Research Article

On the Partial Sums of the q -Generalized Dini Function

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Our objective in this paper is to introduce a q -analog of the generalized Dini function. Also, we investigate the lower bound for the ratio of the q -generalized Dini function to its sequences of partial sums.

1. Introduction and Basic Concepts

Let $\mathcal{A}$ denote the class of functions f that have the following Maclaurin's form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which is analytic and univalent in the open unit disc $\mathcal{U} = \{z: |z| < 1\}$ and satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$.

Special functions play an inspired role in applied mathematics and physics. The widespread use of these functions has attracted many researchers to work in many directions. Lately, many authors studied the geometric properties of some special functions such as starlikeness,

univalence, and convexity, see [1–6]. There are several results related to partial sums of analytic univalent functions that were developed by the authors in [7–9]. Specifically, the authors in [10] investigated the partial sums of the generalized Bessel function, and then, a lot of authors followed them in studying the same problem for different special functions such as Bessel [11, 12], Struve [13], Lommel [14], Wright [15], and Mittag-Leffler [16], see also [17].

Our aim in this study is to develop a q -analog of the generalized Dini function, which is inspired by early studies on analytic and special functions. We also provide lower bounds for the ratio of q -generalized Dini function to its sequences of partial sums, for $m \in \mathbb{N} = \{1, 2, 3, \dots\}$, $(\psi_{\gamma, b, c}^a(z; q))_m = z + \sum_{n=1}^m \zeta_n z^{n+1}$. We will investigate the following:

$$\Re \left\{ \frac{\psi_{\gamma, b, c}^a(z; q)}{(\psi_{\gamma, b, c}^a(z; q))_m} \right\}, \Re \left\{ \frac{(\psi_{\gamma, b, c}^a(z; q))_m}{\psi_{\gamma, b, c}^a(z; q)} \right\}, \Re \left\{ \frac{(\psi_{\gamma, b, c}^a(z; q))'_m}{(\psi_{\gamma, b, c}^a(z; q))'_m} \right\}, \text{ and } \Re \left\{ \frac{(\psi_{\gamma, b, c}^a(z; q))'_m}{(\psi_{\gamma, b, c}^a(z; q))'_m} \right\}. \quad (2)$$

To introduce the main results, we would like to recall some fundamentals and concepts related to geometric function theory and the definition of q -generalized Bessel

function. At first, let us consider the following second-order linear homogenous differential equation (for more details, see [18–20]):

$$z^2 w''(z) + bz w'(z) + [cz^2 - \nu^2 + (1-b)\nu] w(z) = 0, \quad (3)$$

$$(b, c, \nu \in \mathbb{C}).$$

The function $w_{\nu,b,c}(z)$ is known as the generalized Bessel function of the first kind of order ν , which is a particular solution of equation (3). The function $w_{\nu,b,c}(z)$ has the following series representation:

$$w_{\nu,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \Gamma(\nu + n + ((b+1)/2))} \left(\frac{z}{2}\right)^{2n+\nu}, \quad (4)$$

where Γ stands for the Gamma function.

Now, let $q \in (0, 1)$ and $a \in \mathbb{C}$; the q -shifted factorial is defined by

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1} (1 - aq^k), & n \in \mathbb{N}. \end{cases} \quad (5)$$

The limit of $(a; q)_n$ as n tends to infinity exists and is denoted by $(a; q)_{\infty}$:

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k). \quad (6)$$

The multiple q -shifted factorial for complex numbers $a_1, a_2, \dots, a_r$ is defined by

$$(a_1, a_2, \dots, a_r; q)_n = \prod_{j=1}^r (a_j; q)_n. \quad (7)$$

If $aq^{\alpha} \neq q^{-n}$, for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define $(a; q)_{\alpha}$ to be

$$(a; q)_{\alpha} = \frac{(a; q)_{\infty}}{(aq^{\alpha}; q)_{\infty}}. \quad (8)$$

Definition 1. Let $b, \nu \in \mathbb{R}$ and $c \in \mathbb{C}$; the q -generalized Bessel function is defined by

$$\begin{aligned} \omega_{\nu,b,c}(z; q) &= \frac{(q^k; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-c)^n (1-q)^{((b-1)/2)}}{(q; q)_n (q^k; q)_n} \left(\frac{z}{2}\right)^{2n+\nu} \\ &= \frac{1}{(q; q)_{k-1}} \sum_{n=0}^{\infty} \frac{(-c)^n (1-q)^{((b-1)/2)}}{(q, q^k; q)_n} \left(\frac{z}{2}\right)^{2n+\nu}, \end{aligned} \quad (9)$$

where $k = \nu + b + 1/2 \neq 0, -1, -2, \dots$, $z \in \mathcal{U}$ and $q \in (0, 1)$.

Now, we introduce the q -generalized Dini function $\phi_{\nu,b,c}^a(z; q)$ in terms of $\omega_{\nu,b,c}(z; q)$.

Definition 2. Let $a \in \mathbb{R}^+$; the q -generalized Dini function $\phi_{\nu,b,c}^a(z; q)$ is defined by

$$\begin{aligned} \phi_{\nu,b,c}^a(z; q) &= (a - \nu) \omega_{\nu,b,c}(\sqrt{z}; q) + \sqrt{z} \omega'_{\nu,b,c}(z; q), \\ &= \frac{1}{(q; q)_{k-1}} \sum_{n=0}^{\infty} \frac{(-c)^n (1-q)^{((b-1)/2)} (a + 2n)}{(q, q^k; q)_n} \left(\frac{z}{4}\right)^{(n+(v/2))}. \end{aligned} \quad (10)$$

Remark 1. By specializing the value of a, b, c , and q , we see that

- (1) $\lim_{q \rightarrow 1^-} \omega_{\nu,b,c}((1-q)z; q) = \omega_{\nu,b,c}(z)$ is the generalized Bessel function of first kind introduced by Orhan and Yagmur [10]
- (2) By putting $b = c = 1$, then $\omega_{\nu,1,1}(z; q) = J_{\nu}^{(1)}(z; q)$ is the first kind of q -Bessel function given by Annaby and Mansour [21]. Also, $\lim_{q \rightarrow 1^-} \omega_{\nu,1,1}((1-q)z; q) = J_{\nu}(z)$ is the familiar Bessel function defined by Baricz [18].
- (3) By putting $b = -c = 1$, then $\omega_{\nu,1,-1}(z; q) = I_{\nu}^{(1)}(z; q)$ is the first kind of modified q -Bessel function given by Annaby and Mansour [21]. Also, $\lim_{q \rightarrow 1^-} \omega_{\nu,1,-1}((1-q)z; q) = I_{\nu}(z)$ is the modified Bessel function defined by Baricz [18].
- (4) $\lim_{q \rightarrow 1^-} \phi_{\nu,b,c}^a((1-q)^2 z; q) = D_{\nu,a,b,c}(z)$ is the generalized Dini function investigated by Deniz and Gören [3]. Also, by putting $b = c = 1$ in the last expression, we get $d_{\nu,a}(z)$ introduced by Aktaş and Orhan [22]. In addition, by putting $a = 1$, we obtain the Dini function $d_{\nu}(z)$ which is introduced by Baricz et al. [2].

Because the function $\phi_{\nu,b,c}^a(z; q)$ defined by (10) do not belong to the class $\mathcal{A}$, we consider the following normalized form of the q -generalized Dini function, $\psi_{\nu,b,c}^a(z; q): \mathcal{U} \rightarrow \mathbb{C}$, as

$$\begin{aligned} \psi_{\nu,b,c}^a(z; q) &= \frac{2^{\nu}(q; q)_{k-1}}{a(1-q)^{b-1/2}} z^{1-(v/2)} \phi_{\nu,b,c}^a(z; q) \\ &= z + \sum_{n=1}^{\infty} \zeta_n z^{n+1}, \end{aligned} \quad (11)$$

where

$$\zeta_n = \frac{(-c)^n (a + 2n)}{4^n a (q, q^k; q)_n}. \quad (12)$$

Definition 3 (subordination principle, see [23–25]). An analytic function f is said to be subordinate to another analytic function g , written as $f(z) \prec g(z)$ ($z \in \mathcal{U}$), if there exists a Schwarz function ω , which is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in \mathcal{U}$), such that $f(z) = g(\omega(z))$. In particular, if the function g is univalent in $\mathcal{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}). \quad (13)$$

Remark 2. It is observed that the function $\Omega(z) = 1 + z/1 - z$ maps $\mathcal{U}$ conformally into the right half plane such that $\Re\{\Omega(z)\} > 0$. That function plays a great roll in proving our main results.

Here, we would like to mention the following inequalities,

$$(q; q)_n \geq (1 - q)^n, \quad (14)$$

and

$$(q^\alpha; q)_n \geq (1 - q^\alpha)^n, \quad (15)$$

are valid for $q \in (0, 1)$, $\alpha \in \mathbb{R}$, and $n \in \mathbb{N}$.

Lemma 1. Let us consider $a \in \mathbb{R}^+$, $b, \nu \in \mathbb{R}$, $c \in \mathbb{C}$ ($|c| < 4$), and $k = \nu + (b + 1/2)$; the function $\psi_{\nu, b, c}^a(z; q)$, referred by (11), satisfies the following inequalities, for all $z \in \mathcal{U}$:

$$\begin{aligned} |\psi_{\nu, b, c}^a(z; q)| &\leq 1 + \frac{4|c|(a+2)(1-q)(1-q^k) - a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2}, \\ |(\psi_{\nu, b, c}^a(z; q))'| &\leq 1 + \frac{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k) + a|c|^3}{a[4(1-q)(1-q^k) - |c|]^3}. \end{aligned} \quad (16)$$

Proof. By taking into consideration inequalities (14) and (15), we obtain

$$\begin{aligned} |\psi_{\nu, b, c}^a(z; q)| &= \left| z + \sum_{n=1}^{\infty} \frac{(-c)^n(a+2n)}{4^n a(q, q^k; q)_n} z^{n+1} \right| \leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n(a+2n)}{a4^n(q; q)_n(q^k; q)_n} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n(a+2n)}{a4^n(1-q)^n(1-q^k)^n} \\ &= 1 + \frac{|c|}{4(1-q)(1-q^k)} \sum_{n=1}^{\infty} \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} + \frac{|c|}{2a(1-q)(1-q^k)} \sum_{n=1}^{\infty} n \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} \\ &\quad + \frac{4|c|(a+2)(1-q)(1-q^k) - a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2} \end{aligned} \quad (17)$$

and

$$\begin{aligned} |(\psi_{\nu, b, c}^a(z; q))'| &= \left| 1 + \sum_{n=1}^{\infty} \frac{(-c)^n(n+1)(a+2n)}{a4^n(q, q^k; q)_n} z^n \right| \leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n(2n^2 + (a+2)n + a)}{a4^n(q; q)_n(q^k; q)_n} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{(2n^2 + (a+2)n + a)}{a} \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^n \\ &= 1 + \frac{|c|}{4(1-q)(1-q^k)} \sum_{n=1}^{\infty} \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} + \frac{|c|(a+2)}{4a(1-q)(1-q^k)} \sum_{n=1}^{\infty} n \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} \\ &\quad + \frac{|c|}{2a(1-q)(1-q^k)} \sum_{n=1}^{\infty} n^2 \left(\frac{|c|}{4(1-q)(1-q^k)} \right)^{n-1} \\ &= 1 + \frac{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k)}{a[4(1-q)(1-q^k) - |c|]^3}. \end{aligned} \quad (18)$$

Thus, we complete the proof. $\square$

is valid, then

2. Main Results

Theorem 1. Let us consider $a \in \mathbb{R}^+$, $b, \nu \in \mathbb{R}$, and $c \in \mathbb{C}$ ($|c| < 4$), $k = \nu + b + 1/2$, and the function $\psi_{\nu,b,c}^a(z; q)$ be defined by (11) and its partial sum $(\psi_{\nu,b,c}^a(z; q))_m = z + \sum_{n=1}^m \zeta_n z^{n+1}$. If the inequality,

$$16a(1-q)^2(1-q^k)^2 + 2a|c|^2 > 4(3a+2)|c|(1-q)(1-q^k), \quad (19)$$

$$\Re \left\{ \frac{\psi_{\nu,b,c}^a(z; q)}{(\psi_{\nu,b,c}^a(z; q))_m} \right\} > \beta_1 \quad (20)$$

and

$$\Re \left\{ \frac{(\psi_{\nu,b,c}^a(z; q))_m}{\psi_{\nu,b,c}^a(z; q)} \right\} > \beta_2 \quad (21)$$

holds true, for all $z \in \mathcal{U}$, where

$$\beta_1 = \frac{16a(1-q)^2(1-q^k)^2 + 4|c|(2-a)(1-q)(1-q^k) + 2a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2},$$

$$\beta_2 = \frac{a[4(1-q)(1-q^k) - |c|]^2}{16a(1-q)^2(1-q^k)^2 + 4|c|(2-a)(1-q)(1-q^k)}.$$

Proof. From the steps of proving Lemma 1, we observe

$$1 + \sum_{n=1}^{\infty} |\zeta_n| \leq 1 + \frac{4|c|(a+2)(1-q)(1-q^k) - a|c|^2}{a[4(1-q)(1-q^k) - |c|]^2}, \quad (23)$$

which is equivalent to

$$\mu \sum_{n=1}^{\infty} |\zeta_n| \leq 1, \quad (24)$$

where

$$\mu = a[4(1-q)(1-q^k) - |c|]^2 / 4|c|(a+2)(1-q)(1-q^k) - a|c|^2.$$

Now, let us consider

$$\frac{1 + \omega(z)}{1 - \omega(z)} = \mu \left[\frac{\psi_{\nu,b,c}^a(z; q)}{(\psi_{\nu,b,c}^a(z; q))_m} - \beta_1 \right]$$

$$= \frac{1 + \sum_{n=1}^m \zeta_n z^n + \mu \sum_{n=m+1}^{\infty} \zeta_n z^n}{1 + \sum_{n=1}^m \zeta_n z^n}. \quad (25)$$

Therefore,

$$\omega(z) = \frac{\mu \sum_{n=m+1}^{\infty} \zeta_n z^n}{2 + 2 \sum_{n=1}^m \zeta_n z^n + \mu \sum_{n=m+1}^{\infty} \zeta_n z^n} \quad (26)$$

and

$$|\omega(z)| \leq \frac{\mu \sum_{n=m+1}^{\infty} |\zeta_n|}{2 - 2 \sum_{n=1}^m |\zeta_n| - \mu \sum_{n=m+1}^{\infty} |\zeta_n|}. \quad (27)$$

Inequality (20) holds true if $|\omega(z)| \leq 1$ according to the definition of subordination. Then, the upcoming inequality,

$$\sum_{n=1}^m |\zeta_n| + \mu \sum_{n=m+1}^{\infty} |\zeta_n| \leq 1, \quad (28)$$

implies that $|\omega(z)| \leq 1$. It suffices to show that the left-hand side of (28) is bounded above by

$$\mu \sum_{n=1}^{\infty} |\zeta_n|, \quad (29)$$

which is equivalent to

$$(\mu - 1) \sum_{n=1}^m |\zeta_n| \geq 0. \quad (30)$$

On the contrary, to prove inequality (21), we consider

$$\frac{1 + p(z)}{1 - p(z)} = (\mu + 1) \left[\frac{(\psi_{\nu,b,c}^a(z; q))_m}{\psi_{\nu,b,c}^a(z; q)} - \beta_2 \right],$$

$$= \frac{1 + \sum_{n=1}^m \zeta_n z^n - \mu \sum_{n=m+1}^{\infty} \zeta_n z^n}{1 + \sum_{n=1}^m \zeta_n z^n}.$$

Therefore,

$$p(z) = \frac{-(1 + \mu) \sum_{n=m+1}^{\infty} \zeta_n z^n}{2 + 2 \sum_{n=1}^m \zeta_n z^n + (1 - \mu) \sum_{n=m+1}^{\infty} \zeta_n z^n} \quad (32)$$

and

$$|p(z)| \leq \frac{(1 + \mu) \sum_{n=m+1}^{\infty} |\zeta_n|}{2 - 2 \sum_{n=1}^m |\zeta_n| - (\mu - 1) \sum_{n=m+1}^{\infty} |\zeta_n|}. \quad (33)$$

Inequality (21) holds true if $|p(z)| \leq 1$ according to the definition of subordination. Then, the upcoming inequality,

$$\sum_{n=1}^m |\zeta_n| + \mu \sum_{n=m+1}^{\infty} |\zeta_n| \leq 1, \quad (34)$$

implies that $|p(z)| \leq 1$. Since the left-hand side of (34) is bounded above $\mu \sum_{n=1}^{\infty} |\zeta_n|$, thus, we complete the proof. $\square$

Theorem 2. Let us consider $a \in \mathbb{R}^+$, $b, v \in \mathbb{R}$, $c \in \mathbb{C}$ ($|c| < 4$), and $k = v + b + 1/2$, the function $\psi_{v,b,c}^a(z; q): \mathcal{U} \rightarrow \mathbb{C}$, be defined by (11), and its partial sum be $(\psi_{v,b,c}^a(z; q))_m = z + \sum_{n=1}^m \zeta_n z^{n+1}$. If the inequalities,

$$\left. \begin{aligned} 32(a+2)|c|(1-q)^2(1-q^k)^2 + a|c|^3 &> 4(3a+2)|c|^2(1-q)(1-q^k), \\ 32a(1-q)^3(1-q^k)^3 + 4(3a+1)|c|^2(1-q)(1-q^k) &> 8(5a+4)|c|(1-q)^2(1-q^k)^2 + a|c|^3, \end{aligned} \right\} \quad (35)$$

are valid, then

$$\Re \left\{ \frac{(\psi_{v,b,c}^a(z; q))'_m}{((\psi_{v,b,c}^a(z; q))'_m)'} \right\} > \gamma_1 \quad (36) \quad \Re \left\{ \frac{((\psi_{v,b,c}^a(z; q))'_m)'}{(\psi_{v,b,c}^a(z; q))'_m} \right\} > \gamma_2, \quad (37)$$

holds true, for all $z \in \mathcal{U}$, where

and

$$\gamma_1 = \frac{64a(1-q)^3(1-q^k)^3 - 16|c|(5a+4)(1-q)^2(1-q^k)^2 + 5(3a+1)|c|^2(1-q)(1-q^k) - 2a|c|^3}{a[4(1-q)(1-q^k) - |c|]^3} \quad (38)$$

and

$$\gamma_2 = \frac{a[4(1-q)(1-q^k) - |c|]^3}{64a(1-q)^3(1-q^k)^3 + 16|c|(4-a)(1-q)^2(1-q^k)^2 - 8|c|(1-q)(1-q^k)}. \quad (39)$$

Proof. From Lemma 1, we observe that

$$1 + \sum_{n=1}^{\infty} |\eta_n| \leq 1 + \frac{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k) + a|c|^3}{a[4(1-q)(1-q^k) - |c|]^3}, \quad (40)$$

which is equivalent to

where $\eta_n = (-c)^n(n+1)(a+2n)/4^n a(q, q^k; q)_n$ and

$$\lambda \sum_{n=1}^{\infty} |\eta_n| \leq 1, \quad (41)$$

$$\lambda = \frac{a[4(1-q)(1-q^k) - |c|]^3}{32|c|(a+2)(1-q)^2(1-q^k)^2 - 4|c|^2(3a+2)(1-q)(1-q^k) + a|c|^3}. \quad (42)$$

Now, let us consider

$$\begin{aligned} \frac{1+r(z)}{1-r(z)} &= \lambda \left[\frac{(\psi_{\gamma,b,c}^a(z;q))'}{((\psi_{\gamma,b,c}^a(z;q))_m)'} - \gamma_1 \right] \\ &= \frac{1 + \sum_{n=1}^m \eta_n z^n + \lambda \sum_{n=m+1}^{\infty} \eta_n z^n}{1 + \sum_{n=1}^m \eta_n z^n}. \end{aligned} \quad (43)$$

Therefore,

$$r(z) = \frac{\lambda \sum_{n=m+1}^{\infty} \eta_n z^n}{2 + 2 \sum_{n=1}^m \eta_n z^n + \lambda \sum_{n=m+1}^{\infty} \eta_n z^n} \quad (44)$$

and

$$|r(z)| \leq \frac{\lambda \sum_{n=m+1}^{\infty} |\eta_n|}{2 - 2 \sum_{n=1}^m |\eta_n| - \lambda \sum_{n=m+1}^{\infty} |\eta_n|}. \quad (45)$$

Inequality (36) holds true if $|r(z)| \leq 1$ according to subordination principle. Then, the upcoming inequality,

$$\sum_{n=1}^m |\eta_n| + \lambda \sum_{n=m+1}^{\infty} |\eta_n| \leq 1, \quad (46)$$

implies that $|r(z)| \leq 1$. It suffices to show that the left-hand side of (47) is bounded above by

$$\lambda \sum_{n=1}^{\infty} |\eta_n|, \quad (47)$$

which is equivalent to

$$(\lambda - 1) \sum_{n=1}^m |\eta_n| \geq 0. \quad (48)$$

On the contrary, to prove inequality (37), we consider

$$\begin{aligned} \frac{1+s(z)}{1-s(z)} &= (\lambda + 1) \left[\frac{((\psi_{\gamma,b,c}^a(z;q))_m)'}{(\psi_{\gamma,b,c}^a(z;q))'} - \gamma_2 \right], \\ &= \frac{1 + \sum_{n=1}^m \eta_n z^n - \lambda \sum_{n=m+1}^{\infty} \eta_n z^n}{1 + \sum_{n=1}^m \eta_n z^n}. \end{aligned} \quad (49)$$

Therefore,

$$s(z) = \frac{-(1+\lambda) \sum_{n=m+1}^{\infty} \eta_n z^n}{2 + 2 \sum_{n=1}^m \eta_n z^n - (\lambda - 1) \sum_{n=m+1}^{\infty} \eta_n z^n} \quad (50)$$

and

$$|s(z)| \leq \frac{(1+\lambda) \sum_{n=m+1}^{\infty} |\eta_n|}{2 - 2 \sum_{n=1}^m |\eta_n| - (\lambda - 1) \sum_{n=m+1}^{\infty} |\eta_n|}. \quad (51)$$

Inequality (37) holds true if $|s(z)| \leq 1$ according to subordination principle. Then, the upcoming inequality,

$$\sum_{n=1}^m |\eta_n| + \lambda \sum_{n=m+1}^{\infty} |\eta_n| \leq 1, \quad (52)$$

implies that $|s(z)| \leq 1$. Since the left-hand side of (52) is bounded above by $\lambda \sum_{n=1}^{\infty} |\eta_n|$; thus, the proof is now completed. $\square$

Data Availability

No data were used in this paper.

Ethical Approval

This study does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors confirm no conflicts of interest.

Authors' Contributions

The authors contributed to the draft of the manuscript; they have read and approved the final manuscript.

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