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Research Article

Cordial and Total Cordial Labeling of Corona Product of Paths and Second Order of Lemniscate Graphs

Ashraf ELrokh[1] **Mohammed M. Ali Al-Shamiri**[2,3] **Shokry Nada**[1] **and Atef Abd El-hay**[1,4]

[1]*Mathematics and Computer Science Department, Faculty of Science Menoufia University, Menoufia, Egypt*
[2]*Department of Mathematics, Faculty of Science and Arts, Muhayl Asser King Khalid University, Abha, Saudi Arabia*
[3]*Department of Mathematics and Computer, Faculty of Science, Ibb University, Ibb, Yemen*
[4]*Faculty of Computer and Artificial Intelligence, Modern University for Technology and Information, Cairo, Egypt*

Correspondence should be addressed to Atef Abd El-hay; atef_1992@yahoo.com

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A simple graph is called cordial if it admits 0-1 labeling that satisfies certain conditions. The second order of lemniscate graph is a graph of two second order of circles that have one vertex in common. In this paper, we introduce some new results on cordial labeling, total cordial, and present necessary and sufficient conditions of cordial and total cordial for corona product of paths and second order of lemniscate graphs.

1. Introduction

Labelling methods are used for a wide range of applications in different subjects including coding theory, computer science, and communication networks. Graph labeling is an assignment of positive integers on vertices or edges or both of them which fulfilled certain conditions. Hundreds of research studies have been working with different types of labeling graphs [1–11], and a reference for this purpose is the survey written by Gallian [7]. All graphs considered, in this theme, are finite, simple, and undirected. The original concept of cordial graphs is due to Cahit [2]. He proved the following: each tree is cordial; a complete graph K_n is cordial if and only if $n \leq 3$ and a complete bipartite graph $K_{n,m}$ is cordial for all positive integers n and m [3]. Let $G = (V, E)$ be a graph, and let $f\colon V \longrightarrow \{0,1\}$ be a labeling of its vertices, and let the induced edge labeling $f^* E \longrightarrow \{0,1\}$ be given by $f^*(uv) = (f(u) + f(v))(\mathrm{mod} 2)$, where $e = uv (\in E)$ and $u, v \in V$. Let v_0 and v_1 be the numbers of vertices that are labeled by 0 and 1, respectively, and let e_0 and e_1 be the corresponding numbers of edges. Such a labeling is called cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$ hold. A graph is called cordial if it admits a cordial labeling. As an extension of the cordial labeling, we define a total cordial labeling of a graph G with vertex set and edge set as an cordial labeling such that number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1, i.e., $|(v_0 + e_0) - (e_1 + v_1)| \leq 1$. A graph with a total cordial labeling is called a total cordial graph. If the vertices of the graph are assigned values subject to certain conditions, it is known as graph labeling. Following three are the common features of any graph labeling problem: (1) a set of numbers from which vertex labels are assigned; (2) a rule that assigns a value to each edge; and (3) a condition that these values must satisfy.

A path with n vertices and $n-1$ edges is denoted by P_n, and a cycle with n vertices and n edges is denoted by C_n [12]. The second power of a lemniscate graph is defined as the union of two second power of cycles where both have a common vertex; it is denoted by $L^2_{n,m} \equiv C^2_n \sharp C^2_m$ [13]. Obviously, $L^2_{n,m}$ has $n + m - 1$ vertices and $2n + 2m - 4$ edges. The corona product $G_1 \odot G_2$ of two graphs G_i (with n_i vertices and m_i edges), $i = 1, 2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. It is easy to show that $G_1 \odot G_2$ has $n_1(1 + n_2)$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges [7, 14–18]. In this paper, we study the cordial and total cordial of the corona product $P_k \odot L^2_{n,m}$ of paths and second power of lemniscate graphs and show that this is cordial and total cordial for all positive integers k, n, m. The rest of the paper is organized as follows. In Section 1, brief summary of definitions that are useful for the present investigations is presented. Terminologies and notations are introduced in Section 2. The main result is presented in Section 3. Finally, the conclusion of this paper is introduced.

2. Terminology and Notation

Given a path or a cycle with $4r$ vertices, let L_{4r} denote the labeling 0011... 0011 (repeated r-times) and let $L^{'}_{4r}$ denote the labeling 1100... 1100 (repeated r times). The labeling 1001 1001... 1001 (repeated r times) and 0110... 0110 (repeated r times) is denoted by S_{4r} and $S^{'}_{4r}$. Let M_{2r} denote the labeling 0101 . . . 01, zero-one repeated r–times if r is even and 0101...010 if r is odd. Sometimes, we modify labeling by adding symbols at one end or the other (or both). If G and H are two graphs, where G has n vertices, the labeling of the corona $G \odot H$ is often denoted by [A: B_1, B_2, B_3, ..., B_n], where A is the labeling of the n vertices of G, and B_i, $1 \leq i \leq n$, is the labeling of the vertices of the copy of H that is connected to the i^{th} vertex of G. For a given labeling of the corona $G \odot H$, we denote v_i and e_i $(i = 0, 1)$ to represent the numbers of vertices and edges, respectively, labeled by i. Let us denote x_i and a_i to be the numbers of vertices and edges labeled by i for the graph G. Also, we let y_i and b_i be those for H, which are connected to the vertices labeled 0 of G. Likewise, let y_i' and b_i' be those for H, which are connected to the vertices labeled 1 of G. It is easily to verify that $v_0 = x_0 + x_0 y_0 + x_1 y_0'$, $v_1 = x_1 + x_0 y_1 + x_1 y_1'$, $e_0 = a_0 + x_0 b_0 + x_1 b_0' + x_0 y_1 + x_1 y_0'$, and $e_1 = a_1 + x_0 b_1 + x_1 b_1' + x_0 y_0 + x_1 y_1'$. Thus, $v_0 - v_1 = (x_0 - x_1) + x_0 (y_0 - y_1) + x_1 (y_0' - y_1')$ and $e_0 - e_1 = (a_0 - a_1) + x_0 (b_0 - b_1) + x_1 (b_0' - b_1') + x_0 (y_0 - y_1) - x_1 (y_0' - y_1')$. In particular, if we have only one labeling for all copies of H, i.e., $y_i = y_i'$ and $b_i = b_i'$, then $v_0 = x_0 + n y_0$, $v_1 = x_1 + n y_1$, $e_0 = a_0 + n b_0 + x_0 y_1 + x_1 y_0$, and $e_1 = a_1 + n b_1 + x_0 y_0 + x_1 y_1$. Thus, $v_0 - v_1 = (x_0 - x_1) + n(y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + n(b_0 - b_1) + (x_1 - x_0)(y_0 - y_1)$, where n is the order of G. Figure 1 illustrates the condition cordial and total cordial labeling of $P_3 \odot L_{3,7}$.

3. Results and Discussion

In this section, we show that the corona product of paths and second power of lemniscate graphs, $P_k \odot L^2_{n,m}$, is cordial and also total cordial for all $k \geq 1$, $n, m \geq 3$.

Throughout our proofs, the way of labeling $L^2_{n,m}$ starts always from a vertex that next the common vertex and go further opposite to this common vertex. Before considering the general form of the final result, let us first prove it in the following specific case. Our main theorem is as follows.

Theorem 1. *The corona product of paths and second power of lemniscate graphs, $P_k \odot L^2_{n,m}$, is cordial and also total cordial for all $k \geq 1$, $n, m \geq 3$.*

In order to prove this theorem, we will introduce a number of lemmas as follows.

Lemma 1. *$P_k \odot L^2_{3,m}$ is cordial and total cordial for all $k \geq 1$ and $m \geq 3$.*

Proof

Case 1. When $m \geq 3$ and $k = 2r$, $r \geq 1$, one can choose the labeling $[M_{2r}; 00100, 11011, \ldots, (r - \text{times})]$ for $P_{2r} \odot L^2_{3,3}$. Therefore, $x_0 = x_1 = r, a_0 = 0, a_1 = 2r - 1$, $y_0 = 4, y_1 = 1, b_0 = 2, b_1 = 4, y_0' = 1, y_1' = 4, b_0' = 2$, and $b_1' = 4$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$ and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. Thus, $P_{2r} \odot L^2_{3,3}$, $r \geq 1$, is cordial and total cordial.

Case 2. When $m \geq 3$ and $k = 2r + 1$, $r \geq 0$, one can choose the labeling $[M_{2r+1}; 00100, 11011, 00100, 11011, \ldots, (\text{r} - \text{times}), 11100]$ for $P_{2r+1} \odot L^2_{3,3}$. Therefore, $x_0 = r + 1, x_1 = r, a_0 = 0, a_1 = 2r, y_0 = 4, y_1 = 1, b_0 = 2, b_1 = 4, y_0' = 1, y_1' = 4, b_0' = 2, b_1' = 4, y_0^* = 2, y_1^* = 3, b_0^* = 4$, and $b_1^* = 2$, where y_i^* and b_i^* are the numbers of vertices and edges labeled i in $L^2_{3,3}$ that are connected to the last zero in P_{4r+3}. Consequently, it is easy to show that $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. Thus, $P_{2r+1} \odot L^2_{3,3}$, $r \geq 0$, is cordial and total cordial.

Case 3. When $m \equiv 0(\text{mod}4)$ and $k \equiv 0(\text{mod}4)$, that means, $k = 4r, r \geq 1$ and $m = 4t$, $t > 1$, then the labeling $[L_{4r}; 0_3 1_3 M_{4t-4}, 0_3 1_3 M_{4t-4}, 01 L_4 M^{'}_{4t-4}, 01 L_4 M^{'}_{4t-4}, \ldots, (r - \text{times})]$ for $P_{4r} \odot L^2_{3,4t}$ can be applied. Therefore, $x_0 = x_1 = 2r, a_0 = 2r, a_1 = 2r - 1, y_0 = y_1 = 2t + 1, b_0 = 4t + 1, b_1 = 4t, y_0' = y_1' = 2t + 1, b_0' = 4t$, and $b_1' = 4t + 1$. So, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the case $P_{4r} \odot L^2_{3,4}$, the labeling $[L_{4r}; 0_3 1_3, 0_3 1_3, 01 L_4, 01 L_4, \ldots, (r - \text{times})]$ is sufficient and thus $P_{4r} \odot L^2_{3,4t}$ is cordial and also total cordial.

Case 4. When $m \equiv 0(\text{mod}4)$ and $k \equiv 1(\text{mod}4)$ that means$k = 4r + 1, r \geq 0$ and $m = 4t$, $t > 1$, then the labeling $[L_{4r}0; 0_3 1_3 M_{4t-4}, 0_3 1_3 M_{4t-4}, 01 L_4 M^{'}_{4t-4}, 01 L_4 M^{'}_{4t-4}, \ldots, (r - \text{times}), 0_3 L_3 M_{4t-4}]$ for $P_{4r+1} \odot L^2_{3,4t}$ is considered. Therefore, $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 2r, y_0 = y_1 = 2t + 1, b_0 = 4t + 1, b_1 = 4t, y_0' = y_1' = 2t + 1, b_0' = 4t$, and $b_1' = 4t + 1$. Hence, $|v_0 - v_1| = 1$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 0$. For the case $P_{4r+1} \odot L^2_{3,4}$, the labeling $[L_{4r}0; 0_3 1_3, 0_3 1_3, 01 L_4, 01 L_4, \ldots, (r - \text{times}), 1_3 0_3]$ is sufficient and thus $P_{4r+1} \odot L^2_{3,4t}$ is cordial and total cordial.

Case 5. When $m \equiv 0(\text{mod}4)$ and $k \equiv 2(\text{mod}4)$ that means $k = 4r + 2, r \geq 0$, and $m = 4t$, $t > 1$, then the labeling $[L_{4r}10; 0_3 1_3 M_{4t-4}, 0_3 1_3 M_{4t-4}, 01 L_4 M^{'}_{4t-4}, 01 L_4 M^{'}_{4t-4}, \ldots, (r - \text{times}), 01 L_4 M^{'}_{4t-4}, 0_3 1_3 M_{4t-4}]$ for $P_{4r+2} \odot L^2_{3,4t}$ is applied. Therefore, $x_0 = x_1 = 2r + 1, a_0 = 2r + 1, a_1 = 2r, y_0 = y_1 = 2t + 1, b_0 = 4t + 1, b_1 = 4t$, $y_0' = y_1' = 2t + 1, b_0' = 4t$, and $b_1' = 4t + 1$. So, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the case $P_{4r+2} \odot L^2_{3,4}$, the labeling $[L_{4r}10; 0_3 1_3, 0_3 1_3, 01 L_4, 01 L_4, \ldots, (r - \text{times}), 01 L_4, 0_3 1_3]$ is sufficient and thus $P_{4r+2} \odot L^2_{3,4t}$ is cordial and total cordial.

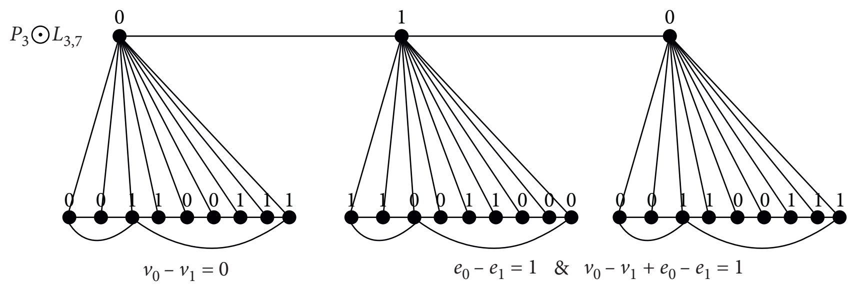

FIGURE 1: Cordial and total cordial labeling of $P_3 \odot L_{3,7}$.

Case 6. When $m \equiv 0(\text{mod}4)$ and $k \equiv 3(\text{mod}4)$ that means $k = 4r + 3, r \geq 0$ and $m = 4t$, $t > 1$, then one can select the labeling $[L_{4r}001; 0_3 1_3 M_{4t-4}, 0_3 1_3 \ M_{4t-4}, 01L_4 M'_{4t-4}, \ 01L_4 M'_{4t-4}, \ldots, (r-\text{times}), 0_3 1_3 M_{4t-4}, 0_3 1_3 M_{4t-4}, 01L_4 M'_{4t-4}]$ for $P_{4r+3} \odot L^2_{3,4t}$.

Therefore, $x_0 = 2r + 2, x_1 = 2r + 1, a_0 = a_1 = 2r + 1, y_0 = y_1 = 2t + 1, b_0 = 4t + 1, b_1 = 4t, y'_0 = y'_1 = 2t + 1, b'_0 = 4t$, and $b'_1 = 4t + 1$. Hence, one can easily show that $|v_0 - v_1| = 1$, $|e_0 - e_1| = 1$ and $|(v_0 + e_0) - (e_1 + v_1)| = 0$. For the case $P_{4r+3} \odot L^2_{3,4}$, the labeling $[L_{4r}001; 0_3 1_3, 0_3 1_3, 01L_4, 01L_4 \ldots, (r-\text{times})]$ is sufficient and thus $P_{4r+3} \odot L^2_{3,4t}$ is cordial and total cordial.

Case 7. When $m \equiv 1(\text{mod}4)$ and $k \equiv 0(\text{mod}4)$ that means $k = 4r, r \geq 1$ and $m = 4t + 1$, $t > 1$, then one can choose the labeling $[L_{4r}; 101L'_4 0M'_{4t-6}0, 101 \ L'_4 0M'_{4t-6}0, 010L_4 1M_{4t-6}1, 010L_4 1M_{4t-6}1, \ldots, (r-\text{times})]$ for $P_{4r} \odot L^2_{3,4t+1}$. Therefore, $x_0 = x_1 = 2r, a_0 = 2r, a_1 = 2r - 1, \quad y_0 = 2t + 1, y_1 = 2t + 2, b_0 = 4t + 2, b_1 = 4t + 1, y'_0 = 2t + 2, y'_1 = 2t + 1, b'_0 = 4t + 2$, and $b'_1 = 4t + 1$. Hence, one can easily show that $|v_0 - v_1| = 0, |e_0 - e_1| = 1$ and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{4r} \odot L^2_{3,5}$, the labeling $[L_{4r}; 01L'_4 0, 01L'_4 0, 10L_4 1, 10L_4 1, \ldots, (r-\text{times})]$ is sufficient and thus $P_{4r} \odot L^2_{3,4t+1}$ is cordial and total cordial.

Case 8. When $m \equiv 1(\text{mod}4)$ and $k \equiv 1(\text{mod}4)$ that means $k = 4r + 1, r \geq 0$ and $m = 4t + 1$, $t > 1$, then one can select the labeling $[L_{4r}0; 101L'_4 0M'_{4t-6}0, 101L'_4 0 \ M'_{4t-6}0, 010L_4 1M_{4t-6}1, 010L_4 1M_{4t-6}1, \ldots, (r-\text{times}), 101L'_4 1M'_{4t-6}1]$ for $P_{4r+1} \odot L^2_{3,4t+1}$. Therefore, $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 2r, y_0 = 2t + 1, y_1 = 2t + 2, b_0 = 4t + 2, b_1 = 4t + 1, y'_0 = 2t + 2, y'_1 = 2t + 1, b'_0 = 4t + 2$, and $b'_1 = 4t + 1$. So, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 0$ and $|(v_0 + e_0) - (e_1 + v_1)| = 0$. For the special case $P_{4r+1} \odot L^2_{3,5}$, the labeling $[L'_{4r}0; 01L'_4 0, 01 \ L'_4 0, 10L_4 1, 10L_4 1, \ldots, (r-\text{times}), 10L_4 1]$ is sufficient and thus $P_{4r+1} \odot L^2_{3,4t+1}$ is cordial and total cordial.

Case 9. When $m \equiv 1(\text{mod}4)$ and $k \equiv 2(\text{mod}4)$ that means $k = 4r + 2, r \geq 0$ and $m = 4t + 1$, $t > 1$, then the labeling $[L_{4r}10; 101L'_4 0M'_{4t-6}0, 101 \ L'_4 0M'_{4t-6} \ 0, 010 \ L_4 1M_{4t-6}1, 010L_4 1M_{4t-6}1, \ldots, (r-\text{times}), 010 \ L_4 1 \ M_{4t-6}0, 101L'_4 0M'_{4t-6}1]$ for $P_{4r+2} \odot L^2_{3,4t+1}$ can be applied. Therefore, $x_0 = 2r + 1, x_1 = 2r + 1, a_0 = 2r + 1, a_1 = 2r, y_0 = 2t + 1, y_1 = 2t + 2, b_0 = 4t + 2, b_1 = 4t + 1, y'_0 = 2t + 2, \ y'_1 = 2t + 1, b'_0 = 4t + 2$, and $b'_1 = 4t + 1$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$ and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{4r+2} \odot L^2_{3,5}$, the labeling $[L_{4r}10; 01L'_4 0, 01L'_4 \ 0, 10L_4 1, 10L_4 1, \ldots, (r-\text{times}), 10L_4 1, 01L'_4 0]$ is sufficient and thus $P_{4r+2} \odot L^2_{3,4t+1}$ is cordial and also total cordial.

Case 10. When $m \equiv 1(\text{mod}4)$ and $k \equiv 3(\text{mod}4)$ that means $k = 4r + 3, r \geq 0$ and $m = 4t + 1$, $t > 1$, then take the labeling $[L_{4r}0_2 1; 101L'_4 0M'_{4t-6}0, 101 \ L'_4 0M'_{4t-6} \ 0, 010L_4 \ 1M_{4t-6}1, 010L_4 1M_{4t-6}1, \ldots, (r-\text{times}), 101L'_4 0M'_{4t-6}1, 101L'_4 0M'_{4t-6}1, , 010L_4 1M_{4t-6}0]$ for $P_{4r+3} L^2_{3,4t+1}$. Therefore, $x_0 = 2r + 2, x_1 = 2r + 1, a_0 = a_1 = 2r + 1, y_0 = 2t + 1, \ y_1 = 2t + 2, b_0 = 4t + 2, b_1 = 4t + 1, y'_0 = 2t + 2, \ y'_1 = 2t + 1, b'_0 = 4t + 2$, and $b'_1 = 4t + 1$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 0$, and $|(v_0 + e_0) - (e_1 + v_1)| = 0$. For the special case $P_{4r+3} \odot L^2_{3,5}$, the labeling $[L_{4r}0_2 1; 01L'_4 0, 01L'_4 0, 10 \ L_4 1, 10L_4 1, \ldots, (r-\text{times}), 01L'_4 0, 10L_4 1, 10L_4 1]$ is sufficient and thus $P_{4r+3} \odot L^2_{3,4t+1}$ is cordial and total cordial.

Case 11. When $m \equiv 2(\text{mod}4)$ and k even that means $m = 4t + 2$, $t > 1$, and $k = 2r, r \geq 1$, then by taking the labeling $[M_{2r}; 0_2 1L'_4 0M'_{4t-4}, 1_2 0 \ L_4 1M_{4t-4}, \ldots, (r-\text{times})]$ for $P_{2r} \odot L^2_{3,4t+2}$, therefore $x_0 = x_1 = r, a_0 = 0, a_1 = 2r - 1, \ y_0 = 2t + 3, y_1 = 2t + 1, b_0 = 4t + 2, b_1 = 4t + 3, \ y'_0 = 2t + 1, y'_1 = 2t + 3, b'_0 = 4t + 2$, and $b'_1 = 4t + 3$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$ and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the case $P_{2r} \odot L^2_{3,6}$, the labeling $[M_{2r}; 0_2 L'_4 01, 1_2 L_4 10, \ldots, (r-\text{times})]$ is sufficient and thus $P_{2r} \odot L^2_{3,6}$ is cordial and total cordial.

Case 12. When $m \equiv 2(\text{mod}4)$ and k odd that means $m = 4t + 2, t > 1$, and $k = 2r + 1, r \geq 1$, then the labeling $[M_{2r+1}; 0_2 1L'_4 0M'_{4t-4}, 1_2 0L_4 1M_{4t-4}, \ldots, (r-\text{times}), 010 \ L_4 1M_{4t-4}]$ for $P_{2r+1} \odot L^2_{3,4t+2}$ is considered. Therefore, $x_0 = r + 1, x_1 = r, a_0 = 0, a_1 = 2r, y_0 = 2t + 3, y_1 = 2t + 1, b_0 = 4t + 2, b_1 = 4t + 3, y'_0 = 2t + 1, y'_1 = 2t + 3, b'_0 = 4t + 2$ and $b'_1 = 4t + 3$. So, $|v_0 - v_1| = 1$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 0$. For the case $P_{2r+1} \odot L^2_{3,6}$, the labeling $[M_{2r+1}; 0_2 L'_4 01, 1_2 L_4 10, \ldots, (r-\text{times}), 01L_4 10]$ is sufficient and thus $P_{2r+1} \odot L^2_{3,6}$ is cordial and total cordial.

Case 13. When $m \equiv 3(\text{mod}4)$ and $k \equiv 0(\text{mod}4)$ that means $m = 4t + 3$, $t \geq 1$, and $k = 4r, r \geq 1$, then the labeling $[L_{4r}; 0_2 M'_{4t+3}, 0_2 M'_{4t+3}, 1_2 M_{4t+3}, 1_2 M_{4t+3}, \ldots, (r-\text{times})]$ for $P_{4r} \odot L^2_{3,4t+3}$ can be applied. Therefore, $x_0 = x_1 = 2r, a_0 = 2r, \quad a_1 = 2r - 1, y_0 = 2t + 3, y$

$_1 = 2t+2, b_0 = 4t+3, b_1 = 4t+4, y_0' = 2t+2, y_1' = 2t+3, b_0{}' = 4t+3$, and $b_1' = 4t+4$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0+e_0) - (e_1+v_1)| = 1$. Thus, $P_{4r} \odot L^2_{3,4t+3}$ is cordial and total cordial.

Case 14. When $m \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$ that means $m = 4t+3$, $t \geq 1$, and $k = 4r+1, r \geq 0$, then one can select the labeling $[L'_{4r}0; 1_2M_{4t+3}, 1_2M_{4t+3}, 0_2M'_{4t+3}, 0_2M'_{4t+3}, \ldots, (r-\text{times}), 1_2M_{4t+3}]$ for $P_{4r+1} \odot L^2_{3,4t+3}$. Therefore, $x_0 = 2r+1, x_1 = 2r, a_0 = 2r+1, a_1 = 2r-1$, $y_0 = 2t+3, y_1 = 2t+2, b_0 = 4t+3, b_1 = 4t+4, y_0' = 2t+2$, $y_1' = 2t+3, b_0' = 4t+3, b_1' = 4t+4, y_0^* = 2t+2, y_1^* = 2t+3, b_0^* = 4t+3$, and $b_1^* = 4t+4$. So, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 0$, and $|(v_0+e_0) - (e_1+v_1)| = 0$. Thus, $P_{4r+1} \odot L^2_{3,4t+3}$ is cordial and total cordial.

Case 15. When $m \equiv 3 \pmod 4$ and $k \equiv 2 \pmod 4$ that means $m = 4t+3$, $t > 1$, and $k = 4r+2, r \geq 0$, then one can take the labeling $[L_{4r}10; 0_31_3M_{4t-4}, 0_31_3M_{4t-4}, 01L_4M'_{4t-4}, 01L_4M'_{4t-4}, \ldots, (r-\text{times}), 01L_4M'_{4t-4}, 0_31_3M4t-4]$ for $P_{4r+2} \odot L^2_{3,4t}$. Therefore, $x_0 = x_1 = 2r+1, a_0 = 2r+1, a_1 = 2r, y_0 = 2t+3, y_1 = 2t+2$, $b_0 = 4t+3, b_1 = 4t+4, y_0' = 2t+2, y_1' = 2t+3, b_0' = 4t+3$, and $b_1' = 4t+4$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0+e_0) - (e_1+v_1)| = 1$. Thus, $P_{4r+2} \odot L^2_{3,4t+3}$ is cordial and total cordial.

Case 16. When $m \equiv 3 \pmod 4$ and $k \equiv 3 \pmod 4$ that means$m = 4t+3$, $t \geq 1$, and $k = 4r+1, r \geq 0$, then one can choose the labeling $[L_{4r}100; 0_2M'_{4t+3}, 0_2M'_{4t+3}, 1_2M_{4t+3}, 1_2M_{4t+3}, \ldots, (r-\text{times}), 1_2 M_{4t+3}, 0_2M'_{4t+3}, 1_2M_{4t+3}]$ for $P_{4r+3} \odot L^2_{3,4t+3}$.

Therefore, $x_0 = 2r+2, x_1 = 2r+1, a_0 = 2r+2, a_1 = 2r$, $y_0 = 2t+3, y_1 = 2t+2, b_0 = 4t+3, b_1 = 4t+4, y_0' = 2t+2$, $y_1' = 2t+3, b_0' = 4t+3, b_1' = 4t+4, y_0^* = 2t+2, y_1^* = 2t+3$, $b_0^* = 4t+3$, and $b_1^* = 4t+4$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 0$, and $|(v_0+e_0) - (e_1+v_1)| = 0$. Thus, $P_{4r+3} \odot L^2_{3,4t+3}$ is cordial and total cordial. □

Lemma 2. *$P_k \odot L^2_{n,m}$ is cordial and total cordial for all $k \geq 1$ and $m > 6$.*

Proof. Let $k = 4r + i'$ ($i' = 0, 1, 2, 3$ and $r \geq 1$) or $k = 2r + j'$ ($j' = 0, 1$ and $r \geq 1$), $n = 4s + i$ and $m = 4t + j$ ($i, j = 1, 2, 3$ and $s, t \geq 2$), then we may use the labeling $A_{i'}$ or $A_{j'}$ for P_k as given in Table 1. For a given value of j with $1 \leq i, j \leq 3$, we may use one of the labeling in the set $\{B_{ij}, B'_{ij}\}$ for $L_{n,m}$, where B_{ij} and B'_{ij} are the labeling of $L^2_{n,m}$ which are connected to the vertices labeled 0 in P_k, while B_{ij} and B'_{ij} are the labeling of P_m which are connected to the vertices labeled 1 in P_k as given in Table 2. Using Table 3 and the formulas $v_0 - v_1 = (x_0 - x_1) + x_0.(y_0 - y_1) + x_1.(y_0' - y_1')$, $e_0 - e_1 = (a_0 - a_1) + x_0.(b_0 - b_1) + x_1.(b_0' - b_1') + x_0.(y_0 - y_1) - x_1.(y_0' - y_1')$, and $(v_0 + e_0) - (e_1 + v_1) = (x_0 - x_1) + 2x_0.(y_0 - y_1) + (a_0 - a_1) + x_0.(b_0 - b_1) + x_1.(b_0' - b_1')$, we can compute the values shown in the last two columns of Table 3. We see that $P_k \odot L^2_{n.m}$ is isomorphic to $P_k \odot L^2_{m,n}$. Since all of these values are 1 or 0, the lemma follows. □

Lemma 3. *$P_k \odot L^2_{4,m}$ is cordial and total cordial for all $k \geq 1$ and $m > 3$.*

Proof

Case 1. When $m \equiv 0 \pmod 4$ and $k = r, r \geq 1$ that means $m = 4t$, $t > 1$, and $k = r$, $r \geq 1$. Then, take the labeling $[1_r; 100L_4M'_{4t-4}, \ldots, (r-\textit{times})]$ for $P_r \odot L^2_{4,4t}$. Therefore, $x_0 = 0, x_1 = r, a_0 = r-1, a_1 = 0, y_0' = 2t+2, y_1' = 2t+1, b_0' = 4t+2$, and $b_1' = 4t+2$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0+e_0) - (e_1+v_1)| = 1$. For the case $P_r \odot L^2_{4,4}$, the labeling $[1_r; 0_31_30, \ldots, (r-\textit{times})]$ is sufficient and thus $P_r \odot L^2_{4,4t}$, $r \geq 1$, is cordial and total cordial.

Case 2. When $m \equiv 1 \pmod 4$ and $k \equiv 0 \pmod 4$ that means $m = 4t+1$, $t > 1$, and $k = 4r, r \geq 1$, then the labeling $[S_{4r}; 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, 10_3L_4 1M_{4t-6}1, 10_3L_41M_{4t-6}1, \ldots, (r-\textit{times})]$ for $P_{4r} \odot L^2_{4,4t+1}$ is applied. Therefore, $x_0 = x_1 = 2r, a_0 = 2r-1, a_1 = 2r, y_0 = 2t+2$, $y_1 = 2t+2, b_0 = b_1 = 4t+3, y_0' = 2t+2, y_1' = 2t+2$, $b_0' = 4t+3$, and $b_1' = 4t+3$. So, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0+e_0) - (e_1+v_1)| = 1$. For the case $P_{4r} \odot L^2_{4,5}$, the labeling $[S_{4r}; 10_2L_41, 10_2L_41, 10_2L_41, 10_2L_41, \ldots, (r-\textit{times})]$ is sufficient and thus $P_{4r} \odot L^2_{4,4t+1}$ is cordial and total cordial.

Case 3. When $m \equiv 1 \pmod 4$ and $k \equiv 1 \pmod 4$ that means $m = 4t+1$, $t > 1$, and $k = 4r+1, r \geq 0$, then one can select the labeling $[S_{4r}0; 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, \ldots, (r-\textit{times}), 10_3L_41M_{4t-6}1]$ for $P_{4r+1} \odot L^2_{4,4t+1}$. Therefore, $x_0 = 2r+1, x_1 = 2r, a_0 = a_1 = 2r$, $y_0 = 2t+2$, $y_1 = 2t+2, b_0 = b_1 = 4t+3, y_0' = 2t+2, y_1' = 2t+2, b_0' = 4t+3$, and $b_1' = 4t+3$. So, $|v_0 - v_1| = 1$. $|e_0 - e_1| = 0$, and $|(v_0+e_0) - (e_1+v_1)| = 1$. For the case $P_{4r+1} \odot L^2_{4,5}$, the labeling $[S_{4r}0; 10_2L_41, 10_2L_41, 10_2L_41, 10_2L_41, \ldots, (r-\textit{times}), 10_2L_41]$ is sufficient and thus $P_{4r+1} \odot L^2_{4,4t+1}$ is cordial and total cordial.

Case 4. When $m \equiv 1 \pmod 4$ and $k \equiv 2 \pmod 4$ that means $m = 4t+1$, $t > 1$, and $k = 4r+2, r \geq 0$, then the labeling $[S_{4r}01; 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, \ldots, (r-\textit{times}), 10_3L_4 1M_{4t-6}1, 10_3L_41M_{4t-6}1]$ for $P_{4r+2} \odot L^2_{4,4t+1}$ is applied. Therefore, $x_0 = x_1 = 2r+1, a_0 = 2r, a_1 = 2r+1$, $y_0 = 2t+2, y_1 = 2t+2, b_0 = b_1 = 4t+3, y_0' = 2t+2, y_1' = 2t+2, b_0' = 4t+3$, and $b_1' = 4t+3$. Hence, $|v_0 - v_1| = 0, |e_0 - e_1| = 1$, and $|(v_0+e_0) - (e_1+v_1)| = 1$. For the case $P_{4r+2} \odot L^2_{4,5}$, the labeling $[S_{4r}01; 10_2L_41, 10_2L_41, 10_2L_41, 10_2L_41, \ldots, (r-\textit{times}), 10_2 L_41, 10_2L_41]$ is sufficient and thus $P_{4r+2} \odot L^2_{4,4t+1}$ is cordial and total cordial.

Case 5. When $m \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$ that means and $m = 4t+1$, $t > 1$, and $k = 4r+3, r \geq 0$, then one can take the labeling $[S_{4r}001; 10_3L_41 M_{4t-6}1, 10_3L_41M_{4t-6}1, 10_3 L_41M_{4t-6}1, 10_3L_41 M_{4t-6}1, \ldots, (r-\textit{times}), 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1, 10_3L_41M_{4t-6}1]$ for $P_{4r+3} \odot L^2_{4,4t+1}$. Therefore, $x_0 = 2r+2, x_1 = 2r+1, a_0 = a_1 = 2r+1, y_0 = 2t+2, y_1 =$

Table 1: Labelling of P_k.

$K = 4r + i'$, $i' = 0, 1, 2, 3$	Labelling of P_k	x_0	x_1	a_0	a_1
$i' = 0$	$A_0 = L_{4r}$	$2r$	$2r$	$2r$	$2r - 1$
$i' = 1$	$A_1 = L_{4r}0$	$2r + 1$	$2r$	$2r$	$2r$
$i' = 2$	$A_2 = L_{4r}10$	$2r + 1$	$2r + 1$	$2r + 1$	$2r$
$i' = 3$	$A_3 = L_{4r}001$	$2r + 2$	$2r + 1$	$2r + 1$	$2r + 1$
$k = 2r + j'$, $i' = 0, 1$					
$i' = 0$	$A_4 = M_{2r}$	r	r	0	$2r - 1$
$i' = 1$	$A_5 = M_{2r+1}$	$r + 1$	r	0	$2r$

Table 2: Labelling of $L^2_{n,m}$.

$n = 4s + i$, $m = 4t + j$, $i, j = 0, 1, 2, 3$	Labelling of $L^2_{n,m}$	y_0	y_1	y_0	y_1
$i = 0$ $j = 0$	$B_{00} = L'_4$ $M_{4s-5}L'_4L'_{4t-4}$	$2s + 2t$ $+1$	$2s + 2t$ -1	$4s + 4t$ $+2$	$4s + 4t$ $+2$
$i = 0$ $j = 0$	$B'_{00} = L_4$ $M'_{4s-5}L_4M'_{4t-4}$	$2s + 2t$ -1	$2s + 2t$ $+1$	$4s + 4t$ $+2$	$4s + 4t$ $+2$
$i = 0$ $j = 1$	$B_{01} = L_4M'_{4s-5}$ $M'_{4t-6}1L'_40$	$2s + 2t$	$2s + 2t$	$4s + 4t$ -1	$4s + 4t$ -1
$i = 0$ $j = 2$	$B_{02} = L'_4M_{4s-5}$ $1L'_40M'_{4t-4}$	$2s + 2t$ $+1$	$2s + 2t$	$4s + 4t$	$4s + 4t$
$i = 0$ $j = 2$	$B'_{02} = L_4M'_{4s-5}$ $0L_41M_{4t-4}$	$2s + 2t$	$2s + 2t$ $+1$	$4s + 4t$	$4s + 4t$
$i = 0$ $j = 3$	$B_{03} = L_4$ $M'_{4s-5}M_{4t+3}$	$2s + 2t$ $+1$	$2s + 2t$ $+1$	$4s + 2t$ $+2$	$4s + 4t$ $+2$
$i = 1$ $j = 1$	$B_{11} = 1L'_40M'_{4s-6}$ $0_2M_{4t-6}0L_41$	$2s + 2t$ $+1$	$2s + 2t$	$4s + 4t$	$4s + 4t$
$i = 1$ $j = 1$	$B'_{11} = 0L_41M_{4s-6}$ $1_2M'_{4t-6}1L'_40$	$2s + 2t$	$2s + 2t$ $+1$	$4s + 4t$	$4s + 4t$
$i = 1$ $j = 2$	$B_{12} = 0L_41M_{4s-6}$ $M'_{4t-6}1L'_40$	$2s + 2t$ $+1$	$2s + 2t$ $+1$	$4s + 4t$ $+1$	$4s + 4t$ $+1$
$i = 1$ $j = 3$	$B_{13} = 1L'_40$ $M'_{4s-6}M_{4t+3}$	$2s + 2t$ $+2$	$2s + 2t$ $+1$	$4s + 4t$ $+2$	$4s + 4t$ $+2$
$i = 1$ $j = 3$	$B'_{13} = 0L_41$ $M_{4s-6}M'_{4t+3}$	$2s + 2t$ $+1$	$2s + 2t$ $+2$	$4s + 4t$ $+2$	$4s + 4t$ $+2$
$i = 2$ $j = 2$	$B_{22} = M_{4s-4}0L_41$ $L'_40M'_{4t-4}$	$2s + 2t$ $+2$	$2s + 2t$ $+1$	$4s + 4t$ $+2$	$4s + 4t$ $+2$
$i = 2$ $j = 2$	$B'_{22} = M'_{4s-4}1L'_40$ L_41M_{4t-4}	$2s + 2t$ $+1$	$2s + 2t$ $+2$	$4s + 4t$ $+2$	$4s + 4t$ $+2$
$i = 2$ $j = 3$	$B_{23} = 0L_41$ $M_{4s-5}M'_{4t+3}$	$2s + 2t$ $+2$	$2s + 2t$ $+2$	$4s + 4t$ $+3$	$4s + 4t$ $+3$
$i = 3$ $j = 3$	$B_{33} = M_{4s+2}$ M_{4t+3}	$2s + 2t$ $+2$	$2s + 2t$ $+3$	$4s + 4t$ $+3$	$4s + 4t$ $+3$
$i = 3$ $j = 3$	$B'_{33} = M_{4s+2}$ M_{4t+3}	$2s + 2t$ $+3$	$2s + 2t$ $+2$	$4s + 4t$ $+3$	$4s + 4t$ $+3$

$2t + 2$, $b_0 = b_1 = 4t + 3$, $y'_0 = 2t + 2$, $y'_1 = 2t + 2$, $b'_0 = 4t + 3$, and $b'_1 = 4t + 3$. So, $|v_0 - v_1| = 1$, $|e_0 - e_1| = 0$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the case $P_{4r+3} \odot L^2_{4,5}$, the labeling $[S_{4r}01; 10_2L_41, 10_2L_41, 10_2L_41, 10_2L_41, \ldots, (r - times), 10_2L_41, 10_2L_41, 10_2L_41]$ is sufficient and thus $P_{4r+3} \odot L^2_{4,4t+1}$ is cordial and total cordial.

Case 6. When $m \equiv 2 (\text{mod} 4)$ and k is even that means $m = 4t + 2$, $t > 1$, and $k = 2r$, $r \geq 1$. Then, one can

TABLE 3: Labelling of $P_k \odot L^2_{n,m}$.

i'/j'	ij	P_k	$L^2_{n,m}$	$\|v_0 - v_1\|$	$\|e_0 - e_1\|$	$\|(v_0 + e_0) - (e_1 + v_1)\|$
0	00	A_4	B_{00}, B_{00}'	0	1	1
1	00	A_5	$B_{00}, B_{00}', .., B_{00}, B_{00}', B_{00}'$	0	1	1
0	01	A_0	$B_{01}, B_{01}, B_{01}, B_{01}$	0	1	1
1	01	A_1	$B_{01}, B_{01}, B_{01}, B_{01}, \ldots, B_{01}$	1	0	1
2	01	A_2	$B_{01}, B_{01}, B_{01}, B_{01}, \ldots, B_{01}, B_{01}$	0	1	1
3	01	A_3	$B_{01}, B_{01}, B_{01} B_{01}, \ldots, B_{01}, B_{01}, B_{01}$	1	0	1
0	02	A_4	B_{02}, B_{02}'	0	1	1
1	02	A_5	$B_{02}, B_{02}', .., B_{02}, B_{02}', B_{02}'$	0	1	1
0	01	A_0	$B_{03}, B_{03}, B_{03}, B_{03}$	0	1	1
1	01	A_1	$B_{03}, B_{03}, B_{03}, B_{03}, \ldots, B_{03}$	1	0	1
2	01	A_2	$B_{03}, B_{03}, B_{03}, B_{03}, \ldots, B_{03}, B_{03}$	0	1	1
3	01	A_3	$B_{03}, B_{03}, B_{03}, B_{03}, \ldots, B_{03}, B_{03}, B_{03}$	1	0	1
0	11	A_4	B_{11}, B_{11}'	0	1	1
1	11	A_5	$B_{11}, B_{11}', .., B_{11}, B_{11}', B_{11}'$	0	1	1
0	12	A_0	$B_{12}, B_{12}, B_{12}, B_{12}$	0	1	1
1	12	A_1	$B_{12}, B_{12}, B_{12}, B_{12}, .., B_{12}$	1	0	1
2	12	A_2	$B_{12}, B_{12}, B_{12}, B_{12}, .., B_{12}, B_{12}$	0	1	1
3	12	A_3	$B_{12}, B_{12}, B_{12}, B_{12}, .., B_{12}, B_{12}, B_{12}$	1	0	1
0	13	A_4	B_{13}, B_{13}'	0	1	1
1	13	A_5	$B_{13}, B_{13}', .., B_{13}, B_{13}', B_{13}'$	0	1	1
0	22	A_4	B_{22}, B_{22}'	0	1	1
1	22	A_5	$B_{22}, B_{22}', .., B_{22}, B_{22}', B_{22}'$	0	1	1
0	23	A_0	$B_{23}, B_{23}, B_{23}, B_{23}$	0	1	1
1	23	A_1	$B_{23}, B_{23}, B_{23}, B_{23}, .., B_{23}$	1	0	1
2	23	A_2	$B_{23}, B_{23}, B_{23}, B_{23}, .., B_{23}, B_{23}$	0	1	1
3	23	A_3	$B_{23}, B_{23}, B_{23}, B_{23}, .., B_{23}, B_{23}, B_{23}$	1	0	1
0	33	A_4	B_{33}, B_{33}'	0	1	1
1	33	A_5	$B_{33}, B_{33}', .., B_{33}, B_{33}', B_{33}'$	0	1	1

choose the labeling $[M_{2r}; 10_3L_41M_{4t-4}, 01_3L_4'0M_{4t-4}', \ldots, (r - times)]$ for $P_{2r} \odot L^2_{4,4t+2}$. Therefore, $x_0 = x_1 = r, a_0 = 0, a_1 = 2r - 1$, $y_0 = 2t + 3$, $y_1 = 2t + 2, b_0 = b_1 = 4t + 3, y_0' = 2t + 2, y_1' = 2t + 3$, and $b_0' = b_1' = 4t + 4$. Hence, one can easily show that $|v_0 - v_1| = 0$. $|e_0 - e_1| = 1$ and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{2r} \odot L^2_{4,6}$, the labeling $[M_{2r}; 10_2L_401, 01_2L_4'10, \ldots, (r - times)]$ is sufficient, and thus $P_{2r} \odot L^2_{4,4t+2}$, $r \geq 1$, is cordial and total cordial.

Case 7. When $m \equiv 2\,(\mathrm{mod}4)$ and k is odd that means $m = 4t + 2$, $t > 1$, and $k = 2r + 1$ where $r \geq 0$, then one can choose the labeling $[M_{2r+1}; 10_3L_41M_{4t-4}, 01_3L_4'0M_{4t-4}', \ldots, (r - times), 01_3L_4'0M_{4t-4}]$ for $P_{2r+1} \odot L^2_{4,4t+2}$. Therefore, $x_0 = r + 1, x_1 = r, a_0 = 0, a_1 = 2r$, $y_0 = 2t + 3$, $y_1 = 2t + 2, b_0 = b_1 = 4t + 3, y_0' = 2t + 2, y_1' = 2t + 3, b_0' = b_1' = 4t + 4, y_0^* = 2\,t + 2, y_1^* = 2t+ 3$, and $b_0^* = b_1^* = 4t + 3$, where y_i^* and b_i^* are the numbers of vertices and edges labeled i in $L^2_{4,4t+2}$ that are connected to the last zero in P_{4r+3}. Consequently, it is easy to show that $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$ and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{2r+1} \odot L^2_{4,6}$, the labeling $[M_{2r}; 10_2L_401, 01_2L_4'10, \ldots, (r - times), 01_2L_4'10]$ is sufficient and thus $P_{2r+1} \odot L^2_{4,4t+2}$, $r \geq 0$, is cordial and total cordial.

Case 8. When $m \equiv 3\,(\mathrm{mod}4)$ and $k \equiv 0\,(\mathrm{mod}4)$ that means $m = 4t + 3$, $t > 1$, and $k = 4r, r \geq 1$, then one can choose the labeling $[L_{4r}; 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, \ldots, (r - times)]$ for $P_{4r} \odot L^2_{4,4t+3}$. Therefore, $x_0 = x_1 = 2r, a_0 = 2r - 1, a_1 = 2r, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5, y_0' = y_1' = 2t + 3$, and $b_0' = b_1' = 4t + 5$. Hence, one can easily show that $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. Thus, $P_{4r} \odot L^2_{4,4t+3}$ is cordial and total cordial.

Case 9. When $m \equiv 3\,(\mathrm{mod}4)$ and $k \equiv 1\,(\mathrm{mod}4)$ that means $m = 4t + 3$, $t > 1$, and $k = 4r + 1, r \geq 0$. Then, the labeling $[L_{4r}0; 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, \ldots, (r - times), 10_3M_{4t}11]$ for $P_{4r+1} \odot L^2_{4,4t+3}$ is considered. Therefore, $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 2r, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5, y_0' = y_1' = 2t + 3$, and $b_0' = b_1' = 4t + 5$. So, $|v_0 - v_1| = 1$, $|e_0 - e_1| = 0$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. Thus, $P_{4r+1} \odot L^2_{4,4t+3}$ is cordial and total cordial.

Case 10. When $m \equiv 3\,(\mathrm{mod}4)$ and $k \equiv 2\,(\mathrm{mod}4)$ that means $m = 4t + 3$, $t > 1$, and $k = 4r + 2, r \geq 0$, then one can select the labeling $[L_{4r}01; 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, \ldots, (r - times), 10_3M_{4t}11, 10_3M_{4t}11]$ for $P_{4r+2} \odot L^2_{4,4t+3}$. Therefore, $x_0 = x_1 = 2r + 1, a_0 = 2r, a_1 = 2r + 1, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5$, $y_0' = y_1' = 2t + 3$, and $b_0' = b_1' = 4t + 5$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. Thus, $P_{4r+2} \odot L^2_{4,4t+3}$ is cordial and total cordial.

Case 11. When $m \equiv 3\,(\mathrm{mod}4)$ and $k \equiv 3\,(\mathrm{mod}4)$ that means $m = 4t + 3$, $t > 1$, and $k = 4r + 3, r \geq 0$, then one can take the labeling $[L_{4r}001; 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11, \ldots, (r - times)$,

$10_3M_{4t}11, 10_3M_{4t}11, 10_3M_{4t}11]$ for $P_{4r+3} \odot L^2_{4,4t+1}$. Therefore, $x_0 = 2r + 2, x_1 = 2r + 1, a_0 = a_1 = 2r + 1, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5, y'_0 = y'_1 = 2t + 3,$ and $b'_0 = b'_1 = 4t + 5$. Hence, $|v_0 - v_1| = 1$, $|e_0 - e_1| = 0$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. Thus, $P_{4r+3} \odot L^2_{4,4t+3}$ is cordial and also total cordial; by this, the lemma was proved. □

Lemma 4. *$P_k \odot L^2_{5,m}$ is cordial and also total cordial for all $k \geq 1$ and $m \geq 3$.*

Proof

Case 1. When $m \equiv 0\,(\text{mod}4)$ that means $m = 4t$, $t \geq 1$, since $P_k \odot L^2_{5,4}$ isomorphic to $P_k \odot L^2_{4,5}$, by Lemma 3, $P_k \odot L^2_{4,5}$ is cordial and total cordial and then $P_k \odot L^2_{5,4}$ is cordial and total cordial. Also, since $P_k \odot L^2_{5,4t}$, $t \geq 1$ isomorphic to $P_k \odot L^2_{4t,5}$, by Lemma 3, $P_k \odot L^2_{4t,5}$ is cordial and total cordial and then $P_k \odot L^2_{5,4t}$ is cordial and total cordial.

Case 2. When $m \equiv 1\,(\text{mod}4)$ and k is even that means $m = 4t + 1$, $t \geq 1$, and $k = 2r$, $r \geq 1$, then one can take the labeling $[M_{2r}; L'_4 1 L'_4 0 M^{'}_{4t-6}0, L_4 0 L_4 1 M_{4t-6}1, \ldots, (r - times)]$ for $P_{2r} \odot L^2_{5,4t+1}$. Therefore, $x_0 = r, x_1 = r, a_0 = 0, a_1 = 2r - 1, y_0 = 2t + 3, y_1 = 2t + 2, b_0 = b_1 = 4t + 4, y'_0 = 2t + 2, y'_1 = 2t + 3,$ and $b'_0 = b'_1 = 4t + 4$. So, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{2r} \odot L^2_{5,5}$, the labeling $[M_{2r}; L_4 L'_4 0, L'_4 L_4 1, \ldots, (r - times)]$ is sufficient and thus $P_{2r} \odot L^2_{5,4t+1}$, $r \geq 1$, is cordial and total cordial.

Case 3. When $m \equiv 1\,(\text{mod}4)$ and k is odd that means $m = 4t + 1$, $t \geq 1$, and $k = 2r + 1$, $r \geq 1$, then the labeling $[M_{2r+1}; L'_4 1 L'_4 0 M^{'}_{4t-6}0, L_4 0 L_4 1 M_{4t-6}1, \ldots, (r - times), L_4 0 L_4 1 M_{4t-6}1]$ for $P_{2r+1} \odot L^2_{5,4t+1}$ can be applied. Therefore, $x_0 = r + 1, x_1 = r, a_0 = 0, a_1 = 2r, y_0 = 2t + 3, y_1 = 2t + 2, b_0 = b_1 = 4t + 4, y'_0 = 2t + 2, y'_1 = 2t + 3 b'_0 = b'_1 = 4t + 4, y^*_0 = 2t + 2, y^*_1 = 2t + 3,$ and $b^*_0 = b^*_1 = 4t + 4$, where y^*_i and b^*_i are the numbers of vertices and edges labeled i in $L_{5,4t+1}$ that are connected to the last zero in P_{2r+1}. Consequently, it is easy to show that $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{2r+1} \odot L^2_{5,5}$, the labeling $[M_{2r+1}; L_4 L'_4 0, L'_4 L_4 1, \ldots, (r - times), L'_4 L_4 1]$ is sufficient and thus $P_{2r+1} \odot L^2_{5,4t+1}$, $r \geq 1$, is cordial and total cordial.

Case 4. When $m \equiv 2\,(\text{mod}4)$ and $k \equiv 0\,(\text{mod}4)$ that means $m = 4t + 2$, $t \geq 1$, and $k = 4r$, $r \geq 1$, then one can choose the labeling $[L_{4r}; L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, \ldots, (r - time)]$ for $P_{4r} \odot L^2_{5,4t+2}$. Therefore, $x_0 = x_1 = 2r, a_0 = 2r, a_1 = 2r - 1, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5, y'_0 = y'_1 = 2t + 3,$ and $b'_0 = b'_1 = 4t + 5$. Consequently, it is easy to show that $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{4r} \odot L^2_{5,6}$, the labeling $[L_4 r; L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, \ldots, (r - time)]$ is sufficient and thus $P_{4r} \odot L^2_{5,4t+2}$ is cordial and total cordial.

Case 5. When $m \equiv 2\,(\text{mod}4)$ and $k \equiv 1\,(\text{mod}4)$ that means $m = 4t + 2$, $t \geq 1$, and $k = 4r + 1$, $r \geq 0$, then one can take the labeling $[L_{4r}0; L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, \ldots, (r - time), L_4 0 L_4 1 M^{'}_{4t-4}]$ for $P_{4r+1} \odot L^2_{5,4t+2}$. Therefore, $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 2r, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5, y'_0 = y'_1 = 2t + 3,$ and $b'_0 = b'_1 = 4t + 5$. So, $|v_0 - v_1| = 1$, $|e_0 - e_1| = 0$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{4r+1} \odot L^2_{5,6}$, the labeling $[L_4 r0; L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, \ldots, (r - time), L'_4 L_4 10]$ is sufficient and thus $P_{4r+1} \odot L^2_{5,4t+2}$ is cordial and total cordial.

Case 6. When $m \equiv 2\,(\text{mod}4)$ and $k \equiv 2\,(\text{mod}4)$ that means $m = 4t + 2$, $t \geq 1$, and $k = 4r + 2$, $r \geq 0$, then one can select the labeling $[L_{4r}10; L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, \ldots, (r - time), L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}]$ for $P_{4r+2} \odot L^2_{5,4t+2}$. Therefore, $x_0 = x_1 = 2r + 1, a_0 = 2r + 1, a_1 = 2r, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5, y'_0 = y'_1 = 2t + 3,$ and $b'_0 = b'_1 = 4t + 5$. Hence, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{4r+2} \odot L^2_{5,6}$, the labeling $[L_4 r10; L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, \ldots, (r - time), L'_4 L_4 10, L'_4 L_4 10]$ is sufficient and thus $P_{4r+2} \odot L^2_{5,4t+2}$ is cordial and total cordial.

Case 7. When $m \equiv 2\,(\text{mod}4)$ and $k \equiv 3\,(\text{mod}4)$ that means $m = 4t + 2$, $t \geq 1$, and $k = 4r + 3$, $r \geq 0$, then the labeling $[[L_{4r}001; L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, \ldots, (r - time), L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}, L_4 0 L_4 1 M^{'}_{4t-4}]$ for $P_{4r+3} \odot L^2_{5,4t+2}$ is considered. Therefore, $x_0 = 2r + 2, x_1 = 2r + 1, a_0 = a_1 = 2r + 1, y_0 = y_1 = 2t + 3, b_0 = b_1 = 4t + 5, y'_0 = y'_1 = 2t + 3,$ and $b'_0 = b'_1 = 4t + 5$. Consequently, it is easy to show that $|v_0 - v_1| = 1$, $|e_0 - e_1| = 0$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. For the special case $P_{4r+3} \odot L^2_{5,6}$, the labeling $[L_4 r001; L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10, \ldots, (r - time), L'_4 L_4 10, L'_4 L_4 10, L'_4 L_4 10]$ is sufficient and thus $P_{4r+3} \odot L^2_{5,4t+2}$ is cordial and also total cordial.

Case 8. When $m \equiv 3\,(\text{mod}4)$ and k is even that means $m = 4t + 3$, $t \geq 1$, and $k = 2r$ where $r \geq 1$, then the labeling $[M_{2r}; 0_3 1 M_{4t+3}, 1_3 0 M_{4t+3} \ldots, (r - times)]$ for $P_{2r} \odot L^2_{5,4t+3}$ can be applied. Therefore, $x_0 = r, x_1 = r, a_0 = 0, a_1 = 2r - 1, y_0 = 2t + 4, y_1 = 2t + 3, b_0 = b_1 = 4t + 6, y'_0 = 2t + 3, y'_1 = 2t + 4,$ and $b'_0 = b'_1 = 4t + 6$. Consequently, it is easy to show that $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0 + e_0) - (e_1 + v_1)| = 1$. Thus, $P_{2r} \odot L^2_{5,4t+3}$, $r \geq 1$, is cordial and total cordial.

Case 9. When $m \equiv 3\,(\text{mod}4)$ and k is odd that means $m = 4t + 3$, $t \geq 1$, and $k = 2r + 1$ where $r \geq 0$, then one can take the labeling $[M_{2r+1}; 0_3 1 M_{4t+3}, 1_3 0 M_{4t+3}, \ldots, (r - times), 1_3 0 M_{4t+3}]$ for $P_{2r+1} \odot L^2_{5,4t+3}$. Therefore, $x_0 = r + 1, x_1 = r, a_0 = 0, a_1 = 2r, y_0 = 2t + 3, y_1 = 2t + 2, b_0 = b_1 = 4t + 4, y'_0 = 2t + 2, y'_1 = 2t + 3 b'_0 = b'_1 = 4t + 4, y^*_0 = 2t + 3, y^*_1 = 2t + 4,$ and $b^*_0 = b^*_1 = 4t + 6$, where y^*_i and b^*_i are the numbers of vertices and edges labeled i in $L_{5,4t+3}$ that are connected to the last zero in P_{2r+1}. So, $|v_0 - v_1| = 0$, $|e_0 - e_1| = 1$, and $|(v_0+e_0)-(e_1+v_1)|=1$. Thus, $P_{2r+1}\odot L^2_{5,4t+3}$, $r\geq 1$, is cordial and total cordial, and by this, the lemma was proved. □

Lemma 5. *$P_k\odot L^2_{6,m}$ is cordial and total cordial for all m,k.*

Proof

Case 1. When $m\equiv 0(\text{mod}4)$, since $P_k\odot L^2_{6,4}$ is isomorphic to $P_k\odot L^2_{4,6}$ and $P_k\odot L^2_{4,6}$ is cordial and total cordial, then $P_k\odot L^2_{6,4}$ cordial and total cordial. Also, since $P_k\odot L^2_{6,4t}$ is isomorphic to $P_k\odot L^2_{4t,6}$ and $P_k\odot L^2_{4t,6}$ is cordial and total cordial, then $P_k\odot L^2_{6,4t}$ is cordial and total cordial.

Case 2. When $m\equiv 1(\text{mod}4)$, i.e., $m=4t+1$, $t\geq 1$, since $P_k\odot L^2_{6,5}$ is isomorphic to $P_k\odot L^2_{5,6}$ and $P_k\odot L^2_{4,6}$ is cordial and total cordial, then $P_k\odot L^2_{5,6}$ is cordial and total cordial. Also, since $P_k\odot L^2_{6,4t+1}$ is isomorphic to $P_k\odot L^2_{4t+1,6}$ and $P_k\odot L^2_{4t+1,6}$ is cordial and total cordial, then $P_k\odot L^2_{6,4t+1}$ is cordial and total cordial.

Case 3. When $m\equiv 2(\text{mod}4)$ and k is even that means $m=4t+2$, $t\geq 1$, and $k=2r$, $r\geq 1$, then one can choose the labeling $[M_{2r};L_4'01L_4'0M'_{4t-4},L_410L_41M_{4t-4},\ldots,(r-times)]$ for $P_{2r}\odot L^2_{6,4t+2}$. Therefore, $x_0=r,x_1=r,a_0=0,a_1=2r-1,y_0=2t+4,y_1=2t+3,b_0=b_1=4t+6,y_0'=2t+3,y_1'=2t+4$, and $b_0'=b_1'=4t+6$. Consequently, it is easy to show that $|v_0-v_1|=0$ and $|e_0-e_1|=1$. For the special case $P_{2r}\odot L^2_{6,6}$, the labeling $[M_{2r};L_4'0L_4'01,L_41L_410,\ldots,(r-times)]$ is sufficient and thus $P_{2r}\odot L^2_{6,4t+2}$, $r\geq 1$, is cordial and also total cordial.

Case 4. When $m\equiv 2(\text{mod}4)$ and k is odd that means $m=4t+2$, $t\geq 1$, and $k=2r+1$, $r\geq 0$, then one can choose the labeling $[M_{2r+1};L_4'01L_4'0M'_{4t-4},L_410L_41M_{4t-4},\ldots,(r-times),L_410L_41M_{4t-4}]$ for $P_{2r+1}\odot L^2_{6,4t+2}$. Therefore, $x_0=r+1,x_1=r,a_0=0,a_1=2r,y_0=2t+2t+1,y_1=2s+2t$, $b_0=b_1=4s+4t$, $y_0=2t+4,y_1=2t+3,b_0=b_1=4t+6$, $y_0'=2t+3,y_1'=2t+4,b_0'=b_1'=4t+6,y_0^*=2t+3$, $y_1^*=2t+4$, and $b_0^*=b_1^*=4t+6$, where y_i^* and b_i^* are the numbers of vertices and edges labeled i in $L^2_{6,4t+2}$ that are connected to the last zero in P_{2r+1}. So, $|v_0-v_1|=0$, $|e_0-e_1|=1$, and $|(v_0+e_0)-(e_1+v_1)|=1$. For the special case $P_{2r+1}\odot L^2_{6,6}$, the labeling $[M_{2r+1};L_4'0L_4'01,L_41L_410,\ldots,(r-times),L_41L_410]$ is sufficient and thus $P_{2r+1}\odot L^2_{6,4t+2}$, $r\geq 1$, is cordial and total cordial.

Case 5. When $m\equiv 3(\text{mod}4)$ and $k\equiv 0(\text{mod}4)$ that means $m=4t+3$, $t\geq 1$, and $k=4r$, $r\geq 1$, then the labeling $[L_{4r};L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},\ldots,(r-time)]$ for $P_{4r}\odot L^2_{6,4t+3}$ is applied. Therefore $x_0=x_1=2r,a_0=2r,a_1=2r-1,y_0=y_1=2t+3,b_0=b_1=4t+7,y_0'=y_1'=2t+4$ and $b_0'=b_1'=4t+7$. Consequently, it is easy to show that $|v_0-v_1|=0$, $|e_0-e_1|=1$, and $|(v_0+e_0)-(e_1+v_1)|=1$. Thus, $P_{4r}\odot L^2_{6,4t+3}$ is cordial and total cordial.

Case 6. When $m\equiv 3(\text{mod}4)$ and $k\equiv 1(\text{mod}4)$ that means $m=4t+3$, $t\geq 1$, and $k=4r+1$, $r\geq 0$, then one can choose the labeling $[L_{4r}0;L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},\ldots,(r-time),L_41M_{4t+3}]$ for $P_{4r+1}\odot L^2_{6,4t+3}$. Therefore, $x_0=2r+1,x_1=2r,a_0=a_1=2r,y_0=y_1=2t+3,b_0=b_1=4t+7,y_0'=y_1'=2t+4$, and $b_0'=b_1'=4t+7$. So, $|v_0-v_1|=1$, $|e_0-e_1|=0$, and $|(v_0+e_0)-(e_1+v_1)|=1$. Thus, $P_{4r+1}\odot L^2_{6,4t+3}$ is cordial and total cordial.

Case 7. When $m\equiv 3(\text{mod}4)$ and $k\equiv 2(\text{mod}4)$ that means $m=4t+3$, $t\geq 1$, and $k=4r+2$, $r\geq 0$, then one can take the labeling $[L_{4r}10;L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},\ldots,(r-time),L_41M_{4t+3},L_41M_{4t+3}]$ for $P_{4r+2}\odot L^2_{6,4t+3}$. Therefore, $x_0=x_1=2r+1,a_0=2r+1,a_1=2r,y_0=y_1=2t+3,b_0=b_1=4t+7,y_0'=y_1'=2t+4$, and $b_0'=b_1'=4t+7$. Hence, $|v_0-v_1|=0$, $|e_0-e_1|=1$, and $|(v_0+e_0)-(e_1+v_1)|=1$. Thus, $P_{4r+2}\odot L^2_{6,4t+3}$ is cordial and total cordial.

Case 8. When $m\equiv 3(\text{mod}4)$ and $k\equiv 3(\text{mod}4)$ that means $m=4t+3$, $t\geq 1$, and $k=4r+3$, $r\geq 0$, then one can select the labeling $[L_{4r}001;L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3},\ldots,(r-time),L_41M_{4t+3},L_41M_{4t+3},L_41M_{4t+3}]$ for $P_{4r+3}\odot L^2_{6,4t+3}$. Therefore, $x_0=2r+2,x_1=2r+1,a_0=a_1=2r+1,y_0=y_1=2t+3$, $b_0=b_1=4t+7,y_0'=y_1'=2t+4$, and $b_0'=b_1'=4t+7$. Consequently, it is easy to show that $|v_0-v_1|=1$, $|e_0-e_1|=0$, and $|(v_0+e_0)-(e_1+v_1)|=1$. Thus, $P_{4r+3}\odot L^2_{6,4t+3}$ is cordial and total cordial; by this, the lemma was proved, and through the proofs of these lemmas, we have completed the proof of our main theorem. □

4. Conclusions

In this paper, we test the cordial and total cordial labeling of corona product of paths and second power of lemniscate graphs. We found that $P_k\odot L^2_{n,m}$ is cordial and also total cordial for all $k\geq 1$, $n,m\geq 3$. In future work, we can improve this work by using the different graphs with other mathematical operations to prove the cordial and total cordial labeling.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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