Research Article

Conditions of Semiscalar Equivalence of One Class 3×3 Matrices of Simple Structure

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This paper solves the problem of classifying matrices over a ring of polynomials up to semiscalarly equivalence transformations. For the selected set of polynomial matrices of simple structure, the so-called oriented by characteristic roots reduced matrix is constructed. The latter, in addition to the triangular form and the presence of invariant factors on the main diagonal, has some predefined properties. Invariants and conditions of semiscalar equivalence are established for such matrices.

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[x]$ the ring of polynomials in an indeterminate x over $\mathbb{C}$. Let $M(n, \mathbb{C})$, $M(n, \mathbb{C}[x])$ denote the algebras of n -by- n matrices over $\mathbb{C}$, $\mathbb{C}[x]$, respectively, and $GL(n, \mathbb{C})$, $GL(n, \mathbb{C}[x])$ their corresponding groups of units (general linear groups). Two matrices $F(x)$, $G(x)$ of $M(n, \mathbb{C}[x])$ are said to be semiscalarly equivalent when there exists a pair $(P, Q(x)) \in GL(n, \mathbb{C}) \times GL(n, \mathbb{C}[x])$ such that (see [1, 2])

$$PF(x)Q(x) = G(x). \quad (1)$$

This is an equivalence relation; its equivalence classes $\{PF(x)Q(x)\}$ are called semiscalarly equivalence classes. Therefore, there is a problem of classification of polynomial matrices up to semiscalar equivalence. If a matrix $F(x)$ has a full rank, then in the class $\{PF(x)Q(x)\}$, there is a lower triangular matrix in which its invariant factors stand on the main diagonal (see [1] Theorem 1 §1 Section 4). A similar result was also obtained in [3, 4]. The specified triangular matrix is defined by the class $\{PF(x)Q(x)\}$ ambiguously; therefore, it cannot be used at once for the establishment of belonging of any matrix to this class. Thus, there is a need to specify in class $\{PF(x)Q(x)\}$ such a matrix, which is determined with a

lesser degree of ambiguity and for which we can find a system of invariants. To do this, it is necessary to clarify the mentioned triangular form of the matrix, that is, to establish such a matrix, which, in addition to the triangular form and the presence on the main diagonal of its invariant factors, has some predetermined properties. The problem of classifying of polynomial matrices of the same size up to semiscalar equivalence contains the matrix pair problem, which is the problem of classifying a pair of square matrices of the same size up to similarity transformations. Donovan and Freislich [5] call a matrix problem “wild” if it contains the matrix pair problem and “tame” otherwise. A certain characterization of tame and wild problems is given by Drozd [6]. Thus, the problem of classifying polynomial matrices up to semiscalar equivalence is wild.

We investigate the problem of determining when two matrices are semiscalarly equivalent. This study aims to construct a simpler form in the class of semiscalarly equivalent matrices and to find its invariants. In this paper, we limit ourselves to the case when a matrix $F(x) \in M(3, \mathbb{C}[x])$ has a simple structure, a complete rank, and a unit first invariant factor. The latter condition is not significant. More precisely, all the obtained results can be applied to a wider class of matrices. It should also be noted

that this article is a continuation of the research initiated by the author [7]. Much useful information on the issues discussed in this article can be found in the monograph [8]. In particular, this monograph also investigates other types of equivalences of matrices and finite sets of matrices over different rings. Some generalizations of the concept of semiscalar equivalence in the case of matrices over quadratic rings and their application to the solution of matrix equations can be found in recent work [9].

2. Preliminary Information

Suppose that in the class $\{PF(x)Q(x)\}$ (see (1)), we have a matrix

$$A(x) = \begin{pmatrix} 1 & 0 & 0 \\ a_1(x) & \phi_1(x) & 0 \\ a_3(x) & a_2(x) & \phi_2(x) \end{pmatrix}, \quad (2)$$

where $\deg a_1(x) < \deg \phi_1(x)$, $\deg a_2(x), \deg a_3(x) < \deg \phi_2(x)$, $\phi_1(x)$ divides $a_2(x)$ and $\phi_2(x)$, that is, 1, $\phi_1(x)$, and $\phi_2(x)$ are invariant factors. Since $A(x)$ has a simple structure, polynomials $\phi_1(x)$, $\phi_2(x)$ do not have multiple roots. We assume that $\deg \phi_1(x), \deg(\phi_2(x)/\phi_1(x)) > 1$. Otherwise, we have $a_1(x) \equiv 0$ or $a_2(x) \equiv 0$. Then, the task is slightly simplified and is the subject of another study. Denote by M_1 and M_2 the sets of roots of polynomials $\phi_1(x)$ and $\phi_{12}(x) = \phi_2(x)/\phi_1(x)$, respectively. The notation $a(x)|b(x)$ means $a(x)$ divides $b(x)$, and $(a(x), b(x))$ is the greatest common divisor of polynomials $a(x)$ and $b(x)$. The corresponding notation will be used in the case of three polynomials. Let also be $a'_2(x) := a_2(x)/\phi_1(x)$ for element $a_2(x)$ of matrix $A(x)$. In the future, we will denote the zero element of the main field and zero rows and columns as 0. We can assume that in the matrix $A(x)$ for some fixed root, $\alpha_0 \in M_1$ is satisfied the condition

$$a_1(\alpha_0) = a_3(\alpha_0) = 0. \quad (3)$$

This is easily achieved by adding some multiples with the numerical coefficients of the first row of a matrix $A(x)$ to its other rows.

Proposition 1. *The greatest common divisor $(a_1(x), a_3(x), \phi_1(x))$ for the class $\{PF(x)Q(x)\}$ (see (1)) does not depend on the choice of the matrix $A(x)$ with condition (3) with a fixed characteristic root $\alpha_0 \in M_1$.*

Proof. Suppose, in addition to $A(x)$, to the class $\{PF(x)Q(x)\}$ belongs some other matrix

$$B(x) = \begin{pmatrix} 1 & 0 & 0 \\ b_1(x) & \phi_1(x) & 0 \\ b_3(x) & b_2(x) & \phi_2(x) \end{pmatrix}, \quad (4)$$

where $\deg b_1(x) < \deg \phi_1(x)$, $\deg b_2(x), \deg b_3(x) < \deg \phi_2(x)$, $b_1(\alpha_0) = b_3(\alpha_0) = 0$.

Then, equality

$$\|s_{ij}\|_1^3 \begin{pmatrix} 1 & 0 & 0 \\ a_1(x) & \phi_1(x) & 0 \\ a_3(x) & a_2(x) & \phi_2(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1(x) & \phi_1(x) & 0 \\ b_3(x) & b_2(x) & \phi_2(x) \end{pmatrix} \|r_{ij}(x)\|_1^3, \quad (5)$$

is satisfied for some $\|s_{ij}\|_1^3 \in GL(3, \mathbf{C})$, $\|r_{ij}(x)\|_1^3 \in GL(3, \mathbf{C}[x])$. If we put $x = \alpha_0$ in the latter equality, then we have $s_{21} = s_{31} = 0$, so $s_{11} \neq 0$. Also from (5), you can write

$$s_{22}a_1(x) + s_{23}a_3(x) = b_1(x)r_{11}(x) + \phi_1(x)r_{21}(x), \quad (6)$$

$$s_{32}a_1(x) + s_{33}a_3(x) = b_3(x)r_{11}(x) + b_2(x)r_{21}(x) + \phi_2(x)r_{31}(x). \quad (7)$$

Since $\phi_1(x)|r_{12}(x), r_{13}(x)$, then $(r_{11}(x), \phi_1(x)) = 1$. Therefore, from (6) and (7), we have that $(a_1(x), a_3(x), \phi_1(x))|b_1(x), b_3(x)$. If we take into account the symmetry of semiscalar equivalence, then we have that $(b_1(x), b_3(x), \phi_1(x))|a_1(x), a_3(x)$. Therefore, $(a_1(x), a_3(x), \phi_1(x)) = (b_1(x), b_3(x), \phi_1(x))$.

If $(a_1(x), a_3(x), \phi_1(x)) = \phi_1(x)$, then $a_1(x) \equiv 0$. This case is the subject of labor research [7]. In this article, we consider the situation when $(a_1(x), a_3(x), \phi_1(x)) \neq \phi_1(x)$. Then, there is a root $\alpha_1 \in M_1$ such that $\|a_1(\alpha_1) \ a_3(\alpha_1)\| \neq 0$. $\square$

3. Reduction to a Special Triangular Form

Proposition 2. *If for matrix $A(x)$, (2) of class $\{PF(x)Q(x)\}$ condition (3) and $(a_1(x), a_3(x), \phi_1(x)) \neq \phi_1(x)$ are satisfied, then in this class, there is such a matrix $B(x)$ (4) that $b_1(\alpha_0) = b_3(\alpha_0) = 0$ and $b_1(\alpha_1) \neq 0$, $b_3(\alpha_1) = 0$, for some $\alpha_1 \in M_1$.*

Proof. If in $A(x)$, for a fixed $\alpha_1 \in M_1$, we have $a_1(\alpha_1) \neq 0$, then the desired matrix is $B(x)$ achieved by adding some multiple (with a numerical coefficient) of the second row of the matrix $A(x)$ to its third row. If $a_1(\alpha_1) = 0$, $a_3(\alpha_1) \neq 0$, then first we choose $s_{23} \neq 0$ so that $1 + s_{23}a'_2(\beta) \neq 0$ for every $\beta \in M_2$. Then, from the congruences

$$\begin{aligned} a_1(x) + s_{23}a_3(x) - b_1(x) &\equiv 0 \pmod{\phi_1(x)}, \\ a'_2(x) - b'_2(x)(1 + s_{23}a'_2(x)) &\equiv 0 \pmod{\phi_{12}(x)}, \\ a_3(x) - b'_2(x)(a_1(x) + s_{23}a_3(x) - b_1(x)) - b_3(x) &\equiv 0 \pmod{\phi_2(x)}, \end{aligned} \quad (8)$$

consistently find $b_1(x)$, $b'_2(x)$, $b_3(x)$ of degrees $\deg b_1(x) < \deg \phi_1(x)$, $\deg b'_2(x) < \deg \phi_{12}(x)$, $\deg b_3(x) < \deg \phi_2(x)$. Next, from elements s_{23} , $b_1(x)$, $b_2(x) = b'_2(x)\phi_1(x)$, and $b_3(x)$, we construct a matrix $B(x)$ of the form (4) and matrices

$$\begin{aligned} \|s_{ij}\|_1^3 &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & s_{23} \\ 0 & 0 & 1 \end{vmatrix}, \\ \|r_{ij}(x)\|_1^3, \end{aligned} \quad (9)$$

where

$$\begin{aligned} r_{11}(x) &= 1, \\ r_{12}(x) &= r_{13}(x) = 0, \\ r_{21}(x) &= \frac{a_1(x) + s_{23}a_3(x) - b_1(x)}{\phi_1(x)}, \\ r_{22}(x) &= 1 + s_{23}a'_2(x), \\ r_{23}(x) &= s_{23}\phi_{12}(x), \\ r_{31}(x) &= \frac{a_3(x) + b'_2(x)(a_1(x) + s_{23}a_3(x) - b_1(x)) - b_3(x)}{\phi_2(x)}, \\ r_{32}(x) &= \frac{a'_2(x) - b'_2(x)(1 + s_{23}a'_2(x))}{\phi_{12}(x)}, \\ r_{33}(x) &= 1 - s_{23}b'_2(x). \end{aligned} \quad (10)$$

It is easy to see that $r_{21}(x), r_{31}(x), r_{32}(x) \in \mathbb{C}[x]$ and $\|r_{ij}(x)\|_1^3 \in \text{GL}(3, \mathbb{C}[x])$. By direct verification, we establish that the matrices $B(x)$, $\|s_{ij}\|_1^3$, and $\|r_{ij}(x)\|_1^3$ together with the original matrix $A(x)$ satisfy equality (5). This means that $B(x)$ belongs to the class $\{PF(x)Q(x)\}$ and also $b_1(\alpha_1) \neq 0$, $b_1(\alpha_0) = b_3(\alpha_0) = 0$. By adding to the third row of the matrix $B(x)$ its second row multiplied by $-b_3(\alpha_1)/b_1(\alpha_1)$, we obtain the matrix we need with zero value of the (3, 1)-element at $x = \alpha_1$.

We can further assume that $b_1(\alpha_1) = 1$ for the matrix $B(x)$ referred to in Proposition 2. $\square$

Proposition 3. *If the elements of matrices $A(x)$ (2), $B(x)$ (4) of class $\{PF(x)Q(x)\}$ satisfy the conditions:*

$$\begin{aligned} a_1(\alpha_0) &= b_1(\alpha_0) = a_3(\alpha_0) = b_3(\alpha_0) = a_3(\alpha_1) = b_3(\alpha_1) = 0, \\ a_1(\alpha_1) &= b_1(\alpha_1) = 1, \end{aligned} \quad (11)$$

for some $\alpha_0, \alpha_1 \in M_1$, then $(a_3(x), \phi_1(x)) = (b_3(x), \phi_1(x))$, and the matrix $\|s_{ij}\|_1^3$ in relation (5) has an upper triangular form.

Proof. The proof of Proposition 1 shows that in the matrix $\|s_{ij}\|_1^3$ from equality (5), there is $s_{21} = s_{31} = 0$. If we put $x = \alpha_1$ in (7), we have $s_{32} = 0$. Then, from (7), it is seen that $(b_3(x), \phi_1(x))|a_3(x)$ since $s_{33} \neq 0$. Given the symmetry of the relation of semiscalar equivalence, we can obtain that

$(a_3(x), \phi_1(x))|b_3(x)$. Therefore, $(a_3(x), \phi_1(x)) = (b_3(x), \phi_1(x))$. $\square$

Proposition 4. *If for elements of matrices $A(x)$ (2), $B(x)$ (4) of class $\{PF(x)Q(x)\}$ are true conditions (8) for some roots $\alpha_0, \alpha_1 \in M_1$, and $(a_3(x), \phi_1(x)) = \phi_1(x)$, then the following conditions are also satisfied:*

- (i) $(a_1(x), \phi_1(x)) = (b_1(x), \phi_1(x))$,
 $(a_1(x) - 1, \phi_1(x)) = (b_1(x) - 1, \phi_1(x))$;
- (ii) Equivalence $a_1(\alpha_i) = a_1(\alpha_j) \Leftrightarrow b_1(\alpha_i) = b_1(\alpha_j)$ is performed for arbitrary $\alpha_i, \alpha_j \in M_1$;
- (iii) If $I_{ij}(A) := (a_1(\alpha_j) - 1)a_1(\alpha_i)/(a_1(\alpha_i) - 1)a_1(\alpha_j)$,
 $I_{ij}(B) := (b_1(\alpha_j) - 1)b_1(\alpha_i)/(b_1(\alpha_i) - 1)b_1(\alpha_j)$
then $I_{ij}(A) = I_{ij}(B)$ for such $\alpha_i, \alpha_j \in M_1$ that $a_1(\alpha_i), a_1(\alpha_j) \neq 0$, $a_1(\alpha_i), a_1(\alpha_j) \neq 1$.

Proof. If we take into account that $(a_3(x), \phi_1(x)) = \phi_1(x)$, then from (6), we can write

$$s_{22}a_1(x) - b_1(x)(s_{11} + s_{12}a_1(x)) \equiv 0 \pmod{\phi_1(x)}, \quad (12)$$

where do we get the first part of the condition (i) Putting in (12) $x = \alpha_1, x = \alpha_i, x = \alpha_j$, we have, respectively,

$$s_{22} - s_{11} - s_{12} = 0, \quad (13)$$

$$s_{22}a_1(\alpha_i) - b_1(\alpha_i)(s_{11} + s_{12}a_1(\alpha_i)) = 0, \quad (14)$$

$$s_{22}a_1(\alpha_j) - b_1(\alpha_j)(s_{11} + s_{12}a_1(\alpha_j)) = 0. \quad (15)$$

Since $s_{11}, s_{22} \neq 0$, then from (13), (14), we obtain that $a_1(\alpha_i) = 1 \Leftrightarrow b_1(\alpha_i) = 1$ for an arbitrary $\alpha_i \in M_1$. Also from (14), (15) follows such equivalence $a_1(\alpha_i) = a_1(\alpha_j) \Leftrightarrow b_1(\alpha_i) = b_1(\alpha_j)$. This fully proves conditions (i) and (ii).

Equality $I_{ij}(A) = I_{ij}(B)$ based on the proved (i), (ii) is obvious for those $\alpha_i, \alpha_j \in M_1$ that $a_1(\alpha_i) = a_1(\alpha_j)$. In the case of $a_1(\alpha_i) \neq a_1(\alpha_j)$, this equality follows from (13)–(15), if we take into account $s_{11}, s_{22} \neq 0$. $\square$

Corollary 1. *Suppose that the conditions of Proposition 4 for matrices $A(x)$ (2), $B(x)$ (4) are satisfied, and let*

$$\begin{aligned} J_{ij}(A) &:= \frac{(a_1(\alpha_j) - 1)a_1(\alpha_i)}{a_1(\alpha_j) - a_1(\alpha_i)}, \\ J_{ij}(B) &:= \frac{(b_1(\alpha_j) - 1)b_1(\alpha_i)}{b_1(\alpha_j) - b_1(\alpha_i)}. \end{aligned} \quad (16)$$

Then, $J_{ij}(A) = J_{ij}(B)$ for such $\alpha_i, \alpha_j \in M_1$ that $a_1(\alpha_i), a_1(\alpha_j) \neq 0$, $a_1(\alpha_i), a_1(\alpha_j) \neq 1$, $a_1(\alpha_i) \neq a_1(\alpha_j)$.

Proof. It is easy to see that $J_{ij}(A) = I_{ij}(A)/I_{ij}(A) - 1$, $J_{ij}(B) = I_{ij}(B)/I_{ij}(B) - 1$. Under Proposition 4, we have $I_{ij}(A) \neq 1$. Therefore, $J_{ij}(A) = J_{ij}(B)$. $\square$

Proposition 5. Suppose that for matrix $A(x)$ (2), the conditions of Proposition 4 hold. Let also α_2 and α_j be a fixed and an arbitrary root of the set M_1 , respectively, for which both values $a_1(\alpha_2), a_1(\alpha_j)$ are different from 0 and 1, and $a_1(\alpha_2) \neq a_1(\alpha_j)$. Then, in the class $\{PF(x)Q(x)\}$, there is a matrix $B(x)$ (4) with the conditions of Proposition 4 and an arbitrarily fixed value $b_1(\alpha_2) \neq 0$ other than 1 and $J_{2j}(A)$ (16).

Proof. Let us choose for an unknown polynomial $b_1(x)$ at $x = \alpha_2$ some value $b_1(\alpha_2) \neq 0$ other than 1 and $J_{2j}(A)$. Consider the equation

$$\frac{(b_1(\alpha_j) - 1)b_1(\alpha_2)}{(b_1(\alpha_2) - 1)b_1(\alpha_j)} = I_{2j}(A), \quad (17)$$

with the unknown $b_1(\alpha_j)$ for each element α_j specified in Proposition, where $I_{2j}(A)$ is defined in (9). Let us write the latter in this form

$$b_1(\alpha_j)(b_1(\alpha_2)(1 - I_{2j}(A)) + I_{2j}(A)) = b_1(\alpha_2). \quad (18)$$

Since $b_1(\alpha_2) \neq I_{2j}(A)/I_{2j}(A) - 1 = J_{2j}(A)$ (see the proof of Corollary 1), it is solvable for each element α_j stated in Proposition. Let us now construct a polynomial $b_1(x)$ of degree $< \deg \phi_1(x)$ by its values $b_1(\alpha_i)$ for each root $\alpha_i \in M_1$ as follows:

$$b_1(\alpha_i) = \begin{cases} 0, & \text{if } a_1(\alpha_i) = 0, \\ 1, & \text{if } a_1(\alpha_i) = 1, \\ b_1(\alpha_2), & \text{if } a_1(\alpha_i) = a_1(\alpha_2), \\ b_1(\alpha_j), & \text{if } a_1(\alpha_i) = a_1(\alpha_j). \end{cases} \quad (19)$$

$$\|r_{ij}(x)\|_1^2 = \left\| \begin{array}{cc} 1 + s_{12}a_1(x) & s_{12}\phi_1(x) \\ \frac{s_{22}a_1(x) - b_1(x)(1 + s_{12}a_1(x))}{\phi_1(x)} & s_{22} - s_{12}b_1(x) \end{array} \right\| \in M(2, \mathbf{C}[x]). \quad (23)$$

From (22), we can find polynomials $b_2(x)b_3(x)$, of degrees $< \deg \phi_2(x)$, since $\det \|r_{ij}(x)\|_1^2 = s_{22} \neq 0$. From the found $b_1(x)$, $b_2(x)b_3(x)$, and $\|1 \ s_{22} \ s_{12}\|_1^T$, we construct a matrix $B(x)$ of the form (4) and a matrix

$$\|s_{ij}\|_1^3 = \left\| \begin{array}{ccc} 1 & s_{12} & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & 1 \end{array} \right\|. \quad (24)$$

We also construct a matrix $\|r_{ij}(x)\|_1^3$, in which the upper left 2×2 -block coincides with the above matrix $\|r_{ij}(x)\|_1^2$ and, in addition, $r_{13}(x) = r_{23}(x) \equiv 0$, $r_{33}(x) = 1$,

Consider the equation

$$\left\| \begin{array}{ccc} 1 & -1 & 1 \\ (a_1(\alpha_2))^{-1} & (b_1(\alpha_2))^{-1} & 1 \\ (a_1(\alpha_j))^{-1} & (b_1(\alpha_j))^{-1} & 1 \end{array} \right\| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \quad (20)$$

relative to $\|x_1 \ x_2 \ x_3\|^T$. If we take into account the equality $I_{2j}(A) = I_{2j}(B)$ for every α_j specified in Proposition, and the inequality $a_1(\alpha_2) \neq b_1(\alpha_2)$, then the rank of the matrix of this equation is 2, and it has a solution $\|1 \ s_{22} \ s_{12}\|^T$ with nonzero all its components. Based on this solution, we write the congruence

$$s_{22}a_1(x) - b_1(x)(1 + s_{12}a_1(x)) \equiv 0 \pmod{\phi_1(x)}, \quad (21)$$

and consider the equation

$$\|b_3(x) \ b_2(x)\| \|r_{ij}(x)\|_1^2 \equiv \|a_3(x) \ a_2(x)\| \pmod{\phi_2(x)}, \quad (22)$$

with unknown $\|b_3(x) \ b_2(x)\|$, where

$$\begin{aligned} r_{31}(x) &= \frac{a_3(x) - b_3(x)(1 + s_{12}a_1(x)) - b_2(x)r_{21}(x)}{\phi_2(x)} \in \mathbf{C}[x], \\ r_{32}(x) &= \frac{a_2(x) - b_3(x)r_{12}(x) - b_2(x)r_{22}(x)}{\phi_2(x)} \in \mathbf{C}[x]. \end{aligned} \quad (25)$$

It is easy to see that the matrix $A(x)$ together with the constructed $B(x)$, non-singular $\|s_{ij}\|_1^3$ and invertible $\|r_{ij}(x)\|_1^3$ satisfy equality (5). In addition, the element $b_1(x)$ of the matrix $B(x)$ has a preselected value $b_1(\alpha_2)$. Therefore, $B(x)$ is the desired matrix. $\square$

Proposition 6. Suppose that for matrices $A(x)$ (2), $B(x)$ (4) with the conditions of Proposition 4, we have $0 \neq a_1(\alpha_2) = b_1(\alpha_2) \neq 1$ for some $\alpha_2 \in M_1$ and

$$\begin{aligned} a'_2(x) &:= \frac{a_2(x)}{\phi_1(x)}, \\ b'_2(x) &:= \frac{b_2(x)}{\phi_1(x)}, \\ \delta_A(x) &:= \det \begin{vmatrix} a_1(x) & 1 \\ a_3(x) & a'_2(x) \end{vmatrix}, \\ \delta_B(x) &:= \det \begin{vmatrix} b_1(x) & 1 \\ b_3(x) & b'_2(x) \end{vmatrix}. \end{aligned} \quad (26)$$

Then, in these matrices $a_1(x) = b_1(x)$, $(a_3(x), \phi_2(x)) = (b_3(x), \phi_2(x))$, $(a'_2(x), \phi_{12}(x)) = (b'_2(x), \phi_{12}(x))$, $(\delta_A(x), \phi_{12}(x)) = (\delta_B(x), \phi_{12}(x))$ and in the upper triangular matrix $\|s_{ij}\|_1^3$ from relation (5), we have $s_{12} = 0$, $s_{11} = s_{22}$.

Proof. For the elements $a_1(x), b_1(x)$ of the matrices $A(x), B(x)$, the congruence (12) is performed. Putting in it $x = \alpha_1$ and $x = \alpha_2$, we obtain, respectively, equality (13) and

$$s_{22}a_1(\alpha_2) - b_1(\alpha_2)(s_{11} + s_{12}a_1(\alpha_2)) = 0. \quad (27)$$

From them, it immediately follows that $s_{12} = 0$, $s_{11} = s_{22}$, because the values $a_1(\alpha_2) = b_1(\alpha_2)$ are different from 0 and 1. Then, if we return to (12), we obtain $a_1(x) = b_1(x)$. If we take into account what has already been proved, then from (5), we can write the congruences as follows:

$$a_3(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) \equiv s_{11}b_3(x) \pmod{\phi_2(x)}, \quad (28)$$

$$a'_2(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) \equiv s_{22}b'_2(x) \pmod{\phi_{12}(x)}, \quad (29)$$

from which it follows

$$\delta_A(x)(s_{33} + s_{13}(x)\delta_B(x) - s_{23}b'_2(x)) \equiv s_{11}\delta_B(x) \pmod{\phi_{12}(x)}. \quad (30)$$

From (28), (29), and (30), it is easy to see that $(a_3(x), \phi_2(x)) = (b_3(x), \phi_2(x))$, $(a'_2(x), \phi_{12}(x)) = (b'_2(x), \phi_{12}(x))$ and $(\delta_A(x), \phi_{12}(x)) = (\delta_B(x), \phi_{12}(x))$, respectively. $\square$

Corollary 2. For matrices $A(x)$ (2), $B(x)$ (4) of class $\{PF(x)Q(x)\}$ with the conditions of Proposition 6, we have the following equivalences:

$$\begin{aligned} a_2(x) &\equiv 0 \Leftrightarrow b_2(x) \equiv 0, \\ a_3(x) &\equiv 0 \Leftrightarrow b_3(x) \equiv 0, \\ \delta_A(\alpha_i) &\equiv 0 \pmod{\phi_{12}(x)} \Leftrightarrow \delta_B(\alpha_i) \equiv 0 \pmod{\phi_{12}(x)}. \end{aligned} \quad (31)$$

Proposition 7. Suppose that for the elements of matrices $A(x)$ (2), $B(x)$ (4), the conditions of Proposition 4 are satisfied, and let $a'_2(x), b'_2(x)$, $\delta_A(x), \delta_B(x)$ are defined in Proposition 6. Then equivalences

$$\begin{aligned} a'_2(\beta_i) &= \delta_A(\beta_i) \Leftrightarrow b'_2(\beta_i) = \delta_B(\beta_i), \\ a'_2(\beta_i)(\delta_A(\beta_i))^{-1} &= a'_2(\beta_j)(\delta_A(\beta_j))^{-1} \\ \Leftrightarrow b'_2(\beta_i)(\delta_B(\beta_i))^{-1} &= b'_2(\beta_j)(\delta_B(\beta_j))^{-1}, \end{aligned} \quad (32)$$

are fulfilled for such $\beta_i, \beta_j \in M_2$ that $\delta_A(\beta_i), \delta_A(\beta_j) \neq 0$.

Proof. Matrices $A(x), B(x)$ satisfy equation (5), where the matrix $\|s_{ij}\|_1^3$ according to Proposition 3 has an upper triangular form. From (5), except (12), we obtain the congruences as follows:

$$a_3(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) + s_{11}\delta_B(x) \equiv a_1(x)(s_{22}b'_2(x) - s_{12}\delta_B(x)) \pmod{\phi_2(x)}. \quad (33)$$

$$a'_2(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) \equiv s_{22}b'_2(x) - s_{12}\delta_B(x) \pmod{\phi_{12}(x)}. \quad (34)$$

If we exclude from (33) and (34) the members containing s_{13}, s_{23}, s_{33} , we get

$$s_{11}a'_2(x)\delta_B(x) - s_{22}b'_2(x)\delta_A(x) + s_{12}\delta_A(x)\delta_B(x) \equiv 0 \pmod{\phi_{12}(x)}. \quad (35)$$

From (12) for $x = \alpha_1$, we have (13), and from (35) for $x = \beta_i$ and $x = \beta_j$, respectively, we obtain

$$s_{11}a'_2(\beta_i)(\delta_A(\beta_i))^{-1} - s_{22}b'_2(\beta_i)(\delta_B(\beta_i))^{-1} + s_{12} = 0, \quad (36)$$

$$s_{11}a'_2(\beta_j)(\delta_A(\beta_j))^{-1} - s_{22}b'_2(\beta_j)(\delta_B(\beta_j))^{-1} + s_{12} = 0. \quad (37)$$

From (13) and (36) follows the first of the required equivalences, and from (36) and (37)-the second. $\square$

Proposition 8. Suppose that in the matrix $A(x)$ (2) with the conditions of Proposition 4, element $a_1(x)$ of the set M_1

$$\|s_{ij}\|_1^3 = \begin{vmatrix} 1 & s_{12} & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

$$\|r_{ij}(x)\|_1^2 = \begin{vmatrix} 1 + s_{12}a_1(x) & s_{12}\phi_1(x) \\ \frac{s_{22}a_1(x) - a_1(x)(1 + s_{12}a_1(x))}{\phi_1(x)} & s_{22} - s_{12}a_1(x) \end{vmatrix} \in M(2, \mathbb{C}[x]).$$

(38)

Obviously, $s_{22} = 1 + s_{12} \neq 0$ and $\|r_{ij}(x)\|_1^2 \in GL(2, \mathbb{C}[x])$. We also construct a matrix $B(x)$ of the form (4), in which $b_1(x) = a_1(x)$ and elements $b_2(x), b_3(x)$ of degrees $< \deg \phi_2(x)$ are defined from the congruence

$$\|b_3(x) \ b_2(x)\| \|r_{ij}(x)\|_1^2 \equiv \|a_3(x) \ a_2(x)\| \pmod{\phi_2(x)}. \quad (39)$$

Based on the elements of matrix $\|r_{ij}(x)\|_1^2$ and elements $r_{13}(x) = r_{23}(x) \equiv 0$,

$$r_{31}(x) = \frac{a_2(x) - b_3(x)r_{12}(x) - b_2(x)r_{22}(x)}{\phi_2(x)} \in \mathbb{C}[x],$$

$$r_{32}(x) = \frac{a_3(x) - b_3(x)r_{11}(x) - b_2(x)r_{21}(x)}{\phi_2(x)} \in \mathbb{C}[x], \quad (40)$$

$$r_{33}(x) = 1,$$

we construct matrix $\|r_{ij}(x)\|_1^3$. Obviously, $\|r_{ij}(x)\|_1^3 \in GL(3, \mathbb{C}[x])$. We make sure that the constructed matrices $\|s_{ij}\|_1^3, B(x)$, and $\|r_{ij}(x)\|_1^3$ together with $A(x)$ satisfy equality (5). Therefore, $B(x)$, like $A(x)$, belongs to class $\{PF(x)Q(x)\}$. The elements of the matrices $A(x), B(x)$ satisfy the congruence (35), where $s_{11} = 1$, and s_{12}, s_{22} are defined above. If we put $x = \beta_0$ in this congruence, we get $b'_2(\beta_0) = -\delta_B(\beta_0)$. $\square$

Proposition 9. Suppose that for matrices $A(x)$ (2), $B(x)$ (4) of the class, $\{PF(x)Q(x)\}$ conditions of Proposition 8 are satisfied, and we have $a'_2(\beta_0) = -\delta_A(\beta_0)$, $b'_2(\beta_0) = -\delta_B(\beta_0)$ for some $\beta_0 \in M_2$. Then, the conclusion of Proposition 6 is true.

acquires only the values 0 and 1. Let also be $a'_2(\beta_0) \neq \delta_A(\beta_0) \neq 0$ for some $\beta_0 \in M_2$, where $a'_2(x), \delta_A(x)$ are defined in Proposition 6. Then, in the class $\{PF(x)Q(x)\}$, there is a matrix $B(x)$ of the form (4) with the conditions of Proposition 4, in which $b_1(x) = a_1(x)$ and $b'_2(\beta_0) = -\delta_B(\beta_0) \neq 0$.

Proof. Denote $c := a'_2(\beta_0)(\delta_A(\beta_0))^{-1}$, $s_{12} := -0,5(c+1)$, $s_{22} := -0,5(c-1)$ and construct matrices

Proof. Congruences (13) and (35) are performed for the elements of matrices $A(x), B(x)$. If in these congruences to put accordingly $x = \alpha_1$ and $x = \beta_0$, then from the received, we will receive that $s_{12} = 0$ and $s_{11} = s_{22}$. The rest of the proof is the same as in the proof of Proposition 6. $\square$

Corollary 3. For matrices $A(x)$ (2), $B(x)$ (4) with the conditions of Proposition 9, we have $a_3(x) \equiv 0 \Leftrightarrow b_3(x) \equiv 0$ and $\phi_{12}(x)|\delta_A(x) \Leftrightarrow \phi_{12}(x)|\delta_B(x)$.

4. Main Results

Definition 1. Matrix $B(x)$ of Proposition 4, in which $b_1(\alpha_i) = 0$ or $b_1(\alpha_i) = 1$ for each $\alpha_i \in M_1$ and $\delta_B(\beta_j) = 0$ or $\delta_B(\beta_j) = b'_2(\beta_j) \neq 0$ for each $\beta_j \in M_2$, is called oriented by the characteristic roots $\alpha_0, \alpha_1 \in M_1$ reduced matrix. Matrix $B(x)$ of Proposition 5, for which

$$b(\alpha_2) = \begin{cases} -1, & \text{if } J_{2j} \neq -1 \text{ for each } \alpha_j \in M_1, \\ \max|J_{2j} + 1|, & \text{in opposite case,} \end{cases} \quad (41)$$

is called oriented by the characteristic roots $\alpha_0, \alpha_1, \alpha_2 \in M_1$ reduced matrix. Matrix $B(x)$ of Proposition 8 is called oriented by the characteristic roots $\alpha_0, \alpha_1 \in M_1, \beta_0 \in M_2$ reduced matrix.

Theorem 1. Suppose that in the oriented by the same characteristic roots $\alpha_0, \alpha_1, \alpha_2 \in M_1$ or $\alpha_0, \alpha_1 \in M_1, \beta_0 \in M_2$ reduced matrices $A(x)$ (2), $B(x)$ (4), we have $a_2(x), a_3(x) \neq 0$, and

$$c(\beta_i) := \begin{cases} a'_2(\beta_i), & \text{if } a'_2(\beta_i) \neq 0, \\ \delta_A(\beta_i), & \text{if } a'_2(\beta_i) = 0, \delta_A(\beta_i) \neq 0, \\ 0, & \text{if } a'_2(\beta_i) = \delta_A(\beta_i) = 0, \end{cases} \quad (42)$$

$$d(\beta_i) := \begin{cases} b'_2(\beta_i), & \text{if } b'_2(\beta_i) \neq 0, \\ \delta_B(\beta_i), & \text{if } b'_2(\beta_i) = 0, \delta_B(\beta_i) \neq 0, \\ 0, & \text{if } b'_2(\beta_i) = \delta_B(\beta_i) = 0, \end{cases}$$

where $\beta_i \in M_2$ (notation $a'_2(x), b'_2(x), \delta_A(x), \delta_B(x)$ see Proposition 6). Matrices $A(x), B(x)$ are semiscalarly equivalent if and only if their elements $a_1(x), b_1(x)$ coincide, and the rest satisfy the following conditions:

$$(i) \begin{aligned} (a'_2(x), \phi_{12}(x)) &= (b'_2(x), \phi_{12}(x)), \\ (a_3(x), \phi_{12}(x)) &= (b_3(x), \phi_{12}(x)), \\ (\delta_A(x), \phi_{12}(x)) &= (\delta_B(x), \phi_{12}(x)); \end{aligned}$$

$$(ii) \text{ for arbitrary such roots } \beta_i \in M_2 \text{ that } a'_2(\beta_i) \neq 0, \text{ we have}$$

$$\delta_A(\beta_i)(a'_2(\beta_i))^{-1} = \delta_B(\beta_i)(b'_2(\beta_i))^{-1}, \quad (43)$$

$$a_3(\beta_i)(a'_2(\beta_i))^{-1} = b_3(\beta_i)(b'_2(\beta_i))^{-1}.$$

(iii) for arbitrary roots $\beta_i, \beta_j \in M_2$, we have

$$\|a'_2(\beta_i) - a'_2(\beta_j) \quad \delta_A(\beta_i) - \delta_A(\beta_j)\| = 0 \Leftrightarrow \|b'_2(\beta_i) - b'_2(\beta_j) \quad \delta_B(\beta_i) - \delta_B(\beta_j)\| = 0. \quad (44)$$

(iv) for arbitrary roots $\beta_i, \beta_j, \beta_l \in M_2$ such that $c(\beta_i), c(\beta_j), c(\beta_l) \neq 0$ and rows $\|a'_2(\beta_i) \quad \delta_A(\beta_i)\|$,

$\|a'_2(\beta_j) \quad \delta_A(\beta_j)\| \quad \|a'_2(\beta_l) \quad \delta_A(\beta_l)\|$ are pairwise different, there is a number $k \neq 0$ such that

$$\begin{aligned} & \frac{k}{c(\beta_i)c(\beta_j)c(\beta_l)} \det \begin{vmatrix} a'_2(\beta_i) & \delta_A(\beta_i) & 1 \\ a'_2(\beta_j) & \delta_A(\beta_j) & 1 \\ a'_2(\beta_l) & \delta_A(\beta_l) & 1 \end{vmatrix} \\ &= \frac{1}{d(\beta_i)d(\beta_j)d(\beta_l)} \det \begin{vmatrix} b'_2(\beta_i) & \delta_B(\beta_i) & 1 \\ b'_2(\beta_j) & \delta_B(\beta_j) & 1 \\ b'_2(\beta_l) & \delta_B(\beta_l) & 1 \end{vmatrix}. \\ & \frac{k}{c(\beta_i)c(\beta_j)} \det \begin{vmatrix} c(\beta_i) & 1 \\ c(\beta_j) & 1 \end{vmatrix} = \frac{k}{d(\beta_i)d(\beta_j)} \det \begin{vmatrix} d(\beta_i) & 1 \\ d(\beta_j) & 1 \end{vmatrix}, \\ & \text{if } \det \begin{vmatrix} a'_2(\beta_i) & \delta_A(\beta_i) \\ a'_2(\beta_j) & \delta_A(\beta_j) \end{vmatrix} = 0. \end{aligned} \quad (45)$$

$$(46)$$

Proof

(i) Necessity

(ii) For semiscalarly equivalent matrices $A(x), B(x)$ the equality $a_1(x) = b_1(x)$, condition (i) and condition (ii) for zero values $a_3(\beta_i), \delta_A(\beta_i)$ follow from Propositions 6 or 9. For the elements of matrices $A(x), B(x)$, the congruences (33) and (34) are satisfied, from which (35) follows. If we consider that $s_{12} = 0$ and $s_{11} = s_{22}$ (see Propositions 6 or 9), then from (35), we have

$$a'_2(x)\delta_B(x) - b'_2(x)\delta_A(x) \equiv 0 \pmod{\phi_{12}(x)}. \quad (47)$$

From the latter equivalence for $a'_2(\beta_i) \neq 0$ ($b'_2(\beta_i) \neq 0$) and nonzero values $\delta_A(\beta_i), \delta_B(\beta_j)$, it

follows the first equality from (43). The second of them follows from the first, since $\delta_A(x) = a_1(x)a'_2(x) - a_3(x)$, $\delta_B(x) = b_1(x)b'_2(x) - b_3(x)$, $a_1(x) = b_1(x)$.

(iii) Equivalence (43) for zero values $a'_2(\beta_i), a'_2(\beta_j)$ and $\delta_A(\beta_i), \delta_A(\beta_j)$ follows from (i). If $a'_2(\beta_i) = a'_2(\beta_j) = 0$ and $a_3(\beta_i) = \delta_A(\beta_i) = \delta_A(\beta_j) = a_3(\beta_j) \neq 0$, then from (33) for $x = \beta_i, x = \beta_j$, we have

$$\begin{cases} s_{33}(b_3(\beta_i))^{-1} - s_{11}(a_3(\beta_i))^{-1} - s_{13} = 0, \\ s_{33}(b_3(\beta_j))^{-1} - s_{11}(a_3(\beta_j))^{-1} - s_{13} = 0, \end{cases} \quad (48)$$

whence $b_3(\beta_i) = b_3(\beta_j)$. Since in this case $\delta_B(\beta_i) = -b_3(\beta_i)$, $\delta_B(\beta_j) = -b_3(\beta_j)$, so

$\delta_B(\beta_i) = \delta_B(\beta_j)$. This proves the equivalence (44) for zero values $a'_2(\beta_i), a'_2(\beta_j)$ and nonzero values $\delta_A(\beta_i), \delta_A(\beta_j)$. Consider now the case where

$a'_2(\beta_i) = a'_2(\beta_j) \neq 0$ and $\delta_A(\beta_i) = \delta_A(\beta_j)$. If we take into account (43), then from (34), where $s_{12} = 0$, we have

$$\begin{cases} s_{33}(b'_2(\beta_i))^{-1} - s_{22}(a'_2(\beta_i))^{-1} + s_{13}\delta_A(\beta_i)(a'_2(\beta_i))^{-1} - s_{23} = 0, \\ s_{33}(b'_2(\beta_j))^{-1} - s_{22}(a'_2(\beta_j))^{-1} + s_{13}\delta_A(\beta_j)(a'_2(\beta_j))^{-1} - s_{23} = 0, \end{cases} \quad (49)$$

whence $b'_2(\beta_i) = b'_2(\beta_j)$. Then, it is easy to trace the chain of implications

$$\begin{aligned} & \|a'_2(\beta_i) \ \delta_A(\beta_i)\| = \|a'_2(\beta_j) \ \delta_A(\beta_j)\| \Rightarrow \\ & \Rightarrow a'_2(\beta_i) \|1 \ \delta_A(\beta_i)(a'_2(\beta_i))^{-1}\| = a'_2(\beta_j) \|1 \ \delta_A(\beta_j)(a'_2(\beta_j))^{-1}\| \Rightarrow \\ & \Rightarrow \|1 \ \delta_B(\beta_i)(b'_2(\beta_i))^{-1}\| = \|1 \ \delta_B(\beta_j)(b'_2(\beta_j))^{-1}\| \Rightarrow \\ & \|b'_2(\beta_i) \ \delta_B(\beta_i)\| = \|b'_2(\beta_j) \ \delta_B(\beta_j)\|. \end{aligned} \quad (50)$$

(iv) Suppose we have three roots $\beta_i, \beta_j, \beta_l \in M_2$, for which $c(\beta_i), c(\beta_j), c(\beta_l) \neq 0$ (respectively, $d(\beta_i), d(\beta_j), d(\beta_l) \neq 0$). For elements of matrices $A(x), B(x)$ the congruences (28), (29) hold. From (28) and (29) follows the congruence (30). From

(29) or (30), depending on whether $c(\beta_n) = a'_2(\beta_n)$ or $c(\beta_n) = \delta_A(\beta_n)$, for $x = \beta_n, n = i, j, l$, we obtain

$$\begin{cases} s_{33}(c(\beta_i) - kd(\beta_i)) + s_{13}c(\beta_i)\delta_B(\beta_i) - s_{23}c(\beta_i)b'_2(\beta_i) = 0, \\ s_{33}(c(\beta_j) - kd(\beta_j)) + s_{13}c(\beta_j)\delta_B(\beta_j) - s_{23}c(\beta_j)b'_2(\beta_j) = 0, \\ s_{33}(c(\beta_l) - kd(\beta_l)) + s_{13}c(\beta_l)\delta_B(\beta_l) - s_{23}c(\beta_l)b'_2(\beta_l) = 0, \end{cases} \quad (51)$$

where $k = s_{11}(s_{33})^{-1} \neq 0$. Since $s_{33} \neq 0$, then

$$\det \begin{vmatrix} c(\beta_i) - kd(\beta_i) & c(\beta_i)\delta_B(\beta_i) & c(\beta_i)b'_2(\beta_i) \\ c(\beta_j) - kd(\beta_j) & c(\beta_j)\delta_B(\beta_j) & c(\beta_j)b'_2(\beta_j) \\ c(\beta_l) - kd(\beta_l) & c(\beta_l)\delta_B(\beta_l) & c(\beta_l)b'_2(\beta_l) \end{vmatrix} = 0 \quad (52)$$

If we take into account that $c(\beta_n)\delta_B(\beta_n) = d(\beta_n)\delta_A(\beta_n)$ and $c(\beta_n)b'_2(\beta_n) = d(\beta_n)a'_2(\beta_n)$ (see (43)), then from (57), we have equality

$$\begin{aligned} & \det \begin{vmatrix} c(\beta_i) & c(\beta_i)\delta_B(\beta_i) & c(\beta_i)b'_2(\beta_i) \\ c(\beta_j) & c(\beta_j)\delta_B(\beta_j) & c(\beta_j)b'_2(\beta_j) \\ c(\beta_l) & c(\beta_l)\delta_B(\beta_l) & c(\beta_l)b'_2(\beta_l) \end{vmatrix} \\ & = k \det \begin{vmatrix} d(\beta_i) & d(\beta_i)\delta_A(\beta_i) & d(\beta_i)a'_2(\beta_i) \\ d(\beta_j) & d(\beta_j)\delta_A(\beta_j) & d(\beta_j)a'_2(\beta_j) \\ d(\beta_l) & d(\beta_l)\delta_A(\beta_l) & d(\beta_l)a'_2(\beta_l) \end{vmatrix}, \end{aligned} \quad (53)$$

from which it follows equality (45). From condition

$$\det \begin{vmatrix} a'_2(\beta_i) & \delta_A(\beta_i) \\ a'_2(\beta_j) & \delta_A(\beta_j) \end{vmatrix} = 0, \quad (54)$$

follows either $c(\beta_i) = a'_2(\beta_i) \neq 0, c(\beta_j) = a'_2(\beta_j) \neq 0$, or $a'_2(\beta_i) = a'_2(\beta_j) = 0, c(\beta_i) = \delta_A(\beta_i) \neq 0, c(\beta_j) = \delta_A(\beta_j) \neq 0$. Then

$$\det \begin{vmatrix} b'_2(\beta_i) & \delta_B(\beta_i) \\ b'_2(\beta_j) & \delta_B(\beta_j) \end{vmatrix} = 0, \quad (55)$$

and $c(\beta_i)d(\beta_i) = c(\beta_i)b'_2(\beta_i), c(\beta_j)d(\beta_j) = c(\beta_j)b'_2(\beta_j)$, or $c(\beta_i)d(\beta_i) = c(\beta_i)\delta_B(\beta_i), c(\beta_j)d(\beta_j) = c(\beta_j)\delta_B(\beta_j)$. Since $s_{33} \neq 0$, then in each case, from system (51), we have

$$\det \begin{vmatrix} c(\beta_i) - kd(\beta_i) & c(\beta_i)d(\beta_i) \\ c(\beta_j) - kd(\beta_j) & c(\beta_j)d(\beta_j) \end{vmatrix} = 0, \quad (56)$$

where it follows (43).

Sufficiency. Let $\beta_1, \dots, \beta_m$ are all those elements from the set M_2 for which $c(\beta_1), \dots, c(\beta_m) \neq 0$, and all rows $\|a'_2(\beta_1) \ \delta_A(\beta_1)\|, \dots, \|a'_2(\beta_m) \ \delta_A(\beta_m)\|$ are pairwise different. Under the conditions of theorem (see (iii)), the rows

$\|b'_2(\beta_1) \ \delta_B(\beta_1)\|, \dots, \|b'_2(\beta_m) \ \delta_B(\beta_m)\|$ are also pairwise different, and each has a nonzero first element. Consider the equation

$$\begin{vmatrix} c(\beta_1) - kd(\beta_1) & c(\beta_1)\delta_B(\beta_1) & -c(\beta_1)b'_2(\beta_1) \\ \vdots & \vdots & \vdots \\ c(\beta_m) - kd(\beta_m) & c(\beta_m)\delta_B(\beta_m) & -c(\beta_m)b'_2(\beta_m) \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = 0, \quad (57)$$

with the unknown $\|x_1 \ x_2 \ x_3\|^T$. If we take into account that $c(\beta_i)\delta_B(\beta_i) = d(\beta_i)\delta_A(\beta_i)$ and $c(\beta_i)b'_2(\beta_i) = d(\beta_i)a'_2(\beta_i)$, $1, \dots, m$, (see condition (ii)), then from (45), it follows that in the matrix of equation (57), every three rows are linearly dependent. Hence, this equation has a nonzero solution. If for some $\beta_i, \beta_j \in M_2$, $1 \leq i, j \leq m$, we have

$$\det \begin{vmatrix} a'_2(\beta_i) & \delta_A(\beta_i) \\ a'_2(\beta_j) & \delta_A(\beta_j) \end{vmatrix} = 0, \quad (58)$$

then from conditions (i), (ii) can be obtained

$$\det \begin{vmatrix} b'_2(\beta_i) & \delta_B(\beta_i) \\ b'_2(\beta_j) & \delta_B(\beta_j) \end{vmatrix} = 0. \quad (59)$$

Then, it follows from (46) that in the matrix of equation (57), the rows for the same β_i, β_j are linearly dependent. Therefore, this equation has a solution $\|x_{10} \ x_{20} \ x_{30}\|^T$ with a nonzero first component x_{10} . Based on this solution, taking into account conditions (i)–(iii), we can write the congruences

$$a'_2(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) - s_{22}b'_2(x) \equiv 0 \pmod{(x - \beta_1) \dots (x - \beta_m)}, \quad (60)$$

$$\delta_A(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) - s_{11}\delta_B(x) \equiv 0 \pmod{(x - \beta_1) \dots (x - \beta_m)}. \quad (61)$$

where $s_{33} = x_{10}$, $s_{11} = s_{22} = kx_{10}$, $s_{13} = x_{20}$, $s_{23} = x_{30}$. It should be borne in mind that $s_{33} + s_{13}\delta_B(\beta_i) - s_{23}b'_2(\beta_i) \neq 0$ for each $\beta_i = \{\beta_1, \dots, \beta_m\}$. It is clear that from (60) and (61), it is possible to pass to (29), (30). If we take into account that $(\phi_1(x), \phi_{12}(x)) = 1$ and $a_1(x) = b_1(x)$, then by adding to the congruence (30) multiplied by $-a_1(x)$ the congruence (29), we proceed to the congruence (28) with the same above-mentioned coefficients $s_{33}, s_{11}, s_{13}, s_{23}$. Based on the elements of matrices $A(x)$, $B(x)$ and elements $s_{33}, s_{11}, s_{22} = s_{11}, s_{13}, s_{23}$, we construct matrices

$$\|s_{ij}\|_1^3 = \begin{vmatrix} s_{11} & 0 & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & s_{33} \end{vmatrix}, \quad (62)$$

$$\|r_{ij}(x)\|_1^3,$$

where

$$\begin{aligned} r_{11}(x) &= s_{11} + s_{13}a_3(x), \\ r_{12}(x) &= s_{13}a_2(x), \\ r_{13}(x) &= s_{13}\phi_2(x), \\ r_{21}(x) &= \frac{a_3(x)(s_{23} - s_{13}b_1(x))}{\phi_1(x)} \in \mathbb{C}[x], \\ r_{22}(x) &= s_{22} + a'_2(x)(s_{23} - s_{13}b_1(x)), \\ r_{23}(x) &= \phi_{12}(x)(s_{23} - s_{13}b_1(x)), \\ r_{33}(x) &= s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x), \\ r_{31}(x) &= \frac{a_3(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) - s_{11}b_3(x)}{\phi_2(x)} \in \mathbb{C}[x], \\ r_{32}(x) &= \frac{a'_2(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) - s_{22}b'_2(x)}{\phi_{12}(x)} \in \mathbb{C}[x]. \end{aligned} \quad (63)$$

It is easy to verify the reversibility of matrix $\|r_{ij}(x)\|_1^3$ and the truth of equality (5) for constructed $\|s_{ij}\|_1^3$, $\|r_{ij}(x)\|_1^3$ and the original matrices $A(x)$, $B(x)$. The theorem is proved. $\square$

Theorem 2. Suppose that in the oriented by the same characteristic roots $\alpha_0, \alpha_1 \in M_1$ the reduced matrices $A(x)$ (2), $B(x)$ (4) (see Definition 1) we have\

$$\begin{aligned} M_{21} &:= \{\beta_p: \beta_p \in M_2, \delta_A(\beta_p) \neq 0\}, \\ M_{22} &:= \{\beta_q: \beta_q \in M_2, \delta_A(\beta_q) = 0\}, \end{aligned} \quad (64)$$

where $\delta_A(x)$ is defined in Proposition 6. Matrices $A(x)$, $B(x)$ are semiscalarly equivalent if and only if their elements $a_1(x), b_1(x)$ coincide, and the rest satisfy the following conditions:

- (i) $(\delta_A(x), \phi_{12}(x)) = (\delta_B(x), \phi_{12}(x))$,
 $(a'_2(x), \phi_{12}(x)) = (b'_2(x), \phi_{12}(x))$,
 $(a_3(x), \phi_{12}(x)) = (b_3(x), \phi_{12}(x))$;
- (ii) for arbitrary roots $\beta_i, \beta_j \in M_{21}$ or $\beta_i, \beta_j \in M_{22}$, we have $a'_2(\beta_i) = a'_2(\beta_j) \Leftrightarrow b'_2(\beta_i) = b'_2(\beta_j)$;
- (iii) for arbitrary roots $\beta_i, \beta_j, \beta_l \in M_{21}$ or $\beta_i, \beta_j, \beta_l \in M_{22}$ such that $a'_2(\beta_i), a'_2(\beta_j), a'_2(\beta_l)$ are nonzero and pairwise different, we have

$$\frac{a'_2(\beta_l)(a'_2(\beta_j) - a'_2(\beta_i))}{a'_2(\beta_j)(a'_2(\beta_l) - a'_2(\beta_i))} = \frac{b'_2(\beta_l)(b'_2(\beta_j) - b'_2(\beta_i))}{b'_2(\beta_j)(b'_2(\beta_l) - b'_2(\beta_i))}. \quad (65)$$

Proof. Necessity. According to Proposition 4, the elements $a_1(x), b_1(x)$ of matrices $A(x), B(x)$ on the set M_1 acquire the same values (0 and 1). Since $\deg a_1(x), \deg b_1(x) < |M_1|$, then $a_1(x) = b_1(x)$.

- (i) Matrices $A(x), B(x)$ satisfy equality (5), in which the matrix $\|s_{ij}\|_1^3$ according to Proposition 3 has an upper triangular form. For the elements of these matrices from (5) follow the congruences (33), (34). If we exclude s_{12} from them, we can obtain a congruence (30). From the latter, given the symmetry of the relation of semiscalar equivalence, it follows $(\delta_A(x), \phi_{12}(x)) = (\delta_B(x), \phi_{12}(x))$. Therefore, $\delta_B(\beta_i) \neq 0$ for every $\beta_i \in M_{21}$. This means that $b'_2(\beta_i) \neq 0$ for the same $\beta_i \in M_{21}$. Also from (34), for $x = \beta_i \in M_{22}$, we have

$$a'_2(\beta_j)(s_{33} - s_{23}b'_2(\beta_j)) - s_{22}b'_2(\beta_j) = 0. \quad (66)$$

whence $a'_2(\beta_j) = 0 \Leftrightarrow b'_2(\beta_j) = 0$. Therefore, $(a'_2(x), \phi_{12}(x)) = (b'_2(x), \phi_{12}(x))$. If we put $\beta_i \in M_{21}$ in (33) and take into account that $b'_2(\beta_i) = \delta_B(\beta_i) = b_1(\beta_i)b'_2(\beta_i) - b_3(\beta_i)$, $s_{22} = s_{11} + s_{12}$, we can write

$$a_3(\beta_i)(s_{33} + (s_{13} - s_{23})b'_2(\beta_i)) - s_{11}b_3(\beta_i) = 0. \quad (67)$$

If you put $x = \beta_j \in M_{22}$ and take into account $b_1(\beta_i)b'_2(\beta_i) = b_3(\beta_i)$, then from (33), we have

$$a_3(\beta_i)(s_{33} - s_{23})b_3(\beta_i) - s_{22}b_3(\beta_i) = 0. \quad (68)$$

From the obtained equalities (67), (68), it follows that $a_3(\beta_i) = 0 \Leftrightarrow b_3(\beta_i) = 0$, $a_3(\beta_j) = 0 \Leftrightarrow b_3(\beta_j) = 0$, whence $(a_3(x), \phi_{12}(x)) = (b_3(x), \phi_{12}(x))$.

- (ii) The required equivalence for zero values $a'_2(\beta_i), a'_2(\beta_j)$ follows from (i), because then the values $b'_2(\beta_i), b'_2(\beta_j)$ are also zero. Let now $a'_2(\beta_i), a'_2(\beta_j) \neq 0$. From (34) for $\beta_i, \beta_j \in M_{21}$, we have

$$\begin{aligned} a'_2(\beta_i)(s_{33} + (s_{13} - s_{23})b'_2(\beta_i)) - s_{11}b'_2(\beta_i) &= 0, \\ a'_2(\beta_j)(s_{33} + (s_{13} - s_{23})b'_2(\beta_j)) - s_{11}b'_2(\beta_j) &= 0. \end{aligned} \quad (69)$$

and for $\beta_i, \beta_j \in M_{22}$, we have

$$a'_2(\beta_i)(s_{33} - s_{23}b'_2(\beta_i)) - s_{22}b'_2(\beta_i) = 0. \quad (70)$$

and (66). From each of the obtained pairs of equalities, it follows that $b'_2(\beta_i) = b'_2(\beta_j)$.

- (iii) Let $\beta_i, \beta_j, \beta_l \in M_{21}$ or $\beta_i, \beta_j, \beta_l \in M_{22}$, and all values $a'_2(\beta_i), a'_2(\beta_j), a'_2(\beta_l)$ are nonzero and pairwise different. Under conditions (i), (ii), $b'_2(\beta_i), b'_2(\beta_j), b'_2(\beta_l)$ are also nonzero and pairwise different. If we put $x = \beta_i, x = \beta_j, x = \beta_l$ in (66), then in each of the cases $\beta_i, \beta_j, \beta_l \in M_{21}$ or $\beta_i, \beta_j, \beta_l \in M_{22}$, we get systems

$$\begin{cases} s_{11}(a'_2(\beta_i))^{-1} - s_{33}(b'_2(\beta_i))^{-1} - s_{13} + s_{23} = 0, \\ s_{11}(a'_2(\beta_j))^{-1} - s_{33}(b'_2(\beta_j))^{-1} - s_{13} + s_{23} = 0, \\ s_{11}(a'_2(\beta_l))^{-1} - s_{33}(b'_2(\beta_l))^{-1} - s_{13} + s_{23} = 0, \end{cases} \quad (71)$$

or

$$\begin{cases} s_{22}(a'_2(\beta_i))^{-1} - s_{33}(b'_2(\beta_i))^{-1} + s_{23} = 0, \\ s_{22}(a'_2(\beta_j))^{-1} - s_{33}(b'_2(\beta_j))^{-1} + s_{23} = 0, \\ s_{22}(a'_2(\beta_l))^{-1} - s_{33}(b'_2(\beta_l))^{-1} + s_{23} = 0, \end{cases} \quad (72)$$

respectively. Since $s_{11}, s_{22}, s_{33} \neq 0$, then from each of the obtained systems, we have

$$\det \begin{vmatrix} (a'_2(\beta_i))^{-1} - (a'_2(\beta_i))^{-1} & (b'_2(\beta_j))^{-1} - (b'_2(\beta_i))^{-1} \\ (a'_2(\beta_j))^{-1} - (a'_2(\beta_i))^{-1} & (b'_2(\beta_l))^{-1} - (b'_2(\beta_i))^{-1} \end{vmatrix} = 0, \quad (73)$$

whence it follows (65).

Sufficiency. Let in matrices $A(x), B(x)$ the elements $a_1(x), b_1(x)$ coincide, and for the rest conditions (i)–(iii) are fulfilled. If $a_2(x) \equiv 0$, then $\phi_{12}(x)\delta_A(x)$ (see definition) and $a_3(x) \equiv 0$. Then, from (i), it follows that $b_2(x) = b_3(x) \equiv 0$. In this case, $A(x) = B(x)$ and everything is proved. Now let $a_2(x) \neq 0$ and $\{\beta_1, \dots, \beta_m\}$ is the maximum subset of the set M_2 for which all values $a'_2(\beta_1), \dots, a'_2(\beta_m)$ are nonzero and $a'_2(\beta_i) \neq a'_2(\beta_j)$ if $\beta_i, \beta_j \in M_{21}$ or $\beta_i, \beta_j \in M_{22}$, $1 \leq i, j \leq m$.

Obviously, $m \geq 1$. Under conditions (i), (ii), we have $b'_2(\beta_1), \dots, b'_2(\beta_m) \neq 0$ and $b'_2(\beta_i) \neq b'_2(\beta_j)$. Let also

$\{\beta_1, \dots, \beta_r\} \subset M_{21}$, $\{\beta_{r+1}, \dots, \beta_m\} \subset M_{22}$, $0 \leq r \leq m$. Consider the equation

$$\begin{pmatrix} (a'_2(\beta_1))^{-1} & 0 & -(b'_2(\beta_1))^{-1} & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (a'_2(\beta_r))^{-1} & 0 & -(b'_2(\beta_r))^{-1} & -1 & 1 \\ 0 & (a'_2(\beta_{r+1}))^{-1} & -(b'_2(\beta_{r+1}))^{-1} & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (a'_2(\beta_m))^{-1} & -(b'_2(\beta_m))^{-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0, \quad (74)$$

relative to the unknown $\|x_1 \ x_2 \ x_3 \ x_4 \ x_5\|^T$. The rank of the matrix of this equation does not exceed 4, because, as follows from (iii), the rank of each of the selected submatrices is not more than 2. This means that the equation

has a nonzero solution. Moreover, it is easy to see that there is a solution $\|x_{10} \ x_{20} \ x_{30} \ x_{40} \ x_{50}\|^T$, where $x_{10}, x_{20}, x_{30} \neq 0$. Based on this solution, the congruences can be written:

$$a'_2(x)(s_{33} + (s_{13} - s_{23})b'_2(x)) - s_{11}b'_2(x) \equiv 0 \pmod{(x - \beta_1) \dots (x - \beta_r)}, \quad (75)$$

$$a'_2(x)(s_{33} - s_{23}b'_2(x)) - s_{22}b'_2(x) \equiv 0 \pmod{(x - \beta_{r+1}) \dots (x - \beta_m)}, \quad (76)$$

where $s_{11} = x_{10}$, $s_{22} = x_{20}$, $s_{33} = x_{30}$, $s_{13} = x_{40}$, $s_{23} = x_{50}$. Since $a'_2(\beta_i) = \delta_A(\beta_i)$, $b'_2(\beta_i) = \delta_B(\beta_i)$, for every $\beta_i \in \{\beta_1, \dots, \beta_r\}$ (see definition and condition (i)), (75) can be written as

$$\begin{aligned} a'_2(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) + s_{12}\delta_B(x) - s_{22}b'_2(x) \\ \equiv 0 \pmod{(x - \beta_1) \dots (x - \beta_r)}, \end{aligned} \quad (77)$$

where $s_{12} = s_{22} - s_{11}$. From the congruences (76) and (77), we can write

$$\begin{aligned} a'_2(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b'_2(x)) + s_{12}\delta_B(x) - s_{22}b'_2(x) \\ \equiv 0 \pmod{(x - \beta_1) \dots (x - \beta_m)}, \end{aligned} \quad (78)$$

since all $\beta_1, \dots, \beta_m$ are different and

$$\delta_A(x) \equiv \delta_B(x) \equiv 0 \pmod{(x - \beta_{r+1}) \dots (x - \beta_m)}. \quad (79)$$

For $\beta_i, \beta_j \in M_{21}$ or $\beta_i, \beta_j \in M_{22}$ under conditions (i), (ii), the following implications are true:

$$\delta_A(\beta_i) = \delta_A(\beta_j) \Leftarrow a'_2(\beta_i) = a'_2(\beta_j) \Rightarrow b'_2(\beta_i) = b'_2(\beta_j) \Rightarrow \delta_B(\beta_i) = \delta_B(\beta_j). \quad (80)$$

Therefore, the congruence (78) is performed modulo $\phi_{12}(x)$, that is, the congruence (34) is true. Since $a'_2(\beta_i) = \delta_A(\beta_i)$, $b'_2(\beta_i) = \delta_B(\beta_i)$ for every $\beta_i \in M_{21}$ and

$\delta_A(\beta_j) = \delta_B(\beta_j) = 0$ for every $\beta_j \in M_{22}$, we can move from (75) to congruence

$$\delta_A(x)(s_{33} + s_{13}(x)\delta_B(x) - s_{23}b'_2(x)) \equiv s_{11}\delta_B(x) \pmod{(x - \beta_1) \dots (x - \beta_m)}, \quad (81)$$

and from it to the congruence (30), if we take into account (80). If we add to the congruence (30), the congruence (34) multiplied by $-a_1(x) = -b_1(x)$ and take into account that $(a_3(x), \phi_1(x)) = (b_3(x), \phi_1(x)) = \phi_1(x)$, $(\phi_1(x), \phi_{12}(x)) = 1$, then we can obtain the congruence (33). According to the above s_{ij} , $i, j = 1, 2, 3$, $i \leq j$, and elements of matrices $A(x), B(x)$, we construct matrices

$$\begin{aligned} \|s_{ij}\|_1^3 &= \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & s_{23} \\ 0 & 0 & s_{33} \end{pmatrix}, \\ \|r_{ij}(x)\|_1^3 & \end{aligned} \quad (82)$$

where

$$\begin{aligned}
r_{11}(x) &= s_{11} + s_{13}a_3(x), \\
r_{12}(x) &= s_{12}\phi_1(x) + s_{13}a_2(x), \\
r_{13}(x) &= s_{13}\phi_2(x), \\
r_{21}(x) &= \frac{s_{22}a_1(x) + s_{23}a_3(x) - b_1(x)(s_{11} + s_{12}a_1(x) + s_{13}a_3(x))}{\phi_1(x)}, \\
r_{22}(x) &= s_{22} + s_{23}a_2'(x) - b_1(x)(s_{12} + s_{13}a_2'(x)), \\
r_{23}(x) &= \phi_{12}(x)(s_{23} - s_{13}b_1(x)), \\
r_{33}(x) &= s_{33} - s_{13}b_3(x) - b_2'(x)(s_{23} - s_{13}b_1(x)), \\
r_{31}(x) &= \frac{a_3(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b_2'(x)) + s_{11}\delta_B(x) + a_1(x)(s_{12}\delta_B(x) - s_{22}b_2'(x))}{\phi_2(x)}, \\
r_{32}(x) &= \frac{a_2'(x)(s_{33} + s_{13}\delta_B(x) - s_{23}b_2'(x)) + s_{12}\delta_B(x) - s_{22}b_2'(x)}{\phi_{12}(x)}.
\end{aligned} \tag{83}$$

The first of these matrices is not singular. In the second matrix $r_{21}(x) \in \mathbb{C}[x]$, since $\phi_1(x)|a_3(x)$, $s_{12} = s_{22} - s_{11}$ and the equal polynomials $a_1(x), b_1(x)$ on the set M_1 acquire only two values (0 and 1). We also have $r_{31}(x), r_{32}(x) \in \mathbb{C}[x]$, considering (33) and (34). We can see that $\|r_{ij}(x)\|_1^3 \in \text{GL}(3, \mathbb{C}[x])$, and moreover, both of these matrices $\|s_{ij}\|_1^3, \|r_{ij}(x)\|_1^3$ together with $A(x), B(x)$ satisfy equality (5). This means that $A(x), B(x)$ are semiscalarly equivalent. This completes the proof of the theorem. $\square$

5. Conclusion

In the class of semiscalarly equivalent polynomial matrices of simple structure, the so-called oriented by characteristic roots reduced matrix is established. It has a lower triangular form, invariant factors on the main diagonal, and some other properties. The left transformation matrix in the transition from one oriented by characteristic roots reduced matrix to another semiscalarly equivalent oriented (by the same characteristic roots) reduced matrix has an upper triangular form (Propositions 3, 6, 9). Theorems 1 and 2 give the necessary and sufficient conditions of the semiscalarly equivalence of oriented by the same characteristic roots reduced matrices. Proof of all statements is constructive. Based on them, we can specify the method of constructing the left and right transformation matrices in the transition from an arbitrary triangular matrix with invariant factors on the main diagonal to the oriented by characteristic roots reduced matrix (Propositions 2, 5, 8). This makes it possible to apply the obtained results to the solution of linear two-sided matrix equations of the type $XA(x) - B(x)Y(x) = 0$ with unknown nonsingular matrix X over $\mathbb{C}$ and invertible $Y(x)$ over $\mathbb{C}[x]$. Matrix equations of this type are often found in mathematical physics, other sections of mathematics, and applied problems.

Data Availability

The data used to support the findings of the study are included within the article as references.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

- [1] P. S. Kazimirskii, *Factorization of Matrix Polynomials*, Naukova Dumka, Kyiv, Ukraine, 1981.
- [2] P. S. Kazimirskii and V. M. Petrychkovych, "On the equivalence of polynomials matrices," in *Theoretical and Applied Problems in Algebra and Differential Equations*, Naukova Dumka, Kyiv, Ukraine, 1977.
- [3] L. Baratchart, "Un theoreme de factorisation et son application a la representation des systemes cuclique causaux," *Comptes Rendus de l'Academie des Sciences - Series I: Mathematics*, vol. 295, no. 3, pp. 223–226, 1982.
- [4] J. A. Dias da Silva and T. J. Laffey, "On simultaneous similarity of matrices and related questions," *Linear Algebra and Its Applications*, vol. 291, no. 1-3, pp. 167–184, 1999.
- [5] P. Donovan and M. R. Freislich, "Some evidence for an extension of the Brauer–Thrall conjecture," *Sonderforschungsbereich Theoretical Mathematica*, vol. 40, pp. 24–26, 1972.
- [6] J. A. Drozd, "Tame and wild matrix problems," *Representation Theory II*, vol. 832, pp. 242–258, 1980.
- [7] B. Z. Shavarovskii, "Oriented by characteristic roots reduced matrices in the class of semiscalarly equivalent," *Journal of Mathematics*, vol. 2021, Article ID 5592756, 6 pages, 2021.
- [8] V. Petrychkovych, *Generalized Equivalence of Matrices and its Collections and Factorization of Matrices over Rings*, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of the NAS of Ukraine, Lviv, Ukraine, 2015.

- [9] N. B. Ladzoryshyn and V. M. Petrychkovych, “Standard form of matrices over quadratic rings with respect to the (z, k) -equivalence and the structure of solutions of bilateral matrix linear equations-equivalence and the structure of solutions of bilateral matrix linear equations,” *Journal of Mathematical Sciences*, vol. 253, no. 1, pp. 54–62, 2021.