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Research Article

On the Minimum General Sum-Connectivity of Trees of Fixed Order and Pendent Vertices

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For a graph G, its general sum-connectivity is usually denoted by $\chi_\alpha(G)$ and is defined as the sum of the numbers $[d_G(u)+d_G(v)]^\alpha$ over all edges uv of G, where $d_G(u), d_G(v)$ represent degrees of the vertices u, v, respectively, and α is a real number. This paper addresses the problem of finding graphs possessing the minimum χ_α value over the class of all trees with a fixed order n and fixed number of pendent vertices n_1 for $\alpha > 1$. This problem is solved here for the case when $4 \le n_1 \le (n+5)/3$ and $\alpha > 1$, by deriving a lower bound on χ_α for trees in terms of their orders and number of pendent vertices.

1. Introduction

We only discuss simple connected graphs in this study. For a graph G, its edge set and vertex set are represented by $E(G)$ and $V(G)$, respectively. Let $d_G(u)$ indicate the degree of a vertex $u \in V(G)$. When there is no uncertainty regarding the graph under consideration, we write $d(u)$ instead of $d_G(u)$. Those (chemical-) graph-theoretical concepts and notation utilized in this article but not described here may be found in related books, such as [1–4].

For a graph G, its general sum-connectivity index $\chi_\alpha(G)$, devised in [5], is defined as,

$$\chi_\alpha(G) = \sum_{uv\in E(G)} (d(u)+d(v))^\alpha. \tag{1}$$

where α is a real number. The special cases $\chi_1(G)$ and $\chi_{(-1/2)}(G)$ of $\chi_\alpha(G)$ yield the first Zagreb index (for example, see [6]) and sum-connectivity index of G [7].

A pendent vertex in a graph is a vertex of degree 1. In this paper, we are concerned with the following problem concerning the general sum-connectivity index.

Problem 1. Find the graphs having the maximum and minim general sum-connectivity index χ_α among all trees of a fixed order and number of pendent vertices.

Cui and Zhong [8] studied the maximal part of problem 1 for $-1 \le \alpha < 0$. The minimal part of problem 1 for $-1 \le \alpha < 0$ was solved in [9]. Tache and Tomescu [10] reported the solution to the maximal part of problem 1 when $\alpha \ge 1$; the minimal part was solved in [11] for $\alpha = 1$. Additional information about the known mathematical aspects of the general sum-connectivity index can be found, for example, in [12–16], in [17] (where several general results give special cases for the general sum-connectivity index), and in the related references given therein.

In this paper, a lower bound on χ_α for trees of order n and number of pendent vertices n_1 is derived for $n \ge 1$ and $\alpha > 1$. As a consequence of the obtained bound, the minimal part of problem 1 is solved when $4 \le n_1 \le (n+5)/3$ and $\alpha > 1$.

2. Main Results

Before proving the main results, we give some definitions and notation that are used in the rest of this section. The star and path graphs with n vertices are represented by S_n and P_n, respectively. A branching vertex of a graph is a vertex of degree greater than 2. For a graph G and a vertex $u \in V(G)$, denote by $N_G(u)$ the set of all those vertices of G that are adjacent to u. The elements of the set $N_G(u)$ are known as the neighbors of u. A pendent path $u_1u_2 \ldots u_k$ of a graph G is a non-trivial path in G such that one of the vertices u_1, u_k is pendent and the other is branching, and each u_i (if $2 \le i \le k-1$) has degree 2.

Transformation 1. Let T be a tree containing at least two branching vertices. Let $u_1u_2\ldots u_r$ be a pendent path of T such that $r \ge 3$, where $d_T(u_1) \ge 3$ and $d_T(u_r) = 1$. Choose a vertex $w \in E(T)$ lying on the unique path between u_1 and another branching vertex of T. Suppose that T' is formed from T by inserting the two edges u_1u_3, u_rw, and dropping the two edges u_2u_3, u_1w. The trees T and $T\prime$ are depicted in Figure 1.

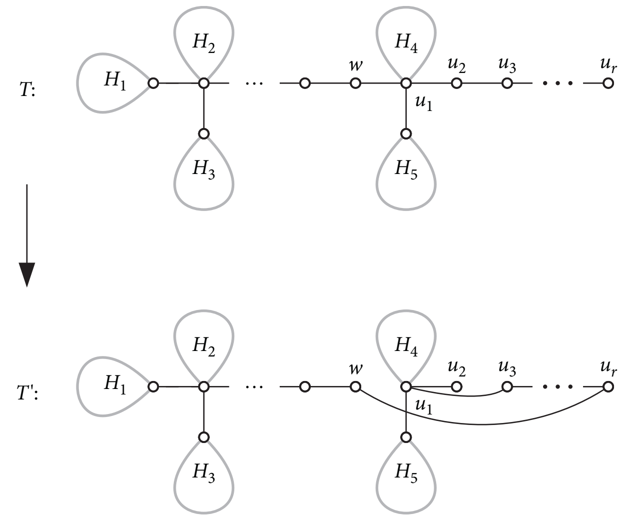

Figure 1: The trees T and $T\prime$ specified in Transformation 1, where the subtree H_i may or may not be trivial, $i = 1, 2, \ldots, 5$.

Lemma 1. *If T and $T\prime$ are the trees specified in Transformation 1, then $\chi_\alpha(T) > \chi_\alpha(T\prime)$ for $\alpha > 1$.*

Proof. Since $d_T(u)_1 \ge 3$ and $d_T(w) \ge 2$, by using the definition of χ_α, one has

$$\begin{aligned}\chi_\alpha(T) - \chi_\alpha(T\prime) &= (d_T(u_1) + d_T(w))^\alpha \\ &\quad - (d_T(u_1) + 1)^\alpha \\ &\quad - (d_T(w) + 2)^\alpha + 3^\alpha \\ &\ge (d_T(w) + 3)^\alpha - 4^\alpha \\ &\quad - (d_T(w) + 2)^\alpha + 3^\alpha \\ &\ge 5^\alpha - 4^\alpha - 4^\alpha + 3^\alpha > 0,\end{aligned} \tag{2}$$

for $\alpha > 1$.

An internal path $u_1u_2\cdots u_k$ of a graph G is a non-trivial path in G such that both the vertices u_1, u_k are branching, and each u_i (if $2 \le i \le k-1$) has degree 2. Denote by $\mathbb{T}(n, n_1)$ the class consisting of all trees T with order n, pendent vertices n_1, and maximum degree 3 such that every pendent (internal) path of T has length one (at least two, respectively). If $T \in \mathbb{T}(n, n_1)$ then after simple calculations, one gets.

$$\chi_\alpha(T) = (n - 2n_1 + 5)4^\alpha + 2(n_1 - 3)5^\alpha. \tag{3}$$

□

Theorem 1. *Let T be a tree with n vertices, among which n_1 are pendent vertices. Then*

$$\chi_\alpha(T) \ge (n - 2n_1 + 5)4^\alpha + 2(n_1 - 3)5^\alpha = \psi_\alpha(n, n_1), \tag{4}$$

where the equality sign in the inequality holds if and only if $T \in \mathbb{T}(n, n_1)$.

Proof. For $n = 1, 2, 3, 4, 5$, the result is straightforwardly verified. Next, suppose that $n \ge 6$. Note that the function f_α defined by $f_\alpha(x) = (x-1)x^\alpha - 2(x-4)5^\alpha + (x-7)4^\alpha$, is strictly increasing for $x \ge 6$ and $\alpha > 1$ because $f'_\alpha(x) > x^\alpha - 2\cdot 5^\alpha + 4^\alpha \ge 6^\alpha - 2\cdot 5^\alpha + 4^\alpha > 0$. Thus, for $n_1 = n - 1$ we have $T = S_n$ and hence

$$\begin{aligned}\chi_\alpha(S_n) - \psi_\alpha(n, n_1) &= (n-1)n^\alpha - 2(n-4)5^\alpha \\ &\quad + (n-7)4^\alpha = f_\alpha(n) \ge f_\alpha(6) \\ &= 5\cdot 6^\alpha - 4\cdot 5^\alpha - 4^\alpha > 0,\end{aligned} \tag{5}$$

for $n \ge 6$ and $\alpha > 1$. In what follows, we assume that $2 \le n_1 \le n-2$ with $n \ge 6$ and we prove the result by using mathematical induction on n_1. If $n_1 = 2$, then $T = P_n$ and

$$\chi_\alpha(P_n) - \psi_\alpha(n, 2) = 2\cdot 5^\alpha - 4\cdot 4^\alpha + 2\cdot 3^\alpha > 0, \tag{6}$$

for $\alpha > 1$. Next, assume that $n_1 \ge 3$ and that the theorem holds for $n_1 - 1$, where $n_1 \le n-2$ and $n \ge 6$. □

Case 1. The tree T does not contain any pendent path of length greater than 1.

In this case, we consider its several subcases.

Subcase 1. The tree T contains at least one pendent vertex whose neighbor has degree greater than 3.

Let $u \in V(T)$ be a vertex of degree s having r pendent neighbors $u_0, u_1, \ldots, u_{r-1}$ and $s - r$ non-pendent neighbors $u_r, u_{r+1}, \ldots, u_{s-1}$, where $s \ge 4$ and $r \ge 1$. Since $n \ge 6$, one has $s - r > 0$. By keeping in mind the inequalities $s - r > 0$, $\alpha > 1$, $s \ge 4$, and inductive hypothesis, we have.

$$\begin{aligned}\chi_\alpha(T) &= \chi_\alpha(T - u_0) + (s+1)^\alpha + (r-1)\left[(s+1)^\alpha - s^\alpha\right] \\ &\quad + \sum_{j=r}^{s-1}\left[\left(s + d_T(u_j)\right)^\alpha - \left(s + d_T(u_j) - 1\right)^\alpha\right] \\ &\ge \chi_\alpha(T - u_0) + (s+1)^\alpha + (r-1)\left[(s+1)^\alpha - s^\alpha\right] \\ &\quad + (s-r)\left[(s+2)^\alpha - (s+1)^\alpha\right] \\ &= \chi_\alpha(T - u_0) + (2r - s)(s+1)^\alpha - (r-1)s^\alpha \\ &\quad + (s-r)(s+2)^\alpha \\ &\ge \left[\psi_\alpha(n, n_1) + 4^\alpha - 2\cdot 5^\alpha\right] + (2r - s)(s+1)^\alpha \\ &\quad - (r-1)s^\alpha + (s-r)(s+2)^\alpha \\ &= \psi_\alpha(n, n_1) + (s-r)\left[(s+2)^\alpha - (s+1)^\alpha\right] \\ &\quad + (r-1)\left[(s+1)^\alpha - s^\alpha\right] + (s+1)^\alpha + 4^\alpha - 2\cdot 5^\alpha \\ &\ge \psi_\alpha(n, n_1) + \left[6^\alpha - 5^\alpha\right] + 5^\alpha + 4^\alpha - 2\cdot 5^\alpha > \psi_\alpha(n, n_1).\end{aligned} \tag{7}$$

Subcase 2. Every pendent vertex of T is adjacent to a vertex of degree 3 and there is at least one vertex of degree 3 having only one pendent neighbor.

Let u be a vertex of degree 3 and u_0 be a pendent neighbor of u. Assume that u_1 and u_2 are the non-pendent neighbors of u. By utilizing the induction hypothesis, we get,

$$\begin{aligned}\chi_\alpha(T) &= \chi_\alpha(T-u_0) + \sum_{i=1}^{2}\left[\left(d_T(u_i)+3\right)^\alpha - \left(d_T(u_i)+2\right)^\alpha\right] \\ &\geq \chi_\alpha(T-u_0) + 2\left(5^\alpha - 4^\alpha\right) + 4^\alpha \geq \psi_\alpha(n,n_1),\end{aligned} \tag{8}$$

where $\chi_\alpha(T) = \psi_\alpha(n,n_1)$ if and only if $d_T(u_1), d_T(u_2) \in \{2\}$ and $T-u_0 \in \mathbb{T}(n-1,n_1-1)$; e.g., if and only if $T \in \mathbb{T}(n, n_1)$.

Subcase 3. Every pendent vertex of T is adjacent to a vertex of degree 3 and there is at least one vertex of degree 3 having one branching neighbor and two pendent neighbors.

Let u be a vertex of degree 3 and u_0, u_1 be its two pendent neighbors. Assume that u_2 is the non-pendent neighbor of u. By utilizing the induction hypothesis, we get

$$\begin{aligned}\chi_\alpha(T) &= \chi_\alpha(T-\{u_0,u_1\}) + \left(d_T(u_2)+3\right)^\alpha \\ &\quad - \left(d_T(u_2)+1\right)^\alpha + 2\cdot 4^\alpha \geq \chi_\alpha(T-\{u_0,u_1\}) + 6^\alpha + 4^\alpha \\ &\geq \psi_\alpha(n-2,n_1-1) + 6^\alpha + 4^\alpha \geq \psi_\alpha(n,n_1) \\ &\quad - 2\cdot 5^\alpha + 6^\alpha + 4^\alpha > \psi_\alpha(n,n_1).\end{aligned} \tag{9}$$

Subcase 4. Every pendent vertex of T is adjacent to a vertex of degree 3, which has one neighbor of degree 2 and two pendent neighbors.

If T has the maximum degree 3 and if T contains no pair of adjacent vertices of degree 3, then $T \in \mathbb{T}(n,n_1)$ and we get $\chi_\alpha(T) = \psi_\alpha(n,n_1)$.

Now, assume that T has the maximum degree 3 and $uv \in E(T)$ such that $d_T(u) = d_T(v) = 3$. Let $w_1 \in V(T)$ be a vertex of degree 3 having two pendent vertices and a vertex w of degree 2. Take $N_T(w) = \{w_1, w_2\}$. Let $T\prime$ be the tree formed from T by deleting the edges uv, w_1w, w_1w, and adding the edges w_1w_2, uw, vw. Then, $T\prime$ is a tree with n vertices and n_1 pendent vertices. Also, note that both the trees T and $T\prime$ have the same degree sequence. Since the maximum degree of T is 3, it holds that $d_T(w_2) = 2$ or 3. On the other hand, we have

$$\chi_\alpha(T) - \chi_\alpha(T\prime) = \left(d_T(w_2)+2\right)^\alpha - \left(d_T(w_2)+3\right)^\alpha + 6^\alpha - 5^\alpha \geq 0. \tag{10}$$

Note that if $d_T(w_2) = 3$ then $\chi_\alpha(T) = \chi_\alpha(T')$ and the tree $T\prime$ contains a vertex (namely w_1) of degree 3 having two pendent neighbors and a neighbor of degree 3, and hence from Subcase 3 it follows that $\chi_\alpha(T) = \chi_\alpha(T\prime) > \psi_\alpha(n,n_1)$.

If $d_T(w_2) = 2$ then $\chi_\alpha(T) - \chi_\alpha(T\prime) > 0$. If $T\prime \in \mathbb{T}(n,n_1)$ then we are done. If $T\prime$ contains at least one pair of adjacent branching vertices then we repeat the above process until we obtain the desired result.

It remains to prove the desired result in the considered case (that is, Subcase 4) when the maximum degree of T is greater than 3. Let $u\prime \in V(T)$ be a vertex of maximum degree Δ, where $\Delta \geq 4$. Take $N_T(u\prime) = \{u_1, u_2, \ldots, u_\Delta\}$ and let $v\prime \in V(T)$ be a vertex of degree 3 having two pendent neighbors and a neighbor of degree 2. Let v_1 be a pendent neighbor of $v\prime$. Without loss of generality, we assume that u_2 lies on the unique $u\prime$-$v\prime$ path. Let $T\prime\prime$ be the tree deduced from T by deleting the edge $u_1u\prime$ and inserting the edge $u_1v\prime$. Observe that the tree $T\prime\prime$ has n vertices among which $n_1 - 1$ are pendent. Thus, by using the the inequalities $d_T(u) = \Delta \geq 4$, $d_T(u_i) \geq 2$, $i \in \{1, 2, \ldots, \Delta\}$ (because of the considered case), and inductive hypothesis, we have

$$\begin{aligned}\chi_\alpha(T) &= \chi_\alpha(T\prime\prime) + \sum_{i=2}^{\Delta}\left[\left(d_T(u_i)+\Delta\right)^\alpha - \left(d_T(u_i)+\Delta-1\right)^\alpha\right] \\ &\quad + \left(d_T(u_1)+\Delta\right)^\alpha - \left(d_T(u_1)+2\right)^\alpha + 4^\alpha - 6^\alpha \\ &\geq \chi_\alpha(T\prime\prime) + (\Delta-1)\left[(\Delta+2)^\alpha - (\Delta+1)^\alpha\right] + (\Delta+2)^\alpha \\ &\quad - 6^\alpha \geq \chi_\alpha(T\prime\prime) + 3\left(6^\alpha - 5^\alpha\right) \\ &\geq \psi_\alpha(n,n_1) + 2\left(4^\alpha - 5^\alpha\right) + 3\left(6^\alpha - 5^\alpha\right) > \psi_\alpha(n,n_1),\end{aligned} \tag{11}$$

for $\alpha > 1$.

Case 2. T contains at least one pendent path of length greater than 1.

First, we suppose that T contains only one branching vertex. Assume also that T contains at least two pendent paths of lengths at least 2. Let $u_0u_1u_2\cdots u_r$ and $v_0v_1v_2\cdots v_s$ be two such pendent paths, where u_0, v_0 are branching vertices and u_r, v_s are pendent vertices. Let $T\prime$ be the tree formed by deleting the edge u_1u_2 and adding the edge u_2v_s. Since $\alpha > 1$ and $n_1 \geq 3$, we have

$$\begin{aligned}\chi_\alpha(T) - \chi_\alpha(T\prime) &= (n_1+2)^\alpha - (n_1+1)^\alpha - \left(4^\alpha - 3^\alpha\right) \\ &\geq 5^\alpha - 4^\alpha - \left(4^\alpha - 3^\alpha\right) > 0,\end{aligned} \tag{12}$$

Thus, from the above discussion, we conclude that if T contains only one branching vertex then

$$\begin{aligned}\chi_\alpha(T) \geq \chi_\alpha(T^\star) &= (n-n_1-2)4^\alpha + (n_1+2)^\alpha \\ &\quad + (n_1-1)(n_1+1)^\alpha + 3^\alpha,\end{aligned} \tag{13}$$

where $T^\star$ is a tree formed by attaching $n_1 - 1$ pendent vertices to one of the pendent vertices of the path graph P_{n-n_1+1}. For $3 \leq n_1 \leq 8$, by direct calculations it is verified that the inequality

$$(n-n_1-2)4^\alpha + (n_1+2)^\alpha + (n_1-1)(n_1+1)^\alpha + 3^\alpha > \psi_\alpha(n,n_1), \tag{14}$$

holds for $\alpha > 1$ Also, for $n_1 \geq 9$ and $\alpha > 1$, one has

$$\begin{aligned}&(n-n_1-2)4^\alpha + (n_1+2)^\alpha + (n_1-1)(n_1+1)^\alpha \\ &\quad + 3^\alpha - \psi_\alpha(n,n_1) > (n_1-1)(n_1+1)^\alpha - 2(n_1-3)5^\alpha,\end{aligned} \tag{15}$$

the right hand side of this inequality is positive because

$$\left(\frac{n_1-1}{n_1-3}\right)\left(\frac{n_1+1}{5}\right)^{\alpha} > \left(\frac{n_1+1}{5}\right)^{\alpha} \geq 2^{\alpha} > 2. \quad (16)$$

Therefore,

$$\chi_\alpha(T) \geq \chi_\alpha(T^\star) > \psi_\alpha(n, n_1). \quad (17)$$

Next, we suppose that T contains at least two branching vertices. By Transformation 1 and Lemma 1 there exists a tree $T^\dagger$ of order n with n_1 pendent vertices such that $T^\dagger$ contains no pendent path of length greater than 1 and $\chi_\alpha(T) > \chi_\alpha(T^\dagger)$. However, by Case 1, it holds that $\chi_\alpha(T^\dagger) \geq \psi_\alpha(n, n_1)$. Therefore, $\chi_\alpha(T) > \psi_\alpha(n, n_1)$. This completes the proof.

Observe that the class $\mathbb{T}(n, n_1)$ is non-empty whenever $n_1 \geq 4$ and $n \geq 3n_1 - 5$. Thus, the next result immediately follows from Theorem 1.

Corollary 1. *For $\alpha > 1$ and $4 \leq n_1 \leq ((n+5)/3)$, among all trees with order n and number of pendent vertices n_1, the element (s) of the class $\mathbb{T}(n, n_1)$ uniquely minimize the general sum-connectivity index χ_α.*

3. Concluding remarks

For a graph G, its general Platt index [18] is defined as

$$Pl_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v) - 2)^{\alpha}, \quad (18)$$

where α is a real number. The graph invariant $Pl_2(G)$ is known as the reformulated first Zagreb index of G [19]. Because of the similarity between the definitions of χ_α and Pl_α, one obtains Theorem 2 (corresponding to Theorem 1) and Corollary 1 (corresponding to Corollary 2).

Theorem 2. *Let T be a tree with n vertices, among which n_1 are pendent vertices. Then*

$$Pl_\alpha(T) \geq (n - 2n_1 + 5)2^{\alpha} + 2(n_1 - 3)3^{\alpha}, \quad (19)$$

where the equality sign in the inequality holds if and only if $T \in \mathbb{T}(n, n_1)$.

Corollary 2. *For $\alpha > 1$ and $4 \leq n_1 \leq ((n+5)/3)$, among all trees with order n and number of pendent vertices n_1, the element (s) of the class $\mathbb{T}(n, n_1)$ uniquely minimize the general Platt index Pl_α.*

Data Availability

Data about this study may be requested from the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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