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Research Article

An Iterative Algorithm for Solving Fixed Point Problems and Quasimonotone Variational Inequalities

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In this paper, we survey a common problem of the fixed point problem and the quasimonotone variational inequality problem in Hilbert spaces. We suggest an iterative algorithm for finding a common element of the solution of a quasimonotone variational inequality and the fixed point of a pseudocontractive operator. Convergence theorems are shown under some mild conditions. Several corollaries are also obtained.

1. Introduction

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H. Let $f: C \longrightarrow H$ be a nonlinear operator. In this paper, our work is closely related to a classical variational inequality of finding a point $x^\dagger \in C$ such that

$$\left\langle f\left(x^\dagger\right), x - x^\dagger \right\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

We use $\mathrm{Sol}(C, f)$ to denote the solution set of (1).

It is well known that variational inequality problems provide a general mathematical framework for a large number of problems arising in optimization [1–8]. For example, constrained optimization problems such as LP and NLP are special cases of variational inequalities, and systems of equations and complementarity problems can be cast as variational inequalities. Thus, variational inequality problems have many applications, including those in transportation networks [9], signal processing [10, 11], regression analysis [12], equilibrium problems [13, 14], fixed point problems [15–19], and complementarity problems [1, 20]. There are numerous iterative algorithms for solving variational inequalities and related problems, (see for examples [21–31]).

Let $\varphi: C \longrightarrow \mathbb{R}$ be a convex function. Letting $f(x) = \nabla \varphi(x)$, the variational inequality (1) is equivalent to the following minimization problem:

$$\min_{x \in C} \varphi(x), \tag{2}$$

which implies that we can use the following projection-gradient algorithm [32–35] to solve variational inequality (1), i.e., an iterative sequence $\{u_n\}$ generated by the recursive form:

$$u_{n+1} = \mathrm{proj}_C\left[u_n - \varsigma_n f(u_n)\right], \tag{3}$$

where $\varsigma_n > 0$ is the step size, and $\mathrm{proj}_C: H \longrightarrow C$ is the metric projection.

The sequence $\{u_n\}$ generated by the projection-gradient algorithm is the convergent provided. f is strongly (pseudo) monotone (see [25, 36]), or f is inverse strongly monotone (see [10, 35]). However, if f is plain monotone, then the sequence $\{u_n\}$ generated by (3) does not necessarily converge. To overcome this flaw, many iterative methods have been proposed, such as the proximal point method [37, 38], Korpelevich's extragradient method [39–41] and its variant forms [42–44], the subgradient extragradient method [45, 46], and Tseng's method [47]. Especially, Bot et al. [48] suggested the following Tseng-type forward-backward-forward algorithm:

$$\begin{cases} v_n = P_C(u_n - \lambda f(u_n)), \\ u_{n+1} = \mu_k(v_n + \lambda(f(u_n) - f(v_n))) + (1-\mu_k)u_n, \quad \forall n \geq 0. \end{cases} \tag{4}$$

Bot et al. [48] proved that the sequence $\{u_n\}$ generated by (4) converges weakly to an element in Sol (C, f) provided f is pseudomonotone and sequentially weakly continuous.

Let $\mathrm{Sol}^d(C, f)$ be the solution set of the dual variational inequality of (1), that is,

$$\mathrm{Sol}^d(C, f) \coloneqq \{u \in C | \langle f(x), x-u \rangle \geq 0, \quad \forall x \in C\}. \tag{5}$$

where $\mathrm{Sol}^d(C, f)$ is the closed convex. If C is convex and f is continuous, then $\mathrm{Sol}^d(C, f) \subset \mathrm{Sol}(C, f)$.

To show the convergence of the sequence $\{u_n\}$, a common condition $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$ has been added, that is,

$$\langle f(x), x-u \rangle \geq 0, \quad \forall u \in \mathrm{Sol}(C, f) \text{ and } x \in C, \tag{6}$$

which is a direct consequence of the pseudomonotonicity of f. But this conclusion (that is, $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$) is false, if f is quasimonotone.

The main purpose of this paper is to introduce a self-adaptive forward-backward-forward algorithm to solve quasimonotone variational inequalities (1) and the fixed point problem of pseudocontractive operators. The algorithm is designed such that the step-sizes are dynamically chosen and its convergence is guaranteed without prior knowledge of the Lipschitz constant of f. We prove that the proposed algorithm converges weakly to a common element of the solution of a quasimonotone variational inequality and the fixed point of a pseudocontractive operator under some additional conditions.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T\colon C \longrightarrow C$ be a nonlinear operator. $\mathrm{Fix}(T)$ is used to denote the set of fixed points of T, i.e., $\mathrm{Fix}(T)\colon = \{x \in C | x = Tx\}$. " $\rightharpoonup$ " and " $\longrightarrow$ " is used to denote weak convergence and strong convergence, respectively. Let $\{u_n\}$ be a sequence in H. $\omega_w(u_n)$ is used to denote the set of all weak cluster points of $\{u_n\}$, i.e., $\omega_w(u_n) = \{u^\dagger\colon \exists \{u_{n_i}\} \subset \{u_n\} \text{ such that } u_{n_i} \rightharpoonup u^\dagger \ (i \longrightarrow \infty)\}$.

Let $f\colon C \longrightarrow H$ be a nonlinear operator. We recall that f is said to be

(i) pseudomonotone if

$$\langle f(x^\dagger), x - x^\dagger \rangle \geq 0 \text{ implies } \langle f(x), x - x^\dagger \rangle \geq 0, \quad \forall x, x^\dagger \in C \tag{7}$$

(ii) quasimonotone if

$$\langle f(x^\dagger), x - x^\dagger \rangle > 0 \text{ implies} \langle f(x), x - x^\dagger \rangle \geq 0, \quad \forall x, x^\dagger \in C \tag{8}$$

(iii) L-Lipschitz continuous if there exists some constant $L > 0$ such that

$$\| f(x) - f(x^\dagger) \| \leq L \| x - x^\dagger \|, \text{for all } x, x^\dagger \in C \tag{9}$$

(iv) sequently weakly continuous if $u_n \rightharpoonup \tilde{x}$ implies that $f(u_n) \rightharpoonup f(\tilde{x})$.

We recall that an operator $T\colon C \longrightarrow C$ is said to be pseudocontractive if

$$\| T(x) - T(x^\dagger) \|^2 \leq \| x - x^\dagger \|^2 + \| (I-T)x - (I-T)x^\dagger \|^2, \tag{10}$$

for all $x, x^\dagger \in C$.

For fixed $x \in H$, there exists a unique $x^\dagger \in C$ satisfying $\|x - x^\dagger\| = \inf\{\|x - \tilde{x}\|\colon \tilde{x} \in C\}$. $x^\dagger$ is denoted by $\mathrm{proj}_C[x]$. The projection proj_C has the following basic property: for given $x \in H$,

$$\langle x - \mathrm{proj}_C[x], y - \mathrm{proj}_C[x] \rangle \leq 0, \quad \forall y \in C. \tag{11}$$

Applying this characteristic inequality, we have the following equivalence relation:

$$x^\dagger \in \mathrm{Sol}(f, C) \Leftrightarrow x^\dagger = \mathrm{proj}_C[x^\dagger - \varsigma f(x^\dagger)], \quad \forall \varsigma > 0. \tag{12}$$

In a Hilbert space H, we have

$$\| \zeta u + (1-\zeta)u^\dagger \|^2 = \zeta \|u\|^2 + (1-\zeta)\|u^\dagger\|^2 - \zeta(1-\zeta)\|u - u^\dagger\|^2, \tag{13}$$

$\forall u, u^\dagger \in H$ and $\forall \zeta \in [0, 1]$.

Lemma 1 (see [44]). *Let C be a nonempty, convex, and closed subset of a Hilbert space H. We assume that $T\colon C \longrightarrow C$ is an L-Lipschitz pseudocontractive operator. Then, for all $\tilde{u} \in C$ and $u^\dagger \in \mathrm{Fix}(T)$, we have*

$$\| u^\dagger - T[(1-\varpi)\tilde{u} + \varpi T(\tilde{u})] \|^2 \leq \| \tilde{u} - u^\dagger \|^2 + (1-\varpi)\| \tilde{u} - T[(1-\varpi)\tilde{u} + \varpi T(\tilde{u})] \|^2, \tag{14}$$

where $0 < \varpi < 1/\sqrt{1+L^2} + 1$.

Lemma 2 (see [14]). *Let C be a nonempty, convex, and closed subset of a Hilbert space H. Let $T\colon C \longrightarrow C$ be a continuous pseudocontractive operator. Then,*

(i) $\mathrm{Fix}(T) \subset C$ is closed and convex

(ii) T is a demiclosedness, i.e., $u_n \rightharpoonup \tilde{z}$ and $T(u_n) \longrightarrow z^{\dagger}$ imply that $T(\tilde{z}) = z^{\dagger}$

3. Main Results

In this section, we introduce our main results. Let C be a nonempty closed convex subset of a real Hilbert space H. We assume that the following conditions are satisfied:

(C1): the operator $f: H \longrightarrow H$ is quasimonotone; κ-Lipschitz continuous and satisfies the following property (P):

$$\left.\begin{array}{l} H \ni x_n \rightharpoonup x^{\dagger} \in H \text{ as } n \longrightarrow \infty \\ \liminf_{n\longrightarrow+\infty} \|f(x_n)\| = 0 \end{array}\right\} \text{imply that } f\left(x^{\dagger}\right) = 0 \tag{15}$$

(C2): the operator $T: H \longrightarrow H$ is pseudocontractive and L-Lipschitz continuous

(C3): $\Gamma: = \mathrm{Sol}^d(C, f) \cap \mathrm{Fix}(T) \neq \emptyset$ and $\{x \in C: f(x) = 0\} \setminus \mathrm{Sol}^d(C, f)$ is a finite set

Remark 1. If the operator f is sequentially weakly continuous, then f satisfies the property (P).

Next, we present an iterative algorithm for finding a common point in Γ. Let $\{\zeta_n\}$, $\{\alpha_n\}$, and $\{\varpi_n\}$ be three sequences in $(0, 1)$. Let $\beta \in (0, 1)$ and $\varsigma_0 > 0$ be two constants.

Algorithm 1. Initialization: let $u_0 \in H$ be an initial guess. We set $n = 0$.

Step 1. Let the n-th iterate u_n be given. We compute

$$\begin{cases} \widehat{v}_n = (1 - \varpi_n) u_n + \varpi_n T(u_n) \\ v_n = (1 - \alpha_n) u_n + \alpha_n T(\widehat{v}_n) \end{cases}. \tag{16}$$

Step 2. Let the n-th step size ς_n be known. We compute

$$w_n = \mathrm{proj}_C [v_n - \varsigma_n f(v_n)], \tag{17}$$

and

$$u_{n+1} = (1 - \zeta_n) v_n + \zeta_n w_n + \zeta_n \varsigma_n [f(v_n) - f(w_n)]. \tag{18}$$

Step 3. We update the $n+1$-th step size by the following form:

$$\varsigma_{n+1} = \begin{cases} \min\left\{\varsigma_n, \dfrac{\beta\|w_n - v_n\|}{\|f(w_n) - f(v_n)\|}\right\}, & \text{if } f(w_n) \neq f(v_n), \\ \varsigma_n, & \textit{else}. \end{cases} \tag{19}$$

We set $n: = n + 1$ and return to step 1.

Based on Algorithm 1, we have the following remark.

Remark 2. (i) By (17), if at some step $w_n = v_n = \mathrm{proj}_C [v_n - \varsigma_n f(v_n)]$, then $v_n \in \mathrm{Sol}(C, f)$. (ii) By (19), $\varsigma_{n+1} \le \varsigma_n$ and $\varsigma_n \ge \min\{\varsigma_0, \beta/\kappa\}$ for all n, so $\lim_{n\longrightarrow\infty} \varsigma_n = \varsigma^{\dagger}$ exists, and $\varsigma^{\dagger} \ge \min\{\varsigma_0, \beta/\kappa\} > 0$.

Next, we prove the convergence of Algorithm 1.

Theorem 1. *Suppose that* $0 < \underline{\alpha} < \alpha_n < \overline{\alpha} < \varpi_n < \overline{\varpi} < 1/\sqrt{1 + L^2} + 1\ (\forall n \ge 0)$ *and* $0 < \liminf_{n\longrightarrow\infty} \zeta_n \le \limsup_{n\longrightarrow\infty} \zeta_n < 1$. *Then, the sequence* $\{u_n\}$ *generated by Algorithm 1 converges weakly to some point in* Γ.

Proof. Let $\tilde{x} \in \Gamma$. Since $\tilde{x} \in \mathrm{Sol}^d(C, f) \subset C$, from (11) and (17), we have

$$\langle w_n - v_n + \varsigma_n f(v_n), w_n - \tilde{x} \rangle \le 0, \tag{20}$$

which yields that

$$\langle w_n - v_n, w_n - \tilde{x} \rangle \le \varsigma_n \langle f(v_n), \tilde{x} - w_n \rangle. \tag{21}$$

Noting that $w_n \in C$ and $\tilde{x} \in \mathrm{Sol}^d(C, f)$, we have

$$\langle f(w_n), \tilde{x} - w_n \rangle \le 0. \tag{22}$$

Combining (21) and (23), we obtain

$$\langle w_n - v_n, w_n - \tilde{x} \rangle + \varsigma_n \langle f(v_n) - f(w_n), w_n - \tilde{x} \rangle \le 0. \tag{23}$$

In Hilbert space H, we have $\langle x - y, x - z \rangle = 1/2 (\|x - y\|^2 + \|x - z\|^2 - \|y - z\|^2)$ for all $x, y, z \in H$. Setting $x = w_n$, $y = v_n$, and $z = \tilde{x}$, we deduce $\langle w_n - v_n, w_n - \tilde{x} \rangle = 1/2 (\|w_n - v_n\|^2 + \|w_n - \tilde{x}\|^2 - \|v_n - \tilde{x}\|^2)$. This together with (1) implies that

$$\begin{aligned} &\frac{1}{2}\left(\|w_n - v_n\|^2 + \|w_n - \tilde{x}\|^2 - \|v_n - \tilde{x}\|^2\right) \\ &\quad + \varsigma_n \langle f(v_n) - f(w_n), w_n - \tilde{x} \rangle \le 0, \end{aligned} \tag{24}$$

and it follows that

$$\begin{aligned} \|w_n - \tilde{x}\|^2 \le{}& \|v_n - \tilde{x}\|^2 - 2\varsigma_n \langle f(v_n) - f(w_n), w_n - \tilde{x} \rangle \\ & - \|w_n - v_n\|^2. \end{aligned} \tag{25}$$

Based on (18), we have

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &= \|(1-\zeta_n)(v_n-\tilde{x})+\zeta_n(w_n-\tilde{x})+\zeta_n\varsigma_n[f(v_n)-f(w_n)]\|^2\\ &= \|(1-\zeta_n)(v_n-\tilde{x})+\zeta_n(w_n-\tilde{x})\|^2+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\langle v_n-\tilde{x}, f(v_n)-f(w_n)\rangle\\ &\quad+2\zeta_n^2\varsigma_n\langle w_n-\tilde{x}, f(v_n)-f(w_n)\rangle.\end{aligned}\tag{26}$$

Using (13) and from (26), we deduce

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &= (1-\zeta_n)\|v_n-\tilde{x}\|+\zeta_n\|w_n-\tilde{x}\|^2-\zeta_n(1-\zeta_n)\|v_n-w_n\|^2\\ &\quad+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2+2\zeta_n^2\varsigma_n\langle w_n-\tilde{x}, f(v_n)-f(w_n)\rangle\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\langle v_n-\tilde{x}, f(v_n)-f(w_n)\rangle.\end{aligned}\tag{27}$$

According to (25) and (27), we obtain

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &\le \|v_n-\tilde{x}\|-\zeta_n(2-\zeta_n)\|v_n-w_n\|^2+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\langle v_n-w_n, f(v_n)-f(w_n)\rangle\\ &\le \|v_n-\tilde{x}\|-\zeta_n(2-\zeta_n)\|v_n-w_n\|^2+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\|v_n-w_n\|\|f(v_n)-f(w_n)\|.\end{aligned}\tag{28}$$

Thanks to (19), $\|f(w_n)-f(v_n)\|\le\beta\|w_n-v_n\|/\varsigma_{n+1}$. This together with (28) implies that

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &\le \|v_n-\tilde{x}\|-\zeta_n(2-\zeta_n)\|v_n-w_n\|^2+\zeta_n^2\beta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}\|w_n-v_n\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\beta\frac{\varsigma_n}{\varsigma_{n+1}}\|v_n-w_n\|^2\\ &= \|v_n-\tilde{x}\|^2-\zeta_n\left[2-\zeta_n-\zeta_n\beta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}-2(1-\zeta_n)\beta\frac{\varsigma_n}{\varsigma_{n+1}}\right]\|v_n-w_n\|^2.\end{aligned}\tag{29}$$

It is noted that $0<\liminf_{n\longrightarrow\infty}\zeta_n\le\limsup_{n\longrightarrow\infty}\zeta_n<1$ and $\lim_{n\longrightarrow\infty}\varsigma_n/\varsigma_{n+1}=1$. Then, we have $\liminf_{n\longrightarrow\infty}\zeta_n[2-\zeta_n-\zeta_n\beta^2\varsigma_n^2/\varsigma_{n+1}^2-2(1-\zeta_n)\beta\varsigma_n/\varsigma_{n+1}]>0$. So, there exists a positive constant θ and a positive integer $\mathcal{N}$ such that when $n\ge\mathcal{N}$,\

$$\zeta_n\left[2-\zeta_n-\zeta_n\beta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}-2(1-\zeta_n)\beta\frac{\varsigma_n}{\varsigma_{n+1}}\right]\ge\theta.\tag{30}$$

In combination with (29), we get

$$\|u_{n+1}-\tilde{x}\|^2\le\|v_n-\tilde{x}\|-\theta\|v_n-w_n\|^2, n\ge\mathcal{N}.\tag{31}$$

By (13) and (16), we obtain

$$\begin{aligned}\|v_n-\tilde{x}\|^2 &= \|(1-\alpha_n)(u_n-\tilde{x})+\alpha_n(T(\hat{v}_n)-\tilde{x})\|^2\\ &= (1-\alpha_n)\|u_n-\tilde{x}\|^2+\alpha_n\|T(\hat{v}_n)-\tilde{x}\|^2\\ &\quad-\alpha_n(1-\alpha_n)\|u_n-T(\hat{v}_n)\|^2.\end{aligned}\tag{32}$$

Using Lemma 1, we have

$$\begin{aligned}\|T(\hat{v}_n)-\tilde{x}\|^2 &= \|T[(1-\varpi_n)u_n+\varpi_n T(u_n)]-\tilde{x}\|^2\\ &\le \|u_n-\tilde{x}\|^2+(1-\varpi_n)\|u_n-T(\hat{v}_n)\|^2.\end{aligned}\tag{33}$$

Substituting (33) into (32), we get

$$\|v_n-\tilde{x}\|^2\le\|u_n-\tilde{x}\|^2+(\alpha_n-\varpi_n)\alpha_n\|u_n-T(\hat{v}_n)\|^2,\tag{34}$$

which results, together with (31), that

$$\|u_{n+1}-\tilde{x}\|^2 \le \|u_n-\tilde{x}\|^2 - (\varpi_n-\alpha_n)\alpha_n\|u_n - T(\hat{v}_n)\|^2 - \theta\|v_n-w_n\|^2, n \ge \mathcal{N}, \tag{35}$$

which implies that

$$(\varpi_n-\alpha_n)\alpha_n\|u_n - T(\hat{v}_n)\|^2 + \theta\|v_n-w_n\|^2 \le \|u_n-\tilde{x}\|^2 - \|u_{n+1}-\tilde{x}\|^2, n \ge \mathcal{N}. \tag{36}$$

By assumption, $\liminf_{n\longrightarrow\infty}(\varpi_n-\alpha_n)\alpha_n > 0$. From (35), we conclude that $\|u_{n+1}-\tilde{x}\| \le \|u_n-\tilde{x}\|, n \ge \mathcal{N}$. Therefore, $\lim_{n\longrightarrow\infty}\|u_n-\tilde{x}\|$ exists, and the sequence $\{u_n\}$ is bounded.

In combination with (36), we derive

$$\lim_{n\longrightarrow\infty}\|u_n - T(\hat{v}_n)\| = 0, \tag{37}$$

$$\lim_{n\longrightarrow\infty}\|v_n - w_n\| = 0. \tag{38}$$

By (16), $v_n - u_n = \alpha_n(T(\hat{v}_n) - u_n)$, it follows from (37) that

$$\lim_{n\longrightarrow\infty}\|v_n - u_n\| = 0. \tag{39}$$

From (38) and the Lipschitz continuity of f, we have

$$\lim_{n\longrightarrow\infty}\|f(v_n) - f(w_n)\| = 0. \tag{40}$$

According to the boundedness of the sequence $\{u_n\}$, we conclude that the sequence $\{v_n\}$ is bounded by (34) and the sequence $\{w_n\}$ is bounded because of $\|w_n\| \le \|v_n\| + \varsigma_n\|f(v_n)\|$ by (17).

Since T is L-Lipschitz continuous, we have

$$\begin{aligned}\|u_n - T(u_n)\| &\le \|u_n - T(\hat{v}_n)\| + \|T(\hat{v}_n) - T(u_n)\|\\ &\le \|u_n - T(\hat{v}_n)\| + L\varpi_n\|u_n - T(u_n)\|.\end{aligned} \tag{41}$$

It follows that

$$\|u_n - T(u_n)\| \le \frac{1}{1-L\varpi_n}\|u_n - T(\hat{v}_n)\|. \tag{42}$$

This together with (37) implies that

$$\lim_{n\longrightarrow\infty}\|u_n - T(u_n)\| = 0. \tag{43}$$

By virtue of (18), (38), and (40), we have

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - v_n\| = 0. \tag{44}$$

Next, we show that $\omega_w(u_n) \subset \Gamma$. Selecting any $x^* \in \omega_w(u_n)$ and letting $\{u_{n_i}\}$ to be a subsequence of $\{u_n\}$ such that $u_{n_i} \rightharpoonup x^*$ as $i \longrightarrow \infty$, from (38) and (39), we have $v_{n_i} \rightharpoonup x^*$ and $w_{n_i} \rightharpoonup x^*$. Taking into account (43) and Lemma 2, we obtain that $x^* \in \mathrm{Fix}(T)$. Next, we show that $x^* \in \mathrm{Sol}(C, f)$. Based on (11) and $w_{n_i} = \mathrm{proj}_C[v_{n_i} - \varsigma_{n_i} f(v_{n_i})]$, we receive

$$\left\langle w_{n_i} - v_{n_i} + \varsigma_{n_i} f(v_{n_i}), w_{n_i} - x^\dagger \right\rangle \le 0, \quad \forall x^\dagger \in C, \tag{45}$$

which yields

$$\begin{aligned}&\frac{1}{\varsigma_{n_i}}\left\langle v_{n_i} - w_{n_i}, u - w_{n_i}\right\rangle + \left\langle f(v_{n_i}), w_{n_i} - v_{n_i}\right\rangle\\ &\quad\le \left\langle f(v_{n_i}), x^\dagger - v_{n_i}\right\rangle, \quad \forall x^\dagger \in C.\end{aligned} \tag{46}$$

Owing to (39), $\lim_{i\longrightarrow\infty}\|v_{n_i} - w_{n_i}\| = 0$. It follows from (46) that

$$\liminf_{i\longrightarrow\infty}\left\langle f(v_{n_i}), x^\dagger - v_{n_i}\right\rangle \ge 0, \quad \forall x^\dagger \in C. \tag{47}$$

There are two possible cases: $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| = 0$ and $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| > 0$.

If $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| = 0$, by $v_{n_i} \rightharpoonup x^*$ and f satisfying (16), we obtain that $f(x^*) = 0$. If $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| > 0$, then there exists an integer $\mathcal{I} > 0$ satisfying $f(v_{n_i}) \ne 0$ for all $i \ge \mathcal{I}$. By (47), we achieve

$$\liminf_{i\longrightarrow+\infty}\left\langle \frac{f(v_{n_i})}{\|f(v_{n_i})\|}, x^\dagger - v_{n_i}\right\rangle \ge 0, \quad \forall x^\dagger \in C. \tag{48}$$

Let $\{\xi_j\}$ be a positive strictly decreasing sequence such that $\xi_j \longrightarrow 0$ as $j \longrightarrow +\infty$. By virtue of (48), there exists a strictly increasing subsequence $\{n_{i_j}\}$ satisfying $n_{i_j} \ge \mathcal{I}$ and $\forall j \ge 0$,

$$\left\langle \frac{f(v_{n_{i_j}})}{\|f(v_{n_{i_j}})\|}, x^\dagger - v_{n_{i_j}}\right\rangle + \xi_j > 0, \quad \forall x^\dagger \in C, \tag{49}$$

which results that

$$\left\langle f(v_{n_{i_j}}), x^\dagger - v_{n_{i_j}}\right\rangle + \xi_j\|f(v_{n_{i_j}})\| > 0, \quad \forall x^\dagger \in C, \forall j \ge 0. \tag{50}$$

We set $\tilde{v}_j = f(v_{n_{i_j}})/\|f(v_{n_{i_j}})\|^2$ for all $j \ge 0$. Then, $\langle f(v_{n_{i_j}}), \tilde{v}_j\rangle = 1$ for each $j \ge 0$. Owing to (50), we have

$$\left\langle f(v_{n_{i_j}}), x^\dagger + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j - v_{n_{i_j}}\right\rangle > 0, \quad \forall x^\dagger \in C, \forall j \ge 0. \tag{51}$$

Since f is quasimonotone on H, by (51), we get

$$\begin{aligned}&\left\langle f\left(x^\dagger + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j\right), x^\dagger + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j - v_{n_{i_j}}\right\rangle \ge 0,\\ &\qquad \forall x^\dagger \in C, \forall j \ge 0.\end{aligned} \tag{52}$$

Since $\lim_{j\longrightarrow+\infty}\xi_j\|f(v_{n_{i_j}})\|\|\tilde{v}_j\| = \lim_{j\longrightarrow+\infty}\xi_j = 0$ and f is Lipschitz continuous, $\lim_{j\longrightarrow\infty} f(x + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j) = f(x)$. Letting $j \longrightarrow +\infty$ in (52), we deduce

$$\left\langle f\left(x^{\dagger}\right), x^{\dagger} - x^{*} \right\rangle \geq 0, \quad \forall x^{\dagger} \in C, \tag{53}$$

which means $x^* \in \mathrm{Sol}^d(C, f)$.

Next, we show that x^* is the unique weak cluster point of $\{u_n\}$ in $\mathrm{Sol}^d(C, f)$. Let $\overline{x} \in \mathrm{Sol}^d(C, f)$ be another weak cluster point of $\{u_n\}$. Then, there exists a sequence $\{u_n\}$ of $\{u_n\}$ satisfying $u_{n_j} \rightharpoonup \overline{x}$ as $j \longrightarrow +\infty$. We note that for all $k \geq 0$,

$$2\langle u_n, x^* - \overline{x}\rangle = \|u_n - \overline{x}\|^2 - \|u_n - x^*\|^2 + \|x^*\|^2 - \|\overline{x}\|^2. \tag{54}$$

We note that $\lim_{n\longrightarrow+\infty}\|u_n - x^*\|$ and $\lim_{n\longrightarrow+\infty}\|u_n - \overline{x}\|$ exist. From (54), $\lim_{n\longrightarrow+\infty}\langle u_n, x^* - \overline{x}\rangle$ exists. Hence,

$$\lim_{i\longrightarrow+\infty}\left\langle u_{n_i}, x^* - \overline{x}\right\rangle = \lim_{j\longrightarrow+\infty}\left\langle u_{n_j}, x^* - \overline{x}\right\rangle. \tag{55}$$

Since $u_{n_i} \rightharpoonup x^*$ and $u_{n_j} \rightharpoonup \overline{x}$, from (55), we have

$$\langle x^*, x^* - \overline{x}\rangle = \langle \overline{x}, x^* - \overline{x}\rangle, \tag{56}$$

which implies that $\|x^* - \overline{x}\|^2 = 0$, and hence $x^* = \overline{x}$. Therefore, $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}^d(C, f)$. By the condition (C3), $\{x \in C, f(x) = 0\} \setminus \mathrm{Sol}^d(C, f)$ is a finite set. Therefore, $\{u_n\}$ has finite weak cluster points in $\mathrm{Sol}(C, f)$ denoted by $q_1, q_2, \ldots, q_m$. We set $N_0 = \{1, 2, \ldots, m\}$ and $\nu = \min\{\|q_j - q_k\|/3, j, k \in N_0, j \neq k\}$. Let q_j, $j \in N_0$ be any weak cluster point in $\mathrm{Sol}(C, f)$ and $\{u^j_{n_i}\}$ be a subsequence of $\{u_n\}$ satisfying $u^j_{n_i} \rightharpoonup q_j$ as $i \longrightarrow +\infty$. Then, we have

$$\lim_{i\longrightarrow+\infty}\left\langle u^j_{n_i}, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle = \left\langle q_j, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle, \quad \forall k \in N_0 \text{ and } k \neq j. \tag{57}$$

By the definition of ν, we have $\forall k \neq j$,

$$\begin{aligned}\left\langle q_j, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle &= \frac{\|q_j - q_k\|}{2} + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|} \\ &> \nu + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|}.\end{aligned} \tag{58}$$

In the light of (57) and (58), there exists an integer int^j_i such that when $i \geq \mathrm{int}^j_i$,

$$u^j_{n_i} \in \left\{x: \left\langle x, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle > \nu + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|}\right\}, \quad k \in N_0, k \neq j. \tag{59}$$

We write

$$Sb_j = \bigcap_{k=1,k\neq j}^{m}\left\{x: \left\langle x, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle > \nu + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|}\right\}. \tag{60}$$

Taking into account (59) and (60), we have $u^j_{n_i} \in Sb_j$ when $i \geq \max\{\mathrm{int}^j_i, j \in N_0\}$.

Now, we show that $u_n \in \cup_{j=1}^m Sb_j$ for a large enough n. If not, there exists a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that $u_{n_l} \notin \cup_{j=1}^m Sb_j$. By the boundedness of $\{u_{n_l}\}$, there exists a subsequence of $\{u_{n_l}\}$ convergent weakly to x^*. Without the loss of generality, we still denote the subsequence as $\{u_{n_l}\}$. According to assumptions, $u_{n_l} \notin \cup_{j=1}^m Sb_j$, so $u_{n_l} \notin Sb_j$ for any $j \in N_0$. Therefore, there exists a subsequence $\{u_{n_{l_s}}\}$ of $\{u_{n_l}\}$ such that when $\forall s \geq 0$,

$$u_{n_{l_s}} \notin \left\{x: \left\langle x, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle > \nu + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|}\right\}, \quad k \in N_0, k \neq j. \tag{61}$$

Thus,

$$x^* \notin \left\{x: \left\langle x, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle > \nu + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|}\right\}, \quad k \in N_0, k \neq j, \tag{62}$$

which implies that $x^* \neq q_j$ and $j \in N_0$. This is impossible. So, for a large enough positive integer N_1, $u_n \in \cup_{j=1}^m Sb_j$ when $n \geq N_1$.

Next, we show that $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}(C, f)$. First, there exists a positive integer $N_2 \geq N_1$ such that $\|u_{n+1} - u_n\| < \nu$ for all $n \geq N_2$. We assume that $\{u_n\}$ has at least two weak cluster points in $\mathrm{Sol}(C, f)$. Then, there exists $\widehat{n} \geq N_2$ such that $u_{\widehat{n}} \in Sb_j$ and $u_{\widehat{n}+1} \in Sb_k$, where $j, k \in N_0$ and $m \geq 2$, that is,

$$u_{\widehat{n}} \in Sb_j = \bigcap_{k=1,k\neq j}^{m}\left\{x: \left\langle x, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle > \nu + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|}\right\}, \tag{63}$$

and

$$u_{\widehat{n}+1} \in Sb_k = \bigcap_{j=1,j\neq k}^{m}\left\{x: \left\langle x, \frac{q_k - q_j}{\|q_k - q_j\|}\right\rangle > \nu + \frac{\|q_k\|^2 - \|q_j\|^2}{2\|q_k - q_j\|}\right\}. \tag{64}$$

Therefore,

$$\left\langle u_{\widehat{n}}, \frac{q_j - q_k}{\|q_j - q_k\|}\right\rangle > \nu + \frac{\|q_j\|^2 - \|q_k\|^2}{2\|q_j - q_k\|}, \tag{65}$$

and

$$\left\langle u_{\widehat{n+1}}, \frac{q_k - q_j}{\|q_k - q_j\|} \right\rangle > \nu + \frac{\|q_k\|^2 - \|q_j\|^2}{2\|q_k - q_j\|}. \tag{66}$$

Combining (65) and (66), we achieve

$$\left\langle u_{\widehat{n}} - u_{\widehat{n+1}}, \frac{q_j - q_k}{\|q_j - q_k\|} \right\rangle > 2\nu. \tag{67}$$

At the same time, we have

$$\|u_{\widehat{n+1}} - u_{\widehat{n}}\| < \nu. \tag{68}$$

Based on (67) and (68), we deduce

$$2\nu < \left\langle u_{\widehat{n}} - u_{\widehat{n+1}}, \frac{q_j - q_k}{\|q_j - q_k\|} g \right\rangle \leq \|u_{\widehat{n}} - u_{\widehat{n+1}}\| < \nu. \tag{69}$$

This leads to a contradiction. Then, $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}(C, f)$. So, the sequence $\{u_n\}$ has the unique weak cluster point $x^* \in \Gamma$. Therefore, the sequence $\{u_n\}$ converges weakly to $x^* \in \Gamma$. This completes the proof. □

Based on Algorithm 1 and Theorem 1, we can obtain the following algorithms and the corresponding corollaries.

Algorithm 2. Initialization: let $u_0 \in H$ be an initial guess. We set $n = 0$.

Step 1. Let the n-th iterate u_n and the n-th step size ς_n be given. We compute

$$w_n = \mathrm{proj}_C[u_n - \varsigma_n f(u_n)], \tag{70}$$

and

$$u_{n+1} = (1 - \zeta_n)u_n + \zeta_n w_n + \zeta_n \varsigma_n [f(u_n) - f(w_n)]. \tag{71}$$

Step 2. We update the $n+1$-th step size by the following form:

$$\varsigma_{n+1} = \begin{cases} \min\left\{\varsigma_n, \dfrac{\beta\|w_n - u_n\|}{\|f(w_n) - f(u_n)\|}\right\}, & \text{if } f(w_n) \neq f(u_n), \\ \varsigma_n, & \text{else.} \end{cases} \tag{72}$$

We set $n: = n + 1$ and return to step 1.

Corollary 1. *We assume that the operator $f: H \longrightarrow H$ is quasimonotone, κ-Lipschitz continuous and satisfies the property (P). Suppose that $Sol^d(C, f) \neq \emptyset$, $\{x \in C: f(x) = 0\} \setminus Sol^d(C, f)$ is a finite set and $0 < \liminf_{n\longrightarrow\infty} \zeta_n \leq \limsup_{n\longrightarrow\infty} \zeta_n < 1$. Then, the sequence $\{u_n\}$ generated by Algorithm 2 converges weakly to some point in $Sol(C, f)$.*

Algorithm 3. Initialization: let $u_0 \in C$ and $\varsigma_0 > 0$. We set $n = 0$.

Step 1. For known u_n, we compute

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T[(1 - \varpi_n)u_n + \varpi_n T(u_n)]. \tag{73}$$

Step 2. We set $n: = n + 1$ and return to step 1.

Corollary 2. *We assume that $T: C \longrightarrow C$ is a pseudocontractive and L-Lipschitz continuous operator. We suppose that $Fix(T) \neq \emptyset$ and $0 < \underline{\alpha} < \alpha_n < \overline{\alpha} < \varpi_n < \overline{\varpi} < 1/\sqrt{1 + L^2} + 1\,(\forall n \geq 0)$. Then, the sequence $\{u_n\}$ generated by Algorithm 3 converges weakly to some point in $Fix(T)$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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