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Research Article

Modified Zagreb Connection Indices for Benes Network and Related Classes

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The study of networks such as Butterfly networks, Benes networks, interconnection networks, David-derived networks through graph theoretical parameters is among the modern trends in the area of graph theory. Among these graph theoretical tools, the topological Indices (TIs) have been frequently used as graph invariants. TIs are also the essential tools for quantitative structure activity relationship (QSAR) as well as quantity structure property relationships (QSPR). TIs depend on different parameters, such as degree and distance of vertices in graphs. The current work is devoted to the derivation of 2-distance based TIs, known as, modified first Zagreb connection index ZC_1^* and first Zagreb connection index (ZC_1) for $r-$ dimensional Benes network and some classes generated from Benes network. The horizontal cylindrical Benes network $(HCB(r))$, vertical cylindrical Benes network $(VCB(r))$, and toroidal Benes network $(TB(r))$ are the three classes generated by identifying the vertices of the first row with the last row, the first column with the last column of the Benes network. The obtained results are also analyzed through graphical tools.

1. Introduction

The study of networks such as Butterfly network [1], Benes network [2, 3], interconnection network [4–6], David-derived network [7] through graph theoretical parameters is among the modern trends in the area of graph theory. In an interconnected network (ICN), the processing nodes are the multiprocessors that are utilized to construct a network based on homogeneously identical processor memory pairs. The transmissions of messages enable programs to be compiled and then executed. Constructive significance to the architectural plan and usage of multiprocessor ICN is on account of economical, reasonable, systematic and more efficient microprocessors and chips [8]. The resemblance of ICN with communication patterns is a natural scenario, which makes them more valuable and influential. Mostly networks are interconnected and due to the dependency on one another, these are networks are needed to be assessed and improved for upcoming work.

To design these networks, graphs are used in a very natural manner, in which the components or processors are distinguished by vertices, and certain communication links such as fiber optic cables etc., are represented by edges. The functionality of the mentioned components is accomplished through incidence functions. It enables to examine networks, its components and the links between components through study of graphs as graphs and networks are similar in the sense of structure.

Butterfly graphs are the elementary graphs in Fourier transform networks that can perform Fast Fourier

Transforms (FFT) very expeditiously. In Butterfly networks ($BF(r)$), the series of interconnection patterns and switching stages permits k inputs to be linked to k outputs. The Benes network comprises back to back connected butterflies. This worthwhile network is familiar for permutation routing [2]. These are remarkable multistage interconnection networks, which are entertained by striking and distinctive topologies for communication networks [3]. The Benes networks are utilized in parallel computing systems such as NEC Cenju-3, IBM, SP1/SP2 and MIT Transit Project. These networks also have applications in the internal structural composition of optical couplers [1, 4]. In an r-dimensional Benes, there are $2r + 1$ number of levels, with each level having $2r$ nodes. An r-dimensional butterfly is organized from the level 0 to r nodes. The adjacently connected butterflies which share the central level to generate a Benes network. An r-dimensional Benes network is denoted by $B(r)$, for example, $B(3)$ is shown in Figure 1.

New representations of the Benes networks are recently constructed by embedding it on the surface of torus and cylinder known as Toroidal Benes network ($TB(r)$), horizontal cylindrical Benes network ($HCB(r)$), and vertical cylindrical Benes network ($VCB(r)$). For further details, see [9].

From now onward, G represents a simple, connected graph with edge and vertex sets by $E(G)$ and $V(G)$, respectively. Moreover, for $v \in V(G)$, d_v and $N(v)$ represent its degree and set of neighbors, respectively. The connection number τ_v is the cardinality of the set of vertices which lie at distance 2 from v. For further details on undefined terminologies, we refer [5, 10, 11]. Various invariants assigned to molecular structures or networks, set up correlations between their physicochemical properties and structures. A class among the graph invariants is the class of topological indices (TIs). These invariants (TIs) are usually dependent on distance and degree and are came up to be beneficial in anticipating the multiple features of structures including networks and molecular graphs. The first and primordial topological index, entitled as the Wiener index, introduced by H. Wiener [12], in 1947, while studying the alkanes. Based on its productive outcomes and predictive ability, numerous TIs of chemical graphs, have been flourished subsequently. The Zagreb connection index (ZCI) is a noteworthy class of TIs and depends upon connection number denoted by τ_v. This connection number expresses the total vertices at distance (edges in a minimal path) two from arbitrary vertex v [13]. This class came into sight in 1972 to quantify the total π-electron energy [14]. After that, researchers took no notice of it for many years. Lately, Ali and Trinajstic [15] reinvestigated the ZCIs and revealed that the ZCI comparatively to classical Zagreb indices come up with finer absolute values of the correlation. Utilizing the connection number ZCI Ali et al. defined the modified first ZCI [15], given as $ZC_1^*(G) = \sum_{v\in V(G)} d_v\tau_v$ and first Zagreb Connection index is defined and denoted as $ZC_1(G) = \sum_{v\in V(G)} \tau_v^2$ [16]. For further details about computation of indices, we refer the readers [17, 18]. In this paper, we compute 2-distance based TIs, known as, modified first Zagreb connection index ZC_1^* and first Zagreb connection index (ZC_1) for $r-$dimensional Benes and Butterfly networks, $HCB(r)$, $VCB(r)$, and $TB(r)$. The obtained results are also analyzed through graphical tools.

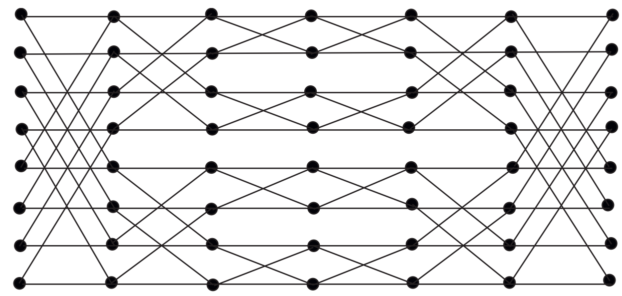

Figure 1: 3-dimensional Benes network.

2. Main Results

Throughout this paper $r \geq 2$. Figures 2 and 3 represent the graphs of $VCB(r)$ and $HCB(r)$, respectively. The graph of $TB(r)$, i.e., embedding of $B(r)$ on Torous is given in Figure 4. Now, we present the results in the following sections.

2.1. Modified Zagreb Connection Indices for Benes and Butterfly Networks.

2.1. Modified Zagreb Connection Indices for Benes and Butterfly Networks. In an $r-$dimensional Benes network $B(r)$, total number of vertices are $2^r(2r+1)$, whereas in $BF(r)$, there are $2^r(r+1)$ total number of vertices. For details of these networks, see [6]. These networks are presented in Figures 5 and 6 for $r = 3$.

Theorem 1. *For $r-$ dimensional Benes network G, we have:*

(1) $ZC_1^*(G) = 2^{r+2}(42 + 20(r-3))$

(2) $ZC_1(G) = 200r.2^r - 269.2^r$

PROOF. Let $[w, i] = [w_1 w_2 \ldots w_r, i]$ be an arbitrary node of Benes network, where i denotes the level $(0 \leq i \leq 2r)$ and $w = w_1 w_2 \ldots w_r$ is an r-bit binary number that denotes the row of the node. Let us denote by V_k, $0 \leq k \leq 2r$, the set of vertices of k th column. For $0 \leq i, j \leq 2r$, if $[w, i]$ and $[w', j]$ are adjacent then $w_j = 0$ implies $w_j' = 1$, and vice versa. We partition the vertices $v = [w, i]$, $0 \leq i \leq 2r$ of benes networks into the different cases depending on τ_v: □

Case 1. (For$i = 0$and$i = 2r$)

Let $v \in V_0$, that is $v = [w_1 w_2 \ldots w_r, 0]$. Then $N(v) = N([w_1 \ldots w_r, 0]) = \{[w_1 \ldots w_r, 1], [w_1' \ldots w_r, 1]\}$. Furthermore, $N([w_1 w_2 \ldots w_r, 1]) = \{[w_1 \ldots w_r, 0], [w_1 \ldots w_r, 2], [w_1' \ldots w_r, 0], [w_1 w_2' \ldots w_r, 2]\}$ and.$N([w_1' \ldots w_r, 1]) = \{[w_1 \ldots w_r, 0], [w_1' \ldots w_r, 2], [w_1' \ldots w_r, 0], [w_1' w_2' \ldots w_r, 2]\}$. So, there are 5 distinct vertices that are at distance 2 from v, that is $\tau_v = 5$ for $v \in V_0$. On the other hand for $v \in V_0$, $d_v = 2$. Thus, $\sum_{v\in V_0} d_v\tau_v = 2.5.2^r = 10.2^r$. The case when $v \in V_{2r}$ is also the same.

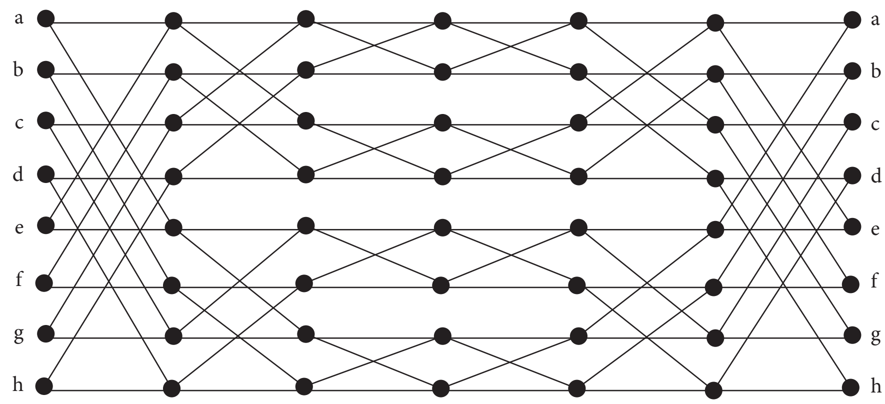

Figure 2: Normal representation of VCB(3).

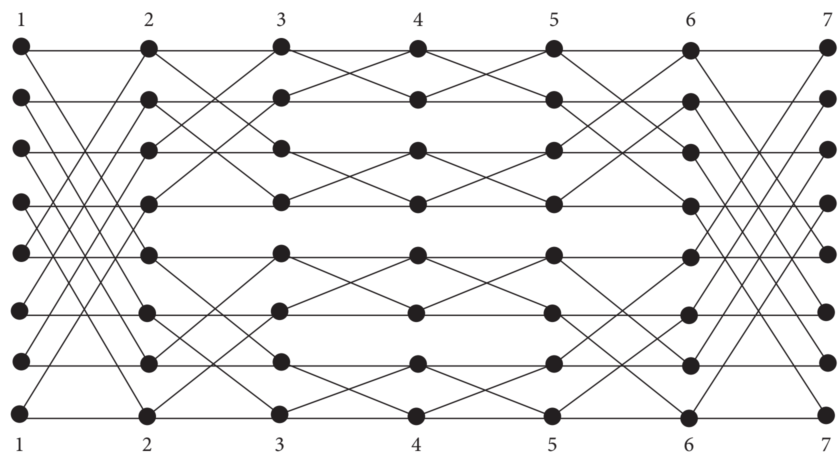

Figure 3: Normal representation of HCB(3).

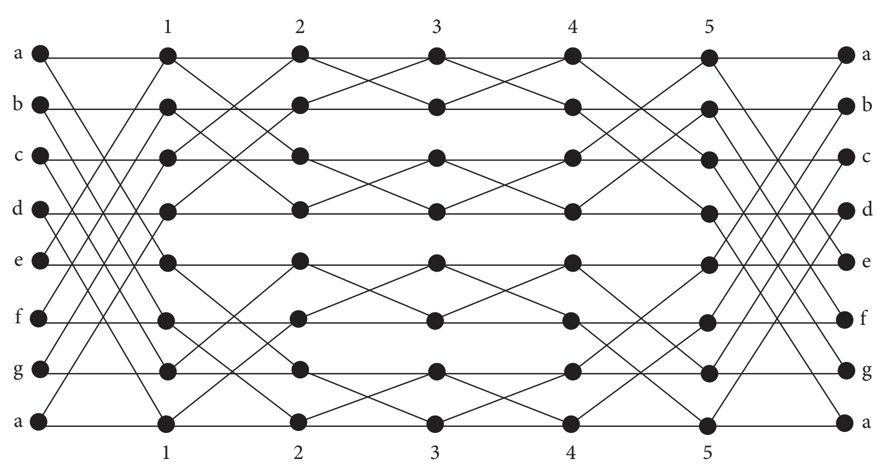

Figure 4: Normal representation of TB(3).

Case 2. (For $i = r$)

Let $v \in V_r$, that is $v = [w_1 \dots w_r, r]$. Then $N(v) = \{[w_1 \dots w_r, r+1], [w_1 \dots w_r', r+1], [w_1 \dots w_r, r-1], [w_1 \dots w_r', r-1]\}$. Furthermore, $N([w_1 \dots w_r, r+1]) = \{[w_1 \dots w_r, r], [w_1 \dots w_r, r+2], [w_1 \dots w_{r-1}' w_r, r+2], [w_1 \dots w_r', r]\}$, $N([w_1 \dots w_r', r+1]) = \{[w_1 \dots w_r, r], [w_1 \dots w_r', r+2], [w_1 \dots w_{r-1}' w_r', r+2], [w_1 \dots w_r', r]\}$, $N([w_1 \dots w_r, r-1]) = \{[w_1 \dots w_r, r], [w_1 \dots w_r, r-2], [w_1 \dots w_{r-1}' w_r, r-2], [w_1 \dots w_r', r]\}$ and $N([w_1 \dots w_r', r-1]) = \{[w_1 \dots w_r, r], [w_1 \dots w_r', r-2], [w_1 \dots w_{r-1}' w_r', r-2], [w_1 \dots w_r', r]\}$.

Thus, $\tau_v = 9$ for $v \in V_r$. On the other hand for $v \in V_r$, $d_v = 4$. So, $\sum_{v \in V_r} d_v \tau_v = 4.9.2^r = 36.2^r$.

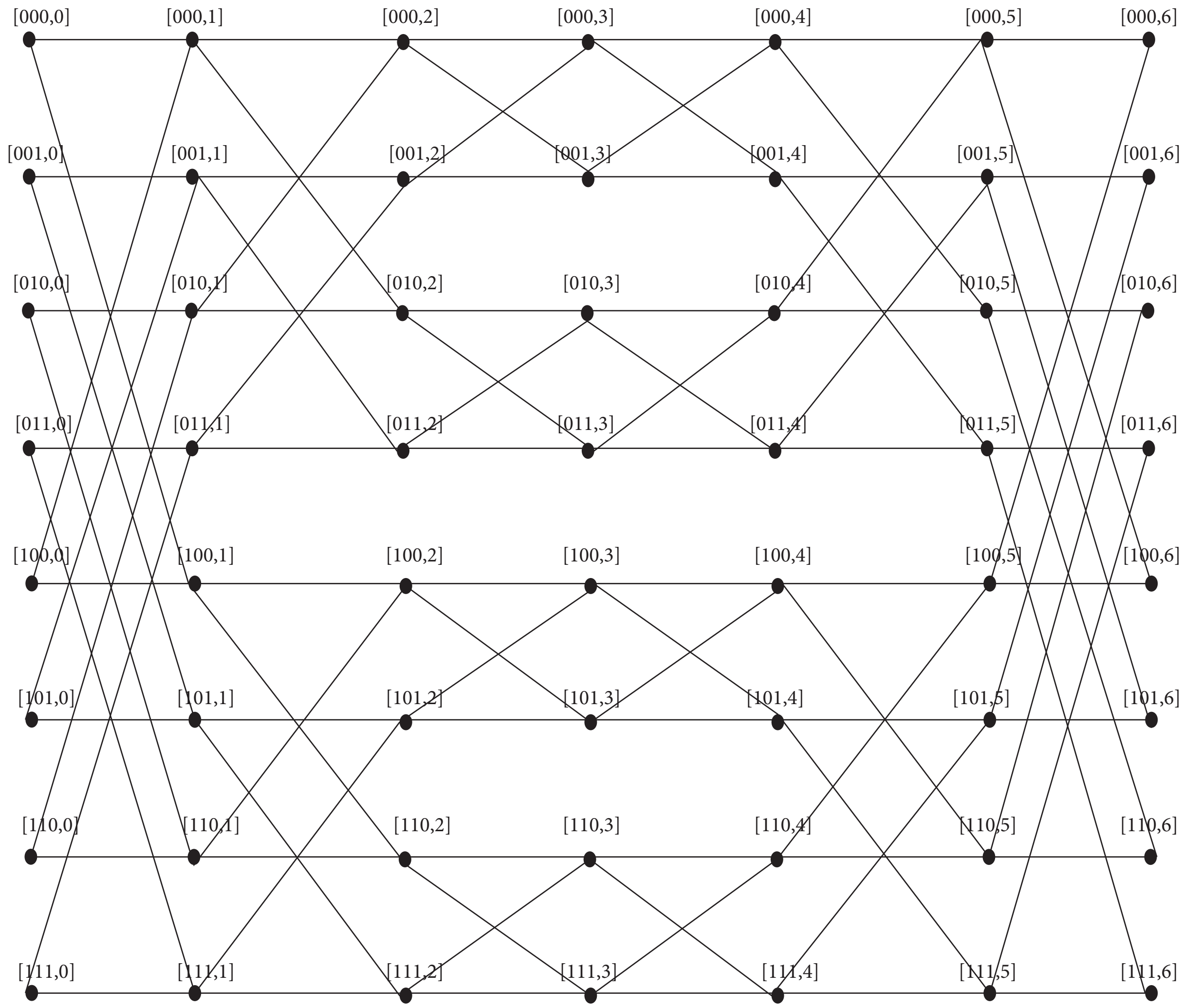

Figure 5: Normal representation of B (3).

Case 3. (For$i = 1, i = 2r - 1$)

Let $v \in V_1$, that is $v = [w_1 w_2 \dots w_r, 1]$. Then $N(v) = \{[w_1 \dots w_r, 2], [w_1 w_2' \dots w_r, 2], [w_1 \dots w_r, 0], [w_1' \dots w_r, 0]\}$. Furthermore, $N([w_1 \dots w_r, 2]) = \{[w_1 \dots w_r, 1], [w_1 \dots w_r, 3], [w_1 w_2' \dots w_r, 1], [w_1 w_2 w_3' \dots w_r, 3]\}$, $N([w_1 w_2' \dots w_r, 2]) = \{[w_1 \dots w_r, 1], [w_1 w_2' \dots w_r, 3], [w_1 w_2' \dots w_r, 1], [w_1 w_2' w_3' \dots w_r, 3]\}$, $N([w_1 w_2 \dots w_r, 0]) = \{[w_1 w_2 \dots w_r, 1], [w_1' w_2 \dots w_r, 1]\}$ and.

$N[w_1' \dots w_r, 0] = \{[w_1 w_2 \dots w_r, 1], [w_1' w_2 \dots w_r, 1]\}$. Thus, $\tau_v = 6$ for $v \in V_1$. On the other hand for $v \in V_1, d_v = 4$. So, $\sum_{v \in V_1} d_v \tau_v = 4.6.2^r = 24.2^r$. The case for $i = 2r - 1$ is similar.

Case 4. (For$i = r - 1, i = r + 1$)

Let $v \in V_{r-1}$, that is $v = [w_1 \dots w_r, r - 1]$. Then $N(v) = \{[w_1 \dots w_r, r], [w_1 \dots w_r', r], [w_1 \dots w_r, r - 2], [w_1 \dots w_{r-1}' w_r, r - 2]\}$. Furthermore, $N([w_1 \dots w_r, r]) = \{[w_1 \dots w_r, r-1], [w_1 \dots w_r, r + 1], [w_1 \dots w_r', r + 1], [w_1 \dots w_r', r - 1]\}$, $N([w_1 \dots w_r', r]) = \{[w_1 \dots w_r, r - 1], [w_1 \dots w_r', r + 1], [w_1 \dots w_r', r - 1], [w_1 \dots w_r, r + 1]\}$, $N([w_1 \dots w_r, r - 2]) = \{[w_1 \dots w_r, r - 1], [w_1 \dots w_{r-2}' w_{r-1} w_r, r - 3], [w_1 \dots w_r, r - 3], [w_1 \dots w_{r-1}' w_r, r - 1]\}$ and $N([w_1 \dots w_{r-1}' w_r, r - 2]) = \{[w_1 \dots w_r, r - 1], [w_1 \dots w_{r-1}' w_r, r - 1], [w_1 \dots w_{r-2}' w_{r-1}' w_r, r - 3], [w_1 \dots w_{r-1}' w_r, r - 3]\}$.

Thus $\tau_v = 8$ for $v \in V_{r-1}$. On the other hand for $v \in V_{r-1}$, $d_v = 4$. So $\sum_{v \in V_{r-1}} d_v \tau_v = 4.8.2^r = 32.2^r$. The case for $i = r + 1$ is similar.

Case 5. (For$1 < i < r - 1, r + 1 < i < 2r - 1$)

Let $v \in V_i$, where $1 < i < r - 1$ or $r + 1 < i < 2r - 1$, that is $v = [w_1 \dots w_r, i]$. Then $N(v) = \{[w_1 \dots w_r, i + 1], [w_1 \dots w_{i+1}' \dots w_r, i + 1], [w_1 \dots w_r, i - 1], [w_1 \dots w_i' \dots w_r, i - 1]\}$. Furthermore, $N([w_1 \dots w_r, i + 1]) = \{[w_1 \dots w_r, i], [w_1 \dots w_r, i + 2], [w_1 \dots w_{i+2}' \dots w_r, i + 2], [w_1 \dots w_{i+1}' \dots w_r, i]\}$, $N([w_1 \dots w_r, i - 1]) = \{[w_1 \dots w_r, i], [w_1 \dots w_r, i - 2], [w_1 \dots w_{i-1}' \dots w_r, i - 2], [w_1 \dots w_i' \dots w_r, i]\}$, $N([w_1 \dots w_{i+1}' \dots w_r, i + 1]) = \{[w_1 \dots w_r, i], [w_1 \dots w_{i+1}' \dots w_r, i + 2], [w_1 \dots w_{i+1}' \dots w_r, i], [w_1 \dots w_{i+1}' w_{i+2}' \dots w_r, i + 2]\}$ and $N([w_1 \dots w_i' \dots w_r, i - 1]) = \{[w_1 \dots w_r, i], [w_1 \dots w_i' \dots w_r, i - 2], [w_1 \dots w_i' \dots w_r, i], [w_1 \dots w_{i-1}' w_i' \dots w_r, i - 2]\}$. Thus $\tau_v = 10$, on the other hand $d_v = 4$ for $v \in V_i$. So $\sum_{v \in V_i} d_v \tau_v = 4.10.2^r (r - 3) = 40.2^r (r - 3)$. Hence $ZC_1^* (G) = \sum_{v \in V(G)} d_v \tau_v = 2.10.2^r + 36.2^r + 2.32.2^r + 2.24.2^r + 2.40 (r - 3)\, 2^r = 2^{r+2} (42 + 20 (r - 3))$.

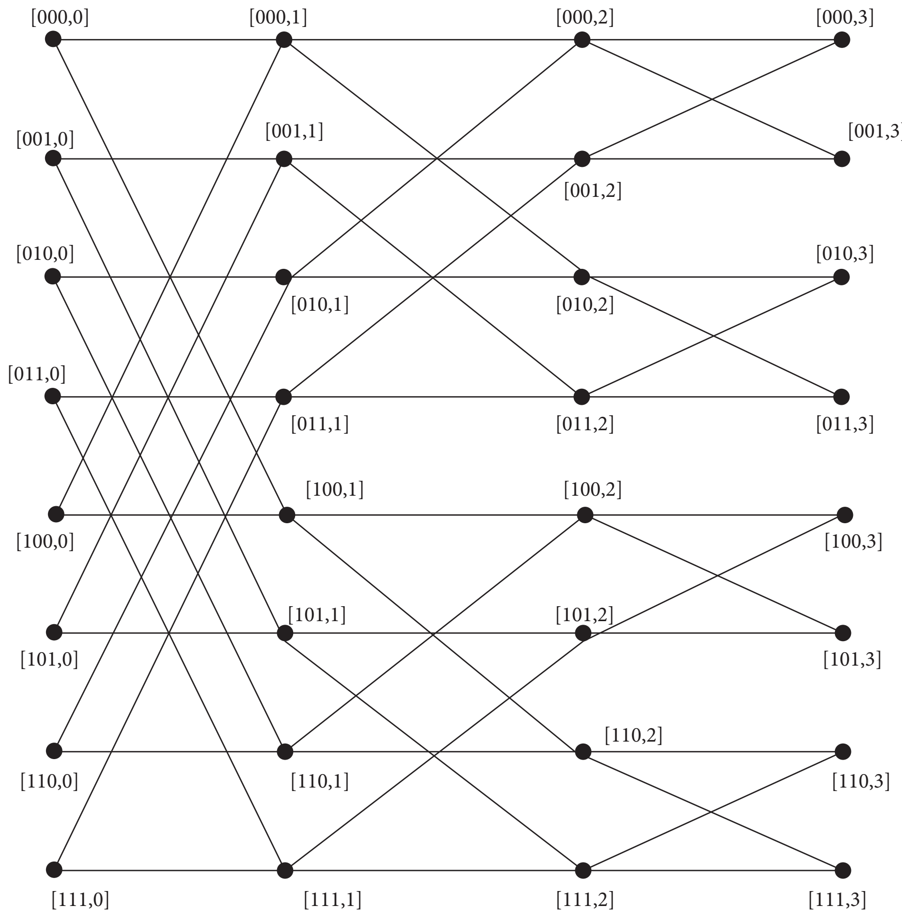

FIGURE 6: Normal representation of BF (3).

Also, $ZC_1(G) = 2.25.2^r + 81.2^r + 2.36.2^r + 2.64.2^r + 2.100.2^r\ (r-3) = 200r.2^r - 269.2^r$.

Theorem 2. *For $r-$ dimensional Butterfly network G, we have*

(1) $ZC_1^*(G) = 2^r(40r - 52)$

(2) $ZC_1(G) = 100r.2^r - 178.2^r$

PROOF. Let $v \in V_r$ in a butterfly network G, that is $v = [w_1 \ldots w_r, r]$. Then $N(v) = \{[w_1 \ldots w_r, r-1], [w_1 \ldots w_r', r-1]\}$. Furthermore, $N([w_1 \ldots w_r, r-1]) = \{[w_1 \ldots w_r', r], [w_1 \ldots w_r, r], [w_1 \ldots w_r, r-2], [w_1 \ldots w_{r-1}' w_r, r-2]\}$ and

$N([w_1 \ldots w_r', r-1]) = \{[w_1 \ldots w_{r-1}' w_r', r-2]t, n[qw_1 \ldots w_r', rh][[_{w_1} \ldots w_r', r-2x]7, C[; w_1 \ldots w_r, r]\}$. Thus, for $v \in V_r, \tau_v = 5$ and $d_v = 2$. The case when $v \in V_0$ is also same.

Moreover, if $v \in V_{r-1}$ that is $v = [w_1 w_2 \ldots w_r, r-1]$. Then $N(v) = \{[w_1 \ldots w_r, r], [w_1 \ldots w_{r-1}' w_r, r-2], [w_1 \ldots w_r, r-2], [w_1 \ldots w_r', r]\}$ and $N([w_1 \ldots w_{r-1}' w_r, r-2]) = \{[w_1 \ldots w_r, r-1], [w_1 \ldots w_{r-1}' w_r, r-1], [w_1 \ldots w_{r-1}' w_r, r-3][w_1 \ldots w_{r-2}' w_{r-1}' w_r, r-3]\}$ $N([w_1 \ldots w_r, r-2]) = \{[w_1 \ldots w_r, r-1], [w_1 \ldots w_{r-1}' w_r, r-1], [w_1 \ldots w_{r-2}' w_{r-1} w_r, r-3], [w_1 \ldots w_r, r-3]\}$, $N([w_1 w_2 \ldots w_r, r]) = \{[w_1 w_2 \ldots_r, r-1], [w_1 \ldots w_r', r-1]\}$ and. $N([w_1 \ldots w_r', r]) = \{[w_1 w_2 \ldots w_r, r-1], [w_1 \ldots w_r', r-1]\}$.

Thus, for $v \in V_{r-1}, \tau_v = 6$ and $d_v = 4$. The case for $v \in V_1$ is also similar.

By following the same pattern, it can be proved that if $v \in V_i$ for $2 \le i \le r-2$, $d_v = 4$ and $\tau_v = 10$. So, $ZC_1^*(G) = 2.5.2^r + 4.6.2^r + 4.10.2^r(r-3) + 2.5.2^r + 4.6.2^r = 2^r(40r - 52)$ and $ZC_1(G) = 25.2^r + 36.2^r + 100.2^r(r-3) + 36.2^r + 25.2^r = 100r.2^r - 178.2^r$. □

2.2. Modified Zagreb Connection Indices for Vertical Cylindrical Representations of Benes Networks. This network is obtained by the identification of last column with first column of Benes network. Following the construction of $VCB(r)$, clearly there are $r2^{r+2}$ vertices in $VCB(r)$. This is a regular graph with degree of each vertex as four [9].

Theorem 3. *For $G = VCB(r)$, we have*

(1) $ZC_1^*(G) = 2^r(200 + 80(r-3))$

(2) $ZC_1(G) = 200r.2^r - 182.2^r$

PROOF. Let G denotes the 4-regular graph of $VCB(r)$ in which $[w, i] = [w_1 w_2 \ldots w_r, i]$ is an arbitrary node, for any

$0 \le i \le 2r-1$. Let V_k, $0 \le k \le 2r-1$ be the set of vertices of k th column in G. For $0 \le i, j \le 2r-1$, if $[w,i]$ and $[w', j]$ are adjacent then $w_j = 0$ implies $w'_j = 1$, and vice versa. We partition the vertices $v = [w,i]$, $0 \le i \le 2r-1$ of G in to the different cases depending on τ_v: □

Case 6. (For$i = 0$)

Let $v \in V_0$, that is $v = [w_1 w_2 \ldots w_r, 0]$, then $N(v) = N[w_1 \ldots w_r, 0] = \{[w_1 \ldots w_r, 1], [w'_1 \ldots w_r, 1], [w_1 \ldots w_r, 2r-1], [w'_1 \ldots w_r, 2r-1]\}$.

Furthermore,
$N([w_1 w_2 \ldots w_r, 1]) = \{[w_1 \ldots w_r, 0], [w_1 \ldots w_r, 2], [w'_1 \ldots w_r, 0], [w_1 w'_2 \ldots w_r, 2]\}$ $N([w'_1 \ldots w_r, 1]) = \{[w_1 \ldots w_r, 0], [w'_1 \ldots w_r, 2], [w'_1 \ldots w_r, 0], [w'_1 w'_2 \ldots w_r, 2]\}$, $N([w_1 \ldots w_r, 2r-1]) = \{[w_1 \ldots w_r, 2r-2], [w_1 w'_2 \ldots w_r, 2r-2], [w'_1 \ldots w_r, 0], [w_1 \ldots w_r, 0]\}$ and. $N([w'_1 \ldots w_r, 2r-1]) = \{[w'_1 \ldots w_r, 2r-2], [w'_1 \ldots w_r, 0], [w'_1 w'_2 \ldots w_r, 2r-1], [w_1 \ldots w_r, 0]\}$. So, $\tau_v = 9$ and therefore, $\sum_{v \in V_0} d_v \tau_v = 4.9.2^r = 36.2^r$.

Case 7. (For$i = 1, i = 2r-1$)

Let $v \in V_1$, that is $v = [w_1 w_2 \ldots w_r, 1]$. Then $N(v) = \{[w_1 \ldots w_r, 2], [w_1 w'_2 \ldots w_r, 2], [w_1 \ldots w_r, 0], [w'_1 \ldots w_r, 0]\}$.

Furthermore,
$N([w_1 w_2 \ldots w_r, 2]) = \{[w_1 \ldots w_r, 1], [w_1 \ldots w_r, 3], [w_1 w_2 w'_3 \ldots w_r, 3], [w_1 w'_2 \ldots w_r, 1]\}$, $N[w_1 w'_2 \ldots w_r, 2] = \{[w_1 \ldots w_r, 1], [w_1 w'_2 \ldots w_r, 3], [w_1 w'_2 \ldots w_r, 1], [w_1 w'_2 w'_3 \ldots w_r, 3]\}$,

$N([w_1 \ldots w_r, 0]) = \{[w_1 \ldots w_r, 2r-1]t, n \; [qw'_1 \ldots w_r, 2r-1h][[_w 1 \ldots w_r, 1x]7, C[; w_1 \ldots w_r, 1]\}$ and $N([w'_1 \ldots w_r, 0]) = \{[w'_1 \ldots w_r, 2r-1], [w_1 \ldots w_r, 2r-1], [w'_1 w'_2 \ldots w_r, 2r-1], [w_1 \ldots w_r, 1]\}$. So, $\tau_v = 8$ and therefore $\sum_{v \in V_1} d_v \tau_v = 4.8.2^r = 32.2^r$. The case for $i = 2r-1$ is similar.

Case 8. (For$1 < i < 2r-1$)

Similarly, if $v \in V_i$, where $1 < i < 2r-1$, then we have $\sum_{v \in V_i} = 2.40(r-3)2^r + 2.32.2^r + 36.2^r = 2^r \quad (80(r-3) + 100)$. Therefore, $\mathrm{ZC}_1^*(G) = \sum_{v \in V(G)} d_v \tau_v = 36.2^r + 2.32.2^r + 2^r(80(r-3)+100) = 2^r(200 + 80(r-3))$ and $ZC_1(G) = 81.2^r + 2.64.2^r + 2.100(r-3).2^r + 2.64.2^r + 81.2^r = 200r.2^r - 182.2^r$.

2.3. Modified Zagreb Connection Indices for Horizontal Cylindrical Representations of Benes Networks. The network $\mathrm{HCB}(r)$ is obtained by identifying vertices of the last row of Benes network with the corresponding vertices of the first row. The total vertices in $\mathrm{HCB}(r)$ are $(2r+1)(2^r-1)$ [9]. By following the same method as in previous theorems, the $(2r+1)(2^r-1)$ vertices are partitioned in terms of degree and τ as shown in Table 1.

Theorem 4. *For $G = HCB(r)$, we have*

(1) $ZC_1^*(G) = 802 + 104r - 72.2^r + 80r.2^r$

(2) $ZC_1(G) = 200r.2^r - 269.2^r - 152r + 2304$

PROOF. From the partition of the vertices of $\mathrm{HCB}(r)$ and Table 1 comprising connection number of the vertices, we have

Table 1: Vertex partition of $\mathrm{HCB}(r)$ corresponding to connection numbers and degrees.

Deg	τ_v	Vertices
3	9	2
6	11	2
6	14	2
6	16	1
6	18	$2(r-3)$
2	5	$2(2^r-34)$
2	7	2 (2)
4	6	$2(2^r-36)$
4	8	2 (2)
4	7	2 (2)
4	8	$2(2^r-6)$
4	10	2 (4)
4	12	2
4	9	(2^r-4)
4	10	$2(2^r-4)(r-33)$
4	14	2 (2)

Table 2: Vertex partition of $\mathrm{TB}(r)$ corresponding to connection numbers and degree.

Deg	τ_v	No. of vertices
6	18	1
6	14	2
6	14	2
6	16	1
6	18	2 (r-3)
4	9	$(2^r$-4)
4	12	2 (2)
4	8	2 $(2^r$-4)
4	11	2 (2)
4	7	2 (2)
4	8	$2(2^r-6)$
4	10	2 (4)
4	12	2
4	9	(2^r-4)
4	10	$2(2^r-4)$ (r-3)
4	14	2 (2)

$\mathrm{ZC}_1^*(G) = \sum_{v \in V(G)} d_v \tau_v = 54 + 132 + 168 + 96 + 216(r-3) + 20(2^r-4) + 56 + 48(2^r-6) + 128 + 112 + 64(2^r-6) + 320 + 96 + 36(2^r-4) + 80(2^r-4)(r-3) + 224 = 1386 + 216r - 648 + 20.2^r - 80 + 48.2^r - 288 + 64.2^r - 384 + 36.2^r - 144 + 80r.2^r - r320 - 240.2^r + 960 = 802 - 104r - 72.2^r + 80r.2^r$.

Similarly, $ZC_1(G) = 162 + 242 + 392 + 256 + 648r - 1944 + 50.2^r - 200 + 196 + 72.2^r - 432 + 256 + 196 + 128.2^r - 768 + 800 + 288 + 81.2^r - 324 + 200.r.2^r - 800r - 600.2^r + 2400 + 784 = 200r.2^r - 269.2^r - 152r + 2304$. □

2.4. Modified Zagreb Connection Indices for Toroidal Representation of Benes Networks. The $\mathrm{TB}(r)$ is obtained by the identification of the vertices of bottom row of $\mathrm{VCB}(r)$ to the vertices of the top row. Here, benes network is embedded on Torus. The total number of vertices in $\mathrm{TB}(r)$ is $2r(2^r-1)$ [9]. Now, we compute ZC_1^* and ZC_1 for this network.

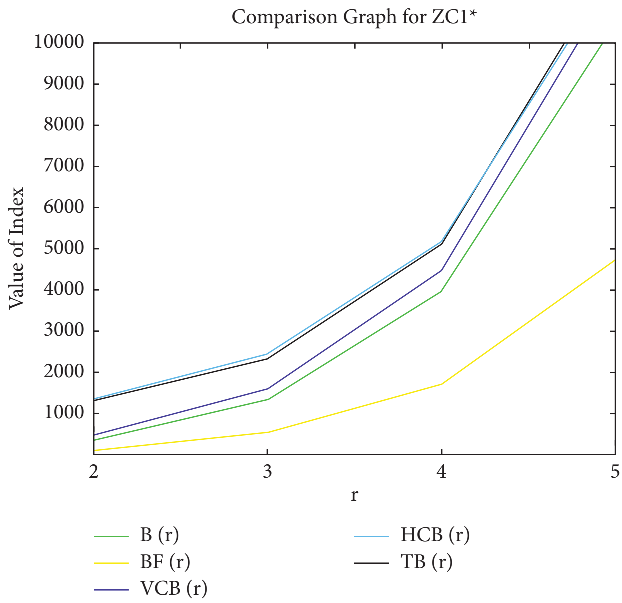

Figure 7: Comparison graph.

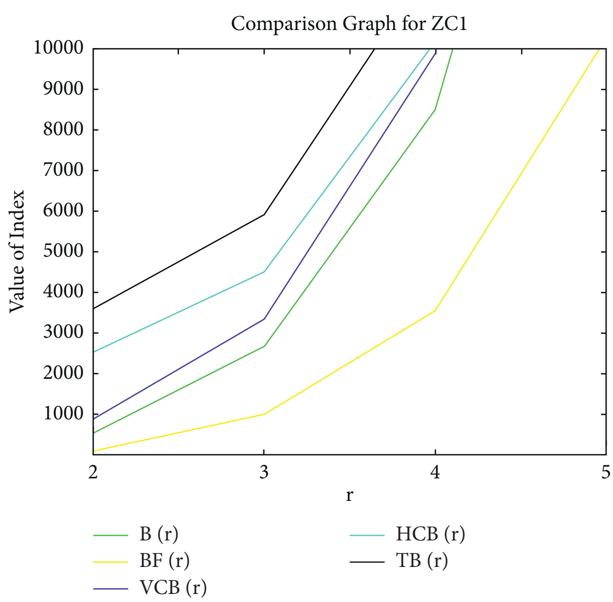

Figure 8: Comparison graph.

Theorem 5. *For $G = TB(r)$, we have*

(1) $ZC_1^*(G) = 80r.2^r - 40.2^r - 104r + 1044$

(2) $ZC_1(G) = 200r.2^r - 182.2^r - 152r + 3020$

PROOF. The $2r(2^r - 1)$ vertices of $\mathrm{TB}(r)$ are partitioned on the basis of degree as: $2r(2^r - 2)$ and $2r$ vertices have degrees 4 and 6, respectively. The connection numbers of vertices of $TB(r)$ are presented in Table 2.

So, we have $ZC_1^*(G) = \sum_{v\in V(G)} d_v\tau_v = 108 + 168 + 168 + 96 + 216(r-3) + 36(2^r - 4) + 192 + 64(2^r - 4) + 176 + 112 + 64(2^r - 6) + 320 + 96 + 36(2^r - 4) + 80(2^r - 4)\quad (r-3) + 224 = 80r.2^r - 40.2^r - 104r + 1044$.

Moreover, $ZC_1(G) = 324 + 392 + 392 + 256 + \quad 648r - 1944 + 81.2^r - 324 + 576 + \quad 128.2^r - 512 + 484 + 196 + 128.2^r - 768 + \quad 800 + 288 + 81.2^r \quad -324 + 200.r.2^r \quad -800r - 600.2^r + 2400 + 784 = 200r.2^r - 182.2^r - 152r + 3020$. □

3. Conclusion

The newly generated structures and networks are always interesting topic to be studied. In [9], several new networks such as $\mathrm{HCB}(r)$, $\mathrm{VCB}(r)$ and $\mathrm{TB}(r)$ have been introduced by using $B(r)$. By keeping in view of the importance to study new networks, we computed 2-distance based TIs for these new classes of networks. Moreover, we have also used graphical tools to describe a comparison among the values of the computed indices. Figures 7 and 8 present the rise in the values of the computed TIs of the networks with respect to the size r of the networks $B(r)$, $\mathrm{BF}(r)$, $\mathrm{VCB}(r)$, $\mathrm{HCB}(r)$, and $\mathrm{TB}(r)$. The current paper will be a step forward towards the study of these networks for general distance and the maximum distance based descriptors.

Data Availability

No additional data set is used to support the study.

Conflicts of Interest

The authors declare no conflicts of interest.

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