Hindawi
Journal of Mathematics
Volume 2022, Article ID 8377463, 13 pages
https://doi.org/10.1155/2022/8377463

Research Article

Structure of k-Quasi-(m, n)-Isosymmetric Operators

Sid Ahmed Ould Ahmed Mahmoud[1] **El Moctar Ould Beiba**[2] **Sidi Hamidou Jah**[3] **and Maawiya Ould Sidi**[1]

[1] *Mathematics Department, College of Science, Jouf University, Sakaka P.O.Box 2014, Saudi Arabia*
[2] *Department of Mathematics and Computer Sciences, Faculty of Sciences and Techniques, University of Nouakchott, Al Aasriya P.O. Box 5026, Nouakchott, Mauritania*
[3] *Department of Mathematics, College of Science Qassim University, P.O. Box 6640, Buraydah 51452, Saudi Arabia*

Correspondence should be addressed to Sid Ahmed Ould Ahmed Mahmoud; sidahmed@ju.edu.sa

Received 24 April 2022; Accepted 3 August 2022; Published 26 September 2022

Academic Editor: Ali Jaballah

The investigation of new operators belonging to some specific classes has been quite fashionable since the beginning of the century, and sometimes it is indeed relevant. In this study, we introduce and study a new class of operators called k-quasi-(m, n)-isosymmetric operators on Hilbert spaces. This new class of operators emerges as a generalization of the (m, n)-isosymmetric operators. We give a characterization for any operator to be k-quasi-(m, n)-isosymmetric operator. Using this characterization, we prove that any power of an k-quasi-(m, n)-isosymmetric operator is also an k-quasi-(m, n)-isosymmetric operator. Furthermore, we study the perturbation of an k-quasi-(m, n)-isosymmetric operator with a nilpotent operator. The product and tensor products of two k-quasi-(m, n)-isosymmetries are investigated.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$ and let

$$\mathbb{C}[u, v] = \left\{ P(u, v) = \sum_{\substack{0 \le k \le m \\ 0 \le l \le n}} c_{k,l} u^k v^l, \quad c_{k,l} \in \mathbb{C} \right\}. \tag{1}$$

For $R \in \mathcal{B}(\mathcal{H})$, we will write $\operatorname{ran}(R)$, $\ker(R)$, and R^* the range, the kernel (or null space), and the adjoint of R, respectively. Also, $\sigma_p(R), \sigma_{ap}(R), \sigma(R)$, and $\sigma_s(R)$ denote the point spectrum, the approximate spectrum, the spectrum, and the surjective spectrum R.

The hereditary functional calculus defines

$$P(R) = \sum_{\substack{0 \le k \le m \\ 0 \le l \le n}} c_{k,l} R^{*l} R^k, \quad \text{for } P \in \mathbb{C}[u, v]. \tag{2}$$

It is easy to check that for $P \in \mathbb{C}[u, v]$ and $Q \in \mathbb{C}[u, v]$, we have

$$PQ(R) = \sum_{k,l} c_{k,l} R^{*l} Q(R) R^k = \sum_{k,l} d_{k,l} R^{*l} P(R) R^k. \tag{3}$$

Recall that an operator $R \in \mathcal{B}(\mathcal{H})$ is called a hereditary root or simply root of $P \in \mathbb{C}[u, v]$ if $P(R) = 0$. For more details on the hereditary functional calculus, we refer the reader to [4, 5].

In recent years, the concepts of m-isometric operators and the related classes of operators, namely, n-quasi-m-isometries, (m, C)-isometries, and n-quasi-(m, C)-isometries have received substantial attention. Several authors have been introduced, and these classes of operators are studied intensively in the papers [12–18], [20–23, 28, 32]. It has been proved that some products of m-isometries are again m-isometry [8, 11], the powers of an m-isometry are m- isometries, and the perturbation of m-isometries by nilpotent operators has been studied in [6, 9, 10]. The dynamics of m-isometries has been explored in [7]. Almost all of these properties have been extended to n-quasi-m-isometric operators, (m, C)-isometries, and n-quasi-(m, C)-isometries. The reader can refer to the papers [12–14, 17, 18, 20–23, 32] for more details.

Let $R \in \mathscr{B}(\mathscr{H})$ and m, n, and k be positive integers.

(1) R is called m-isometry [1–3] if it is a root of $P(u, v) = (vu - 1)^m$, that is,

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} R^{*k} R^k = 0. \tag{4}$$

(2) R is called m-symmetry [15, 27] if it is a root of $P(u, v) = (v - u)^m$, that is,

$$\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} R^{*(m-k)} R^k = 0. \tag{5}$$

(3) R is called k-quasi-m-isometry [19, 20, 31] if it is a root of $P(u, v) = v^k (vu - 1)^m u^k$, that is,

$$R^{*k} \left(\sum_{0 \leq j \leq m} (-1)^j \binom{m}{j} R^{*j} R^j \right) R^k = 0. \tag{6}$$

(4) R is called k-quasi-n-symmetric [33] if it is a root of $P(u, v) = v^k (v - u)^m u^k$, that is,

$$R^{*k} \left(\sum_{0 \leq j \leq n} (-1)^j \binom{n}{j} R^{*n-j} R^j \right) R^k = 0. \tag{7}$$

(5) R is called (m, n)-isosymmetric [29, 30] if it is a root of

$$P(u, v) = (vu - 1)^m (v - u)^n, \tag{8}$$

that is,

$$\begin{aligned} &\sum_{0 \leq j \leq m} (-1)^j \binom{m}{j} R^{*(m-j)} \left(\sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} R^{*(n-k)} R^k \right) R^{m-j}. \\ &= \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} R^{*(n-k)} \left(\sum_{0 \leq j \leq m} (-1)^j \binom{m}{j} R^{*(m-j)} R^{m-j} \right) R^k \\ &= 0. \end{aligned} \tag{9}$$

For $n, m \in \mathbb{N}$, set $\alpha_n(u, v) = (v - u)^n$, $\beta_m(u, v) = (vu - 1)^m$, and

$$\gamma_{m,n}(u, v) = \beta_m(u, v)\alpha_n(u, v) = \alpha_n(u, v)\beta_m(u, v). \tag{10}$$

For $R \in \mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned} \alpha_n(R) &= \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} R^{*n-k} R^k, \\ \beta_m(R) &= \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} R^{*m-k} R^{m-k}, \\ \gamma_{m,n}(R) &= \begin{cases} \sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} R^{*n-k} \beta_m(R) R^k, \\ \text{or} \\ \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} R^{*m-k} \alpha_n(R) R^{m-k}. \end{cases} \end{aligned} \tag{11}$$

It is easy to see that

$$\begin{cases} \gamma_{m+1,n}(R) = R^* \gamma_{m,n}(R) R - \gamma_{m,n}(R), \\ \gamma_{m,n+1}(R) = R^* \gamma_{m,n}(R) - \gamma_{m,n}(R) R. \end{cases} \tag{12}$$

It is well known that a common way to prepare a scientific study is to introduce some new mathematical objects and then state several results related to them. The investigation of new operators belonging to some specific classes has been quite fashionable since the beginning of the century, and sometimes it is indeed relevant. The motivation of this study is to introduce and study the concept of k-Quasi-(m, n)-isosymmetric operators on Hilbert spaces. This new class of operators emerges as a generalization of the (m, n)-isosymmetric operators. It is proved that there is an operator which is a k-quasi-$(n; m)$-isosymmetric operator, but not (n, m)-isosymmetric, and thus, the proposed new class is larger than the class of (n, m)-isosymmetric operators. In section two, we give a matrix characterization of k-quasi-(m, n)-isosymmetric operators in terms of (m, n)-isosymmetric operators. We give some results related to this class by using this matrix representation. In section three, we investigate some spectral properties of k-quasi-(m, n)-isosymmetric operators; in particular, we explore conditions for k-quasi-(n, m)-isosymmetric operators to be k-quasi-m-isometric operators or k-quasi-n-symmetric operators. Finally, in section forth, we study the sum of an k-quasi-(m, n)-isosymmetric operator with a nilpotent operator. We also study the product and tensor product of two k-quasi-(m, n)-isosymmetric operators.

2. Structure of k-Quasi-(m, n)-Isosymmetric Operators

In the present section, we give the definition and basic properties of k-quasi-(m, n)-isosymmetric operators. The obtained results improve and generalize some works on m-isometries, n-quasi-m-isometric, n-symmetries, and k-quasi-n-symmetric operators.

Definition 1. An operator $R \in \mathscr{B}(\mathscr{H})$ is said to be k-quasi-(m,n)-isosymmetric operator if R is a root of the polynomial

$$P_{m,n,k}(u,v) = v^k \gamma_{m,n}(u,v)u^k, \tag{13}$$

for some positive integers m, n, and k, or equivalently,

$$\begin{aligned} &R^{*k}\left(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}R^{*(m-j)}\left(\sum_{0\leq r\leq n}(-1)^r\binom{n}{r}R^{*(n-r)}R^r\right)R^{m-j}\right)R^k \\ &= R^{*k}\cdot\left(\sum_{0\leq r\leq n}(-1)^r\binom{n}{r}R^{*(n-r)}\left(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}R^{*(m-j)}R^{m-j}\right)R^r\right)R^k \\ &= 0. \end{aligned} \tag{14}$$

Example 1

(i) Every m-isometric operator is an k-quasi-(m,n)-isosymmetric, and every n-symmetric operator is an k-quasi-(m,n)-isosymmetric operator
(ii) Every (m,n)-isosymmetric operator is an k-quasi-(m,n)-isosymmetric operator
(iii) Every k-quasi-m-isometric operator is an k-quasi-(m,n)-isosymmetric operator
(iv) Every k-quasi-n-symmetric operator is an k-quasi-(m,n)-isosymmetric operator

Remark 1. Since $q \geq k$, $r \geq m$, and $s \geq n$, the polynomial $v^k(vu-1)^m(v-u)^n u^k$ divides $v^q(vu-1)^r(v-u)^s u^q$; it follows that if r is a k-quasi-(m,n)-isosymmetric operator, then it is a q-quasi-(r,s)-isosymmetric operator.

Remark 2

(1) If $n = m = k = 1$, 1-quasi-$(1,1)$-isosymmetric operator is a quasi-isosymmetric, i.e., an operator R is quasi-isosymmetric if and only if

$$R^*\left(R^{*2}R - R^*R^2 - R^* + R\right)R = R^{*3}R^2 - R^{*2}R^3 - R^{*2}R + R^*R^2 = 0. \tag{15}$$

(2) An operator R is quasi-$(2,1)$-isosymmetric if and only if

$$R^*\left(R^{*3}R^2 - R^{*2}R^3 - 2R^{*2}R + 2R^*R^2 + R^* - R\right)R = 0. \tag{16}$$

(3) An operator T is quasi-$(1,2)$-isosymmetric if and only if

$$R^*\left(R^{*3}R - 2R^{*2}R^2 + R^*R^3 - R^{*2} + 2R^*R - R^2\right)R = 0. \tag{17}$$

Remark 3. In the following example, we show that there is an operator which is k-quasi-(m,n)-isosymmetric, but not (m,n)-isosymmetric for some positive integers $n, m, k \geq 1$, and therefore, the proposed new class is large than the class of (n,m)-isosymmetric operators.

Example 2. Let $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{B}(\mathbb{C}^3)$. The direct calculation shows that

$$R^*\left(R^{*2}R - R^*R^2 - R^* + R\right)R = 0 \text{ and } R^{*2}R - R^*R^2 - R^* + R \neq 0. \tag{18}$$

Thus, R is a quasi-isosymmetric but not isosymmetric operator.

Example 3

(1) Let $R = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. A simple calculation shows that

$$\alpha_n(R) \neq 0,\ \beta_m(R) \neq 0,\ \text{and}\ \gamma_{m,n,k}(R) = 0. \tag{19}$$

Thus, R is a k-quasi-(m,n)-isosymmetric operator; however, R is neither m-isometry nor n-symmetry.

(2) Let $R = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. A simple calculation shows that

$$\begin{aligned} &R^*(R^*R - I)R \neq 0,\quad R^*(R^* - R)R \neq 0 \text{ and} \\ &R^*\left(R^{*2}R - R^*R^2 - R^* + R\right)R = 0. \end{aligned} \tag{20}$$

Thus, R is a quasi-isosymmetric operator; however, R is neither quasi-isometry nor quasi-symmetry.

Remark 4. The following inclusions hold:

$$\begin{aligned}&[m-\text{isometry}]\nsubseteq[k-\text{quasi}\;-m-\;\text{isometry}]\nsubseteq[k-\text{quasi}-(m,n)-\text{isosymmetry}],\\&[n-\text{symmetry}]\nsubseteq[k-\text{quasi}-n-\text{symmetry}]\nsubseteq[k-\text{quasi}-(m,n)-\text{isosymmetry}],\\&[(m,n)-\text{isosymmetry}]\nsubseteq[k-\text{quasi}-(m,n)-\text{isosymmetry}].\end{aligned}\tag{21}$$

Proposition 1. *Let $R\in\mathscr{B}(\mathscr{H})$, then the following statements are equivalent:*

(1) R is k-quasi-(m,n)-isosymmetric operator

(2) $\langle\gamma_{m,n}(R)y\,|y\rangle=0,\ \forall\, y\in\overline{ran(R^k)}$

Proof

$$\begin{aligned}&R\text{ is a }k-\text{quasi}-(m,n)-\text{isosymmetric}\\&\Leftrightarrow R^{*k}\gamma_{m,n}(R)R^k=0,\\&\Leftrightarrow\langle R^{*k}\gamma_{m,n}(R)R^kx\,|\,x\rangle=0,\ \forall\, x\in\mathscr{H}\\&\Leftrightarrow\langle\gamma_{m,n}(R)R^kx\,|\,R^kx\rangle=0,\ \forall\, x\in\mathscr{H}\\&\Leftrightarrow\langle\gamma_{m,n}(R)y\,|\,y\rangle=0,\ \forall\, y\in\overline{\text{ran}(R^k)}.\end{aligned}\tag{22}$$

□

Corollary 1. *Let $R\in\mathscr{B}(\mathscr{H})$ with k and q are two non-negative integers such that$\overline{ran(R^k)}=\overline{ran(R^q)}$, then R is k-quasi-$(m,n)$-isosymmetric operator if and only if R is q-quasi-(m,n)-isosymmetric operator.*

Proof. Straightforward from Proposition 1. □

Theorem 1. *Let $R\in\mathscr{B}(\mathscr{H})$ be k -quasi- (m,n) -isosymmetric operator. If $ker(R^{*q})=ker(R^{*(q+1)})$ for some $1\le q\le k-1$, then R is q -quasi- (m,n) -isosymmetric.*

Proof. From the assumptions $\ker(R^{*q})=\ker(R^{*(q+1)})$ and $q\le k-1$, it follows that $\ker(R^{*q})=\ker(R^{*k})$. Thus, $\overline{\text{ran}(R^k)}=\overline{\text{ran}(R^q)}$. Applying Corollary 1, we get the desired conclusion. □

Proposition 2. *Let $\mathscr{M}$ be a closed subspace of $\mathscr{H}$ which reduces R . If R is k -quasi-(m,n)-isosymmetric, then $R|_{\mathscr{M}}$ is k -quasi- (m,n) -isosymmetric.*

Proof. Let $S=R|_{\mathscr{M}}$ be the restriction of R to $\mathscr{M}$. On the one hand, we have

$$\gamma_{m,n}(S)=\gamma_{m,n}(R)|_{\mathscr{M}}.\tag{23}$$

On the other hand, we have $\overline{\text{ran}(S^k)}^{\mathscr{M}}\subset\overline{\text{ran}(R^k)}$, where $\overline{\text{ran}(S^k)}^{\mathscr{M}}$ is the closer of $\text{ran}(S^k)$ in $\mathscr{M}$. Thus, $\langle\gamma_{m,n}(S)y|y\rangle=0,\ \forall\, y\in\overline{\text{ran}(S^k)}$ since $\langle\gamma_{m,n}(R)y|y\rangle=0$, $\forall\, y\in\overline{\text{ran}(R^k)}$. Therefore, by statement (2) of Proposition 1, $S=R|_{\mathscr{M}}$ is k-quasi-(m,n)-isosymmetric on $\mathscr{M}$. □

The following theorem characterizes the members of k-quasi-(m,n)-isosymmetric operators.

Theorem 2. *Let $R\in\mathscr{B}(\mathscr{H})$ such that $\mathscr{H}\neq\overline{ran(R^k)}$. Then, the following properties are equivalent:*

(1) Risk-quasi-(m,n)-isosymmetric operator

(2) $R=\begin{pmatrix}R_1&R_2\\0&R_3\end{pmatrix}on\mathscr{H}=\overline{ran(R^k)}\oplus ker(R^{*k})$, where-$R_1=R|_{\overline{ran(R^k)}}$is an$(m,n)$-isosymmetric operator and$R_3^k=0$

Proof. (1)⇒(2). By taking into account the matrix representation related to the decomposition $\mathscr{H}=\overline{\text{ran}(R^k)}\oplus\ker(R^{*k})$ as $R=\begin{pmatrix}R_1&R_2\\0&R_3\end{pmatrix}$, we get

$$\begin{aligned}\begin{pmatrix}R_1&0\\0&0\end{pmatrix}&=RP_{|\overline{\text{ran}(R^k)}}\\&=P_{|\overline{\text{ran}(R^k)}}RP_{|\overline{\text{ran}(R^k)}},\end{aligned}\tag{24}$$

where $P_{|\overline{\text{ran}(R^k)}}$ is the orthogonal projection onto $\overline{\text{ran}(R^k)}$.

From the condition that R is k-quasi-(m,n)-isosymmetric operator, we have

$$P_{|\overline{\text{ran}(R^k)}}\left(\sum_{0\le j\le m}(-1)^j\binom{m}{k}R^{*j}\alpha_n(R)R^j\right)P_{|\overline{\text{ran}(R^k)}}=0,\tag{25}$$

or

$$P_{|\overline{\text{ran}(R^k)}}\left(\sum_{0\le r\le n}(-1)^r\binom{n}{r}R^{*(n-r)}\beta_m(R)R^r\right)P_{|\overline{\text{ran}(R^k)}}=0.\tag{26}$$

From this, we deduce that

$$\sum_{0\le j\le m}(-1)^j\binom{m}{j}R_1^{*j}\alpha_n(R_1)R_1^j=0,\tag{27}$$

or

$$\sum_{0\le r\le n}(-1)^r\binom{n}{r}R_1^{*(n-r)}\beta_m(R_1)R_1^r=0.\tag{28}$$

Therefore, R_1 is (m,n)-an isosymmetric operator.

On the other hand, let $z = z_1 + z_2 \in \mathcal{H} = \overline{\operatorname{ran}(R^k)} \oplus \ker(R^{*k})$. The direct calculation shows that

$$\begin{aligned}\langle R_3^k z_2 | z_2\rangle &= \langle R^k\Big(I - P_{|\overline{\operatorname{ran}(R^k)}}\Big)z, \Big(I - P_{|\overline{\operatorname{ran}(R^k)}}\Big)z\rangle \\ &= \langle\Big(I - P_{|\overline{\operatorname{ran}(R^k)}}\Big)z, R^{*k}\Big(I - P_{|\overline{\operatorname{ran}(R^k)}}\Big)z\rangle \\ &= 0.\end{aligned} \tag{29}$$

So that, $R_3^k = 0$.

(2)⇒(1) Assume that $R = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix}$ onto $\mathcal{H} = \overline{\operatorname{ran}(R^k)} \oplus \ker(R^{*k})$, with

$$\begin{aligned}\gamma_{m,n}(R_1) &= \sum_{0\le r\le n} (-1)^r \binom{n}{r} R_1^{*\,n-r}\beta_m(R_1)R_1^r \\ &= \sum_{0\le j\le m} (-1)^j \binom{m}{k} R_1^{*\,m-j}\alpha_n(R_1)R_1^{m-j} \\ &= 0,\end{aligned} \tag{30}$$

and $R_3^k = 0$.

Direct calculation shows that $R^k = \begin{pmatrix} R_1^k & \sum_{0\le j\le k-1} R_1^j R_2 R_3^{k-1-j} \\ 0 & R_3^k \end{pmatrix} = \begin{pmatrix} R_1^k & w_k \\ 0 & 0 \end{pmatrix}$ and therefore, $R^{*k} = \begin{pmatrix} R_1^{*k} & 0 \\ w_k^* & 0 \end{pmatrix}$. Moreover,

$$R^k R^{*k} = \begin{pmatrix} R_1^k R_1^{*k} + w_k w_k^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_k & 0 \\ 0 & 0 \end{pmatrix}, \tag{31}$$

where $C_k = R_1^k R_1^{*k} + w_k w_k^* = C_k^*$.

$$\begin{aligned}&R^k R^{*k}\left(\sum_{0\le j\le m} (-1)^j \binom{m}{j} R^{*j}\alpha_n(R)R^j\right)R^k R^{*k} \\ &= \begin{pmatrix} C_k & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_{m,n}(R_1) & A \\ B & D \end{pmatrix}\begin{pmatrix} C_k & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} C_k\alpha_{m,n}(R_1)C_k & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0.\end{aligned} \tag{32}$$

This means that

$$\begin{aligned}&\langle R^k R^{*k}\left(\sum_{0\le j\le m} (-1)^j \binom{m}{j} R^{*j}\alpha_n(R)R^j\right)R^k R^{*k}x | x\rangle = 0 \\ &\Rightarrow \langle R^{*k}\left(\sum_{0\le j\le m} (-1)^j \binom{m}{j} R^{*j}\alpha_n(R)R^j\right)R^k R^{*k}x | R^{*k}x\rangle = 0 \\ &\Rightarrow R^{*k}\left(\sum_{0\le j\le m} (-1)^j \binom{m}{j} R^{*j}\alpha_n(R)R^j\right)R^k = 0 \quad \text{on } \overline{\operatorname{ran}\big(R^{*k}\big)} = \ker\big(R^k\big)^\perp.\end{aligned} \tag{33}$$

Obviously, $R^{*k}\left(\sum_{0\le j\le m} (-1)^j \binom{m}{j} R^{*j}\alpha_n(R)R^j\right)R^k = 0$ on $\ker(R^k)$, and consequently,

$$R^{*k}\left(\sum_{0\le j\le m} (-1)^j \binom{m}{j} R^{*j}\alpha_n(R)R^j\right)R^k = 0, \tag{34}$$

on $\mathcal{H} = \ker(R^k) \oplus \ker(R^k)^\perp$.

Therefore, R is a k-quasi-(m,n)-isosymmetric operator. □

Corollary 2. *If $R \in \mathcal{B}(\mathcal{H})$ is an k-quasi-(m,n)-isosymmetric such that $\operatorname{ran}(R^k)$ is dense, then R is an (m,n)-isosymmetric.*

Proof. This is a direct consequence of Theorem 2. □

Corollary 3. *If $R \in \mathcal{B}(\mathcal{H})$ is an invertible k-quasi-(m,n)-isosymmetric operator, then R^{-1} is a k-quasi-(m,n)-isosymmetric operator.*

Proof. Under the assumption that R is an invertible k-quasi-(m,n)-isosymmetric operator, it follows that R is an (m,n)-isosymmetric operator, and so is R^{-1} by Theorem 2.4 in [24]. Therefore, R^{-1} is a k-quasi-(m,n)-isosymmetric operator. □

Corollary 4. *Let $R \in \mathcal{B}(\mathcal{H})$ be a k-quasi-(m,n)-isosymmetric operator such that $\overline{\operatorname{ran}(R^k)} \neq \mathcal{H}$. If the restriction $R_1 = R|_{\overline{\operatorname{ran}(R^k)}}$ is invertible, then R is similar to a direct sum of an (m,n)-isosymmetric operator and a nilpotent operator with an index of nilpotence less than or equal k.*

Proof. Consider the matrix decomposition of R as

$$T = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\operatorname{ran}\big(R^k\big)} \oplus \ker\big(R^{*k}\big). \tag{35}$$

Then, R_1 is an (m,n)-isosymmetric operator by Theorem 2. Since R_1 is invertible, we have $0 \notin \sigma(R_1)$, which implies $\sigma(R_1) \cap \sigma(R_3) = \varnothing$. By Rosenblum's corollary [26], it follows that there exists $A \in \mathcal{B}(\mathcal{H})$ for which $R_1 A - A R_3 = R_2$. Therefore, we have

$$R = \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ 0 & R_3 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} R_1 & 0 \\ 0 & R_3 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} (R_1 \oplus R_3) \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}. \quad (36)$$

□

It was proved that power of m-isometric (resp m-symmetric) operator is again m-isometric (resp m-symmetric) operator. The following corollary shows that the same property holds for k-quasi-(m,n)-isosymmetric operators.

Corollary 5. *If $R \in \mathcal{B}(\mathcal{H})$ is a k-quasi-(m,n)-isosymmetric operator, then R^q is also a k-quasi-(m,n)-isosymmetric operator for any positive integer q.*

Proof. Let $\overline{\operatorname{ran}(R^k)} = \mathcal{H}$, then R is an (m,n)-isosymmetric and so is R^q for any $q \in \mathbb{N}$ by Theorem 2.4 in [24]. If $\overline{\operatorname{ran}(R^k)} \neq \mathcal{H}$, we can use the decomposition of $R = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\operatorname{ran}(R^k)} \oplus \ker(R^{*k})$, where R_1 is an (m,n)-isosymmetric operator and so is R_1^q. On the other hand, observing that

$$R^q = \begin{pmatrix} R_1^q & \sum_{0 \le j \le q-1} R_1^j R_2 R_3^{q-1-j} \\ 0 & R_3^q \end{pmatrix}, \quad (37)$$

it follows that R^q is k-quasi-(m,n)-isosymmetric operator by applying Theorem 2. □

Theorem 3. *Let $\mathbf{R} = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. If R_1 is a surjective (m,n)-isosymmetric operator and $R_3^k = 0$, then $\mathbf{R}$ is similar to an k-quasi-(m,n)-isosymmetric operator.*

Proof. Under the assumptions on R_1 and R_3, we have $\sigma_s(R_1) \cap \sigma_{ap}(R_3) = \emptyset$. From the axiom (c) in Theorem 3.5.1 in [16], it follows that there exists some operator $A \in \mathcal{B}(\mathcal{H})$ such that $R_1 A - A R_3 = R_2$. Hence,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix} = \begin{pmatrix} R_1 & 0 \\ 0 & R_3 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}. \quad (38)$$

From this, we deduce that R is similar to $W = \begin{pmatrix} R_1 & 0 \\ 0 & R_3 \end{pmatrix}$.

By using the facts that R_1 is a (m,n)-isosymmetric and $R_3^k = 0$, we can obtain

$$W^{*k}\left(\sum_{0 \le j \le m} (-1)^j \binom{m}{j} W^{*j} \alpha_n(W) W^j\right) W^k = \begin{pmatrix} R_1^{*k}\left(\sum_{0 \le j \le m} (-1)^j \binom{m}{j} R_1^{*j} \alpha_n(R_1) R_1^j\right) R_1^k & 0 \\ 0 & 0 \end{pmatrix} = 0. \quad (39)$$

Consequently, $\mathbf{R}$ is similar to a k-quasi-(m,n)-isosymmetric operator. □

3. Spectral Properties

In this section, we investigate some spectral properties of k-quasi-(m,n)-isosymmetric operators.

Let $P \in \mathbb{C}[u,v]$ and $R \in \mathcal{B}(\mathcal{H})$. Let $u, v \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$ be such that $Ru = \lambda u$ and $Rv = \mu v$, we get by a little calculation that

$$\langle P(R)u | v\rangle = P(\lambda, \overline{\mu})\langle u | v\rangle. \quad (40)$$

It follows from (40) that if R is a root of p and λ is a spectrum point of R, then $p(\lambda, \overline{\lambda}) = 0$.

We have the following theorem.

Theorem 4. *Let $P \in \mathbb{C}[u,v]$ and $R \in \mathcal{B}(\mathcal{H})$. If $\lambda, \mu \in \sigma_{ap}(R)$ and $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be two sequences of unit vectors such that $(R-\lambda)u_n \longrightarrow 0$ and $(R-\mu)v_n \longrightarrow 0$ as $n \longrightarrow \infty$, then*

$$\langle P(R)u_n, v_n\rangle - p(\lambda, \overline{\mu})\langle u_n | v_n\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (41)$$

Proof. For $u \in \mathcal{H}$, $\alpha \in \mathbb{C}$, and $r \in \mathbb{N}$, we have

$$(R^r - \alpha^r)u = \left(\sum_{0 \le j \le r-1} R^j \alpha^{r-j-1}\right)(R-\alpha)U. \quad (42)$$

Thus, we get

$$\|(R^r - \alpha^r)u\| \le \left(\sum_{0 \le j \le r-1} \|R\|^j |\alpha|^{r-j-1}\right)\|(R-\alpha)u\|. \quad (43)$$

Let $m, n \in \mathbb{N}$. Set

$$w_{k,n} = R^n u_k - \lambda^n u_k,\ z_{k,m} = R^m v_k - \mu^m v_k. \quad (44)$$

Applying (43) to λ and μ, respectively, we obtain

$$\|w_{k,n}\| \le \left(\sum_{0 \le j \le n-1} \|R\|^j |\lambda|^{n-j-1}\right)\|Ru_k - \lambda u_k\|,$$
$$\|z_{k,m}\| \le \left(\sum_{0 \le j \le m-1} \|R\|^j |\mu|^{m-j-1}\right)\|Rv_k - \mu v_k\|. \quad (45)$$

Thus, $w_{k,n} \longrightarrow 0$ and $z_{k,m} \longrightarrow 0$ as $k \longrightarrow 0$.
On the other hand, it is easy to verify

$$\begin{aligned}&\langle R^{*m}R^n u_k, v_k\rangle - \overline{\mu}^m\lambda^n\langle u_k, v_k\rangle \\ &\quad = \langle w_{k,n}, z_{k,m}\rangle + \langle w_{k,n}, \mu^m y_k\rangle + \langle \lambda^n u_k, z_{k,m}\rangle.\end{aligned} \tag{46}$$

Since the right side of the previous equality tends to 0, we obtain

$$\langle R^{*m}R^n u_k, v_k\rangle - \overline{\mu}^m\lambda^n\langle u_k, v_k\rangle \longrightarrow 0. \tag{47}$$

Taking linear combinations, we get

$$\langle P(R)u_k , v_k\rangle - P(\lambda, \overline{\mu})\langle u_k, v_k\rangle \longrightarrow 0. \tag{48}$$

□

Corollary 6. *Let $P \in \mathbb{C}[u,v]$ and $R \in \mathscr{B}(\mathscr{H})$. Let $\lambda, \mu \in \sigma_{ap}(R)$ and $(u_k)_{k\in\mathbb{N}} \subset \mathscr{H}$ and $(v_k)_{k\in\mathbb{N}} \subset \mathscr{H}$ such that $\|u_k\| = \|v_k\| = 1$, $Ru_k - \lambda u_k \longrightarrow 0$, and $Rv_k - \mu v_k \longrightarrow 0$. If R is a root of P, then one of the following statements holds:*

(1) $\langle u_k | t v_k\rangle \longrightarrow 0$ ask $\longrightarrow \infty$

(2) $P(\lambda, \overline{\mu}) = 0$

Proof. Since R is a root of P, we have $\langle P(R)u_k , v_k\rangle = 0$, $\forall k$. This yields, from (41), the desired conclusion. □

Corollary 7. *Let $P \in \mathbb{C}[u,v]$ and $R \in \mathscr{B}(\mathscr{H})$. Let $u, v \in \mathscr{H}$ be nonzero vectors and $\lambda, \mu \in \mathbb{C}$ be such that $Ru = \lambda u$ and $Rv = \mu v$. If R is a root of P, then one of the following two statements holds:*

(1) $\langle u, v\rangle = 0$

(2) $P(\lambda, \overline{\mu}) = 0$

Proof. Take $u_k = u$ and $v_k = v$ in Corollary 6. □

As a consequence of Theorem 4, we have the following two results due to Stankus [30].

Corollary 8 *(Proposition 21 [30]). Let$R \in \mathscr{B}(\mathscr{H})andP \in \mathbb{C}[u,v]$. Then,*

$$\sigma_{ap}(R) \subset \left\{\lambda \in \mathbb{C}, P(\lambda, \overline{\lambda}) \in \overline{W(P(R))}\right\}. \tag{49}$$

Corollary 9 *(Corollary 22 in [30]). Let$R \in \mathscr{B}(\mathscr{H})andP \in \mathbb{C}[u,v]$. IfRis a root of P, then*

$$\sigma_{ap}(R) \subset \left\{\lambda \in \mathbb{C}, P(\lambda, \overline{\lambda}) = 0\right\}. \tag{50}$$

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert spaces, $R \in \mathscr{B}(\mathscr{H})$ and $S \in \mathscr{B}(\mathscr{K})$ and

$$P(u,v) = \sum_{k,l\geq 0} c_{k,l} v^l u^k \in \mathbb{C}[u,v]. \tag{51}$$

Consider the bounded linear transformation

$$\Phi_{R,S,P}\colon \mathscr{B}(\mathscr{H},\mathscr{K}) \longrightarrow \mathscr{B}(\mathscr{H},\mathscr{K}), \tag{52}$$

defined by

$$\Phi_{R,S,P}(X) = \sum_{k,l\geq 0} c_{k,l} S^l X R^k \text{ for } X \in \mathscr{B}(\mathscr{H},\mathscr{K}). \tag{53}$$

We also define the maps $A_R, L_S\colon \mathscr{B}(\mathscr{H},\mathscr{K}) \longrightarrow \mathscr{B}(\mathscr{H},\mathscr{K})$ by

$$A_R(X) = XR, \quad L_S(X) = SX \text{ for } X \in \mathscr{B}(\mathscr{H},\mathscr{K}). \tag{54}$$

A_R and L_S commute. Indeed, for $X \in \mathscr{B}(\mathscr{H},\mathscr{K})$, we have

$$\begin{aligned} A_R L_S(x) &= A_R(SX) \\ &= SXR \\ &= L_S(XA) \\ &= L_S(A_R(X)) \\ &= L_S A_R(X), \end{aligned} \tag{55}$$

Lemma 1. *Let $\mathscr{H}$, $\mathscr{K}$, S , R , A_T , L_S, and P be as above. Then,*

$$\Phi_{R,S,P} = P(A_R, L_S). \tag{56}$$

Proof. It is easy to check that for $m \in \mathbb{N}$, we have

$$A_R^m = A_{R^m} \text{ and } L_S^m = L_{S^m}. \tag{57}$$

Thus, for $n, m \in \mathbb{N}$, we have

$$A_R^m L_S^n(X) = S^n X R^m, \quad X \in \mathscr{B}(\mathscr{H},\mathscr{K}). \tag{58}$$

This yields

$$\begin{aligned}\Phi_{R,S,P}(X) &= \sum_{m,n\geq 0} c_{m,n} A_R^m L_S^n(X) \\ &= P(A_R, L_S)(X) \text{ for } X \in \mathscr{B}(\mathscr{H}).\end{aligned} \tag{59}$$

Therefore, $\Phi_{R,S,P} = P(A_R, L_S)$. □

We need the following lemmas.

Lemma 2 *(Lemma 0.11 in [25]). If A and B are commuting operators on the Banach space$\mathscr{X}$, then$\sigma(P(A,B)) \subset P(\sigma(A), \sigma(B)) = \{P(\lambda,\mu),\ \lambda \in \sigma(A), \mu \in \sigma(B)\}$for every polynomial$P \in \mathbb{C}[x,y]$.*

Lemma 3 *(Lemma 27 in [30]). Let $R \in \mathscr{B}(\mathscr{H})$, $P(u,v) = \sum_{m,n\geq 0} c_{m,n} v^n u^m \in \mathbb{C}[u,v]$, $Q \in \mathbb{C}[u,v]$, then $PQ(R) = \sum_{m,n\geq 0} c_{m,n} R^{*n} Q(R) R^m$.*

Remark 5. From (57), Lemma 3 is equivalent to

$$\Phi_{R,R^*,P}(Q(R)) = P(A_R, L_{R^*})(Q(R)) = PQ(R). \tag{60}$$

Remark 6. It is easy to verify that if R is an isomorphism, then L_R and A_R are also isomorphisms and we have $L_R^{-1} = L_{R^{-1}}$ and $A_R^{-1} = A_{R^{-1}}$. Since, for $\lambda \in \mathbb{C}$, $L_{R-\lambda} = L_R - \lambda$, and $A_{R-\lambda} = A_R - \lambda$, we get $\sigma(L_R) \subset \sigma(R)$ and $\sigma(A_R) \subset \sigma(R)$.

Proposition 3. *Let S , $R \in \mathscr{B}(\mathscr{H})$, and$P \in \mathbb{C}[u, v]$, then*

$$\sigma\left(\Phi_{R,S,P}\right) \subset P(\sigma(R), \sigma(S)) = \{P(\lambda, \mu),\ \lambda \in \sigma(R), \mu \in \sigma(S)\}. \tag{61}$$

Proof. From (57), we have $\sigma(\Phi_{R,S,P}) = \sigma(P(A_R, L_S))$. Thus, by Lemma 2, we obtain $\sigma(\Phi_{R,S,P}) \subset P(\sigma(A_R), \sigma(L_S))$. This yields, from Remark 6,

$$\sigma\left(\Phi_{R,S,P}\right) \subset P(\sigma(R), \sigma(S)). \tag{62}$$

□

Corollary 10. *Let $R \in \mathscr{B}(\mathscr{H})$ and $P \in \mathbb{C}[u, v]$, then*

$$\sigma\left(\Phi_{R,R^*,P}\right) \subset \{P(\lambda, \overline{\mu}),\ \lambda, \mu \in \sigma(R)\}. \tag{63}$$

Proof. Apply Proposition 3 by taking $S = R^*$. □

Proposition 4. *Let $R \in \mathscr{B}(\mathscr{H})$ and $P, Q \in \mathbb{C}[u, v]$. If $PQ(R) = 0$, then either $0 \in \sigma(\Phi_{R,R^*,P})$ or $Q(R) = 0$.*

Proof. Suppose that $0 \notin \sigma(\Phi_{R,R^*,p})$. This means that $\Phi_{R,R^*,P}$ is invertible. Thus, from (60), we obtain $Q(R) = 0$. □

From the following, we give a sufficient condition for a k-quasi-(m, n)-isosymmetric operator to be k-quasi m-isometric operator or k-quasi-n-symmetric operator.

Theorem 5. *Let $R \in \mathscr{B}(\mathscr{H})$ be a k -quasi- (m, n) -isosymmetric operator. The following statements hold:*

(1) If $\sigma(R) \cap \sigma(R^*) = \varnothing$, then R is ak-quasi-m-isometry

(2) If $0 \notin \{\lambda\overline{\mu} - 1, \lambda, \mu \in \sigma(R)\}$, then R is ak-quasi-n-symmetric operator

Proof

(1) Set $P_{m,n,k} = PQ$, where $P(u, v) = (v - u)^n$ and $Q(u, v) = v^k(vu - 1)^m u^k$. Since $\sigma(R) \cap \sigma(R^*) = \varnothing$, we have $0 \notin \{P(\lambda, \overline{\mu}), \lambda, \mu \in \sigma(R)\}$. Thus, from (63), we get $0 \notin \sigma(\Phi_{R,R^*,P})$. This yields, by Proposition 4, $Q(R) = 0$. Therefore, R is a k-quasi--$m$isometry.

(2) Set $P_{m,n,k} = PQ$, where $P(u, v) = (vu - 1)^m$ and $Q(u, v) = v^k(v - u)^n u^k$. Since $0 \notin \{\lambda\overline{\mu} - 1, \lambda\mu \in \sigma(R)\}$, similarly, as in (1), we obtain that $0 \notin \sigma(\Phi_{R,R^*,P})$. Applying again Proposition 4, we obtain $Q(R) = 0$. Thus, R is a k-quasi-n-symmetric operator. □

Theorem 6. *Let $R \in \mathscr{B}(\mathscr{H})$ be k -quasi- (m, n) -isosymmetric operator, then R has the single-valued extension property (SEVP).*

Proof. If $\mathscr{H} = \overline{\text{ran}(R^k)}$, then R is an (m, n)-isosymmetric operator, and therefore R has SVEP by Theorem 2.20 in [24]. If $\mathscr{H} \neq \overline{R(R^k)}$, we use the matrix decomposition of R as

$$R = \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix} on\, \mathscr{H} = \overline{\text{ran}\left(R^k\right)} \oplus \ker\left(R^{*k}\right). \tag{64}$$

From Theorem 2, R_1 is a (m, n)-isosymmetric operator and R_3 is a nilpotent operator. Hence, R_1 and R_3 have SVEP; then, by simple calculations, we see that R has SVEP as required. □

Rachid [24] showed that if R is (m, n)-isosymmetric operator, then $\sigma(R) \subset \partial\mathbb{D} \cup \mathbb{R}$, where $\partial\mathbb{D}$ is the unit circle. Now, we extend this result to k-quasi-(m, n)-isosymmetric operators.

Theorem 7. *Let R be n -quasi- (m, n) -isosymmetric operator, then $\sigma_{ap}(R) \subset \partial\mathbb{D} \cup \mathbb{R}$.*

Proof. Let $\gamma \in \sigma_{ap}(R)$, then there exists a sequence $(w_r)_{r\geq 1} \subset \mathscr{H}$, with $\|w_r\| = 1$ such that $(R - \gamma I)w_r \longrightarrow 0$ as $p \longrightarrow \infty$. We have $(R^j - \gamma^j I)w_r \longrightarrow 0$ as $r \longrightarrow \infty$ for all positive integers$j$. From the condition that R is an k-quasi-(m, n)-isosymmetric operator, one has

$$
\begin{aligned}
0 &= \left\langle R^{*k}\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}R^{*m-j}\alpha_n(R)R^{m-j}\right)R^k w_r | w_r\right\rangle\\
&= \left\langle \sum_{0\le j\le m}(-1)^j\binom{m}{j}\alpha_n(R)R^{m+k-j}w_r | R^{m+k-j}w_r\right\rangle\\
&= \left\langle \sum_{0\le j\le m}(-1)^j\binom{m}{j}\alpha_n(R)\left(\left(R^{m+k-j}-\gamma^{m+k-j}\right)w_r+\gamma^{m+k-j}w_r\right) | \left(R^{m+k-j}-\gamma^{m+k-j}\right)w_r+\gamma^{m+k-j}w_r\right\rangle\\
&= |\gamma|^{2k}\left(|\gamma|^2-1\right)^m\langle\alpha_n w_r|w_r\rangle \quad (r\longrightarrow\infty)\\
&= |\gamma|^{2k}\left(|\gamma|^2-1\right)^m\left\langle\sum_{0\le j\le n}(-1)^j\binom{m}{j}R^{*n-j}R^j w_r|w_r\right\rangle \quad (r\longrightarrow\infty)\\
&= |\gamma|^{2k}\left(|\gamma|^2-1\right)^m\left\langle\sum_{0\le j\le n}(-1)^j\binom{m}{j}R^j w_r|R^{n-j}w_r\right\rangle \quad (r\longrightarrow\infty)\\
&= |\gamma|^{2k}\left(|\gamma|^2-1\right)^m\left\langle\sum_{0\le j\le n}(-1)^j\binom{m}{j}\left(R^j-\gamma^j+\gamma^j\right)w_r|\left(R^{n-j}-\gamma^{n-j}+\gamma^{n-j}\right)w_r\right\rangle \quad (r\longrightarrow\infty)\\
&= |\gamma|^{2k}\left(|\gamma|^2-1\right)^m(\gamma-\overline{\gamma})^n\|w_r\|^2\\
&= |\gamma|^{2k}\left(|\gamma|^2-1\right)^m(2\mathrm{Im}(\gamma))^n.
\end{aligned}
\tag{65}
$$

Consequently, $\gamma=0$ or $|\gamma|=1$ or $\gamma\in\mathbb{R}$. This completes the proof. □

Proposition 5. *Let R be k -quasi- (m,n) -isosymmetric operator. If $\gamma=a+ib$ with $b\neq 0$ is an approximate eigenvalue of R , then $\overline{\gamma}$ is an approximate eigenvalue of R^* .*

Proof. Assume that $\gamma=a+ib\in\sigma_{ap}(R)$ with $b\neq 0$, then there exists $(w_r)_r\in\mathscr{H}:\|w_r\|=1$ such that $(R-\gamma)w_r\longrightarrow$ as $r\longrightarrow\infty$. By taking into account that R is a k-quasi-(m,n)-isosymmetric operator, it follows that

$$
\begin{aligned}
0 &= R^{*k}\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}R^{*m-j}\alpha_n(R)R^{m-j}\right)R^k w_r\\
&= R^{*k}\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}R^{*m-j}\alpha_n(R)\left(R^{m+k-j}-\gamma^{m+k-j}+\gamma^{m+k-j}\right)w_r\right)\\
&= R^{*k}\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}R^{*m-j}\alpha_n(R)\left(R^{m+k-j}-\gamma^{m+k-j}\right)w_r\right)+R^{*k}\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}R^{*m-j}\alpha_n(R)\gamma^{m+k-j}w_r\right)\\
&= R^{*k}\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}R^{*m-j}\alpha_n(R)\left(S^{m+k-j}-\gamma^{m+k-j}\right)w_r\right)+\gamma^n R^{*k}\left(I-\gamma R^*\right)^m\alpha_n(R)w_r.
\end{aligned}
\tag{66}
$$

By observing that

$$
\begin{aligned}
\alpha_n(R)w_r &= \sum_{0\le j\le n}(-1)^j\binom{n}{j}R^{n-j}R^j w_r\\
&= \sum_{0\le j\le n}(-)^j\binom{n}{j}R^{n-j}\left(R^j-\gamma^j+\gamma^j\right)w_r\\
&= \sum_{0\le j\le n}(-)^j\binom{n}{j}R^{n-j}\left(R^j-\gamma^j\right)w_r\\
&\quad+(\gamma-R^*)^n w_r,
\end{aligned}
\tag{67}
$$

and $\lim_{r\longrightarrow\infty}(R^s-\gamma^s)w_r=0$ for all positive integers s, we get from the above relations that

$$
\gamma^k R^{*k}\left(I-\gamma R^*\right)^m(\gamma-R^*)^n w_r\longrightarrow 0,\ \text{as}\, r\longrightarrow\infty. \tag{68}
$$

If $(I-\gamma R^*)$ is bounded from below, then so is $(I-\gamma R^*)^m$ and therefore there exists a positive constant $K>0$ such that

$$
\left\|\left(I-\gamma R^*\right)^m w\right\|\ge K\|w\|,\quad \forall\, w\in\mathscr{H}. \tag{69}
$$

From this, we deduce that

$$\|(I-\gamma R^*)^m R^{*k}(\gamma - R^*)^n w_r| \geq K\left\|R^{*k}(\gamma - R^*)^n w_r\right\|. \quad (70)$$

Consequently, $\|R^{*k}(\gamma - R^*)^n w_r\| \longrightarrow 0$, as $r \longrightarrow \infty$. So, we have

$$\begin{aligned} 0 &= \lim_{r\longrightarrow\infty} \langle R^{*k}(\gamma - R^*)^n w_r \mid w_r\rangle \\ &= \lim_{r\longrightarrow\infty} \langle w_r | (\overline{\gamma} - R)^n R^k w_r\rangle \\ &= \lim_{r\longrightarrow\infty} \langle w_r \mid (\overline{\gamma} - R)^n \left(R^k - \gamma^k\right) w_r\rangle \\ &+ \lim_{r\longrightarrow\infty} \langle w_r | \gamma^k (\overline{\gamma} - R)^n w_r\rangle \\ &= \overline{\gamma}^k \lim_{r\longrightarrow\infty} \langle w_r \mid (\overline{\gamma} - R)^n w_r\rangle \\ &= \overline{\gamma}^k (\gamma - \overline{\gamma})^n. \end{aligned} \quad (71)$$

Thus, $\overline{\gamma}^n = 0$ or $\gamma - \overline{\gamma} = 0$, which are a contradiction. Hence, $I - \gamma R^*$ is not bounded from below. In view of Theorem 7, we have $|\gamma| = 1$ and so $I - \gamma R^* = \gamma(\overline{\gamma} - R^*)$, which implies that $\overline{\gamma} - R^*$ is not bounded from below. This proves the statement of the proposition. □

4. Products and Perturbation of k-Quasi-(N, m)-Isosymmetric Operators

In this section, we study the perturbation of an k-quasi-(m,n)-isosymmetric operator by a nilpotent operator and we study the product and tensor product of two k-quasi-(m,n)-isosymmetric operators.

Lemma 4. *For $p, m, n \in \mathbb{N}$, the following identity holds:*

$$\left(v^p u^p - 1\right)^m \left(v^p - u^p\right)^n = \sum_{0\leq r\leq m(p-1)} \sum_{0\leq j\leq n(p-1)} \lambda_r \mu_j v^{(m+n)(p-1)-(r+j)} (vu-1)^m (v-u)^n u^{j+m(p-1)-r}, \quad (72)$$

where λ_r and μ_j are some constants.

Proof. See the proof of statement (ii) in Theorem 3.3 in [14]. □

The following theorem shows that the power of a k-quasi-(m,n)-isosymmetric operator is again a k-quasi-(m,n)-isosymmetric which is similar to that of Corollary 5 but with another proof.

Theorem 8. *If R is k -quasi- (m,n) -isosymmetric, then R^p is k -quasi - (m,n) -isosymmetric for any $p \in \mathbb{N}_0$.*

Proof. Two different proofs of this statement will be given.

First Proof. We need to prove that $(R^p)^{*k}\gamma_{m,n}(R^p)(R^p)^k = 0$ for any $p \in \mathbb{N}$.

In fact, from Lemma 4, we obtain that

$$\begin{aligned} (R^p)^{*k}\gamma_{m,n}(R^p)(R^p)^k &= (R^p)^{*k}\left(\sum_{0\leq r\leq m(p-1)} \sum_{0\leq j\leq n(p-1)} \lambda_r\mu_j (R^{*p})^{(m+n)(p-1)-(r+j)} \gamma_{mn}(R)(R^p)^{j+m(p-1)-r}\right)(R^p)^k \\ &= \left(\sum_{0\leq r\leq m(p-1)} \sum_{0\leq j\leq n(p-1)} \lambda_r\mu_j (R^{*p})\right)^{(m+n)(p-1)-(r+j)} \left((R^p)^{*k}\gamma_{mn}(R)(R^p)^k\right)(R^p)^{j+m(p-1)-r}. \end{aligned} \quad (73)$$

Since R is k-quasi-(m,n)-isosymmetric, we have $R^{*k}\gamma_{m,n}(R)R^k = 0$ and therefore

$$(R^p)^{*k}\gamma_{m,n}(R^p)(R^p)^k = 0. \quad (74)$$

This means that R^p is k-quasi-(m,n)-isosymmetric for any positive integer p.

Second Proof. Let $P(u,v) = P_{m,n,k}(u^p, v^p)$. It is easy to verify that $P_{m,n,k}(R^p) = P(R)$. A little calculation shows that there exists a polynomial Q such that $P = QP_{m,n,k}$. Thus, by (60), we get

$$P_{m,n,k}(R^p) = P(R) = \Phi_{R,R^*,Q}\left(P_{m,n,k}(R)\right), \quad (75)$$

which yields $P_{m,n,k}(R^p) = 0$ since $P_{m,n,k}(R) = 0$. Therefore, R^p is k-quasi-(m,n)-isosymmetric operator. □

Lemma 5. *Let $R, S \in \mathscr{B}(\mathscr{H})$ such that $[R, S] = [R^*, S] = 0$. Then, the following identity holds:*

$$\gamma_{m,n}(RS) = \sum_{0\leq k\leq m} \sum_{0\leq j\leq n} \binom{m}{k}\binom{n}{j} R^{*j+k}\gamma_{m-k,n-j} \cdot (R)\gamma_{k,j}(S)R^k S^{n-j}. \quad (76)$$

Proof. In view of the following identity (see [14]),

$$(cdab-1)^m(cd-ab)^n = \sum_{0\le k\le m}\sum_{0\le j\le n}\binom{m}{k}\binom{n}{j}c^{j+k} \cdot (ca-1)^{m-k}(c-a)^{n-j}(ad-1)^k \cdot (d-b)^j a^k b^{n-k}, \tag{77}$$

it follows by taking $a=R, c=R^*, b=S$, and $d=S^*$ with $RS=SR$ and $RS^*=S^*R$ that

$$\gamma_{m,n}(RS) = \sum_{0\le k\le m}\sum_{0\le j\le n}\binom{m}{k}\binom{n}{j}R^{*j+k}\gamma_{m-k,n-j} \cdot (R)\gamma_{k,j}(S)R^kS^{n-j}. \tag{78}$$

□

Theorem 9. *Let $R\in\mathscr{B}(\mathscr{H})$ and $S\in\mathscr{B}(\mathscr{H})$ such that $[R,S]=[R,S^*]=0$. If R is an k-quasi-(m,n)-isosymmetric and S is an k'-quasi-m'-isometric and n'-symmetric, then RS is an $k_0=\max\{k,k'\}$-quasi-$(m+m'-1,n+n'-1)$-isosymmetric.*

Proof. We prove that $(RS)^{*k_0}\gamma_{m+m'-1,n+n'-1}(RS)(RS)^{k_0}=0$.

In fact, by taking into account Lemma 5, we get

$$\begin{aligned}(RS)^{*k_0}\gamma_{m+m'-1,n+n'-1}(RS)(RS)^{k_0} &= (RS)^{*k_0}\left(\sum_{0\le k\le m+m'-1}\sum_{0\le j\le n+n'-1}\binom{m+m'-1}{k}\binom{n+n'-1}{j}R^{*j+k}\gamma_{m+m'-1-k,n+n'-1-j}\right.\\ &\quad\left.\cdot(R)\gamma_{k,j}(S)R^kS^{n+n'-1-j}\right)(RS)^{k_0}\\ &= \sum_{0\le k\le m+m'-1}\sum_{0\le j\le n+n'-1}\binom{m+m'-1}{k}\binom{n+n'-1}{j}R^{*j+k}R^{*k_0}\gamma_{m+m'-1-k,n+n'-1-j}\\ &\quad\cdot(R)R^{k_0}\times S^{*k_0}\gamma_{k,j}(S)R^kS^{n+n'-1-j}S^{k_0}.\end{aligned} \tag{79}$$

We have the following observations:

(i) If $k\ge m'$ or $j\ge n'$, then $S^{*k_0}\gamma_{k,j}(S)S^{k_0}=0$.

(ii) If $k\le m'-1$ and $j\le n'-1$, then $m+m'-1-k\ge m$ and $n+n'-1-j\ge n$. So that

$$R^{*k_0}\gamma_{m+m'-1-k,n+n'-1-j}(R)R^{k_0}=0. \tag{80}$$

Therefore,

$$(RS)^{*k_0}\gamma_{m+m'-1,n+n'-1}(RS)(RS)^{k_0}=0. \tag{81}$$

Hence, RS is an $k_0=\max\{k,k'\}$-quasi-$(m+m'-1, n+n'-1)$-isosymmetric. □

Corollary 11. *Let $R\in\mathscr{B}(\mathscr{H})$ and $S\in\mathscr{B}(\mathscr{H})$ such that $[R,S]=[R,S^*]=0$. If R is an k-quasi-(m,n)-isosymmetric and S is an k'-quasi-m'-isometric and n'-symmetric, then R^pS^q is an $k_0=\max\{k,k'\}$-quasi-$(m+m'-1,n+n'-1)$-isosymmetric for any positive integers p and q.*

Proof. In view of Theorem 8, we have R^p as k-quasi-(m,n)-isosymmetric for any positive integer p. Similarly, from Theorem 12 in [18] and corollary 3.1 in [27], we have S^q is k'-quasi-m'-isometric and n'-symmetric for any positive integer q.

Applying Theorem 9, we get R^pS^q is an $k_0=\max\{k,k'\}$-quasi-$(m+m'-1,n+n'-1)$-isosymmetric for any positive integers p and q. □

Corollary 12. *Let $R\in\mathscr{B}(\mathscr{H})$ and $S\in\mathscr{B}(\mathscr{H})$. If R is an k_1-quasi-(m_1,n_1)-isosymmetric and S is an k_2-quasi-m_2-isometric and n_2-symmetric, then $R\otimes S$ is $k_0=\max\{k_1,k_2\}$-quasi-$(m+m_1-1,n+n_1-1)$-isosymmetric.*

Proof. It is well known that $R\otimes S=(R\otimes I)(I\otimes S)$ and moreover

$$[R\otimes I, I\otimes S]=[(R\otimes I)^*, I\otimes S]=0. \tag{82}$$

On the other hand, R is an k_1-quasi-(m_1,n_1)-isosymmetric if and only if $R\otimes I$ is an k_1-quasi-(m_1,n_1)-isosymmetric and S is an k_2-quasi-m_2-isometric and n_2-symmetric if and only if $I\otimes S$ is an k_2-quasi-m_2-isometric and n_2-symmetric. From Theorem 9, it follows that $R\otimes S$ is a k_0-quasi-$(m+m_1-1,n+n_1-1)$-isosymmetric. □

Lemma 6. *Let $R,S\in\mathscr{B}(\mathscr{H})$ such that $[R,S]=[R^*,S]=0$, then the following identity holds:*

$$\gamma_{m,n}(R+Q)=\sum_{0\le j\le n}\sum_{i+l+k=m}\binom{m}{i,l,k}\binom{n}{j} \times(R^*+Q^*)^iQ^{*l}\gamma_{k,n-j}(R)\alpha_j(Q)R^lQ^i. \tag{83}$$

Proof. By the equation (see [14]),

$$((c+d)(a+b)-1)^m((c+d)-(a+d))^n =, \sum_{0\le j\le n}\sum_{i+l+k=m}\binom{n}{j}\binom{m}{i,l,k}(c+d)^i d^l (ca-1)^h (c-a)^{n-j}(d-b)^j a^l b^i. \quad (84)$$

If $a=R$, $C=R^*$, $B=S$, and $d=S^*$ with $[R,S]=[R^*,S]=0$, we get

$$\gamma_{m,n}(R+Q)=\sum_{0\le j\le n}\sum_{i+l+k=m}\binom{m}{i,l,k}\binom{n}{j}\times(R^*+Q^*)^i Q^{*l}\gamma_{k,n-j}(R)\alpha_j(Q)R^l Q^i. \quad (85)$$

□

Theorem 10. *Let $R,Q\in\mathscr{B}(\mathscr{H})$ be doubly commuting. If R is an k -quasi- (m,n) -isosymmetric and Q is a nilpotent operator of order q , then $R+Q$ is a $(k+q)$ -quasi- $(m+2q-2,n+2q-1)$ -isosymmetric operator.*

Proof. We need to show that $(R+Q)^{*k+q}\gamma_{m+2q-2,n+2q-2}(R+Q)(R+Q)^{k+q}=0$.

Note that by Lemma 6, we have

$$\gamma_{m+2q-2,n+2q-1}(R+Q)=\sum_{i+l+k=m+2q-2}\sum_{0\le j\le n+2q-1}\binom{m+2q-2}{i,l,k}\binom{n+2q-1}{j}\times(R^*+Q^*)^i Q^{*l}\gamma_{k,n+2q-1-j}(R)\alpha_j(Q)R^l Q^i. \quad (86)$$

However,

$$\begin{aligned}&(R+Q)^{*k+q}\gamma_{m+2q-2,n+2q-2}(R+Q)(R+Q)^{k+q}\\&=\left(\sum_{0\le r\le k+q}\binom{k+q}{r}R^{*(k+q-r)}Q^{*r}\right)\left(\sum_{i+l+k=m+2q-2}\sum_{0\le j\le n+2q-1}\binom{m+2q-2}{i,l,k}\binom{n+2q-1}{j}\right.\\&\left.\times(R^*+Q^*)^i Q^{*l}\gamma_{k,n+2q-1-j}(R)\alpha_j(Q)R^lQ^i\right)\times\left(\sum_{0\le r\le k+q}\binom{k+q}{r}R^{k+q-r}Q^r\right).\end{aligned} \quad (87)$$

Now, observe that if $j\ge 2q$ or $i\ge q$ or $l\ge q$, then $\alpha_j(Q)=0$ or $Q^i=0$ or $Q^{*l}=0$ and hence

$$(R^*+Q^*)^i Q^{*l}\gamma_{k,n+2q-1-j}(R)\alpha_j(Q)R^lQ^i=0. \quad (88)$$

However, if $j\le 2q-1$, $i\le q-1$, and $l\le q-1$, we obtain

$$\begin{aligned}k&=m+2q-1-i-l\ge m-2q-1-q+1-q+1\\&\ge m \text{ and } n+2q-1-j\ge n.\end{aligned} \quad (89)$$

From the fact that R is a k-quasi-n-(m,n)-isosymmetric, we get

$$\begin{aligned}R^{*(k+q-r)}\gamma_{k,n+2q-1}(R)R^{k+q-r}&=0 \quad \text{for } r=0,\ldots,q,\\R^{*(k+q-r)}Q^{*r}\gamma_{k,n+2q-1-j}(R)R^{k+q-r}Q^r&=0 \quad \text{for } r=q+1,\ldots,k+q.\end{aligned} \quad (90)$$

Consequently, we obtain $(R+Q)^{*k+q}\gamma_{m+2q-2,n+2q-2}(R+Q)(R+Q)^{k+q}=0$.

Therefore, $R+Q$ is $(k+q)$-quasi-$(m+2q-2,n+2q-1)$-isosymmetric operator. □

Corollary 13. *Let $M,N\in\mathscr{B}(\mathscr{H})$ such that $[M,N]=[M^*,N]=0$. If M is an k -quasi- (m,n) -isosymmetric, then the operator $S=\begin{pmatrix}M & N\\ 0 & M\end{pmatrix}\in\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$ is $(k+2)$ -quasi- $(m+2,n+3)$ -isosymmetric.*

Proof. Consider $R=\begin{pmatrix}M & 0\\ 0 & M\end{pmatrix}$ and $Q=\begin{pmatrix}0 & N\\ 0 & 0\end{pmatrix}$. Clearly, R is k-quasi-(m,n)-isosymmetric and $Q^2=0$ (i.e., Q is 2-nilpotent). On the other hand, since

$$[M,N]=[M^*,N]=0, \quad (91)$$

it follows that $[R,Q]=[R^*,Q]=0$. In view of Theorem 10, we deduce that $S=R+Q$ is $(k+2)$-quasi-$(m+2,n+3)$-isosymmetric. □

Corollary 14. *Let $R\in\mathscr{B}(\mathscr{H})$ be k -quasi- (m,n) -isosymmetric and $Q\in\mathscr{B}(\mathscr{H})$ be q -nilpotent. Then, $R\otimes I+I\otimes Q$ is $(k+q)$-quasi-$(m+2q-2,n+2q-1)$-isosymmetric.*

Proof. Observe that $R\otimes I\in\mathscr{B}(\overline{\mathscr{H}\otimes\mathscr{H}})$ is k-quasi-(m,n)-isosymmetric and $I\otimes Q\in\mathscr{B}(\overline{\mathscr{H}\otimes\mathscr{H}})$ is q-nilpotent. Moreover,

$$[R\otimes I,\ I\otimes Q]=[(R\otimes I)^*,\ I\otimes Q]=0. \quad (92)$$

Applying Theorem 10, we deduce that $R\otimes I+I\otimes Q$ is $(k+q)$-quasi-$(m+2q-2,n+2q-1)$-isosymmetric. □

Data Availability

Data sharing is not applicable to this study as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was funded by the Deanship of Scientific Research at Jouf University under grant no. DSR-2021-03-0334.

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