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Research Article

Digital Hopf Spaces and Their Duals

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In this article, we study the fundamental notions of digital Hopf and co-Hopf spaces based on pointed digital images. We show that a digital Hopf space, a digital associative Hopf space, a digital Hopf group, and a digital commutative Hopf space are unique up to digital homotopy type; that is, there is only one possible digital Hopf structure up to digital homotopy type on the underlying digital image. We also establish an equivalent condition for a digital image to be a digital Hopf space and investigate the difference between ordinary topological co-Hopf spaces and their digital counterparts by showing that any digital co-Hopf space is a digitally contractible space focusing on deep-learning methods in imaging science.

1. Introduction

1.1. Hopf and Co-Hopf Spaces. In many cases in algebraic topology, it is feasible to introduce a natural group structure on some set whose elements consist of homotopy classes of the base point preserving continuous functions from a pointed topological space to another. It is well known that if (X, x_0) is a group-like space or if (W, w_0) has a cogroup structure, then we can construct a group structure on the set $[(W, w_0), (X, x_0)]$ of homotopy classes of continuous functions from (W, w_0) to (X, x_0) as inherited from various functors.

A Hopf space (X, m_X) is a pointed topological space (X, x_0) along with a base point preserving continuous function $m_X: X \times X \longrightarrow X$, called a multiplication on X, for which the constant map $c_{x_0}: X \longrightarrow X$ is a homotopy identity.

It is well known that every topological group is a Hopf space and that no even-dimensional sphere of nonnegative dimension is a Hopf space except for the zero sphere S^0. What about the odd-dimensional spheres? The answer to this question is that the spheres in dimensions 1, 3, and 7 are the only spheres that are Hopf spaces. In the general case, as compared to a topological group, Hopf spaces may lack associativity and inverses.

As the dual notion of a Hopf space, a pair (C, φ_C) consisting of a pointed space $C = (C, c_0)$ and a continuous function $\varphi_C: C \longrightarrow C \vee C$, called a comultiplication on C, preserving the base point is called a co-Hopf space if $\pi_1 \circ \varphi_C$ and $\pi_2 \circ \varphi_C$ are homotopic to the identity function 1_C on C, where π_1 and π_2 are the projections $C \vee C \longrightarrow C$ onto the first and second summands, respectively; see [1–11] for further details on related topics. One of the reasons for the importance of co-Hopf spaces is that we can obtain a (not necessarily abelian) group structure on the set $[(C, c_0), (Y, y_0)]$ of pointed homotopy classes just like with Hopf spaces.

1.2. Digital Homotopy Sets. Just like classical homotopy theory in algebraic topology, in the realm of digital geometry and computer science, we can construct a covariant functor $[(W, w_0, k_W), -]$ and a contravariant functor $[-, (X, x_0, k_X)]$ from the category of pointed digital images to the category of sets (resp., groups) and functions (resp., group homomorphisms), where (W, w_0, k_W) and (X, x_0, k_X) are pointed digital images with k_W-adjacency and k_X-adjacency relations, respectively. We refer to the papers [12] (Theorem 4.14) and [13] as a special case of the sets of digital homotopy classes from a digital image to another.

The digital fundamental groups and the digital homotopy groups allow us to have much more efficient homotopy characterization of digital images. Indeed, we can handle the fundamental properties of pointed digital images more easily by using the group structure on the set $[(W, w_0, k_W), (X, x_0, k_X)]$ of digital homotopy classes of base point preserving digital continuous functions from (W, w_0, k_W) to (X, x_0, k_X) if the pointed digital image (X, x_0, k_X) with k_X-adjacency has the structure of a digital Hopf group.

1.3. Motivations. Historically, a Hopf space was named after Heinz Hopf [14] as the Eckmann–Hilton dual of a co-Hopf space [15]. In the early 20th century, a lot of interesting results on Lie groups as the particular case of Hopf spaces have been widely developed by suitable methods for CW-complexes from the homology and cohomology viewpoints in algebraic topology. The pointed Hopf spaces have been the direct outgrowth of compact Lie groups in classical homotopy theory; see [15–17]. Recently, the digital version of a Hopf space has been investigated by several authors; see [18–21].

The aforementioned statements serve as some motivation for research on this topic. Therefore, we need to develop and investigate the basic properties of digital Hopf spaces, digital associative Hopf spaces, digital Hopf groups, digital commutative Hopf spaces, and their duals from the digital homotopy points of view as an application to computer science, focusing on deep-learning methods in imaging science. It is shown in [22] that the definition of a path in algebraic topology is coherent with respect to the one used for digital images with an adjacency relation.

In the same vein, digital counterparts of geometry or topology deal with discrete sets, which are recognized to be digitized images of the n-dimensional Euclidean space. For application of the powerful digital homotopy-theoretical tools, it is favorable to reformulate the digital counterparts of Hopf spaces and co-Hopf spaces in the digital homotopy categories as seen in the realms of digital geometry and computer science, based on the deep-learning methods in imaging science.

1.4. Organization of the Paper. This article is organized as follows. In Section 2, we introduce and explain the fundamental notions of digital topology or digital images. In Section 3, we show that a digital Hopf space, a digital associative Hopf space, a digital Hopf group, and a digital commutative Hopf space are unique up to digital homotopy type; that is, there is only one possible digital Hopf structure up to digital homotopy type on the underlying digital image. We also find an equivalent condition for a digital image to be a digital Hopf space. In Section 4, we investigate the differences between ordinary topological co-Hopf spaces and digital co-Hopf spaces by showing that any digital co-Hopf space is digitally contractible. We also show that any strictly digital co-Hopf space is a set having only one element as a base point in which the result is exactly the same as the Eckmann–Hilton dual of a topological group. A summary and further research direction will be given at the end of this paper.

2. Preliminaries

Let $\mathbb{Z}$ be the set (or a topological space or the ring) of integers, and let X be a finite subset of $\mathbb{Z}^n$ for $n \geq 1$. A binary digital image (or digital image for short) is a pair (X, k_X), where k_X indicates some adjacency relation. For a positive integer u with $1 \leq u \leq n$, an adjacency relation of a digital image in $\mathbb{Z}^n$ is defined as follows.

Definition 1 (see [23]). Let u and n be positive integers with $u \leq n$. Then, two distinct points $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in $\mathbb{Z}^n$ are l_u-adjacent if

(i) there exist at most u distinct indices i such that $|a_i - b_i| = 1$;

(ii) for all indices j, if $|a_j - b_j| \neq 1$, then $a_j = b_j$.

The number l_u is the cardinal number $k_X = k(u, n)$, $1 \leq u \leq n$, of the set of elements b of $\mathbb{Z}^n$ which is adjacent to a given element a of $X \subsetneq \mathbb{Z}^n$, and is called an adjacency relation k_X defined on $\mathbb{Z}^n$. A digital image $X \subsetneq \mathbb{Z}^n$ with an adjacency relation k_X on $\mathbb{Z}^n$ is sometimes denoted as (X, k_X) to designate the adjacency relation. For example,

(i) $l_1 = 2 = k(1, 1)$, in $\mathbb{Z}^1$;

(ii) $l_1 = 4 = k(1, 2)$, $l_2 = 8 = k(2, 2)$ in $\mathbb{Z}^2$;

(iii) $l_1 = 6 = k(1, 3)$, $l_2 = 18 = k(2, 3)$, $l_3 = 26 = k(3, 3)$ in $\mathbb{Z}^3$;

(iv) $l_1 = 8 = k(1, 4)$, $l_2 = 32 = k(2, 4)$, $l_3 = 64 = k(3, 4)$, $l_4 = 80 = k(4, 4)$ in $\mathbb{Z}^4$, and so on.

Definition 2 (see [24]). A digital image $X \subset \mathbb{Z}^n$ with an adjacency relation k_X is called k_X-connected if for any pair of distinct elements $\{a, b\}$ of X, there is a set $A = \{a_0, a_1, \ldots, a_s\} \subset X$ of $s + 1$ distinct points satisfying that $a = a_0, a_s = b$, and a_i and a_{i+1} are k_X-adjacent for $i = 0, 1, 2, \ldots, s - 1$ for $s \geq 1$.

Definition 3 (see [12, 24]). Let $X \subset \mathbb{Z}^{n_1}$ and $Y \subset \mathbb{Z}^{n_2}$ be the digital images with k_X-adjacency and k_Y-adjacency relations, respectively. A map $f: X \longrightarrow Y$ between digital images is said to be (k_X, k_Y)-continuous if, for any k_X-connected subset A of X, the image $f(A)$ is also a k_Y-connected subset of Y.

Definition 4 (see [25]). For the nonnegative integers c and d with $c < d$, a digital interval is a set of the type

$$[c, d]_{\mathbb{Z}} = \{z \in \mathbb{Z} | c \leq z \leq d\}, \tag{1}$$

with the 2-adjacency relation in the set of all nonnegative integers.

Definition 5 (see [12, 26]). Let $X := (X, k_X)$ be a digital image along with the k_X-adjacency relation. Then, a (2 k_X)-continuous function $f: [0, m]_{\mathbb{Z}} \longrightarrow X$ is called a digital k_X-path in X.

Definition 6 (see [12, 23]). Let $X: = (X, k_X)$ and $Y: = (Y, k_Y)$ be digital images, and let $f, g: X \longrightarrow Y$ be (k_X, k_Y)-continuous functions. Then, f is said to be digital (k_X, k_Y)-homotopic to g if there exist an integer m and a function $H: X \times [0, m]_{\mathbb{Z}} \longrightarrow Y$ satisfying that

(i) $H(x, 0) = f(x)$ and $H(x, m) = g(x)$ for all $x \in X$;

(ii) the function $H_x: [0, m]_{\mathbb{Z}} \longrightarrow Y$, $x \in X$, which is given by $H_x(t) = H(x, t)$ is $(2, k_Y)$-continuous for all $t \in [0, m]_{\mathbb{Z}}$;

(iii) the function $H_t: X \longrightarrow Y$, $t \in [0, m]_{\mathbb{Z}}$, which is given by $H_t(x) = H(x, t)$ is (k_X, k_Y)-continuous for all $x \in X$.

In this case, the function H is called a digital (k_X, k_Y)-homotopy between f and g, denoted by

$$H: f \simeq_{(k_X, k_Y)} g. \tag{2}$$

It is well known that the relation $\simeq_{(k_X, k_Y)}$ is an equivalence relation on the set of all (k_X, k_Y)-continuous functions from X to Y. For a (k_X, k_Y)-continuous function $f: X \longrightarrow Y$, we denote $[f]$ by

$$[f] = \left\{ g: X \longrightarrow Y | f \simeq_{(k_X, k_Y)} g \right\}, \tag{3}$$

which is called the digital homotopy class represented by f.

Definition 7 (see [12, 26]). A pointed digital image with k_X-adjacency is a triple $X: = (X, x_0, k_X)$. In this situation, the element x_0 of X is called the base point of (X, x_0, k_X). A base point preserving digital continuous function $f: (X, x_0, k_X) \longrightarrow (Y, y_0, k_Y)$ is a (k_X, k_Y)-continuous function $f: (X, k_X) \longrightarrow (Y, k_Y)$ with $f(x_0) = y_0$.

Definition 8 (see [25]). A digital image $X: = (X, x_0, k_X)$ is said to be digitally contractible if the identity function $1_X: X \longrightarrow X$ on X is digitally null-homotopic; that is, it is digital (k_X, k_X)-homotopic to the constant function $c_{x_0}: X \longrightarrow X$ at x_0.

Definition 9 (see [26]). Let $X: = (X, k_X)$ and $Y: = (Y, k_Y)$ be digital images. Then, a (k_X, k_Y)-continuous function $f: X \longrightarrow Y$ is said to be a (k_X, k_Y)-homotopy equivalence if there exists a (k_X, k_Y)-continuous function $g: Y \longrightarrow X$ such that

$$\begin{aligned} g \circ f &\simeq_{(k_X, k_X)} 1_X, \\ f \circ g &\simeq_{(k_Y, k_Y)} 1_Y. \end{aligned} \tag{4}$$

In this case, we say that X and Y have the same digital homotopy type which is denoted by

$$X \simeq_{(k_X, k_Y)} Y. \tag{5}$$

Convention: In this paper, we work on the pointed digital category $\mathcal{D}_*$ of pointed digital images and base point preserving digital continuous functions. Thus, any digital image in this paper has a base point, and all maps (resp., homotopies) are base point preserving digital continuous functions (resp., pointed digital homotopies) between pointed digital images.

3. Digital Hopf Spaces

From now on, the base points and the adjacency relations of digital images will sometimes be omitted for our notational convenience unless we specifically state otherwise.

Definition 10 (see [27]). Let $X: = (X, x_0, k_X)$ and $Y: = (Y, y_0, k_Y)$ be the pointed digital images. Then, the normal product adjacency $NP(k_X, k_Y)$ on the Cartesian product,

$$X \times Y: = \left(X \times Y, x_0 \times y_0, NP\left(k_X, k_Y\right)\right), \tag{6}$$

is defined as follows: if $x, x' \in X$ and $y, y' \in Y$, then the two elements (x, y) and (x', y') of $X \times Y$ are $NP(k_X, k_Y)$-adjacent in $X \times Y$ if and only if

(i) $x = x'$, and y and y' are k_Y-adjacent;

(ii) x and x' are k_X-adjacent, and $y = y'$; or

(iii) x and x' are k_X-adjacent, and y and y' are k_Y-adjacent.

Note that the normal product adjacency relation makes the Cartesian product $X \times Y$ into a digital image with the adjacency relation $NP(k_X, k_Y)$ whose base point is $x_0 \times y_0$.

Let s and t be the integers with $1 < s \leq t$, and let $\left\{(X_i, k_{X_i}) | i = 1, 2, \ldots, t\right\}$ be a family of digital images (X_i, k_{X_i}) for $i = 1, 2, \ldots, t$. Then, as a generalization of the normal product adjacency, we define the following.

Definition 11 (see [28, 29]). Let $x_i, x_i' \in X_i$ for $i = 1, 2, \ldots, t$. Then, the two elements $x = (x_1, x_2, \ldots, x_t)$ and $x' = (x_1', x_2', \ldots, x_t')$ of $\prod_{i=1}^{t} X_i$ are $NP_s(k_{X_1}, k_{X_2}, \ldots, k_{X_t})$-adjacent if and only if

(i) for at least 1 and at most s indices i, x_i, and x_i' are k_{X_i}-adjacent;

(ii) for all other indices j, $x_j = x_j'$.

The adjacency relation $NP_s(k_{X_1}, k_{X_2}, \ldots, k_{X_t})$ is said to be the generalized normal product adjacency on the Cartesian product $\prod_{i=1}^{t} X_i$.

3.1. Notation. We now fix the notations in this paper:

(i) $c_{x_0}: X \longrightarrow X$ and $c_{y_0}: Y \longrightarrow Y$ are the constant functions whose values are x_0 and y_0, respectively.

(ii) $k_{X \times Y}$ is the normal product adjacency on $X \times Y$; that is, $k_{X \times Y}: = NP(k_X, k_Y)$, and in particular $k_{X \times X}: = NP(k_X, k_X)$.

(iii) $k_{X \times Y \times Z}$ is the generalized normal product adjacency on $X \times Y \times Z$; that is, $k_{X \times Y \times Z}: = NP_3(k_X, k_Y, k_Z)$, and in particular $k_{X \times X \times X}: = NP_3(k_X, k_X, k_X)$.

(iv) $T_{X \times Y}: X \times Y \longrightarrow Y \times X$ is the switching function sending (x, y) to (y, x).

(v) $\Delta_X\colon X \longrightarrow X \times X$ is the diagonal function sending x to $x \times x$ as a $(k_X, k_{X\times X})$-continuous function.

We now consider the digital versions of the Hopf space, homotopy associativity, homotopy commutativity, and Hopf group in algebraic topology as follows—see [15, 30, 31] for the terminologies in classical homotopy theory.

Definition 12 (see [18]). A digital image $X := (X, x_0, k_X)$ with a k_X-adjacency relation is called a digital Hopf space if there is a $(k_{X\times X}, k_X)$-continuous function,

$$m_X\colon X \times X \longrightarrow X, \tag{7}$$

such that the diagram in Figure 1 is digital homotopy commutative; that is,

$$m_X \circ \left(c_{x_0} \times 1_X\right) \circ \Delta_X \simeq_{(k_X,k_X)} 1_X \simeq_{(k_X,k_X)} m_X \circ \left(1_X \times c_{x_0}\right) \circ \Delta_X. \tag{8}$$

In this case, the $(k_{X\times X}, k_X)$-continuous function

$$m_X\colon X \times X \longrightarrow X \tag{9}$$

is called a digital multiplication on (X, x_0, k_X), and the constant function c_{x_0} is said to be a digital homotopy identity.

Definition 13 (see [21]). Let $X := (X, x_0, k_X)$ and $Y := (Y, y_0, k_Y)$ be pointed digital Hopf spaces. Then, a (k_X, k_Y)-continuous function $f\colon X \longrightarrow Y$ is said to be a digital Hopf function if

$$f \circ m_X \simeq_{(k_{X\times X}, k_Y)} m_Y \circ (f \times f). \tag{10}$$

Definition 14 (see [21]). A digital multiplication $m_X\colon X \times X \longrightarrow X$ on a pointed digital Hopf space $X := (X, x_0, k_X)$ is said to be digital homotopy associative if the diagram in Figure 2 commutes up to digital homotopy. A digital Hopf space $X = (X, x_0, k_X)$ with a digital homotopy associative multiplication $m_X\colon X \times X \longrightarrow X$ is called a digital associative Hopf space.

Definition 15 (see [21]). Let $X := (X, x_0, k_X)$ be a digital Hopf space with a digital multiplication $m_X\colon X \times X \longrightarrow X$. A (k_X, k_X)-continuous function

$$\nu_X\colon X \longrightarrow X \tag{11}$$

is called a digital homotopy inverse if the diagram in Figure 3 commutes up to digital homotopy.

Remark 1. We should be careful with the term 'inverse' at this moment because an inverse is usually understood to be an element of a group or more generally of a set with a binary operation. The inverse in Definition 15 is a function that produces the homotopy inverse elements.

We note that, in general, there are many kinds of digital homotopy inverses and that one of the digital homotopy inverses $\nu_X\colon X \longrightarrow X$ can be constructed as a (k_X, k_X)-continuous function so that the triangles in Definition 15 are digital homotopy commutative. More precisely, we let $[f]$ and $[g]$ be digital homotopy classes represented by (k_W, k_X)-continuous functions $f, g\colon W \longrightarrow X$, respectively. If $W := (W, w_0, k_W)$ is a pointed digital image, and if $X := (X, x_0, k_X)$ is a pointed digital Hopf space with a digital multiplication $m_X\colon X \times X \longrightarrow X$, then we define a binary operation "$\boxplus$" on the set $[(W, w_0), (X, x_0)]$ of all digital homotopy classes of base point preserving digital continuous functions from W to X by the digital homotopy class of the composite ([21], Definition 3.7):

$$f \boxplus g\colon W \xrightarrow{\Delta_W} W \times W \xrightarrow{f \times g} X \times X \xrightarrow{m_X} X. \tag{12}$$

Let $\nu_X\colon X \longrightarrow X$ be a digital homotopy inverse on a pointed digital Hopf space $X := (X, x_0, k_X)$. Then, every element of the set $[(W, w_0), (X, x_0)]$ with the binary operation "$\boxplus$" is invertible, and the inverse $[f]^{-1}$ of $[f]$ is given by

$$[f]^{-1} = [\nu_X \circ f], \tag{13}$$

where $f\colon W \longrightarrow X$ is any digital (k_W, k_X)-continuous function; see [21] (Theorem 3.24) for more details.

Definition 16 (see [21]). A digital multiplication $m_X\colon X \times X \longrightarrow X$ on a pointed digital Hopf space $X := (X, x_0, k_X)$ is said to be digital homotopy commutative if the diagram in Figure 4 is digital homotopy commutative. Here,

$$T_{X\times X}\colon X \times X \longrightarrow X \times X \tag{14}$$

is the switching function. A digital Hopf space $X = (X, x_0, k_X)$ with a digital homotopy commutative multiplication $m_X\colon X \times X \longrightarrow X$ is called a digital commutative Hopf space.

A topological Hopf group [30] is a pointed homotopy associative Hopf space with a homotopy inverse satisfying the group axioms up to homotopy. We remark that any topological group is a topological Hopf group. We consider a digital version of a topological Hopf group as follows.

Definition 17. A digital Hopf group is a digital Hopf space (X, x_0, k_X) furnished with an associative multiplication $m_X\colon X \times X \longrightarrow X$ and an inverse $\nu_X\colon X \longrightarrow X$ up to digital homotopy.

Example 1. Let $e = (1, 0)$, $a = (0, 1)$, $b = (-1, 0)$, and $c = (0, -1)$ be the elements of $\mathbb{Z}^2$, and let $X := \{e, a, b, c\} \subsetneq \mathbb{Z}^2$ be a digital image with the l_1-adjacency relation defined on $\mathbb{Z}^2$. If we define a digital multiplication

$$m_X = \odot\colon X \times X \longrightarrow X, \tag{15}$$

by the rule of Table 1, then the pointed digital image (X, e, l_1) becomes a group as well as a digital Hopf group; see also [20] (Example 3.8).

The following results show that a digital Hopf space, a digital associative Hopf space, a digital Hopf group, and a digital commutative Hopf space are unique up to the digital homotopy type; that is, there is only one possible digital Hopf structure up to digital homotopy on the underlying digital image.

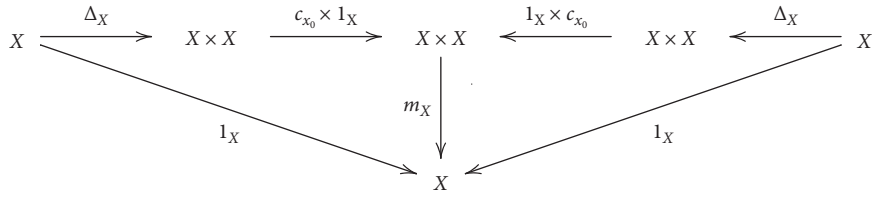

Figure 1: A commutative diagram.

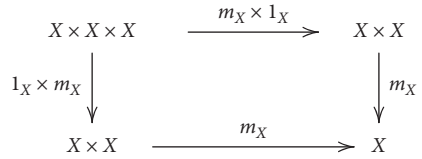

Figure 2: A commutative diagram.

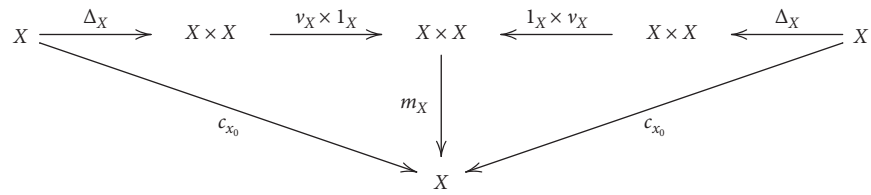

Figure 3: A commutative diagram.

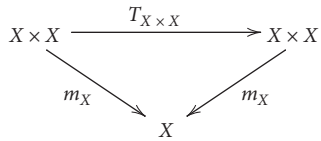

Figure 4: A commutative diagram.

Table 1: A digital multiplication $m_X = \odot : X \times X \longrightarrow X$.

Binary operation	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Theorem 1. *Let $X: = (X, x_0, k_X)$ be a digital Hopf group with a multiplication $m_X: X \times X \longrightarrow X$ and an inverse $v_X: X \longrightarrow X$ up to digital homotopy, and let $Y: = (Y, y_0, k_Y)$ be any digital image. If $f: X \longrightarrow Y$ is a (k_X, k_Y)-homotopy equivalence, then (Y, y_0, k_Y) becomes a digital Hopf group.*

Proof. Since $f: X \longrightarrow Y$ is a (k_X, k_Y)-homotopy equivalence, there exists a (k_Y, k_X)-continuous function $g: Y \longrightarrow X$ such that

$$\begin{aligned} g \circ f &\simeq_{(k_X,k_X)} 1_X, \\ f \circ g &\simeq_{(k_Y,k_Y)} 1_Y. \end{aligned} \tag{16}$$

If we define a function

$$m_Y: Y \times Y \longrightarrow Y \tag{17}$$

to be the composite

$$m_Y: Y \times Y \xrightarrow{g \times g} X \times X \xrightarrow{m_X} X \xrightarrow{f} Y \tag{18}$$

of digital functions, then $m_Y: Y \times Y \longrightarrow Y$ is a digital multiplication on $Y: = (Y, y_0, k_Y)$. Indeed, since the diagram in Figure 5 is digital homotopy commutative, we obtain

$$\begin{aligned} m_Y \circ \left(c_{y_0} \times 1_Y\right) \circ \Delta_Y &\simeq_{(k_Y,k_Y)} \left(f \circ m_X \circ (g \times g)\right) \circ \left(c_{y_0} \times 1_Y\right) \circ \Delta_Y \\ &= f \circ m_X \circ \left(c_{x_0} \times 1_X\right) \circ \Delta_X \circ g \\ &\simeq_{(k_Y,k_Y)} f \circ 1_X \circ g \\ &\simeq_{(k_Y,k_Y)} f \circ g \\ &\simeq_{(k_Y,k_Y)} 1_Y, \end{aligned} \tag{19}$$

and similarly

$$m_Y \circ \left(1_Y \times c_{y_0}\right) \circ \Delta_Y \simeq_{(k_Y,k_Y)} 1_Y. \tag{20}$$

Therefore, (Y, y_0, k_Y) is a digital Hopf space.

Secondly, we show that the diagram in Figure 6 is digital homotopy commutative. Indeed, we obtain

$$\begin{aligned} m_Y \circ (f \times f) &\simeq_{(k_{X\times X},k_Y)} \left(f \circ m_X \circ (g \times g)\right) \circ (f \times f) \\ &= f \circ m_X \circ (g \circ f \times g \circ f) \\ &\simeq_{(k_{X\times X},k_Y)} f \circ m_X \circ \left(1_X \times 1_X\right) \\ &= f \circ m_X, \end{aligned} \tag{21}$$

and similarly

$$m_X \circ (g \times g) \simeq_{(k_{Y\times Y},k_X)} g \circ m_Y. \tag{22}$$

Thus, $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ are digital Hopf functions.

Thirdly, we note that five faces out of the six faces of the hexahedron in Figure 7 are digital homotopy commutative, with the exception of the front face colored in blue. Indeed, using the homotopy relation (20), we can show that the left-hand face of the hexahedron is digital homotopy commutative as follows.

$$\begin{aligned} \left(1_X \times m_X\right) \circ (g \times g \times g) &= g \times \left(m_X \circ (g \times g)\right) \\ &\simeq_{(k_{Y\times Y\times Y},k_{X\times X})} g \times \left(g \circ m_Y\right) \\ &= (g \times g)\left(1_Y \times m_Y\right), \end{aligned} \tag{23}$$

and the same applies for the other four faces. For the back face to be homotopy commutative, we need to be associative. The reader should be reminded that this is from Definition 17. For the right face, refer to the homotopy at (21). The top face is from the previous page and the homotopy between $f \circ g$ and 1_Y. In addition, the bottom face is from the definition of m_Y.

We now prove that the front face is also commutative up to digital homotopy as follows:

$$\begin{aligned} &m_Y \circ (m_Y \times 1_Y) \\ &\simeq \left(f \circ m_X \circ (g \times g)\right) \circ \left(\left(f \circ m_X \circ (g \times g)\right) \times 1_Y\right) \\ &\simeq \left(f \circ m_X \circ (g \times g)\right) \circ \left(\left(f \circ m_X \circ (g \times g)\right) \times (f \circ g)\right) \\ &= \left(f \circ m_X \circ (g \times g)\right) \circ \left((f \times f) \circ \left(m_X \times 1_X\right) \circ ((g \times g) \times g)\right) \\ &= f \circ m_X \circ ((g \times g) \circ (f \times f)) \circ \left(\left(m_X \times 1_X\right) \circ ((g \times g) \times g)\right) \\ &= f \circ m_X \circ (g \circ f \times g \circ f) \circ \left(\left(\mathrm{m_X} \times 1_X\right) \circ ((\mathrm{g} \times \mathrm{g}) \times \mathrm{g})\right) \\ &\simeq f \circ m_X \circ \left(1_X \times 1_X\right) \circ \left(\left(m_X \times 1_X\right) \circ ((g \times g) \times g)\right) \\ &= f \circ \left(m_X \circ \left(m_X \times 1_X\right)\right) \circ (g \times g \times g) \\ &\simeq f \circ \left(m_X \circ \left(1_X \times m_X\right)\right) \circ (g \times g \times g) \\ &= f \circ \left(m_X \circ \left(1_X \times 1_X\right) \circ \left(1_X \times m_X\right)\right) \circ (g \times g \times g) \\ &\simeq f \circ \left(m_X \circ (g \circ f \times g \circ f) \circ \left(1_X \times m_X\right)\right) \circ (g \times g \times g) \\ &= f \circ m_X \circ ((g \times g) \circ (f \times f)) \circ \left(\left(1_X \times m_X\right) \circ (g \times (g \times g))\right) \\ &= \left(f \circ m_X \circ (g \times g)\right) \circ \left((f \times f) \circ \left(1_X \times m_X\right) \circ (g \times (g \times g))\right) \\ &= \left(f \circ m_X \circ (g \times g)\right) \circ \left((f \circ g) \times \left(f \circ m_X \circ (g \times g)\right)\right) \\ &\simeq \left(f \circ m_X \circ g \times g\right) \circ \left(1_Y \times \left(f \circ m_X \circ (g \times g)\right)\right) \\ &\simeq m_Y \circ \left(1_Y \times m_Y\right), \end{aligned} \tag{24}$$

where $\simeq$ means the digital $(k_{Y\times Y\times Y}, k_Y)$-homotopy. Thus, $m_Y: Y \times Y \longrightarrow Y$ is an associative multiplication up to digital homotopy.

We finally define a (k_Y, k_Y)-continuous function

$$\nu_Y: Y \longrightarrow Y, \tag{25}$$

by the digital homotopy class of the compositions

$$\nu_Y: Y \xrightarrow{g} X \xrightarrow{\nu_X} X \xrightarrow{f} Y, \tag{26}$$

so that the diagram in Figure 8 commutes up to digital homotopy. Indeed, since the diagram in Figure 9 is strictly commutative, by using the digital homotopy inverse ν_X, we have

$$\begin{aligned} &m_Y \circ \left(1_Y \times \nu_Y\right) \circ \Delta_Y \\ &\simeq \left(f \circ m_X \circ (g \times g)\right) \circ \left(1_Y \times f \circ \nu_X \circ g\right) \circ \Delta_Y \\ &= f \circ m_X \circ \left(g \times \left(g \circ f \circ \nu_X \circ g\right)\right) \circ \Delta_Y \\ &\simeq f \circ m_X \circ \left(g \times \left(1_X \circ \nu_X \circ g\right)\right) \circ \Delta_Y \\ &= f \circ m_X \circ \left(g \times \left(\nu_X \circ g\right)\right) \circ \Delta_Y \\ &= f \circ m_X \circ \left(1_X \times \nu_X\right) \circ (g \times g) \circ \Delta_Y \\ &= f \circ m_X \circ \left(1_X \times \nu_X\right) \circ \Delta_X \circ g \\ &\simeq f \circ c_{x_0} \circ g \\ &= c_{y_0}, \end{aligned} \tag{27}$$

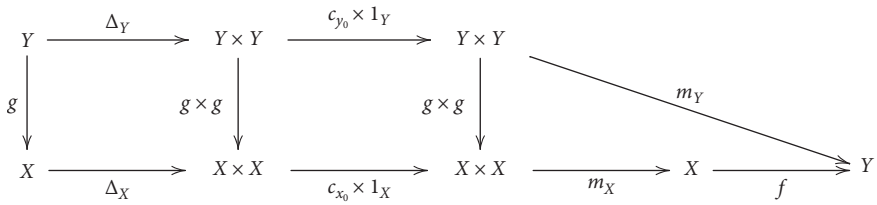

Figure 5: A commutative diagram.

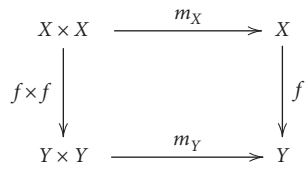

Figure 6: A commutative diagram.

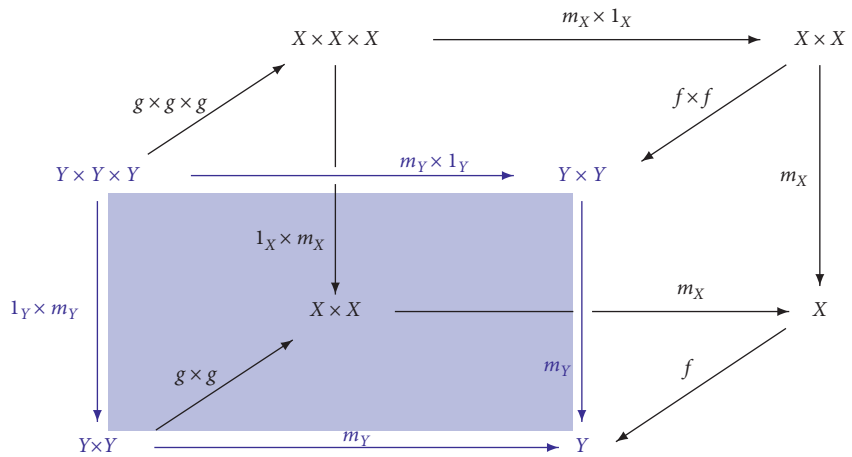

Figure 7: A commutative diagram.

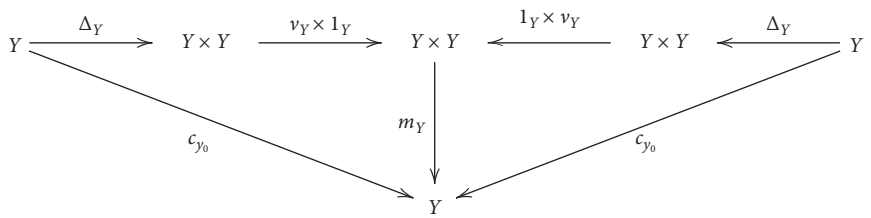

Figure 8: A commutative diagram.

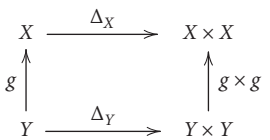

Figure 9: A commutative diagram.

and similarly

$$m_Y \circ (\nu_Y \times 1_Y) \circ \Delta_Y \simeq c_{y_0}, \tag{28}$$

where $\simeq$ means the digital (k_Y, k_Y)-homotopy. Therefore,

$$\nu_Y \colon Y \longrightarrow Y \tag{29}$$

is a digital homotopy inverse.

Using the relations from (19) through (28), (Y, y_0, k_Y) is a digital Hopf group, as required.□ □

Example 2. Let X be the digital Hopf group in Example 1 and let

$$(Y, e, l_1) \coloneqq \left(X \cup \left\{(2,0) \in \mathbb{Z}^2\right\}, e, l_1\right). \tag{30}$$

Then, we can show that

$$X \simeq_{(k_X, k_Y)} Y \tag{31}$$

and that the digital image (Y, e, l_1) becomes a digital Hopf group, where $l_1 = 4$. Indeed, let $f \colon X \longrightarrow Y$ be an inclusion and let $g \colon Y \longrightarrow X$ be a function given by

$$g(y) = \begin{cases} y, & \text{if } y \in X, \\ e, & \text{if } y = (2,0). \end{cases} \tag{32}$$

Then, it can be seen that f and g are (k_X, k_Y)- and (k_Y, k_X)-continuous functions, respectively, and that

$$g \circ f = 1_X \colon X \longrightarrow X. \tag{33}$$

Let $H \colon Y \times [0,1]_{\mathbb{Z}} \longrightarrow Y$ be the function given by

$$H(y,t) = \begin{cases} y, & \text{if } t = 0, \\ f \circ g(y), & \text{if } t = 1, \end{cases} \tag{34}$$

for all $y \in Y$. Then, we see that H is a $(k_{Y \times [0,1]_{\mathbb{Z}}}, k_Y)$-continuous function and that

$$H \colon f \circ g \simeq_{\left(k_{Y \times [0,1]_{\mathbb{Z}}}, k_Y\right)} 1_Y, \tag{35}$$

where

$$k_{Y \times [0,1]_{\mathbb{Z}}} = NP_2(k_Y, l_1) \tag{36}$$

with $l_1 = 2$. By (33) and (35), we have the result.

A binary operation is said to be commutative if changing the order of the operands does not change the result. It is one of the fundamental properties of many binary operations in mathematics, and many mathematical proofs depend on it. We now show that digital homotopy commutativity is digital homotopy invariant.

Theorem 2. *Let $X: = (X, x_0, k_X)$ be a digital commutative Hopf space with a multiplication $m_X \colon X \times X \longrightarrow X$ up to digital homotopy, and let $Y: = (Y, y_0, k_Y)$ be any digital image. If $f \colon X \longrightarrow Y$ is a (k_X, k_Y)-homotopy equivalence, then Y becomes a digital commutative Hopf space.*

Proof. We need to show that the diagram in Figure 10 is digital homotopy commutative. Since the diagram in Figure 11 is strictly commutative, using the digital homotopy commutativity, we have

$$\begin{aligned} m_Y \circ T_{Y \times Y} &\simeq (f \circ m_X \circ (g \times g)) \circ T_{Y \times Y} \\ &= f \circ m_X \circ T_{X \times X} \circ (g \times g) \\ &\simeq f \circ m_X \circ (g \times g) \\ &\simeq m_Y, \end{aligned} \tag{37}$$

where $\simeq$ means the digital $(k_{Y \times Y}, k_Y)$-homotopy. Thus, $Y: = (Y, y_0, k_Y)$ is a digital commutative Hopf space.

As usual, we define the digital projections as follows. □

Definition 18 (see [32]). The first and second digital projections

$$\begin{aligned} p_1 \colon X \times Y &\longrightarrow X, \\ p_2 \colon X \times Y &\longrightarrow Y \end{aligned} \tag{38}$$

are defined by

$$\begin{aligned} p_1(x,y) &= x, \\ p_2(x,y) &= y \end{aligned} \tag{39}$$

as the $(k_{X \times Y}, k_X)$-continuous and $(k_{X \times Y}, k_Y)$-continuous functions, respectively.

We note that the digital products have the following property.

Remark 2. Let $X: = (X, x_0, k_X)$ and $Y: = (Y, y_0, k_Y)$ be digital images. Then, for every digital image $Z: = (Z, z_0, k_Z)$ with digital continuous functions

$$\begin{aligned} q_1 \colon Z &\longrightarrow X, \\ q_2 \colon Z &\longrightarrow Y, \end{aligned} \tag{40}$$

there exists a unique $(k_Z, k_{X \times Y})$-continuous function $\varphi \colon Z \longrightarrow X \times Y$ given by

$$\varphi(z) = (q_1(z), q_2(z)), \tag{41}$$

such that the diagram in Figure 12 is commutative. The $(k_Z, k_{X \times Y})$-continuous function $\varphi \colon Z \longrightarrow X \times Y$ is denoted by $[q_1, q_2]$. In particular, if $X = Y = Z$, and q_1 and q_2 are the identity function 1_X on X, then

$$\varphi = (1_X, 1_X) \colon X \longrightarrow X \times X \tag{42}$$

is the digital diagonal function

$$\Delta_X \colon X \longrightarrow X \times X, \tag{43}$$

sending x to $x \times x$.

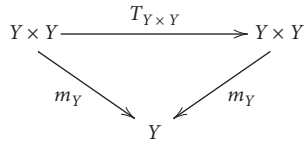

Figure 10: A commutative diagram.

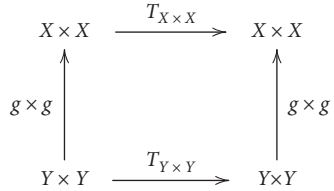

Figure 11: A commutative diagram.

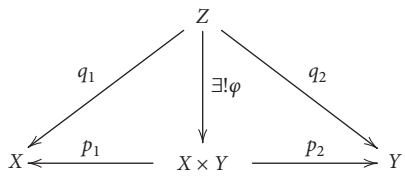

Figure 12: A commutative diagram.

Definition 19 (see [30, 33]). Let $X := (X, x_0)$ and $Y := (Y, y_0)$ be the digital images with $k(u,n)$-adjacency relations in $\mathbb{Z}^n$. The digital wedge product $X \vee Y$ is given by

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y, \tag{44}$$

with the $k(u,2n)$-adjacency relation whose base point is (x_0, y_0) in $\mathbb{Z}^{2n}$. Here, the normal product adjacency in Definition 10 (or the Cartesian product in graph theory [34]) is assumed.

In particular, if $p_1 : X \times X \longrightarrow X$ and $p_2 : X \times X \longrightarrow X$ are the digital projections in Definition 18, then we can consider the projections $\breve{p}_1 : X \vee X \longrightarrow X$ and $\breve{p}_2 : X \vee X \longrightarrow X$ defined by the restrictions $\breve{p}_1 = p_1|_{X\vee X}$ and $\breve{p}_2 = p_2|_{X\vee X}$; that is,

$$\begin{aligned} \breve{p}_1(x, x_0) &= x = \breve{p}_2(x_0, x), \\ \breve{p}_1(x_0, x) &= x_0 = \breve{p}_2(x, x_0), \end{aligned} \tag{45}$$

for all (x, x_0) and (x_0, x) in the digital wedge product $X \vee X$.

Lemma 1. *For every digital image $X := (X, x_0)$, the function*

$$\left[\breve{p}_1, \breve{p}_2\right] : X \vee X \longrightarrow X \times X, \tag{46}$$

is equal to the inclusion function

$$i : X \vee X \longrightarrow X \times X, \tag{47}$$

of $X \vee X$ into $X \times X$.

Proof. Since both digital continuous functions $[\breve{p}_1, \breve{p}_2]$ and i make the two diagrams in Figures 13 and 14 commute, by uniqueness, we have

$$\left[\breve{p}_1, \breve{p}_2\right] = i : X \vee X \longrightarrow X \times X, \tag{48}$$

as required. □

Definition 20. The digital folding map

$$\nabla_X : X \vee X \longrightarrow X, \tag{49}$$

is a digital $(k_{X\vee X}, k_X)$-continuous function defined by

$$\nabla_X(x, x_0) = x = \nabla_X(x_0, x), \tag{50}$$

for all (x, x_0) and (x_0, x) in $X \vee X$.

We now find an equivalent condition for a digital image to be a digital Hopf space as follows.

Lemma 2. *A digital image $X := (X, x_0, k_X)$ is a digital Hopf space with a digital multiplication*

$$m_X : X \times X \longrightarrow X, \tag{51}$$

if and only if the diagram in Figure 15 is commutative up to digital homotopy.

Proof. Let

$$j_1, j_2 : X \longrightarrow X \vee X, \tag{52}$$

be the digital continuous functions given by

$$\begin{aligned} j_1(x) &= (x, x_0), \\ j_2(x) &= (x_0, x), \end{aligned} \tag{53}$$

respectively. Then, it can be shown that the diagram in Figure 16 is a commutative diagram up to digital homotopy. Indeed, we have

$$\begin{aligned} m_X \circ i \circ j_1(-) &= m_X \circ i(-, x_0) \\ &= m_X(-, x_0) \\ &= m_X \circ \left(1_X \times c_{x_0}\right)(-, -) \\ &= m_X \circ \left(1_X \times c_{x_0}\right) \circ \Delta_X(-) \\ &\simeq_{(k_X, k_X)} 1_X(-), \end{aligned} \tag{54}$$

that is,

$$m_X \circ i(-, x_0) \simeq_{(k_X, k_X)} 1_X(-) = \nabla_X(-, x_0), \tag{55}$$

and similarly

$$m_X \circ i(x_0, -) \simeq_{(k_X, k_X)} 1_X(-) = \nabla_X(x_0, -). \tag{56}$$

Conversely, we have

$$\begin{aligned} 1_X = \nabla_X \circ j_1 &\simeq_{(k_X, k_X)} m_X \circ i \circ j_1 = m_X \circ \left(1_X \times c_{x_0}\right) \circ \Delta_X, \\ 1_X = \nabla_X \circ j_2 &\simeq_{(k_X, k_X)} m_X \circ i \circ j_2 = m_X \circ \left(c_{x_0} \times 1_X\right) \circ \Delta_X, \end{aligned} \tag{57}$$

that is,

$$m_X\colon X \times X \longrightarrow X, \tag{58}$$

is a digital multiplication as required. □

Let $A\colon = (A, a_0, k_A), B\colon = (B, b_0, k_B)$, and $X\colon = (X, x_0, k_X)$ be the digital images, and let $f\colon A \longrightarrow X$ and $g\colon B \longrightarrow X$ be (k_A, k_X)-continuous and (k_B, k_X)-continuous functions, respectively. Then, we have the $(k_{A\vee B}, k_{X\vee X})$-continuous function

$$f\vee g\colon A\vee B \longrightarrow X\vee X, \tag{59}$$

defined by

$$(f\vee g)(a, b) = (f(a), g(b)), \tag{60}$$

where each ordered pair should contain at least one entry from the base points a_0, b_0, and x_0 of A, B, and X, respectively. □

Theorem 3. *The digital image $X\colon = (X, x_0, k_X)$ is a digital Hopf space if and only if for any (k_A, k_X)-continuous function $f\colon A \longrightarrow X$ and any (k_B, k_X)-continuous function $g\colon B \longrightarrow X$, there exists a $(k_{A\times B}, k_X)$-continuous function $s\colon A \times B \longrightarrow X$ such that*

$$\nabla_X \circ (f\vee g) \simeq_{(k_{A\vee B}, k_X)} s \circ j, \tag{61}$$

where $j\colon A\vee B \longrightarrow A \times B$ is the inclusion function of $A\vee B$ into $A \times B$.

Proof. Let $m_X\colon X \times X \longrightarrow X$ be the digital multiplication on $X\colon = (X, x_0, k_X)$. Then, we can define a $(k_{A\times B}, k_X)$-continuous function

$$s\colon A \times B \longrightarrow X, \tag{62}$$

by

$$s = m_X \circ (f \times g)\colon A \times B \xrightarrow{f\times g} X \times X \xrightarrow{m_X} X. \tag{63}$$

We thus have

$$\begin{aligned} s \circ j(-, b_0) &\simeq_{(k_{A\vee B}, k_X)} m_X \circ (f \times g) \circ j(-, b_0) \\ &= m_X \circ (f \times g)(-, b_0) \\ &= m_X \circ (f(-), x_0) \\ &= m_X \circ \left(1_X \times c_{x_0}\right)(f(-), f(-)) \\ &= m_X \circ \left(1_X \times c_{x_0}\right) \circ \Delta_X (f(-)) \\ &\simeq_{(k_X, k_X)} 1_X (f(-)) \\ &= f(-) \\ &= \nabla_X \circ (f(-), x_0) \\ &= \nabla_X \circ (f\vee g)(-, b_0), \end{aligned} \tag{64}$$

for all $(-, b_0)$ in $A\vee B$, and similarly

$$s \circ j(a_0, -) \simeq_{(k_{A\vee B}, k_X)} \nabla_X \circ (f\vee g)(a_0, -), \tag{65}$$

for all $(a_0, -)$ in $A\vee B$.

Conversely, for a digital folding map

$$\nabla_X\colon X\vee X \longrightarrow X, \tag{66}$$

there is a $(k_{X\times X}, k_X)$-continuous function $m_X\colon X \times X \longrightarrow X$ such that the diagram in Figure 17 is digital homotopy commutative. By Lemma 2, (X, x_0, k_X) becomes a digital Hopf space furnished with the digital multiplication m_X. □

4. A Difference between Ordinary Co-Hopf Spaces and Digital Co-Hopf Spaces

We now consider the digital version of a co-Hopf space in algebraic topology as follows; see [15, 30, 31] for the terminology in classical homotopy theory.

Definition 21. A digital image $C\colon = (C, c_0, k_C)$ with a k_C-adjacency relation is called a digital co-Hopf space if there is a $(k_C, k_{C\vee C})$-continuous function

$$\varphi_C\colon C \longrightarrow C\vee C, \tag{67}$$

such that the two triangles in Figure 18 are digital homotopy commutative, where 1_C is the identity function on C, and

$$\pi_1, \pi_2\colon C\vee C \longrightarrow C, \tag{68}$$

are the first projection and the second projection, respectively. In this case, the $(k_C, k_{C\vee C})$-continuous function $\varphi_C\colon C \longrightarrow C\vee C$ above is called a digital comultiplication on (C, c_0, k_C).

Let $[f]$ and $[g]$ be digital homotopy classes represented by (k_C, k_Y)-continuous functions $f, g\colon C \longrightarrow Y$, respectively. If $C\colon = (C, c_0, k_C)$ is a pointed digital co-Hopf space with a digital comultiplication $\varphi_C\colon C \longrightarrow C\vee C$, and if $Y\colon = (Y, y_0, k_Y)$ is a digital image, then we also define a binary operation "$\circledast$" on the set $[(C, c_0), (Y, y_0)]$ of all digital homotopy classes of base point preserving digital continuous functions from C to Y by the digital homotopy class of the composite

$$f \circledast g\colon C \xrightarrow{\varphi} C\ C\vee C \xrightarrow{f\vee g} Y\vee Y \xrightarrow{\nabla} Y\ Y, \tag{69}$$

that is,

$$[f] \circledast [g] = [f \circledast g] = \left[\nabla_Y \circ (f\vee g) \circ \varphi_C\right], \tag{70}$$

where $\nabla_Y\colon Y\vee Y \longrightarrow Y$ is the digital folding map. We note that the binary operation "$\circledast$" makes the set $[(C, c_0), (Y, y_0)]$ into an algebraic set just like the aforementioned binary operation "$\boxplus$" in Section 3. Note that there are many kinds of homotopy comultiplications with many distinctive properties in classical homotopy theory; see [2], and [3] (Definitions 3.1, 3.5, and 3.12) for further details.

We note that a topological co-Hopf space needs not be contractible even though it is a finite dimensional space. Take the spheres and the finite wedge products of spheres as examples.

The following theorem shows that there is a disparity or a difference between ordinary topological co-Hopf spaces and digital co-Hopf spaces.

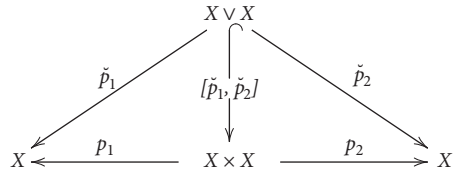

Figure 13: A commutative diagram.

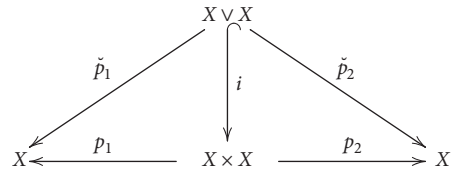

Figure 14: A commutative diagram.

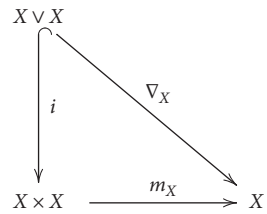

Figure 15: A commutative diagram.

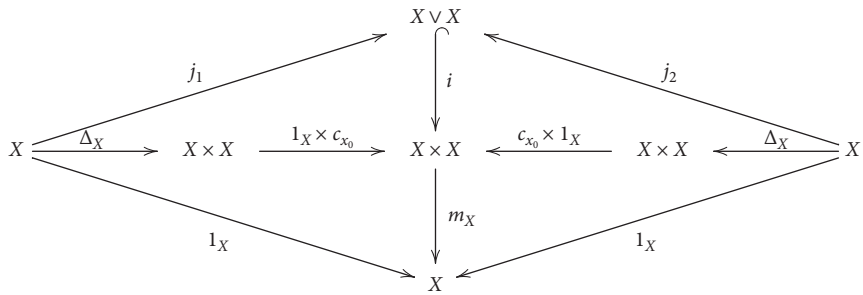

Figure 16: A commutative diagram.

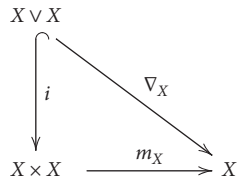

Figure 17: A commutative diagram.

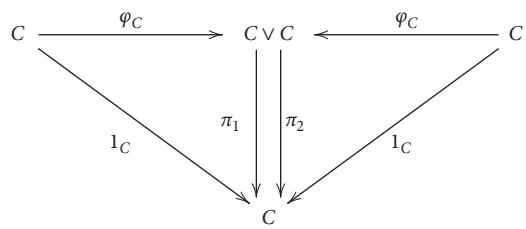

Figure 18: A commutative diagram.

Theorem 4. *Any digital co-Hopf space is digitally contractible.*

Proof. Let $C := (C, c_0, k_C)$ be a digital co-Hopf space together with a digital comultiplication.

$$\varphi_C : C \longrightarrow C \vee C. \tag{71}$$

If x is any element of C, then

$$\varphi_C(x) = (c, d), \tag{72}$$

where either c or d is the base point c_0 of C. We thus have

$$\begin{aligned} 1_C(-) &\simeq_{(k_C,k_C)} \pi_1 \circ \varphi_C(-) \\ &= \pi_1(c, d) \\ &= c, \\ 1_C(-) &\simeq_{(k_C,k_C)} \pi_2 \circ \varphi_C(-) \\ &= \pi_2(c, d) \\ &= d, \end{aligned} \tag{73}$$

in the digital co-Hopf space $C := (C, c_0, k_C)$. If $c = c_0$, then

$$1_C(-) \simeq_{(k_C,k_C)} c_{c_0}(-) = c_0, \tag{74}$$

where $c_{c_0} : C \longrightarrow C$ is the constant function at c_0. Similarly, if $d = c_0$, then

$$1_C(-) \simeq_{(k_C,k_C)} c_{c_0}(-). \tag{75}$$

Thus, C is a contractible digital image. □

Theorem 4 asserts that the set $\mathcal{S}$ of all digital homotopy classes of digital comultiplications $\varphi_C : C \longrightarrow C \vee C$ is the set consisting of a single element; that is, $|\mathcal{S}| = 1$.

A topological group is a group G together with a topology on G such that the binary operation and the inverse function are continuous functions with respect to the topology on G. A Hopf group is a generalization of the topological group which is obtained by replacing the strict equality by the classical homotopy relation.

We now replace the digital homotopy "$\simeq_{(k_C,k_C)}$" in Definition 21 by the strict equality "=" as follows. □

Definition 22. A digital image $C := (C, c_0, k_C)$ along with k_C-adjacency relation is called a strictly digital co-Hopf space if there is a $(k_C, k_{C\vee C})$-continuous function $\varphi_C : C \longrightarrow C \vee C$ such that $\pi_1 \circ \varphi_C = 1_C$ and $\pi_2 \circ \varphi_C = 1_C$.

Remark 3. By Theorem 4 and by restricting the digital homotopy to the strict equality, we can see that any strictly digital co-Hopf space is a set consisting of a single element as the base point; see also [35] (Lemma 3.8).

5. Summary

We have investigated the fundamental notions of digital Hopf spaces, digital Hopf groups, digital co-Hopf spaces, and digital co-Hopf groups based on digital images. We have proven that digital Hopf spaces, digital associative Hopf spaces, digital Hopf groups, and digital commutative Hopf spaces are unique up to the digital homotopy type; that is, there is only one possible digital Hopf structure up to digital homotopy on the underlying digital image.

We have established a necessary and sufficient condition for a digital image to be a digital Hopf space and investigated the relationship or the difference between ordinary topological co-Hopf spaces and digital co-Hopf spaces by showing that any digital co-Hopf space is digitally contractible. We have also noticed that any strictly digital co-Hopf space is a set having only one element as a base point in which the result is exactly the same as the Eckmann–Hilton dual of a topological group as an application in computer science focusing on the deep-learning methods in imaging science.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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