beta + 1)}{\Gamma (\alpha)} (x - a)^{\beta + \alpha},
\]

which concludes that (8) holds.

**Corollary 1.**

(1) For the complex \(\alpha, \beta (\beta \neq -1, -2, -3, \ldots)\), there is

\[
D_{a}^{\alpha} (x - a)^{\beta} = \frac{\partial^{m}}{\partial x^{m}} (x - a)^{\beta} = \frac{\partial^{m}}{\partial x^{m}} (x - a)^{\beta} = \frac{\partial^{m}}{\partial x^{m}} (x - a)^{\beta}
\]

(2) \(D_{a}^{\alpha} (x - a)^{m-n} = 0, n = 1, 2, \ldots\)

**Proof**

(1) For \(Re > 0, \) Remark 1. indicates that corollary naturally holds.

For \(Re > 0, \) \(f(x) = (x - a)^{\beta}, \beta \neq -1, -2, -3, \ldots, x - a > 0\), we have

\[
\lim_{\varepsilon \to 0+} \sum_{j=0}^{m-1} \frac{f^{(j)} (a + \varepsilon) (x - a - \varepsilon)^{j-a}}{\Gamma (j + 1 - a)} = \lim_{\varepsilon \to 0+} \sum_{j=0}^{m-1} \frac{(x - a)^{j-a} (1 - \varepsilon)^{j-a}}{\Gamma (j + 1 - a)}
\]

\[
= \lim_{\varepsilon \to 0+} \sum_{j=0}^{m} \frac{(x - a)^{j-a} (1 - \varepsilon)^{j-a}}{\Gamma (j + 1 - a)} \sum_{l=0}^{\infty} \frac{(1 + \alpha - j)_{l} \varepsilon^{l}}{l!} (x - a)^{j-a}
\]

\[
= \lim_{\varepsilon \to 0+} \sum_{j=0}^{\infty} \frac{(x - a)^{j-a} (1 - \varepsilon)^{j-a}}{\Gamma (j + 1 - a)} \sum_{l=0}^{\infty} \frac{(1 + \alpha - j)_{l} \varepsilon^{l}}{l!} (x - a)^{j-a}.
\]


Due to $\beta + l - j \neq 0$ for $l = 0, 1, 2, \ldots, j = 0, 1, 2, \ldots, m - 1$, we have by Definition 2.

$$N. \lim_{\epsilon \to 0^+} \sum_{j=0}^{m-1} \frac{f^{(j)}(a + \epsilon)(x - a - \epsilon)^{j-a}}{\Gamma(j+1-a)} = 0, \quad (45)$$

$$a_j^{m-a}[(x-a)^{\beta}]^{(m)} = (\beta - m + 1)_{m}a_j^{m-a}(x-a)^{\beta-m}$$

$$= \frac{(\beta - m + 1)_{m}}{\Gamma(\beta - m + 1 + m - a)}(x - a)^{(\beta-m)+(m-a)}$$

$$= \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - a)}(x - a)^{\beta-a}. \quad (46)$$

Hence, the first part of the corollary holds.

(2) From (41) and $B(m-\alpha, n-1) = \left\{ \begin{array}{ll} \Gamma(m-\alpha)\Gamma(\alpha-n+1)/\Gamma(m-n+1), & m \geq n, \\ 0, & m < n, \end{array} \right.$ we have

$$a_j^{m-a}(x-a)^{\alpha-n} = \frac{B(m-\alpha, n-1)(x-a)^{m-n}}{\Gamma(m-\alpha)} \left\{ \begin{array}{ll} \Gamma(\alpha-n+1)(x-a)^{m-n} \\ \Gamma(m-n+1), \; m \geq n, \\ 0, \; m < n. \end{array} \right. \quad (47)$$

Then,

$$a_j^{\alpha}(x-a)^{\alpha-n} = \frac{d^m}{dx^m} \left[a_j^{m-a}(x-a)^{\alpha-n}\right] = \left\{ \begin{array}{ll} \frac{d^m}{dx^m} \left[\frac{\Gamma(\alpha-n+1)(x-a)^{m-n}}{\Gamma(m-n+1)}\right], \; m \geq n \\ \frac{d^m}{dx^m} [0], \; m < n \end{array} \right. = 0. \quad (48)$$

Therefore, the second part of the corollary holds. □

**Theorem 2.** For complex $\alpha, \beta (\beta \neq -1, -2, -3, \ldots)$ and $p = 1, 2, \ldots$, there is

$$x \int_a^\alpha (x-a)^{\beta} \ln^p(x-a) = \frac{(x-a)\beta+\alpha}{\Gamma(\alpha)} \sum_{j=0}^{p} \binom{p}{j} B_{0,j}(\alpha, \beta+1)\ln^{p-j}(x-a), \quad (49)$$

or

$$x \int_a^\alpha (x-a)^{\beta} \ln^p(x-a) = \frac{\Gamma(\beta+1)(x-a)^{\beta+\alpha}}{\Gamma(\alpha+\beta+1)} \sum_{j=0}^{p} \binom{p}{j} H_{0,j}(\beta+1, \alpha+\beta+1)\ln^{p-j}(x-a), \quad (50)$$

for Re > 0 and

$$a \int_a^\alpha (s-a)^{\beta} \ln^p(s-a) = \frac{a \int_a^\alpha (x-a)^{\beta} \ln^p(x-a)}{\Gamma(-\alpha)} \sum_{j=0}^{p} \binom{p}{j} B_{0,j}(-\alpha, \beta+1)\ln^{p-j}(x-a), \quad (51)$$
or

$$RL_x^a D_x^\beta (s-a)^\beta \ln^p (s-a) = \frac{\Gamma (\beta + 1) (x-a)^{\beta - a}}{\Gamma (\beta + 1 - a)} \sum_{j=0}^{p} \binom{p}{j} H_j^p (\beta + 1, \beta + 1 - a) \ln^{p-j} (x-a).$$  \tag{52}$$

**Proof.** By (21) and exchanging the order of integration and then summing, we have

$$\lim_{\epsilon \to 0+} \int_{a+\epsilon}^{x} (x-s)^{\alpha - 1} (s-a)^\beta \ln^p (s-a) ds$$

$$= \lim_{\epsilon \to 0+} \int_{a+\epsilon}^{x} (x-a)^{\alpha - 1} \sum_{l=0}^{\infty} \frac{(1-\alpha)_l (s-a)^{\beta l}}{l!} \ln^p (s-a) ds$$

$$= \lim_{\epsilon \to 0+} \sum_{l=0}^{\infty} \frac{(1-\alpha)_l}{l!} \int_{a+\epsilon}^{x} (s-a)^{\beta l} \ln^p (s-a) ds.$$  \tag{53}$$

From (36), we have

$$\sum_{l=0}^{\infty} \frac{(1-\alpha)_l (x-a)^{\alpha - 1}}{l!} \int_{a+\epsilon}^{x} (s-a)^{\beta l} \ln^p (s-a) ds = p! (x-a)^{\alpha + \beta} \sum_{l=0}^{\infty} \frac{(1-\alpha)_l}{l!} \sum_{j=0}^{p} \frac{(-1)^j}{(l+\beta+1)^{j+1}} (p-j)! \ln^{p-j} (x-a)$$

$$- p! \sum_{l=0}^{\infty} \frac{(1-\alpha)_l (x-a)^{\alpha - 1}}{l!} \sum_{j=0}^{p} \frac{(-1)^j \epsilon^{j+\beta l+1}}{(p-j)! (l+\beta+1)^{j+1}} \ln^{p-j} \epsilon$$

$$= p! (x-a)^{\alpha + \beta} \sum_{j=0}^{p} \frac{(-1)^j}{(p-j)!} \ln^{p-j} (x-a) \sum_{l=0}^{\infty} \frac{(1-\alpha)_l}{l!} \sum_{j=0}^{p} \frac{(-1)^j \epsilon^{j+\beta l+1}}{(p-j)! (l+\beta+1)^{j+1}} \ln^{p-j} \epsilon.$$  \tag{54}$$

By (16), (35), (53), (54), Lemma 3, and Definition 2, we get

$$\int_{a}^{x} (x-a)^{\beta} \ln^p (x-a)$$

$$= \frac{p! (x-a)^{\alpha + \beta}}{\Gamma (\alpha)} \sum_{j=0}^{p} (-1)^j \ln^{p-j} (x-a) \sum_{l=0}^{\infty} \frac{(1-\alpha)_l}{l!} \sum_{j=0}^{p} \frac{(-1)^j \epsilon^{j+\beta l+1}}{(p-j)! (l+\beta+1)^{j+1}} \ln^{p-j} \epsilon$$

$$= \frac{(x-a)^{\alpha + \beta}}{\Gamma (\alpha)} \sum_{j=0}^{p} \binom{p}{j} B_{0,j} (\alpha, \beta + 1) \ln^{p-j} (x-a)$$

$$= \frac{\Gamma (\beta + 1) (x-a)^{\beta - a}}{\Gamma (\alpha + \beta + 1)} \sum_{j=0}^{p} \binom{p}{j} H_j^p (\beta + 1, \alpha + \beta + 1) \ln^{p-j} (x-a),$$  \tag{55}$$

which implies that (49) and (50) hold.

It’s worth noting that if we use the derivative method with parameter, then
\[ aJ_x^a[(x-a)^\beta \ln^p (x-a)] = \frac{d^p}{d\beta^p} [aJ_x^a(x-a)^\beta] \]

\[ = \frac{d^p}{d\beta^p} \left[ \frac{B(\alpha, \beta+1)}{\Gamma(\alpha)} (x-a)^{\alpha \beta} \right] \]

\[ = \frac{(x-a)^{\alpha \beta}}{\Gamma(\alpha)} \sum_{j=0}^{p} \binom{p}{j} B_{0,j}(\alpha, \beta+1) \cdot \ln^{p-j}(x-a), \]

(56)

which is exactly the same as the derivation above.

By (16) and (49), we have

\[ aD_x^m[(x-a)^\beta \ln^p (x-a)] = \frac{d^m}{dx^m} [aJ_x^a[(x-a)^\beta \ln^p (x-a)]] \]

\[ = \frac{d^m}{dx^m} \left[ \frac{(x-a)^{\beta+m-a}}{\Gamma(\alpha)} (x-a)^{\alpha \beta} \right] \]

\[ = \frac{1}{\Gamma(m-a)} \sum_{j=0}^{p} \binom{p}{j} B_{0,j}(-\alpha, \beta+1) \ln^{p-j}(x-a) \]

(57)

In fact, \( aD_x^m[(s-a)^\beta \ln^p (s-a)] \) can be obtained from (29) and (57), but the expression is more complex. Here, we still use the following derivative method with parameter.

\[ aD_x^\beta[(x-a)^\beta \ln^p (x-a)] = \frac{d^p}{dx^p} [aD_x^\beta(x-a)^\beta] \]

\[ = \frac{d^m}{dx^m} \left[ \frac{B(-\alpha, \beta+1)}{-\alpha} (x-a)^{-\alpha} \right] \]

\[ = \frac{(x-a)^{-\alpha}}{-\alpha} \sum_{j=0}^{p} \binom{p}{j} B_{0,j}(-\alpha, \beta+1) \ln^{p-j}(x-a) \]

\[ = \frac{\Gamma(\beta+1)(x-a)^{\beta+\alpha}}{\Gamma(\beta+1-a)} \sum_{j=0}^{p} \binom{p}{j} H_j^{\beta}(\beta+1, \beta+1-a) \ln^{p-j}(x-a), \]

(58)

so, we get (51) and (52).

\[ \square \]

\textbf{Proof.} For \( f(x) = (x-a)^\beta \ln^p (x-a) \), by similar to the proof of Corollary 1, we find that (45) still holds. Hence, \( aD_x^m[(s-a)^\beta \ln^p (s-a)] = \frac{d^m}{dx^m} [aD_x^\beta[(s-a)^\beta \ln^p (s-a)] \]

(29), we get

\[ aD_x^\beta[(s-a)^\beta \ln^p (s-a)] = \frac{d^p}{dx^p} [aD_x^\beta(s-a)^\beta] \ln^{p-m-a}(s-a)^{\beta-m} \ln^{p-j}(x-a) \]

\[ = \frac{d^m}{dx^m} \left[ (x-a)^{\beta+m-a} \ln^p (x-a) \right] \]

(59)

\[ = \frac{m}{\Gamma(m-a)} \sum_{l=0}^{p} \binom{p}{l} (1+\beta-m)_a J_x^{m-a}[(x-a)^{\beta-m} \ln^{p-l}(x-a)]. \]

(60)
Moreover, from (49), we have
\[
a^D_x (x-a)^{\beta-m} \ln^{p-l} (x-a) = \frac{(x-a)^{\beta-a}}{\Gamma (m-a)} \sum_{j=0}^{p-l} \binom{p-l}{j} B_{0,j} (m-a, \beta-m+1) \ln^{p-l-j} (x-a).
\] (61)

Substituting (61) into (60), we get
\[
\frac{C}{a} D^{(x-a)^{\beta}} [\ln^{p} (x-a)] = a^D_x \left[ (x-a)^{\beta} \ln^{p} (x-a) \right]^{(m)}
\]
\[
= \frac{(x-a)^{\beta-a}}{\Gamma (m-a)} \sum_{j=0}^{p} \binom{p}{j} B_{0,j} (m-a, \beta-m+1) \ln^{p-j} (x-a).
\] (62)

By (40) and (62), we have
\[
\frac{C}{a} D^{(x-a)^{\beta}} [\ln^{p} (x-a)] = \frac{(x-a)^{\beta-a}}{\Gamma (-\alpha)} \sum_{j=0}^{p} \binom{p}{j} B_{0,j} (-\alpha, \beta+1) \ln^{p-j} (x-a)
\]
\[
= \frac{(s-a)^{\beta-a}}{\Gamma (s-a) (-\alpha)} \ln^{p-j} (x-a).
\] (63)

This completes the proof of the corollary. □

**Theorem 3.** For the complex Re $a > 0$ and $k = 1, 2, 3, \ldots$, there is
\[
\frac{C}{a} D^{(x-a)^{\beta}} [\ln^{p} (x-a)] = \frac{(-1)^k (x-a)^{\alpha-k} (y + \psi (1 + \alpha - k) - H_{k-1} - \ln (x-a))}{(k-1)! \Gamma (1 + \alpha - k)},
\] (64)
for $k - \alpha \neq 1, 2, \ldots$

and
\[
\frac{C}{a} D^{(x-a)^{\alpha}} [\ln^{p} (x-a)] = \frac{(-1)^k (x-a)^{-\alpha-k} (y + \psi (1 - \alpha - k) - H_{k-1} - \ln (x-a))}{(k-1)! \Gamma (1 - \alpha - k)},
\] (65)
for $\alpha + k \neq 1, 2, \ldots$.

**Proof.** For Re $a > 0$, $k = 1, 2, 3, \ldots$, $x - a > 0$, by (21) and (26), we have
\[
\frac{C}{a} D^{(x-a)^{\alpha}} [\ln^{p} (x-a)] = \frac{(-1)^{k-1} \Gamma (a) (H_{k-1} - \gamma - \psi (\alpha + 1 - k))}{(k-1)! \Gamma (\alpha + 1 - k)} \frac{1 - a^{k-1} \ln (x-a)}{k-1!}
\]
\[
= \frac{(-1)^k (x-a)^{\alpha-k} (y + \psi (\alpha + 1 - k) - H_{k-1} - \ln (x-a))}{(k-1)! \Gamma (\alpha + 1 - k)},
\] (66)

which implies that (64) holds. By (16), (64), and (30), we get
\[
\alpha D_x^n (x-a)^{-k} = \frac{d^m}{dx^m} \left[ a x^{m-a} (x-a)^{-k} \right]
\]

\[
= \frac{d^m}{dx^m} \left[ \frac{(-1)^k (x-a)^{m-a-k} (y + \psi (m-a+1-k) - H_{k-1} - \ln (x-a))}{(k-1)! m m + \ln (x-a)} \right]
\]

\[
= \frac{(-1)^k (x-a)^{-a-k} (y + \psi (m-a+1-k) - H_{k-1} - \ln (x-a))}{(k-1)! m m + \ln (x-a)}
\]

\[
= \frac{(-1)^k (x-a)^{-a-k} (y + \psi (m-a+1-k) - H_{k-1} - \ln (x-a) + \bar{H}_{xk} (m))}{(k-1)! m + \ln (x-a)}
\]

where \( \bar{H}_x (m) = \sum_{j=1}^{m} n (1/a - j) \). Using

\[
\psi (m-a+1-k) + \bar{H}_{axk} (m) = \psi (1-a-k),
\]

and (67) yields that (65) holds.

**Remark 2.** Due to

\[
\lim_{a \rightarrow n} \frac{(-1)^k \psi (1-a-k)}{(k-1)! m + \ln (x-a)} = (-1)^k (k) ,
\]

\[
\lim_{a \rightarrow n} \frac{(-1)^k (x-a)^{-a-k} (y - H_{k-1} - \ln (x-a))}{(k-1)! m + \ln (x-a)} = 0,
\]

for \( n = 1, 2, \ldots \), we have

\[
\lim_{a \rightarrow n} \alpha D_x^n (x-a)^{-k} = (-1)^k (k) (x-a)^{m-k} = (x-a)^-(a).
\]

Under the definition given in this paper, whether \( \beta \neq -1, -2, \ldots \), or \( \beta = -1, -2, \ldots \), there is always

\[
\lim_{a \rightarrow n} \alpha D_x^n (x-a)^{\beta} = (d^\beta/dx^\beta)((x-a)^\beta).
\]

So \( \lim_{a \rightarrow n} \alpha D_x^n (x-a)^{\beta} \) is just the usual derivative \( (d^\beta/dx^\beta)((x-a)^\beta) \), which implies that the definition in this paper is reasonable. For example, taking \( f(x) = (x-a)^{-k} \), \( \alpha \neq 1, 2, \ldots \), there is
Corollary 3. For \( \alpha \neq 1, 2, \ldots, k = 1, 2, 3, \ldots \), there is
\[
\frac{C_a D_x^\alpha (x-a)^{-k}}{(k-1)!} = \frac{(-1)^k (x-a)^{-a-k} (y + \psi (1 - a - k) - H_{k,m-1} - \ln (x-a))}{(k-1)! (1 - \alpha - k)},
\]
(72)
and
\[
\frac{GL_a D_x^\alpha (x-a)^{-k}}{(k-1)!} = \frac{(-1)^k (H_{k-1} - H_{k,m-1}) (x-a)^{-a-k}}{(k-1)! (1 - \alpha - k)},
\]
(73)
By (72) and (73), we see that \( a D_x^\alpha (x-a)^{-k} \) and \( GL_a D_x^\alpha (x-a)^{-k} \) are not equal for \( \text{Re} x > 0 \).

Remark 3. When \( \beta \neq -1, -2, \ldots \), then \( (d^{\beta-1}/dx^{\beta-1}) [a D_x^\alpha (x-a)^{\beta}] = (d^{\beta-1}/dx^{\beta-1}) [GL_a D_x^\alpha (x-a)^{\beta}] \). However, when \( \beta = -1, -2, \ldots \), the relationship is broken. For example, by (64) and (30), we have
\[
\frac{d^{\beta-1}}{dx^{\beta-1}} [a D_x^\alpha (x-a)^{-1}] = \frac{(x-a)^{-a-1} \ln (x-a) - (y + \psi (-a)) (x-a)^{-a-1}}{\Gamma (-a)},
\]
(74)
for \( \alpha \neq 1, 2, \ldots \).

Theorem 4. For \( p, k = 1, 2, \ldots \), there is
\[
\frac{a D_x^\alpha \ln^p (x-a)}{\Gamma (\alpha + 1 - k) (p-1)! (p + 1)} = \frac{(-1)^{k-1} \ln^{p+1} (x-a)}{\Gamma (\alpha + 1 - k) (k-1)! (p + 1)} (x-a)^{-a-k},
\]
(75)
for \( \text{Re}a > 0 \), where

\[
C_j(a, k) = \sum_{l=0, l \neq k-1}^{\infty} \frac{(-1)^l}{l! (l - k + 1)^{l+1}}.
\]

(76)

\[
aD_x^n [(x-a)^{-\alpha} \ln^\beta (x-a)] = (x-a)^{-\alpha-k} \sum_{j=0}^{p} \frac{(-1)^j}{(p-j)!} R_j(a, k) \ln^{p-j} (x-a) + \frac{(-1)^{k-1} Q(a, k) (x-a)^{-\alpha-k} \ln^{p+1} (x-a)}{(p+1)(k-1)!},
\]

(77)

with

\[
R_j(a, k) = C_j(m-a, k) \sum_{l=0}^{m} \frac{(p-j)}{l} a_{m,l} (1-a-k) \ln^{-l}(x-a),
\]

(78)

\[
Q(a, k) = \frac{1}{\Gamma(m-a+1-k)} \sum_{l=0}^{m} \frac{(p+1)}{l} a_{m,l} (1-a-k) \ln^{-l}(x-a).
\]

Proof. By (21), we have

\[
\begin{align*}
\mathbb{N} \times \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{x} (s-a)^{-\alpha-k} \ln^\beta (s-a) ds &= \mathbb{N} \times \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{x} (x-a)^{-\alpha-k} \ln^\beta (s-a) ds \\
&= \mathbb{N} \times \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{x} \frac{(1-a)_{l} (s-a)^{a-l-1}}{l!} \ln^\beta (s-a) ds.
\end{align*}
\]

(79)

From (36), (37), and (79), we get

\[
aD_x^n [(x-a)^{-\alpha} \ln^\beta (x-a)] = \frac{1}{\Gamma(a)} \mathbb{N} \times \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{x} (s-a)^{-\alpha-k} \ln^\beta (s-a) ds
\]

\[
= p! (x-a)^{-\alpha-k} \sum_{l=0, l \neq k-1}^{\infty} \frac{(1-a)_{l} (s-a)^{a-l-1}}{\Gamma(a) l!} \sum_{j=0}^{p} \frac{(-1)^j \ln^{p-j} (x-a)}{(l - k + 1)^{l+1} (p-j)!} + \frac{(1-a)_{k-1} (x-a)^{-\alpha-k} \ln^{p+1} (x-a)}{\Gamma(a) (k-1)! (p+1)}.
\]

(80)

By \((1-a)_{l}/\Gamma(a) = ((-1)^{l}/\Gamma(a - l))\) and exchanging the sum order, we see that (75) holds.

Using (16) and (29) yields that

\[
aD_x^n [(x-a)^{-\alpha} \ln^\beta (x-a)] = \frac{d^m}{dx^m} \left[ aD_x^{m-a} [(x-a)^{-\alpha} \ln^\beta (x-a)] \right]
\]

\[
= \frac{d^m}{dx^m} \left[ p! \sum_{j=0}^{p} \frac{(-1)^j C_j(m-a, k) \ln^{p-j} (x-a)}{(p-j)!} + \frac{(-1)^{k-1} \ln^{p+1} (x-a)}{\Gamma(m-a+1-k)(k-1)!(p+1)} (x-a)^{m-a-k} \right]
\]

(81)

\[
= (x-a)^{-\alpha-k} \sum_{j=0}^{p} \frac{(-1)^j C_j(m-a, k) m}{(p-j)!} \sum_{l=0}^{(p-j)!} \frac{p-j}{l} a_{m,l} (1-a-k) \ln^{p-j-l} (x-a)
\]

\[+ \frac{(-1)^{k-1} (x-a)^{-\alpha-k}}{\Gamma(m-a+1-k)(k-1)!(p+1)} \sum_{l=0}^{m} \frac{p+1}{l} a_{m,l} (1-a-k) \ln^{p+1-l} (x-a).\]
Hence, we conclude that (77) holds. \[\Box\]

Remark 4

(1) For \(n = 1, 2, \ldots\), noting that
\[
\lim_{a \to n+1} (m - a) = 1,
\]
\[
\lim_{a \to n+1} C_j (m - a, k) = \begin{cases} 
\frac{1}{(1 - k)^{2m}}, & k = 2, 3, \ldots, \\
0, & k = 1,
\end{cases}
\]
\[
\lim_{a \to n+1} \Gamma (m - a + 1 - k) = \begin{cases} 
\infty, & k = 2, 3, \ldots, \\
1, & k = 1,
\end{cases}
\]
and employing (38), (39), and (77), we have

\[
a^{D_x^n}_{\alpha} \left[ f^n_x (x - a)^{-k} \right] = a^{D_x^n}_{\alpha} \left[ \frac{(-1)^k (x - a)^{\alpha - k} (\psi + \psi (1 + \alpha - k) - H_{k, 1})}{(k - 1)\Gamma (1 + \alpha - k)} - \frac{(-1)^k (x - a)^{\alpha - k} \ln (x - a)}{k - 1})\Gamma (1 + \alpha - k) \right]
\]
\[
= 0 - \frac{(-1)^k (x - a)^{-k} (B_{0,0} (-\alpha, -\alpha - k + 1) + B_{0,1} (-\alpha, -\alpha - k + 1))}{(k - 1)\Gamma (1 + \alpha - k)\Gamma (-\alpha)}
\]
\[
= (x - a)^{-k},
\]

where \(B_{0,0} (-\alpha, -\alpha - k + 1) = 0\) and \(((-1)^k B_{0,1} (-\alpha, -\alpha - k + 1) = 0\) are used. Again, this shows that the definitions in this paper are reasonable.

4. Fractional Derivatives of Some Elementary Functions

In this section, we shall study the fractional order derivatives of some elementary functions. First of all, we give the following general theorem.

\[
a^{D_x^n}_{\alpha} \left[ (x - a)^{-n} f (x) \right] = \sum_{l=0}^{n-1} \frac{(-1)^{n-1} f^{(l)} (a) (\psi + \psi (1 + l - \alpha - n) - H_{n-1} - \ln (x - a))}{l! (n - l - 1)\Gamma (1 - \alpha - n + l) (x - a)^{\alpha - l}} + \sum_{l=n}^{\infty} \frac{f^{(l)} (a) (l - n)! (x - a)^{\alpha - l}}{l! (l - n + 1 - \alpha)},
\]

\[
a^{D_x^n}_{\alpha} \left[ (x - a)^{-n} \ln^p (x - a) f (x) \right]
\]
\[
= \frac{1}{(x - a)^{\alpha - m}} \sum_{l=0}^{n-1} \frac{f^{(l)} (a) (x - a)^l}{l!} \sum_{p=0}^{\infty} \binom{-1}{p} R_j (a, n - l) \ln^{p-j} (x - a) + \frac{(-1)^{k-1} Q (a, n - l) \ln^{p+1} (x - a))}{(p + 1) (n - l - 1)!}
\]
\[
+ \frac{1}{\Gamma (-\alpha) (x - a)^{\alpha - m}} \sum_{l=0}^{\infty} \binom{-1}{p} \frac{f^{(l)} (a) (x - a)^l}{l!} \sum_{j=0}^{\infty} \binom{p}{j} B_{0,j} (-\alpha, l - n + 1) \ln^{p-j} (x - a).
\]

Theorem 5. If \(f (x)\) is the analytic function, i.e.,
\[
f (x) = \sum_{l=0}^{\infty} f^{(l)} (a) (x - a)^l, \quad l!
\]
then the following formulas are true.
In particular,
\[ aD_x^n \left( x - a \right)^{-n} e^{b(x-a)} = \frac{-1}{\Gamma (-n)} \sum_{l=0}^{n-1} \frac{(-1)^l b^l (\ln (x - a) - \psi (1 + l - \alpha - n) + H_{n-l-1})}{l! (n - l - 1)!} (x - a)^{\alpha + n - l} \cdot \sum_{l=0}^{n-1} \frac{(-1)^l b^l (\ln (x - a) - \psi (1 + l - \alpha - n) + H_{n-l-1})}{l! (n - l - 1)!} (x - a)^{\alpha + n - l}, \]

where we have used the following equations
\[ (\alpha)_k = \Gamma (\alpha + k)/\Gamma (\alpha), \quad \Gamma (\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad \psi (x) = \Gamma ^{-1} (x), \quad H_n = H_n(x) = \sum_{k=1}^{n-1} \frac{1}{x-k}, \]

and
\[ aD_x^n \left( x - a \right)^{-n} \sin b(x-a) = \frac{-1}{\Gamma (-n)} \sum_{l=0}^{n-1} \frac{(-1)^l b^l (\ln (x - a) - \psi (1 + l - \alpha - n) + H_{n-l-1})}{l! (n - l - 1)!} (x - a)^{\alpha + n - l}, \]

\[ aD_x^n \left( x - a \right)^{-n} \cos b(x-a) = \frac{-1}{\Gamma (-n)} \sum_{l=0}^{n-1} \frac{(-1)^l b^l (\ln (x - a) - \psi (1 + l - \alpha - n) + H_{n-l-1})}{l! (n - l - 1)!} (x - a)^{\alpha + n - l}, \]

where we have used the following equations
\[ \cot (x - a) = \frac{1}{x - a} + \sum_{k=1}^{\infty} \frac{(-1)^k B_k (x - a)^{2k-1}}{(2k)!}, \]
\[ \csc (x - a) = \frac{1}{x - a} - 2 \sum_{k=1}^{\infty} \frac{(-1)^k B_k (x - a)^{2k-1}}{(2k)!}, \]
\[ \coth (x - a) = \frac{1}{x - a} + 2 \sum_{k=1}^{\infty} \frac{2^k B_k (x - a)^{2k-1}}{(2k)!}, \]
\[ \csch (x - a) = \frac{1}{x - a} - 2 \sum_{k=1}^{\infty} \frac{2^k B_k (x - a)^{2k-1}}{(2k)!}. \]

For the fractional derivative, there is the following Leibniz derivative formula
\[ aD_x^n [f(x)g(x)] = \sum_{k=0}^{n} \binom{n}{k} aD_x^{\alpha - k} f(x)D_x^k g(x) \quad x \in (a,b), \]

where \( \binom{\alpha}{k} = (\Gamma (\alpha + 1)/\Gamma (\alpha + 1 - k))aD_x^\alpha \) is any of \( aD_x^\alphaC \), \( aD_x^\alphaG \), or \( aD_x^\alphaD \) for \( f(x) \) and \( D_x^k g(x) \) are continuous in \( (a,b) \). Since \( aD_x^\alpha f = ((x-a)^{\alpha}/\Gamma (1-\alpha)) = (\Gamma (\alpha)/(x-a)^{-\alpha} \sin \pi \alpha), \) by (90), we have
\[ aD^a [g(x)] = aD^a [1 \cdot g(x)] = \Gamma (\alpha + 1) \sum_{k=0}^{\infty} \frac{x^{k-a} \sin (a-k) \pi}{k! (\alpha - k) \pi} D^k g(x). \] (93)

Using (92), we obtain that \( (92) \) does not hold for \( aD^a (x-a)^{-n} = aD^a (x-a)^{-n} = aD^a (x-a)^{-n} \), which is not consistent with Corollary 3 so that (92) does not hold for \( g(x) = (x-a)^{-n} \). Therefore, we cannot calculate \( (x-a)^{-n} \).

If \( g(x) \) is the analytic function, the Leibniz derivative formula (92) is still valid for the Riemann–Liouville fractional integral and derivatives [5]. For example, if \( f(x) = (x-a)^{-n} \), \( g(x) \) is an analytic function in Theorem 5, the following formula still holds:

\[
\begin{align*}
\alpha D^k_a [(x-a)^{-n} g(x)] &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \alpha D^k_a (x-a)^{-n} D^k g(x) \\
&= \frac{(-1)^n \Gamma (\alpha + 1)}{(n-1)! (x-a)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{\left( \Gamma (\alpha + 1) \right) (\alpha + k - n) - H_{n-1} - \ln (x-a)) (x-a)^k}{(x-a)^{\alpha+n}} g^{(k)}(x) \\
&= \frac{(-1)^n \Gamma (\alpha + 1) \sin (\alpha \pi)}{\pi (n-1)! (x-a)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{(-1)^k (x-a)^{\alpha+n}}{k!} (\alpha + k - n)_{n-1} (\alpha + k - n) - H_{n-1} - \ln (x-a)) (x-a)^k g^{(k)}(x).
\end{align*}
\] (94)

By (94), we have

\[
\begin{align*}
\alpha D^k_a [(x-a)^{-n} e^{b(x-a)}] &= \frac{(-1)^n \Gamma (\alpha + 1) e^{b(x-a)}}{(n-1)! (x-a)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{b^k (\alpha + k - n) - H_{n-1} - \ln (x-a)) (x-a)^k}{(x-a)^{\alpha+n}} g^{(k)}(x) \\
\alpha D^k_a [(x-a)^{-n} \sin b(x-a)] &= \frac{(-1)^n \Gamma (\alpha + 1)}{(n-1)! (x-a)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{b^k (\alpha + k - n) - H_{n-1} - \ln (x-a)) \sin (bx + (k\pi/2)) (x-a)^k}{(x-a)^{\alpha+n}} \\
\alpha D^k_a [(x-a)^{-n} \cos b(x-a)] &= \frac{(-1)^n \Gamma (\alpha + 1)}{(n-1)! (x-a)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{b^k (\alpha + k - n) - H_{n-1} - \ln (x-a)) \cos (bx + (k\pi/2)) (x-a)^k}{(x-a)^{\alpha+n}}.
\end{align*}
\] (95)

or
Based on the derivative formula of elementary functions, (94) is more convenient to calculate. The numerical calculation shows that (95) and (96) are exactly same as (89), but (89) is the calculated fastest and (95) is the calculated slowest.

\[
\frac{\partial^\alpha}{\partial x^\alpha} \left[ (x-a)^{-n} g(x) \right] = \frac{\partial^\alpha}{\partial x^\alpha} \left[ (x-a) g(x) \right]
\]

\[
= \Gamma(1+\alpha) \sin(\alpha \pi) \sum_{k=0}^{\infty} (\frac{(-1)^k}{k!}) \left( \frac{\Gamma(1+\alpha) x^{-\alpha n}}{\pi (x-a)^{\alpha+1}} \right) \left( \frac{(x-a)(x)}{k!(x-a)^k} \right)
\]

is given by

\[
N(t) = N_0 \Gamma(\mu) t^{\mu-1} E_{\alpha,\beta} (\mu \alpha, \beta, t^\alpha)
\]

where \( E_{\alpha,\beta} (\mu \alpha, \beta, t^\alpha) \) is the generalized Mittag-Leffler function and it is defined as

\[
E_{\alpha,\beta} (\mu \alpha, \beta, t^\alpha) = \sum_{k=0}^{\infty} \frac{t^k}{(k \alpha + \beta)}
\]

Remark 5. Based on the derivative formula of elementary function, (94) is more convenient to calculate \( aD_x^\alpha \left[ (x-a)^{-n} g(x) \right] \), but the calculation efficiency is not as fast as (87).

5. Applications to the Fractional Differential and Integral Equations

At last, as an application of the definitions of the fractional derivative and integrals given by the neutron limit, this section is devoted to study the solution of fractional differential and integral equations.

Theorem 6 (see [1]). If \( \text{Re} \mu > 0 \) and \( \text{Re} \mu > 0 \) then the solution of fractional integral equation.

\[
N(t) + \int_0^t N(x) \ dx = 0, \quad t > 0,
\]

is given by

\[
N(t) = N_0 \Gamma(\mu) t^{\mu-1} E_{\alpha,\beta} (\mu \alpha, \beta, t^\alpha),
\]

where \( E_{\alpha,\beta} (\mu \alpha, \beta, t^\alpha) \) is the generalized Mittag-Leffler function and it is defined as

Through the analytical continuation, when \( \mu < 0, \mu \neq 0, -1, -2, \ldots \), the above conclusion is also established. For \( \mu = 0, -1, -2, \ldots \), (99) is not a solution of the fractional integral equation (98), but we have the following result.

Theorem 7. If \( \text{Re} \alpha > 0, n = 1, 2, \ldots \), then the solution of the following fractional integral equation

\[
N(x) + \int_0^x N(x) \ dx = 0, \quad x > a \geq 0,
\]

is given by
\[
N(x) = \frac{N_0}{(x-a)^n} - \frac{N_0 (-1)^n c^a(x-a)^a}{(x-a)^n (n-1)!} E_{a,1+a-n}^\gamma (-c^a(x-a)^a) + \frac{(-1)^n N_0 \ln(x-a) - \gamma + H_{n-1}}{(n-1)!} c^a(x-a)^a E_{a,1+a-n}^\gamma (-c^a(x-a)^a),
\]

where
\[
E_{a,\beta}^\gamma (t) = \sum_{k=0}^{\infty} \frac{\psi(ka+\beta)t^k}{\Gamma(ka+\beta)}.
\]

Proof. Applying the operator \((-c^a)^a f_x^k\) to both sides of (101), we have by (64)

\[
(-c^a)^a f_x^k N(t) - (-c^a)^a f_x^{k+1} N(x) = N_0 (-c^a)^a f_x^k (x-a)^n
\]

= \begin{align*}
&\frac{(-1)^n N_0 (y + \psi(1+ka-n) - H_{n-1} - \ln(x-a)) (-c^a(x-a)^a)^k}{(n-1)! (x-a)^n \Gamma(1+ka-n)}, \quad k = 1, 2, \ldots, \\
&N_0 (x-a)^n, \quad k = 0.
\end{align*}

Summing up the above expression with respect to \(k\) from 0 to \(\infty\) gives rise to

\[
N(x) = \frac{N_0}{(x-a)^n} + \frac{(-1)^n N_0 (y + \psi(1+ka-n) - H_{n-1} - \ln(x-a)) (-c^a(x-a)^a)^k}{(n-1)! (x-a)^n \Gamma(1+ka-n)}
\]

= \frac{N_0}{(x-a)^n} - \frac{N_0 (-1)^n c^a(x-a)^a}{(x-a)^n (n-1)!} E_{a,1+a-n}^\gamma (-c^a(x-a)^a)

- \frac{(-1)^n N_0 (y - \ln(x-a) - H_{n-1}) c^a(x-a)^a E_{a,1+a-n}^\gamma (-c^a(x-a)^a)}{(n-1)! (x-a)^n}

Hence, we deduce that (102).

Theorem 8. If \(\text{Re} a > 0, n = 1, 2, \ldots\), then the solution of the following fractional differential equation

\[
\begin{cases}
a D_x^a N(x) + c^a N(x) - \frac{N_0}{(x-a)^n} = 0, & x > a \geq 0, \quad \frac{D_x^{a-k}}{\Gamma(a-k)} N(a+) = b_k, \quad k = 1, 2, \ldots, m,
\end{cases}
\]

is given by

\[
N(x) = \frac{N_0 (-1)^n (y - H_{n-1} - \ln(x-a)) E_{a,1+a-n}^\gamma (-c^a(x-a)^a)}{(n-1)! (x-a)^n} + \frac{N_0 (-1)^n E_{a,1+a-n}^\gamma (-c^a(x-a)^a)}{(n-1)! (x-a)^n}
\]

+ \sum_{j=1}^{m} b_j (x-a)^{a-j} E_{a,1+a-j}^\gamma (-c^a(x-a)^a).

Proof. Applying the operator \(a f_x^k\) to both sides of (106), using (64) and

\[
a f_x^k \left[a D_x^a N(x)\right] = N(x) - \sum_{j=1}^{m} \frac{b_j (x-a)^{a-j}}{\Gamma(a+1-j)} N(x)
\]

(108)
we get the following integral equation.

\[
N(x) - g_{Rn}(x, a) = -c_a^x f_x^a N(x), \quad x > a \geq 0,
\]

where

\[
g_{Rn}(x, a) = \frac{N_0(-1)^n(x-a)^{\alpha-n}(y+\psi(1+\alpha-n)-H_{n-1})}{(n-1)!\Gamma(1+\alpha-n)} - \frac{N_0(-1)^n(x-a)^{\alpha-n} \ln(x-a)}{(n-1)!\Gamma(1+\alpha-n)} + \sum_{j=1}^{m} \frac{b_j(x-a)^{\alpha-j}}{\Gamma(a+1-j)}.
\]

Applying the operator \((-c_a)^k a^k f_x^a (x-a)^{-k}\) to both sides of (109), and then using (7), (58), and Lemma 3 yield that

\[
(-c_a)^k a^k N(x) - (-c_a)^{k+1} a^{k+1} N(x) = G_{nk}(x, a),
\]

where

\[
G_{nk}(x, a) = (-c_a)^k a^k g_0(x, a) = \frac{N_0(-1)^n(y+\psi(1+\alpha-n)-H_{n-1})(-c_a)^k}{(n-1)!\Gamma(1+\alpha-n)} \frac{\Gamma(a-n+1)}{\Gamma(a-n+1+ka)} (x-a)^{\alpha-n+ka}
\]

\[
+ \sum_{j=1}^{m} \frac{b_j(-c_a)^k}{\Gamma(a+1-j)} \frac{\Gamma(a-j+1)}{\Gamma(a-j+1+ka)} (x-a)^{\alpha-j+ka} - \frac{N_0(-1)^n(-c_a)^k}{(n-1)!\Gamma(1+\alpha-n)} \frac{(x-a)^{\alpha+n+ka}}{\Gamma(a-n+1+ka)} \ln(x-a)
\]

\[
= \frac{N_0(-1)^n(y+\psi(1+ka+\alpha-n)-H_{n-1})(-c_a)^k (x-a)^{\alpha-n+ka}}{(n-1)!\Gamma(1+\alpha-n+1+ka)} + \sum_{j=1}^{m} \frac{b_j(-c_a)^k (x-a)^{\alpha-j+ka}}{\Gamma(a-j+1+ka)}
\]

\[
- \frac{N_0(-1)^n(-c_a)^k (x-a)^{\alpha-n+ka}}{(n-1)!\Gamma(1+\alpha-n+1+ka)} \ln(x-a), \quad k = 1, 2, \ldots
\]

Summing up the above expression with respect to \(k\) from 0 to \(\infty\) yields that

\[
N(x) = \frac{N_0(-1)^n(x-a)^{\alpha-n}(y+\psi(1+\alpha-n)-H_{n-1})}{(n-1)!\Gamma(1+\alpha-n)} - \frac{N_0(-1)^n(x-a)^{\alpha-n} \ln(x-a)}{(n-1)!\Gamma(1+\alpha-n)} + \sum_{j=1}^{m} \frac{b_j(x-a)^{\alpha-j}}{\Gamma(a+1-j)}
\]

\[
+ \sum_{k=1}^{\infty} \frac{N_0(-1)^n(y+\psi(1+ka+\alpha-n)-H_{n-1})(-c_a)^k (x-a)^{\alpha-n+ka}}{(n-1)!\Gamma(ka+\alpha-n+1)}
\]

\[
+ \sum_{k=1}^{\infty} \sum_{j=1}^{m} \frac{b_j(-c_a)^k (x-a)^{\alpha-j+ka}}{\Gamma(a-j+1+ka)} - \sum_{k=1}^{\infty} \frac{N_0(-1)^n(-c_a)^k (x-a)^{\alpha+n+ka}}{(n-1)!\Gamma(a-n+1+ka)} \ln(x-a)
\]

\[
= \frac{N_0(-1)^n(y-H_{n-1} - \ln(x-a))E_{a,1+\alpha-n}(-c_a^x (x-a)^\alpha)}{(n-1)!\Gamma(1+\alpha-n)} + \frac{N_0(-1)^nE_{a,1+\alpha-n}(-c_a^x (x-a)^\alpha)}{(n-1)!\Gamma(1+\alpha-n)} + \sum_{j=1}^{m} \frac{b_j(x-a)^{\alpha-j}E_{a,1+\alpha-j}(-c_a^x (x-a)^\alpha)}{(n-1)!\Gamma(1+\alpha-n)}.
\]

**Theorem 9.** If \(\Re a > 0, n = 1, 2, \ldots\), then the solution of the following fractional differential equation
thus, we get the following integral equation.

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{c}{a} D_x^a N(t) + c^a N(x) - \frac{N_0}{(x-a)^m} = 0, & x > a \geq 0, N^{(m-k)}(a+0) = b_k, \ k = 1, 2, \ldots, m, \\
\end{array} \right.
\end{aligned}
\]  \hspace{1cm} (114)

is given by

\[
N(t) = \frac{N_0}{(n-1)!} (y - H_{n-1} - \ln (x-a)) E_{\alpha,m-n} (-c^a (x-a)^a) + \frac{N_0}{(n-1)!} (x-a)^{n-a_i} \sum_{j=1}^{m} b_j (x-a)^{m-j} E_{\alpha,m-j} (-c^a (x-a)^a).
\]  \hspace{1cm} (115)

**Proof.** Applying the operator \( a^{\frac{RL}{\alpha}} f_a \) to both sides of (114), using (64) imply that

\[
a^{\frac{RL}{\alpha}} \frac{[c^a D_x^a N(x)]}{a^{\frac{RL}{\alpha}} a^{\frac{RL}{\alpha}} N^{(m)}(x)} = a^{\frac{RL}{\alpha}} a^{\frac{RL}{\alpha}} N^{(m)}(x)
\]

\[
= \frac{1}{\Gamma (m)} \lim_{\epsilon \to 0^+} \int a^{\frac{RL}{\alpha}} (x-s)^{m-1} N^{(m)}(s) ds
\]

\[
= -\frac{N_0}{\Gamma (m+1-j)} \sum_{j=1}^{m} (x-a)^{m-j} N^{(m-j)}(a+\epsilon)
\]

\[
= -\frac{N_0}{\Gamma (m+1-j)} N^{(m-j)}(a+0)
\]

\[
= -\frac{\sum_{j=1}^{m} b_j (x-a)^{m-j}}{(m-j)!}.
\]  \hspace{1cm} (116)

thus, we get the following integral equation.

\[
N(x) + c^a a^{\frac{RL}{\alpha}} N(x) = g_{C,n}(x, a), \quad x > a \geq 0,
\]  \hspace{1cm} (117)

where

\[
g_{C,n}(x, a) = \frac{N_0 (-1)^n (x-a)^{\alpha-n} (y + \psi(1+a-n) - H_{n-1})}{(n-1)! \Gamma (1+a-n)} - \frac{N_0 (-1)^n (x-a)^{\alpha-n} \ln (x-a)}{(n-1)! \Gamma (1+a-n)} + \sum_{j=1}^{m} b_j (x-a)^{m-j} \frac{(m-j)!}{(m-j)!},
\]  \hspace{1cm} (118)

then, according to Theorem 8, we get (115). \hspace{1cm} \square

**Theorem 10.** If \( \text{Re} a > 0, n = 1, 2, \ldots \), then the solution of the following fractional differential equation

\[
\left\{ \begin{array}{l}
a^\frac{RL}{\alpha} D_x^a N(x) + c^a N(x) - \frac{N_0 \ln^p (x-a)}{(x-a)^n} = 0, & x > a \geq 0, a^\frac{RL}{\alpha} D_x^{a+k} N(a+0) = b_k, \ k = 1, 2, \ldots, m,
\end{array} \right.
\]  \hspace{1cm} (119)

is given by
\[ N(t) = \frac{p!}{(x-a)^{\alpha+n}} \sum_{j=0}^{p} \frac{(-1)^j C_j(a, n) \ln^{p-j}(x-a)}{(p-j)!} + \frac{(-1)^{n-1} \ln^{p+1}(x-a)}{\Gamma(\alpha+n)(n-1)!(p+1)(x-a)^{\alpha+n}} \]

\[ + \frac{p! \Gamma(a+1-n)}{(x-a)^{\alpha+n}} \sum_{j=0}^{p} \frac{(-1)^j C_j(a, n) \ln^{p-j}(x-a)}{(p-j)!} \sum_{l=0}^{\infty} \left( \frac{p-j}{l} \right) E_{a,n,l}(-c^\alpha(x-a)^\alpha) \ln^{p+1-l}(x-a) \]

\[ + \frac{(-1)^{n-1}}{(n-1)!(p+1)(x-a)^{\alpha+n}} \sum_{j=0}^{p+1} \left( \frac{p+1}{j} \right) E_{a,n,j}(-c^\alpha(x-a)^\alpha) \ln^{p+1-j}(x-a) + \sum_{j=1}^{m} b_j(x-a)^{\alpha-j} E_{a,1,\alpha-j}(-c^\alpha(x-a)^\alpha), \]

where

\[ E_{a,n,l}(t) = \sum_{k=1}^{\infty} \frac{H_k^\alpha(a+1-n, ka+\alpha+1-n)t^k}{\Gamma(ka+\alpha+1-n)}. \]

**Proof.** Applying the operator \( {}_xJ^\alpha \) to both sides of (118), using (75) and (108), we get

\[ N(x) + c^\alpha {}_xJ_x^\alpha N(x) = g_{R,n,p}(x, \alpha), \quad x > a \geq 0, \]

where

\[ g_{R,n,p}(x, \alpha) = p! \sum_{j=0}^{p} \frac{(-1)^j C_j(a, n) (x-a)^{-\alpha-n} \ln^{p-j}(x-a)}{(p-j)!} + \frac{(-1)^{n-1} (x-a)^{-\alpha-n} \ln^{p+1}(x-a)}{\Gamma(a+1-n)(n-1)!(p+1)} + \sum_{j=1}^{m} \frac{b_j(x-a)^{\alpha-j}}{\Gamma(a+1-j)}. \]

for Re\( \alpha > 0. \)

Exactly according to the discussion of Theorems 8 and 9, we obtain

\[ N(t) = \sum_{k=0}^{\infty} (-c^\alpha)^k a^\alpha \Gamma(ka + \alpha + n + 1) \]

\[ + \frac{(-1)^{n-1} (x-a)^{-\alpha-n} \ln^{p+1}(x-a)}{\Gamma(a+1-n)(n-1)!(p+1)} + \sum_{j=1}^{m} \frac{b_j(x-a)^{\alpha-j}}{\Gamma(a+1-j)} \]

(124)

By Lemma 3 and (124) we obtain (120).

Let us substitute the solutions of (102), (107), (115), and (120) into their respective equations (101), (106), (114), and (119), and make their error curves (see (1)–(4) in Figure 1). Without loss of generality.

Here, let \( b_k = 0, k = 1, 2, \ldots, m. \) In (1)–(4) of Figure 1, the figure above is the solution curve and the figure below is the error curve. The numerical results show that the solutions of the fractional differential and integral equations obtained above satisfy their respective equations. It can be
seen from (2) and (3) of Figure 1 that although the expressions of (107) and (115) are different, their graphs are almost the same. This shows that equations (106) and (114) do not describe a fundamentally different problem.

Finally, we verify the correctness of the above results by numerical solutions of fractional differential equations. According to the method in literature [5], for (106) and (119), the following iterative algorithms are, respectively, as follows:

$$\begin{align*}
N_k &= h^a \left( \frac{N_0}{x_k - a} - c^a N_{k-1} \right) - \sum_{j=0}^{k} w_j^{(a)} N_{k-j}, \quad (125) \\
N_k &= h^a \left( \frac{N_0\ln^p(a_0 + hk - a)}{(x_k - a)^p} - c^a N_{k-1} \right) - \sum_{j=0}^{k} w_j^{(a)} N_{k-j}, \quad (126)
\end{align*}$$

for $k = m, m + 1, \ldots$, where

$$x_k = a_0 + hk, N_k = N(x_k), w_j^{(a)} = (-1)^j \left( \begin{array}{c} a \\ j \end{array} \right), \quad a_0 > a. \quad (127)$$

Without loss of generality, let $N_0 = 1, N_k = N(x_k), k = 1, 2, \ldots, 3m$, where $N(x)$ is the analytic solution. The analytical solutions of (107) or (106) and the numerical
solution obtained by (125) are shown in (1) in Figure 2. The analytical solutions of (120) or (119) and the numerical solution obtained by (126) are shown in (2) in Figure 2.

As can be seen from Figure 2, the analytical solution obtained in this paper is basically consistent with the numerical solution.

6. Conclusions

As defined in this paper, the following function is consistent with traditional fractional integrals and derivatives:

\[(x-a)^\beta f(x), (x-a)^\beta \ln^p(x-a)f(x),\]

where \(f(x)\) is the analytic function, \(\beta \neq -1, -2, \ldots\), \(p = 1, 2, \ldots\), \(f(a) \neq 0\). The fractional integrals and derivatives of the following function can be obtained:

\[(x-a)^{-k}f(x), (x-a)^{-k}\ln^p(x-a)f(x),\]

where \(f(x)\) is the analytic function, \(k = 1, 2, \ldots\), \(p = 1, 2, \ldots\), \(f(a) \neq 0\). The following Leibniz derivative formula of the Riemann–Liouville fractional derivative still holds:

\[
\begin{align*}
\alphaD_x^\alpha [(x-a)^{-n}g(x)] &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{\alpha!}{k!} D_x^{\alpha-k} (x-a)^{-n}D^k g(x) \\
&= \frac{(-1)^{n-1}\Gamma(1+\alpha)\sin(\alpha\pi)}{\pi(n-1)! (x-a)^{n\alpha}} \\
&\times \sum_{k=0}^{\infty} (-1)^k \frac{(1 + \alpha + k - n)_{n-1}(y + \psi(1 + \alpha + k - n - H_{n-1} - \ln(x-a))(x-a)^k) g^{(k)}(x)}{k!}
\end{align*}
\]
and the properties

\[aD_x^{usef}[ (x-a)^{-k} f(x)] = \frac{d^k}{dx^k}[aD_x^{\alpha}[ (x-a)^{-k} f(x)]],\]

\[\lim_{\alpha \to 0} aD_x^{usef}[ (x-a)^{-k} f(x)] = \frac{d^n}{dx^n}[ (x-a)^{-k} f(x)],\]

\[\lim_{\alpha \to n} aD_x^{usef}[ (x-a)^{-k} \ln^p (x-a) f(x)] = \frac{d^n}{dx^n}[ (x-a)^{-k} \ln^p (x-a) f(x)],\]

\[aD_x^\alpha(x-a)^{a-k} = 0, \quad k = 1, 2, \ldots,\]

are true.

Therefore, we present the definitions of the fractional derivative and integrals given by the neutrix limit in our paper. Based on it, the solutions to some of the differential and integral equations given in the previous section cannot be expressed by traditional methods, and the solutions to these differential and integral equations are also expressed by the definitions in this paper.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This work is supported by the National Natural Science Foundation of China (61379009 and 61771010) and the Natural Science Foundation of Shandong Province (ZR2021MA017).

**References**


