Hindawi
Journal of Mathematics
Volume 2022, Article ID 8686041, 23 pages
https://doi.org/10.1155/2022/8686041

Research Article

The Iterative Method for Generalized Equilibrium Problems and a Finite Family of Lipschitzian Mappings in Hilbert Spaces

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Received 9 February 2022; Revised 1 April 2022; Accepted 7 April 2022; Published 12 May 2022

Academic Editor: Xiaolong Qin

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In this research, we introduced the S-mapping generated by a finite family of contractive mappings, Lipschitzian mappings and finite real numbers using the results of Kangtunyakarn (2013). Then, we prove the strong convergence theorem for fixed point sets of finite family of contraction and Lipschitzian mapping and solution sets of the modified generalized equilibrium problem introduced by Suwannaut and Kangtunyakarn (2014). Finally, numerical examples are provided to illustrate our main theorem.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F: C \times C \longrightarrow \mathbb{R}$ be bifunction. *The equilibrium problem* for F is to determine its equilibrium point, i.e., the set

$$EP(F) = \{x \in C: F(x, y) \geq 0, \quad \forall y \in C\}. \tag{1}$$

Equilibrium problems were first introduced by Muu and Oettli [1] in 1992. It contains various problems such as variational inequality problem, fixed point problem, optimization problem and Nash equilibrium problem. Iterative methods for the equilibrium problems are widely studied, see, for example, [2–9].

If we take $F(x, y) = \langle y - x, Bx\rangle$, where $B: C \longrightarrow H$ is a nonlinear mapping, then the equilibrium problem (1) is equivalent to finding an element $x \in C$ such that

$$\langle y - x, Bx\rangle \geq 0, \quad \forall y \in C, \tag{2}$$

which is well-known as *the variational inequality problem*. The solution set of the problem (2) is denoted by $VI(C, A)$.

Variational inequality problem were first defined and studied by Stampacchia [10] in 1964. The variational inequality theory is an important tool based on studying a wide class of problems such as economics, optimization, operations research and engineering sciences. Several iterative algorithms have been used for solving variational inequality problem and related optimization problems (see [11–15] and the references therein).

Let $CB(H)$ be the family of all nonempty closed bounded subsets of H and $\mathscr{H}(\cdot,\cdot)$ be the Hausdorff metric on $CB(H)$ defined as

$$\mathscr{H}(M, N) = \max\left\{\sup_{m\in M} d(m, N), \sup_{n\in N} d(M, n)\right\}, \quad \forall M, N \in CB(H), \tag{3}$$

where $d(m, N) = \inf_{n\in N} d(m, n), d(M, n) = \inf_{n\in N} d(m, n)$ and $d(m, n) = \|m - n\|$.

A multivalued mapping $V: H \longrightarrow CB(H)$ is said to be $\mathscr{H}$-Lipschitz continuous if there exists a constant $\omega > 0$ such that

$$\mathscr{H}(V(p), V(q)) \leq \omega\|p - q\|, \quad \forall p, q \in C. \tag{4}$$

Let $V: H \longrightarrow CB(H)$ a multi-valued mapping, $\phi: C \longrightarrow \mathbb{R}$ be a real-valued function and $\Psi: H \times C \times C \longrightarrow \mathbb{R}$ an equilibrium-like function, that is, $\Psi(z, x, y) + \Psi(z, y, x) = 0$ for every $(z, x, y) \in H \times C \times C$ satisfying the following properties:

$(H1)\,(z,x)\mapsto\Psi(z,x,y)$ is an upper semicontinuous function from $H\times C\longrightarrow\mathbb{R}$, for all fixed $y\in C$, that is, for $(z,x)\in H\times C$, whenever $z_n\longrightarrow z$ and $x_n\longrightarrow x$ as $n\longrightarrow\infty$,

$$\limsup_{n\longrightarrow\infty}\Psi(z_n,x_n,y)\le\Psi(z,x,y);. \tag{5}$$

$(H2)\,x\mapsto\Psi(z,x,y)$ is a concave function, for all fixed $(z,y)\in H\times C$;

$(H3)\,y\mapsto\Psi(z,x,y)$ is a convex function, for all fixed $(z,x)\in H\times C$.

In 2009, Ceng et al. [16] introduced *the generalized equilibrium problem* (GEP) as follows:

$$(GEP)\begin{cases}\text{Find } x\in C \text{ and } z\in V(x) \text{such that,}\\ \Psi(z,x,y)+\phi(y)-\phi(x)\ge 0,\quad \forall y\in C.\end{cases} \tag{6}$$

Furthermore, $(GEP)_s(\Psi,\phi)$ denotes the solution set of the generalized equilibrium problem.

In 2012, Kangtunyakarn [7] investigated the strong convergence theorem using CQ method for two solution sets of the generalized equilibrium problem (GEP) and fixed point problem of nonlinear mappings.

In 2014, by modifying the generalized equilibrium problem (6), Suwannaut and Kangtunyakarn [17] introduced *the modified generalized equilibrium problem* $(MGEP)$ as follows:

$$(MGEP)\begin{cases}\text{Find } x\in C \text{ and } z\in V(I-\rho A)x,\quad \forall\rho>0,\\ \Psi(z,x,y)+\phi(y)-\phi(x)+\langle y-x,Ax\rangle\ge 0,\quad \forall y\in C.\end{cases} \tag{7}$$

where A is a self-mapping on C. Also, $(MGEP)_s(\Psi,\phi,A)$ represents the solution set of $(MGEP)$. If $A=0$, (7) reduces to (6). They also obtain the strong convergence theorem under some mild conditions.

Definition 1. Let W be a self-mapping on C. Then W is called

(i) nonexpansive if

$$\|Wu-Wv\|\le\|u-v\|,\quad \forall u,v\in C;. \tag{8}$$

(ii) contractive if there exists $\tau\in(0,1)$ such that

$$\|Wu-Wv\|\le\tau\|u-v\|,\quad \forall u,v\in C;. \tag{9}$$

(iii) inverse-strongly monotone if there exists a real number $\omega>0$ such that

$$\langle u-v,Wu-Wv\rangle\ge\omega\|Wu-Wv\|^2,\quad \forall u,v\in C. \tag{10}$$

It is well-known that $I-W$ is demiclosed if W is a nonexpansive mapping, see [18]. Moreover, $\mathrm{Fix}(W)$ is used to represent the set of fixed points of W.

Definition 2. (see [19]). A mapping $W\colon C\longrightarrow C$ is called ν-strictly pseudo-contractive if there exists a constant $\nu\in[0,1)$ such that

$$\|Wu-Wv\|^2\le\|u-v\|^2+\nu\|(I-W)u-(I-W)v\|^2,\quad \forall u,v\in C. \tag{11}$$

Browder and Petryshyn [19] introduced and studied the class of strictly pseudo-contractive mapping as an important generalization of the class of nonexpansive mappings. It is trivial to prove that every nonexpansive mapping is strictly pseudo-contractive.

Definition 3. A mapping $W\colon C\longrightarrow C$ is called L-Lipschitzian if there exists $L>0$ satisfying the following inequality:

$$\|Wu-Wv\|\le L\|u-v\|,\quad \forall u,v\in C. \tag{12}$$

Note that if $0<L<1$, W becomes a contractive mapping. If $L=1$, W is said to be a nonexpansive mapping. In fact, all four classes of mappings mentioned in Definitions 1 and 2 are subclasses of Lipchitzian mapping.

Over the past decades, many mathematicians are interested in studying the fixed point of finite family of nonlinear mappings and their properties, (see [6–8, 17, 20–23]).

In 2009, Kangtunyakarn and Suantai [24] defined K-mapping for a finite family of nonexpansive mappings. Let $K\colon C\longrightarrow C$ be defined by

$$\begin{aligned}U_1&=\lambda_1T_1+(1-\lambda_1)I,\\ U_2&=\lambda_2T_2U_1+(1-\lambda_2)U_1,\\ U_3&=\lambda_3T_3U_2+(1-\lambda_3)U_2,\\ &\vdots\\ U_{N-1}&=\lambda_{N-1}T_{N-1}U_{N-2}+(1-\lambda_{N-1})U_{N-2},\\ K=U_N&=\lambda_NT_NU_{N-1}+(1-\lambda_N)U_{N-1},\end{aligned} \tag{13}$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings and $\lambda_i\in[0,1]$, $i=1,2,\ldots,N$. Moreover, under some control conditions, $\mathrm{Fix}(K)=\cap_{i=1}^N\mathrm{Fix}(T_i)$ and K is a nonexpansive mapping.

Later, Kangtunyakarn and Suantai [6] introduced the S-mapping for a finite family of nonexpansive mappings. Let $S\colon C\longrightarrow C$ be defined by

$$\begin{aligned}U_0&=I,\\ U_1&=\alpha_1^1T_1U_0+\alpha_2^1U_0+\alpha_3^1I,\\ U_2&=\alpha_1^2T_2U_1+\alpha_2^2U_1+\alpha_3^2I,\\ U_3&=\alpha_1^3T_3U_2+\alpha_2^3U_2+\alpha_3^3I,\\ &\vdots\\ U_{N-1}&=\alpha_1^{N-1}T_{N-1}U_{N-2}+\alpha_2^{N-1}U_{N-2}+\alpha_3^{N-1}I,\\ S=U_N&=\alpha_1^NT_NU_{N-1}+\alpha_2^NU_{N-1}+\alpha_3^NI,\end{aligned} \tag{14}$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings and $\alpha_i=(\alpha_1^i,\alpha_2^i,\alpha_3^i)\in I\times I\times I$, where $I=[0,1]$ and $\alpha_1^i+\alpha_2^i+\alpha_3^i=1$ for every $j=1,2,\ldots,N$. Moreover, under some control conditions, $\mathrm{Fix}(S)=\cap_{i=1}^N\mathrm{Fix}(T_i)$ and S is a nonexpansive mapping.

If we take $\alpha_3^j = 0$, $\forall j = 1, 2, \ldots, N$, then the S-mapping reduces to the K-mapping.

In 2013, using the concept of S-mapping, Kangtunyakarn [25] introduced S^A-mapping for a finite family of nonexpansive mappings and strictly pseudo-contractive mappings as follows. Let $S^A: C \longrightarrow C$ be defined by

$$\begin{aligned}
U_0 &= I,\\
U_1 &= T_1\left(\alpha_1^1 W_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\right),\\
U_2 &= T_2\left(\alpha_1^2 W_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\right),\\
U_3 &= T_3\left(\alpha_1^3 W_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\right),\\
&\ \vdots\\
U_{N-1} &= T_{N-1}\left(\alpha_1^{N-1} W_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\right),\\
S^A = U_N &= T_N\left(\alpha_1^N W_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I\right),
\end{aligned} \tag{15}$$

where $\{T_i\}_{i=1}^N: C \longrightarrow C$ be a finite family of nonexpansive mappings and $\{W_i\}_{i=1}^N: C \longrightarrow C$ be a finite family of strictly pseudo-contractive mappings, I is the identity mapping and $\alpha_i = (\alpha_1^i, \alpha_2^i, \alpha_3^i) \in I \times I \times I$, where $I = [0, 1]$ and $\alpha_1^i + \alpha_2^i + \alpha_3^i = 1$ for every $j = 1, 2, \ldots, N$. Also, under some control conditions, $\mathrm{Fix}(S^A) = \cap_{i=1}^N \mathrm{Fix}(T_i) \cap \cap_{i=1}^N \mathrm{Fix}(W_i)$ and S^A is a nonexpansive mapping.

If $T_i \equiv I$, for every $i = 1, 2, \ldots, N$, then the S^A-mapping becomes the S-mapping.

Based on the previous research work, we give our theorem for MGEP and S-mapping for Lipschitzian mappings and some important results as follows:

(i) We first establish Lemmas 2 and 3 showing fixed point results and some properties of S-mapping for Lipschitzian mappings under some control conditions.

(ii) We prove a strong convergence theorem of the sequences generated by iterative scheme for finding a common solution of generalized equilibrium problems and fixed point problem for a finite family of contractive mappings and Lipschitzian mappings.

(iii) We give some illustrative numerical examples supporting our main theorem and our examples show that our main result is not true is some conditions fail. Moreover, the main theorem can be used to approximate the value of pi.

2. Preliminaries

Throughout this work, the notations "$\rightharpoonup$" and "$\longrightarrow$" denote weak convergence and strong convergence, respectively.

Lemma 1 (see [26]). *Let $\{u_n\}$ be a sequence of nonnegative real numbers satisfying*

$$u_{n+1} \le (1 - \beta_n) u_n + \eta_n, \quad \forall n \ge 0, \tag{16}$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\eta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \beta_n = \infty$,

(2) $\limsup_{n \longrightarrow \infty} (\eta_n / \beta_n) \le 0$ or $\sum_{n=1}^{\infty} |\eta_n| < \infty$.

Then, $\lim_{n \longrightarrow \infty} u_n = 0$.

Theorem 1 (see [16]). *Let $\phi: C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. Let $V: H \longrightarrow CB(H)$ be $\mathscr{H}$-Lipschitz continuous with constant ω, and $\Psi: H \times C \times C \longrightarrow \mathbb{R}$ be an equilibrium-like function satisfying $(H1) - (H3)$. Let $t > 0$ be a constant. For each $z \in C$, take $p_z \in T(z)$ arbitrarily and define a mapping $S_t: C \longrightarrow C$ as follows:*

$$S_t(z) = \left\{ u \in C \colon \Psi(p_z, u, v) + \phi(v) - \phi(u) + \frac{1}{t}\langle u - z, v - u \rangle \ge 0, \quad \forall v \in C \right\}. \tag{17}$$

Then, the following hold:

(a) S_t is single-valued;

(b) S_t is firmly nonexpansive (that is, for any $u, v \in C$, $\|S_t u - S_t v\|^2 \le \langle S_t u - S_t v, u - v \rangle$) if

$$\Psi(p_1, S_t(z_1), S_t(z_2)) + \Psi(p_2, S_t(z_2), S_t(z_1)) \le 0. \tag{18}$$

for all $(z_1, z_2) \in C \times C$ and all $p_i \in V(z_i)$, $i = 1, 2$;

(c) $\mathrm{Fix}(S_t) = (GEP)_s(\Psi, \phi)$;

(d) $(GEP)_s(\Psi, \phi)$ is closed and convex.

Definition 4 (see [6]). Let C be a nonempty closed convex subset of a real Banach space. For every $i = 1, 2, \ldots, N$, let $\{T_i\}_{i=1}^N, \{W_i\}_{i=1}^N$ be a finite family of a η_i-contractive mapping and L_i-Lipschitzian mapping of C into itself, respectively, with $L_i \ge 1$ and $\eta_i L_i \le 1$. For every $i = 1, 2, \ldots, N$, let $\alpha_i = (\alpha_1^i, \alpha_2^i, \alpha_3^i) \in I \times I \times I$, where $I = [0, 1]$ and $\alpha_1^i + \alpha_2^i + \alpha_3^i = 1$. Define a mapping $S: C \longrightarrow C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= T_1\left(\alpha_1^1 W_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\right),\\
U_2 &= T_2\left(\alpha_1^2 W_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\right),\\
U_3 &= T_3\left(\alpha_1^3 W_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\right),\\
&\ \vdots\\
U_{N-1} &= T_{N-1}\left(\alpha_1^{N-1} W_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\right),\\
S = U_N &= T_N\left(\alpha_1^N W_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I\right).
\end{aligned} \tag{19}$$

This mapping S is called the S-mapping generated by $T_1, T_2, \ldots, T_N$, $W_1, W_2, \ldots, W_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2. *For every $i = 1, 2, \ldots, N$, $\{T_i\}_{i=1}^N$ be a finite family of a η_i-contractive mapping and $\{W_i\}_{i=1}^N$ be L_i-Lipschitzian mapping of C into itself, respectively, with $L_i \geq 1$, $\eta_i L_i \leq 1$ and $\cap_{i=1}^N Fix(T_i) \cap \cap_{i=1}^N Fix(W_i) \neq \emptyset$. For every $i = 1, 2, \ldots, N$, let $\alpha_i = (\alpha_1^i, \alpha_2^i, \alpha_3^i) \in I \times I \times I$, where $I = [0, 1]$ and $\alpha_1^i + \alpha_2^i + \alpha_3^i = 1$. Let S be the S-mapping generated by $W_1, W_2, \ldots, W_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then there hold the following statement:*

(i) $Fix(S) = \cap_{i=1}^N Fix(T_i) \cap \cap_{i=1}^N Fix(W_i)$;

(ii) S is a nonexpansive mapping.

Proof. First, it is clear that $\cap_{i=1}^N \mathrm{Fix}(T_i) \cap \cap_{i=1}^N \mathrm{Fix}(W_i) \subseteq \mathrm{Fix}(S)$.

Next, claim that $\mathrm{Fix}(S) \subseteq \cap_{i=1}^N \mathrm{Fix}(T_i) \cap \cap_{i=1}^N \mathrm{Fix}(W_i)$.

Let $x \in \mathrm{Fix}(S)$ and $y \in \cap_{i=1}^N \mathrm{Fix}(T_i) \cap \cap_{i=1}^N \mathrm{Fix}(W_i)$. By Definition 4, we have

$$
\begin{aligned}
\|x - y\|^2 &= \|Sx - y\|^2 \\
&= \left\|T_N\left(\alpha_1^N W_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I\right)x - y\right\|^2 \\
&\leq \eta_N \left\|\alpha_1^N (W_N U_{N-1}x - y) + \alpha_2^N (U_{N-1}x - y) + \alpha_3^N (x - y)\right\|^2 \\
&\leq \eta_N \left[\alpha_1^N \left\|W_N U_{N-1}x - y\right\|^2 + \alpha_2^N \left\|U_{N-1}x - y\right\|^2 + \alpha_3^N \|x - y\|^2 - \alpha_1^N \alpha_3^N \left\|W_N U_{N-1}x - x\right\|^2 - \alpha_2^N \alpha_3^N \left\|U_{N-1}x - x\right\|^2\right] \\
&\leq \eta_N \left[\alpha_1^N L_N \left\|U_{N-1}x - y\right\|^2 + \alpha_2^N \left\|U_{N-1}x - y\right\|^2 + \alpha_3^N \|x - y\|^2 - \alpha_1^N \alpha_3^N \left\|W_N U_{N-1}x - x\right\|^2 - \alpha_2^N \alpha_3^N \left\|U_{N-1}x - x\right\|^2\right] \\
&= \eta_N \left(\alpha_1^N L_N + \alpha_2^N\right) \left\|U_{N-1}x - y\right\|^2 + \eta_N \alpha_3^N \|x - y\|^2 \\
&\quad - \eta_N \alpha_1^N \alpha_3^N \left\|W_N U_{N-1}x - x\right\|^2 \\
&\quad - \eta_N \alpha_2^N \alpha_3^N \left\|U_{N-1}x - x\right\|^2 \\
&\quad - \eta_N \alpha_1^N \alpha_3^N \left\|W_N U_{N-1}x - x\right\|^2 - \eta_N \alpha_2^N \alpha_3^N \left\|U_{N-1}x - x\right\|^2 \\
&\leq \left(1 - \alpha_3^N\right) \left\|U_{N-1}x - y\right\|^2 + \alpha_3^N \|x - y\|^2 \\
&\quad - \eta_N \alpha_1^N \alpha_3^N \left\|W_N U_{N-1}x - x\right\|^2 - \eta_N \alpha_2^N \alpha_3^N \left\|U_{N-1}x - x\right\|^2 \\
&\leq \left(1 - \alpha_3^N\right) \left\|T_{N-1}\left(\alpha_1^{N-1} W_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\right)x - y\right\|^2 \\
&\quad + \alpha_3^N \|x - y\|^2 \\
&\leq \left(1 - \alpha_3^N\right) \left[\eta_{N-1} \left\|\alpha_1^{N-1} (W_{N-1} U_{N-2}x - y) + \alpha_2^{N-1} (U_{N-2}x - y) + \alpha_3^{N-1} (x - y)\right\|^2\right] \\
&\quad + \alpha_3^N \|x - y\|^2 \\
&\leq \left(1 - \alpha_3^N\right) \left[\eta_{N-1} \left(\alpha_1^{N-1} \left\|W_{N-1} U_{N-2}x - y\right\|^2 + \alpha_2^{N-1} \left\|U_{N-2}x - y\right\|^2 + \alpha_3^{N-1} \|x - y\|^2 - \alpha_1^{N-1} \alpha_3^{N-1} \left\|W_{N-1} U_{N-2}x - x\right\|^2\right.\right. \\
&\quad \left.\left. - \alpha_2^{N-1} \alpha_3^{N-1} \left\|U_{N-2}x - x\right\|^2\right)\right] \\
&\quad + \alpha_3^N \|x - y\|^2 \\
&\leq \left(1 - \alpha_3^N\right) \left[\eta_{N-1} \left(\alpha_1^{N-1} L_{N-1} \left\|U_{N-2}x - y\right\|^2 + \alpha_2^{N-1} \left\|U_{N-2}x - y\right\|^2 + \alpha_3^{N-1} \|x - y\|^2 - \alpha_1^{N-1} \alpha_3^{N-1} \left\|W_{N-1} U_{N-2}x - x\right\|^2\right.\right. \\
&\quad \left.\left. - \alpha_2^{N-1} \alpha_3^{N-1} \left\|U_{N-2}x - x\right\|^2\right)\right] \\
&\quad + \alpha_3^N \|x - y\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(1-\alpha_3^N\right)\Big[\eta_{N-1}L_{N-1}\left(1-\alpha_3^{N-1}\right)\left\|U_{N-2}x-y\right\|^2+\eta_{N-1}\alpha_3^{N-1}\|x-y\|^2-\eta_{N-1}\alpha_1^{N-1}\alpha_3^{N-1}\left\|W_{N-1}U_{N-2}x-x\right\|^2 \\
&\quad -\eta_{N-1}\alpha_2^{N-1}\alpha_3^{N-1}\left\|U_{N-2}x-x\right\|^2\Big] \\
&\quad +\alpha_3^N\|x-y\|^2 \\
&\leq \left(1-\alpha_3^N\right)\left(1-\alpha_3^{N-1}\right)\left\|U_{N-2}x-y\right\|^2+\left(1-\alpha_3^N\right)\left(1-\left(1-\alpha_3^{N-1}\right)\right)\|x-y\|^2 \\
&\quad -\left(1-\alpha_3^N\right)\eta_{N-1}\alpha_1^{N-1}\alpha_3^{N-1}\left\|W_{N-1}U_{N-2}x-x\right\|^2 \\
&\quad -\left(1-\alpha_3^N\right)\eta_{N-1}\alpha_2^{N-1}\alpha_3^{N-1}\left\|U_{N-2}x-x\right\|^2+\alpha_3^N\|x-y\|^2 \\
&\leq \prod_{i=N-1}^{N}\left(1-\alpha_3^i\right)\left\|U_{N-2}x-y\right\|^2+\left(1-\prod_{i=N-1}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2 \\
&\quad -\left(1-\alpha_3^N\right)\eta_{N-1}\alpha_1^{N-1}\alpha_3^{N-1}\left\|W_{N-1}U_{N-2}x-x\right\|^2 \\
&\quad -\left(1-\alpha_3^N\right)\eta_{N-1}\alpha_2^{N-1}\alpha_3^{N-1}\left\|U_{N-2}x-x\right\|^2 \\
&\leq \prod_{i=N-1}^{N}\left(1-\alpha_3^i\right)\left\|U_{N-2}x-y\right\|^2+\left(1-\prod_{i=N-1}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2 \\
&\ \vdots \\
&\leq \prod_{i=3}^{N}\left(1-\alpha_3^i\right)\left\|U_2x-y\right\|^2+\left(1-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2 \\
&\quad -\prod_{i=4}^{N}\left(1-\alpha_3^i\right)\eta_3\alpha_1^3\alpha_3^3\left\|W_3U_2x-x\right\|^2-\prod_{i=4}^{N}\left(1-\alpha_3^i\right)\eta_3\alpha_2^3\alpha_3^3\left\|U_2x-x\right\|^2 \\
&\leq \prod_{i=3}^{N}\left(1-\alpha_3^i\right)\left\|U_2x-y\right\|^2+\left(1-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2 \\
&\leq \prod_{i=2}^{N}\left(1-\alpha_3^i\right)\left\|U_1x-y\right\|^2+\left(1-\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2 \\
&\quad -\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_1^2\alpha_3^2\left\|W_2U_1x-x\right\|^2-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_2^2\alpha_3^2\left\|U_1x-x\right\|^2 \\
&\leq \prod_{i=1}^{N}\left(1-\alpha_3^i\right)\|x-y\|^2+\left(1-\prod_{i=1}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2 \\
&\quad -\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\eta_1\alpha_1^1\alpha_3^1\left\|W_1x-x\right\|^2 \\
&=\|x-y\|^2-\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\eta_1\alpha_1^1\alpha_3^1\left\|W_1x-x\right\|^2.
\end{aligned}
\tag{20}
$$

From (20), it yields that

$$\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\eta_1\alpha_1^1\alpha_3^1\left\|W_1x-x\right\|^2\leq 0. \tag{21}$$

This implies that $x=W_1x$, that is,

$$x\in \operatorname{Fix}\left(W_1\right), \tag{22}$$

Then, by Definition 4, we obtain

$$\begin{aligned}U_1x&=T_1\left(\alpha_1^1W_1U_0x+\alpha_2^1U_0x+\alpha_3^1x\right)\\&=T_1\left(\alpha_1^1x+\alpha_2^1x+\alpha_3^1x\right)\\&=T_1x.\end{aligned} \tag{23}$$

Again, from (20), we get

$$\begin{aligned}\|x-y\|^2&\leq\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\left\|U_1x-y\right\|^2+\left(1-\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2\\&\quad-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_2^2\alpha_3^2\left\|U_1x-x\right\|^2\\&\leq\|x-y\|^2-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_2^2\alpha_3^2\left\|U_1x-x\right\|^2.\end{aligned} \tag{24}$$

which follows that

$$\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_2^2\alpha_3^2\left\|U_1x-x\right\|^2\leq 0. \tag{25}$$

We deduce that

$$U_1x=x. \tag{26}$$

that is, $x\in\operatorname{Fix}(U_1)$.
From (23) and (26), we have

$$x\in\operatorname{Fix}\left(T_1\right). \tag{27}$$

By (22) and (27), it yields that

$$x\in\operatorname{Fix}\left(T_1\right)\cap\operatorname{Fix}\left(W_1\right). \tag{28}$$

From (20) and (26), we derive that

$$\begin{aligned}\|x-y\|^2&\leq\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\left\|U_1x-y\right\|^2+\left(1-\prod_{i=2}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2\\&\quad-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_1^2\alpha_3^2\left\|W_2U_1x-x\right\|^2\\&=\|x-y\|^2-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_1^2\alpha_3^2\left\|W_2x-x\right\|^2.\end{aligned} \tag{29}$$

which implies that

$$\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\eta_2\alpha_1^2\alpha_3^2\left\|W_2x-x\right\|^2\leq 0, \tag{30}$$

Then we obtain $W_2x=x$, that is,

$$x\in\operatorname{Fix}\left(W_2\right). \tag{31}$$

By the definition of U_2, (26) and (31), we get

$$\begin{aligned}U_2x&=T_2\left(\alpha_1^2W_2U_1x+\alpha_2^2U_1x+\alpha_3^2x\right)\\&=T_2\left(\alpha_1^2W_2x+\alpha_2^2x+\alpha_3^2x\right)\\&=T_2\left(\alpha_1^2x+\alpha_2^2x+\alpha_3^2x\right)\\&=T_2x.\end{aligned} \tag{32}$$

By (20), we have that

$$\begin{aligned}\|x-y\|^2&\leq\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\left\|U_2x-y\right\|^2+\left(1-\prod_{i=3}^{N}\left(1-\alpha_3^i\right)\right)\|x-y\|^2\\&\quad-\prod_{i=4}^{N}\left(1-\alpha_3^i\right)\eta_3\alpha_2^3\alpha_3^3\left\|U_2x-x\right\|^2\\&\leq\|x-y\|^2-\prod_{i=4}^{N}\left(1-\alpha_3^i\right)\eta_3\alpha_2^3\alpha_3^3\left\|U_2x-x\right\|^2.\end{aligned} \tag{33}$$

which follows that

$$\prod_{i=4}^{N}\left(1-\alpha_3^i\right)\eta_3\alpha_2^3\alpha_3^3\left\|U_2x-x\right\|^2\leq 0. \tag{34}$$

Thus, we get

$$U_2x=x. \tag{35}$$

By (32) and (35), we obtain

$$x\in\operatorname{Fix}\left(T_2\right). \tag{36}$$

By (31) and (35), it follows that

$$x\in\operatorname{Fix}\left(T_2\right)\cap\operatorname{Fix}\left(W_2\right). \tag{37}$$

By using the same method described above, we easily obtain that $x\in\operatorname{Fix}(T_i)\cap\operatorname{Fix}(W_i)$ and $U_ix=x$, for each $i=1,2,\ldots,N-1$.
From (20), we obtain

$$\begin{aligned}\|x-y\|^2&\leq\left(1-\alpha_3^N\right)\left\|U_{N-1}x-y\right\|^2+\alpha_3^N\|x-y\|^2\\&\quad-\eta_N\alpha_1^N\alpha_3^N\left\|W_NU_{N-1}x-x\right\|^2\\&\leq\|x-y\|^2-\eta_N\alpha_1^N\alpha_3^N\left\|W_Nx-x\right\|^2.\end{aligned} \tag{38}$$

which implies that

$$\eta_N\alpha_1^N\alpha_3^N\left\|W_Nx-x\right\|^2\leq 0. \tag{39}$$

Hence, we have $W_Nx=x$, that is,

$$x\in\operatorname{Fix}\left(W_N\right). \tag{40}$$

By the definition of U_N and (40), it yields that

$$\begin{aligned} x = Sx = U_N x &= T_N\left(\alpha_1^N W_N U_{N-1}x + \alpha_2^N U_{N-1}x + \alpha_3^N x\right) \\ &= T_N\left(\alpha_1^N W_N x + \alpha_2^N x + \alpha_3^N x\right) \\ &= T_N\left(\alpha_1^N x + \alpha_2^N x + \alpha_3^N x\right) \\ &= T_N x. \end{aligned} \tag{41}$$

which follows that

$$x \in \mathrm{Fix}(T_N). \tag{42}$$

Then, we obtain $x \in \mathrm{Fix}(T_N) \cap \mathrm{Fix}(W_N)$, that is,

$$x \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(W_i). \tag{43}$$

Therefore, we can conclude that

$$\mathrm{Fix}(S) \subseteq \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(W_i). \tag{44}$$

Finally, by applying the similar method of (20), S is a nonexpansive mapping. □

Lemma 3. *For each $i = 1, 2, \ldots, N$, let $\{T_i\}_{i=1}^N$ be a finite family of η_i-contractive mappings and $\{W_i\}_{i=1}^N$ be a finite family of L_i-Lipschitzian mappings of C into itself, respectively, with $\eta_i L_i \leq 1$, $\eta = \max_{k=2,3,\ldots,N} \eta_k$ and $L = \max_{k=2,3,\ldots,N} L_k$. For each $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ satisfying the following conditions: $\alpha_i^{n,j} \longrightarrow \alpha_i^j$ as $n \longrightarrow \infty$, for $i = 1, 3$ and $\sum_{n=1}^{\infty} |\alpha_i^{n+1,j} - \alpha_i^{n,j}| < \infty$, for $i = 1, 3$. For every $n \in \mathbb{N}$, let S and S_n be the S-mapping generated by $W_1, W_2, \ldots, W_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and generated by $W_1, W_2, \ldots, W_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. Then, for any bounded sequences $\{x_n\}$ in C, there hold the following statement:*

(i) $\lim_{n\longrightarrow\infty} \|S_n x_n - S x_n\| = 0$;

(ii) $\sum_{n=1}^{\infty} \|S_n x_{n-1} - S_{n-1} x_{n-1}\| < \infty$.

Proof. Let $\{x_n\}$ be a bounded sequence in C. For fixed $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$, let U_k and $U_{n,k}$ be generated by $W_1, W_2, \ldots, W_N$, $T_1, T_2, \ldots, T_N$ and $W_1, W_2, \ldots, W_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively.

First, we will show that (i) holds. For every $n \in \mathbb{N}$, we get

$$\begin{aligned} \|U_{n,1}x_n - U_1 x_n\| &= \left\|T_1\left(\alpha_1^{n,1} W_1 x_n + \left(1 - \alpha_1^{n,1}\right)x_n\right) - T_1\left(\alpha_1^1 W_1 x_n + \left(1 - \alpha_1^1\right)x_n\right)\right\| \\ &\leq \eta_1 \left\|\left(\alpha_1^{n,1} W_1 x_n + \left(1 - \alpha_1^{n,1}\right)x_n\right) - \left(\alpha_1^1 W_1 x_n + \left(1 - \alpha_1^1\right)x_n\right)\right\| \\ &= \eta_1 \left|\alpha_1^{n,1} - \alpha_1^1\right| \|W_1 x_n - x_n\|. \end{aligned} \tag{45}$$

For $k \in \{2, 3, \ldots, N\}$, we have

$$\begin{aligned} \|U_{n,1}x_n - U_1 x_n\| &= \left\|T_1\left(\alpha_1^{n,1} W_1 x_n + \left(1 - \alpha_1^{n,1}\right)x_n\right) - T_1\left(\alpha_1^1 W_1 x_n + \left(1 - \alpha_1^1\right)x_n\right)\right\| \\ &\leq \eta_1 \left\|\left(\alpha_1^{n,1} W_1 x_n + \left(1 - \alpha_1^{n,1}\right)x_n\right) - \left(\alpha_1^1 W_1 x_n + \left(1 - \alpha_1^1\right)x_n\right)\right\| \\ &= \eta_1 \left|\alpha_1^{n,1} - \alpha_1^1\right| \|W_1 x_n - x_n\| \\ &\leq \eta_k \Big[\alpha_1^{n,k} \|W_k U_{n,k-1} x_n - W_k U_{k-1} x_n\| + \left|\alpha_1^{n,k} - \alpha_1^k\right| \|W_k U_{k-1} x_n\| \\ &\quad \alpha_2^{n,k} \|U_{n,k-1} x_n - U_{k-1} x_n\| + \left|1 - \alpha_1^{n,k} - \alpha_3^{n,k} - 1 + \alpha_1^k + \alpha_3^k\right| \|U_{k-1} x_n\| \\ &\quad + \left|\alpha_3^{n,k} - \alpha_3^k\right| \|x_n\|\Big] \\ &\leq \eta_k \Big[\alpha_1^{n,k} L_k \|U_{n,k-1} x_n - U_{k-1} x_n\| + \left|\alpha_1^{n,k} - \alpha_1^k\right| \|W_k U_{k-1} x_n\| \\ &\quad + \alpha_2^{n,k} \|U_{n,k-1} x_n - U_{k-1} x_n\| + \left(\left|\alpha_1^{n,k} - \alpha_1^k\right| + \left|\alpha_3^{n,k} - \alpha_3^k\right|\right) \|U_{k-1} x_n\| \end{aligned}$$

$$
\begin{aligned}
&+\left|\alpha_3^{n,k}-\alpha_3^k\right|\left\|x_n\right\|\Big]\\
&\le \eta_k\left(L_k+1\right)\left\|U_{n,k-1}x_n-U_{k-1}x_n\right\|+\eta_k\left|\alpha_1^{n,k}-\alpha_1^k\right|\left(\left\|W_kU_{k-1}x_n\right\|+\left\|U_{k-1}x_n\right\|\right)\\
&\quad+\eta_k\left|\alpha_3^{n,k}-\alpha_3^k\right|\left(\left\|U_{k-1}x_n\right\|+\left\|x_n\right\|\right)\\
&\le \eta\left(L+1\right)\left\|U_{n,k-1}x_n-U_{k-1}x_n\right\|+\eta\left|\alpha_1^{n,k}-\alpha_1^k\right|\left(\left\|W_kU_{k-1}x_n\right\|+\left\|U_{k-1}x_n\right\|\right)\\
&\quad+\eta\left|\alpha_3^{n,k}-\alpha_3^k\right|\left(\left\|U_{k-1}x_n\right\|+\left\|x_n\right\|\right).
\end{aligned}
\tag{46}
$$

By (45) and (46), we get

$$
\begin{aligned}
&\left\|S_nx_n-Sx_n\right\|\\
&=\left\|U_{n,N}x_n-U_Nx_n\right\|\\
&\le \eta\left(L+1\right)\left\|U_{n,N-1}x_n-U_{N-1}x_n\right\|\\
&\quad+\eta\left|\alpha_1^{n,N}-\alpha_1^N\right|\left(\left\|W_NU_{N-1}x_n\right\|+\left\|U_{N-1}x_n\right\|\right)\\
&\quad+\eta\left|\alpha_3^{n,N}-\alpha_3^N\right|\left(\left\|U_{N-1}x_n\right\|+\left\|x_n\right\|\right)\\
&\le \eta\left(L+1\right)\Big[\eta\left(L+1\right)\left\|U_{n,N-2}x_n-U_{N-2}x_n\right\|\\
&\quad+\eta\left|\alpha_1^{n,N-1}-\alpha_1^{N-1}\right|\left(\left\|W_{N-1}U_{N-2}x_n\right\|+\left\|U_{N-2}x_n\right\|\right)\\
&\quad+\eta\left|\alpha_3^{n,N-1}-\alpha_3^{N-1}\right|\left(\left\|U_{N-2}x_n\right\|+\left\|x_n\right\|\right)\Big]\\
&\quad+\eta\left|\alpha_1^{n,N}-\alpha_1^N\right|\left(\left\|W_NU_{N-1}x_n\right\|+\left\|U_{N-1}x_n\right\|\right)\\
&\quad+\eta\left|\alpha_3^{n,N}-\alpha_3^N\right|\left(\left\|U_{N-1}x_n\right\|+\left\|x_n\right\|\right)\\
&=\left(\eta\left(L+1\right)\right)^2\left\|U_{n,N-2}x_n-U_{N-2}x_n\right\|\\
&\quad+\eta^2\left(L+1\right)\left|\alpha_1^{n,N-1}-\alpha_1^{N-1}\right|\left(\left\|W_{N-1}U_{N-2}x_n\right\|+\left\|U_{N-2}x_n\right\|\right)\\
&\quad+\eta^2\left(L+1\right)\left|\alpha_3^{n,N-1}-\alpha_3^{N-1}\right|\left(\left\|U_{N-2}x_n\right\|+\left\|x_n\right\|\right)\\
&\quad+\eta\left|\alpha_1^{n,N}-\alpha_1^N\right|\left(\left\|W_NU_{N-1}x_n\right\|+\left\|U_{N-1}x_n\right\|\right)\\
&\quad+\eta\left|\alpha_3^{n,N}-\alpha_3^N\right|\left(\left\|U_{N-1}x_n\right\|+\left\|x_n\right\|\right)\\
&=\left(\eta\left(L+1\right)\right)^2\left\|U_{n,N-2}x_n-U_{N-2}x_n\right\|\\
&\quad+\sum_{j=N-1}^{N}\eta^{N-j+1}\left(L+1\right)^{N-j}\left|\alpha_1^{n,j}-\alpha_1^j\right|\left(\left\|W_jU_{j-1}x_n\right\|+\left\|U_{j-1}x_n\right\|\right)\\
&\quad+\sum_{j=N-1}^{N}\eta^{N-j+1}\left(L+1\right)^{N-j}\left|\alpha_3^{n,j}-\alpha_3^j\right|\left(\left\|U_{j-1}x_n\right\|+\left\|x_n\right\|\right)
\end{aligned}
$$

$$\begin{aligned}
&\vdots \\
&\leq (\eta (L+1))^{N-1}\left\|U_{n,1}x_n - U_1x_n\right\| \\
&\quad + \sum_{j=2}^{N} \eta^{N-j+1}(L+1)^{N-j}\left|\alpha_1^{n,j} - \alpha_1^{j}\right|\left(\left\|W_jU_{j-1}x_n\right\| + \left\|U_{j-1}x_n\right\|\right) \\
&\quad + \sum_{j=2}^{N} \eta^{N-j+1}(L+1)^{N-j}\left|\alpha_3^{n,j} - \alpha_3^{j}\right|\left(\left\|U_{j-1}x_n\right\| + \left\|x_n\right\|\right) \\
&\leq (\eta (L+1))^{N-1}\eta_1\left|\alpha_1^{n,1} - \alpha_1^{1}\right|\left\|W_1x_n - x_n\right\| \\
&\quad + \sum_{j=2}^{N} \eta^{N-j+1}(L+1)^{N-j}\left|\alpha_1^{n,j} - \alpha_1^{j}\right|\left(\left\|W_jU_{j-1}x_n\right\| + \left\|U_{j-1}x_n\right\|\right) \\
&\quad + \sum_{j=2}^{N} \eta^{N-j+1}(L+1)^{N-j}\left|\alpha_3^{n,j} - \alpha_3^{j}\right|\left(\left\|U_{j-1}x_n\right\| + \left\|x_n\right\|\right).
\end{aligned} \tag{47}$$

By (47) and the fact that $\alpha_i^{n,j} \longrightarrow \alpha_i^j$ as $n \longrightarrow \infty$, for every $i = 1, 3$ and $j = 1, 2, \ldots, N$, we can deduce that $\lim_{n\longrightarrow\infty}\|S_nx_n - Sx_n\| = 0$.

Finally, we shall prove that (ii) holds. For any $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\left\|U_{n,1}x_{n-1} - U_{n-1,1}x_{n-1}\right\| \\
&= \left\|T_1\left(\alpha_1^{n,1}W_1x_{n-1} + \left(1-\alpha_1^{n,1}\right)x_{n-1}\right) - T_1\left(\alpha_1^{n-1,1}W_1x_{n-1} + \left(1-\alpha_1^{n-1,1}\right)x_{n-1}\right)\right\| \\
&\leq \eta_1\left\|\left(\alpha_1^{n,1}W_1x_{n-1} + \left(1-\alpha_1^{n,1}\right)x_{n-1}\right) - \left(\alpha_1^{n-1,1}W_1x_{n-1} + \left(1-\alpha_1^{n-1,1}\right)x_{n-1}\right)\right\| \\
&= \eta_1\left\|\left(\alpha_1^{n,1} - \alpha_{n-1,1}^{1}\right)W_1x_{n-1} - \left(\alpha_1^{n,1} - \alpha_1^{n-1,1}\right)x_{n-1}\right\| \\
&= \eta_1\left|\alpha_1^{n,1} - \alpha_1^{n-1,1}\right|\left\|W_1x_{n-1} - x_{n-1}\right\|.
\end{aligned} \tag{48}$$

For $k \in \{2, 3, \ldots, N\}$ and the similar argument as (46), we have

$$\begin{aligned}
&\left\|U_{n,k}x_{n-1} - U_{n-1,k}x_{n-1}\right\| \\
&\leq \eta(L+1)\left\|U_{n,k-1}x_{n-1} - U_{n-1k-1}x_{n-1}\right\| \\
&\quad + \eta\left|\alpha_1^{n,k} - \alpha_1^{n-1,k}\right|\left(\left\|W_kU_{n-1,k-1}x_{n-1}\right\| + \left\|U_{n-1,k-1}x_{n-1}\right\|\right) \\
&\quad + \eta\left|\alpha_3^{n,k} - \alpha_3^{n-1,k}\right|\left(\left\|U_{n-1,k-1}x_{n-1}\right\| + \left\|x_{n-1}\right\|\right).
\end{aligned} \tag{49}$$

From (48), (49) and the same method as (47), we obtain

$$\begin{aligned}
&\left\|S_nx_{n-1} - S_{n-1}x_{n-1}\right\| \\
&\leq (\eta (L+1))^{N-1}\eta_1\left|\alpha_1^{n,1} - \alpha_1^{n-1,1}\right|\left\|W_1x_{n-1} - x_{n-1}\right\| \\
&\quad + \sum_{j=2}^{N} \eta^{N-j+1}(L+1)^{N-j}\left|\alpha_1^{n,j} - \alpha_1^{n-1,j}\right|\left(\left\|W_jU_{n-1,j-1}x_{n-1}\right\| + \left\|U_{n-1,j-1}x_{n-1}\right\|\right) \\
&\quad + \sum_{j=2}^{N} \eta^{N-j+1}(L+1)^{N-j}\left|\alpha_3^{n,j} - \alpha_3^{n-1,j}\right|\left(\left\|U_{n-1,j-1}x_{n-1}\right\| + \left\|x_{n-1}\right\|\right).
\end{aligned} \tag{50}$$

Hence, $\sum_{n=1}^{\infty} |\alpha_i^{n+1,j} - \alpha_i^{n,j}| < \infty$, for every $i = 1, 3$ and $j = 1, 2, \ldots, N$, we have $\sum_{n=1}^{\infty} \|S_n x_{n-1} - S_{n-1} x_{n-1}\| < \infty$. □

3. Strong Convergence Theorem

Theorem 2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of η_i-contractive mappings and $\{W_i\}_{i=1}^N$ be a finite family of L_i-Lipschitzian mappings of C into itself, respectively, with $\eta_i L_i \leq 1$, for all $i = 1, 2, \ldots, N$. Assume that $\Omega: = \cap_{i=1}^N Fix(T_i) \cap \cap_{i=1}^N Fix(W_i) \cap \cap_{i=1}^{\overline{N}} (MGEP)_s (\Psi_i, \phi, A) \neq \varnothing$. For each $i = 1, 2, \ldots, \overline{N}$, $V_i: H \longrightarrow CB(H)$ be $\mathscr{H}$-Lipschitz continuous with coefficients μ_i, $\Psi_i: H \times C \times C \longrightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)–(H3). Let $\phi: C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function and $A: C \longrightarrow C$ be an λ-inverse strongly monotone mapping. For every $n \in \mathbb{N}$, let S_n be the S-mapping generated by $W_1, W_2, \ldots, W_n$, $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [b_1, b_2] \subset [0, 1]$, for all $j = 1, 2, \ldots, N$. For every $i = 1, 2, \ldots, \overline{N}$, let $\{z_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in V_i (I - r_1^i A) x_1$, there exists sequences $\{p_n^i\} \in H$ and $\{z_n\}$, $\{c_n^i\} \subseteq C$ such that*

$$\begin{cases} p_n^i \in V_i\left(I - r_n^i A\right) z_n, \|p_n^i - p_{n+1}^i\| \leq \left(1 + \dfrac{1}{n}\right) \mathscr{H}\left(V_i\left(I - r_n^i A\right) z_n, V_i\left(I - r_{n+1}^i A\right) z_{n+1}\right), \\ \Psi_i\left(p_n^i, c_n^i, y\right) + \phi(y) - \phi\left(c_n^i\right) + \dfrac{1}{r_n^i} \langle c_n^i - z_n, y - c_n^i \rangle + \langle A z_n, y - c_n^i \rangle \geq 0, \quad \forall y \in C, \\ v_n = \sum_{i=1}^{\overline{N}} a_n^i c_n^i, \\ z_{n+1} = \gamma_n f(v_n) + \beta_n z_n + \delta_n S_n v_n, \quad \forall n \geq 1, \end{cases} \tag{51}$$

where $f: C \longrightarrow C$ is a contraction mapping with a constant ξ and $\{\gamma_n\}$, $\{\beta_n\}$, $\{\delta_n\} \subseteq (0, 1)$ with $\gamma_n + \beta_n + \delta_n = 1$, $\forall n \geq 1$. Suppose the following statement are true:

(i) $\lim_{n \longrightarrow \infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $0 < \tau \leq \beta_n, \delta_n \leq \upsilon < 1$;

(iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, *for each* $i = 1, 2, \ldots, \overline{N} - 1$ *and* $0 < \eta \leq a_n^{\overline{N}} \leq 1$ *with* $\sum_{n=1}^{\overline{N}} a_n^i = 1$;

(iv) $0 < \varepsilon \leq r_n^i \leq \omega < 2\lambda$, *for every* $n \in \mathbb{N}$ *and* $i = 1, 2, \ldots, \overline{N}$;

(v) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty$, $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty$, $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$, *for each* $i = 1, 2, \ldots, \overline{N}$ *and* $j = 1, 2, \ldots, N$;

(vi) For each $i = 1, 2, \ldots, \overline{N}$, *there exists* $\rho_i > 0$ *such that*

$$\begin{aligned} &\Psi_i\left(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)\right) + \Psi_i\left(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)\right) \\ &\quad \leq -\rho_i \left\|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\right\|^2, \end{aligned} \tag{52}$$

for every $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in C \times C$ *and* $w_j^i \in V_i(x_j)$, *for* $j = 1, 2$, *where* $\Theta_i = \{r_n^i: n \geq 1\}$.

Then $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$ converge strongly to $z^* \in \Omega$, for all $i = 1, 2, \ldots, \overline{N}$.

Proof. The proof will be splited into six steps. □

Step 1. Claim that $I - r_n^i A$ is nonexpansive, for each $i = 1, 2, \ldots, \overline{N}$.

From (51), we get

$$\begin{aligned} &\Psi_i\left(p_n^i, c_n^i, y\right) + \phi(y) - \phi\left(c_n^i\right) \\ &\quad + \frac{1}{r_n^i} \langle c_n^i - \left(I - r_n^i A\right) z_n, y - c_n^i \rangle \geq 0, \end{aligned} \tag{53}$$

for every $y \in C$ and $i = 1, 2, \ldots, \overline{N}$. From (53) and Theorem 1, it yields that

$$c_n^i = T_{r_n^i}\left(I - r_n^i A\right) z_n, \quad \forall i = 1, 2, \ldots, \overline{N}. \tag{54}$$

Put $r^i \in \Theta_i$ for all $i = 1, 2, \ldots, \overline{N}$. From (52), we have

$$\begin{aligned} &\Psi_i\left(w_1^i, T_{r^i}(x_1), T_{r^i}(x_2)\right) + \Psi_i\left(w_2^i, T_{r^i}(x_2), T_{r^i}(x_1)\right) \\ &\quad \leq -\rho_i \left\|T_{r^i}(x_1) - T_{r^i}(x_2)\right\|^2 \leq 0, \end{aligned} \tag{55}$$

for all $(x_1, x_2) \in C \times C$ and $w_j^i \in V_i(x_j)$, $j = 1, 2$.

From (55), it implies that Theorem 1 holds.

Let $u, v \in C$. Since A is λ-inverse strongly monotone with $r_n^i \in (0, 2\lambda)$, it deduces that

$$\begin{aligned}
&\left\|\left(I-r_n^iA\right)u-\left(I-r_n^iA\right)v\right\|^2\\
&=\left\|u-v-r_n^i\left(Au-Av\right)\right\|^2\\
&=\|u-v\|^2-2r_n^i\langle u-v,Au-Av\rangle+\left(r_n^i\right)^2\|Au-Av\|^2\\
&\le\|u-v\|^2-2\lambda r_n^i\|Au-Av\|^2+\left(r_n^i\right)^2\|Au-Av\|^2\\
&=\|u-v\|^2+r_n^i\left(r_n^i-2\lambda\right)\|Au-Av\|^2\\
&\le\|u-v\|^2.
\end{aligned}\tag{56}$$

Thus $I-r_n^iA$ is a nonexpansive mapping, for all $i=1,2,\ldots,\overline{N}$.

Step 2. Prove that $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$, $\forall i=1,2,\ldots,\overline{N}$ are bounded.

Let $z\in\Omega$. From Theorem 1, observe that

$$\begin{aligned}
\left\|v_n-z\right\|&\le\sum_{i=1}^{\overline{N}}a_n^i\left\|c_n^i-z\right\|\\
&=\sum_{i=1}^{\overline{N}}a_n^i\left\|T_{r_n^i}\left(I-r_n^iA\right)z_n-z\right\|\le\left\|z_n-z\right\|.
\end{aligned}\tag{57}$$

By nonexpansiveness of S_n, we derive that

$$\begin{aligned}
\left\|z_{n+1}-z\right\|&\le\gamma_n\left\|f\left(v_n\right)-z\right\|+\beta_n\left\|z_n-z\right\|+\delta_n\left\|S_nv_n-z\right\|\\
&\le\gamma_n\left(\left\|f\left(v_n\right)-f(z)\right\|+\|f(z)-z\|\right)+\beta_n\left\|z_n-z\right\|+\delta_n\left\|v_n-z\right\|\\
&\le\gamma_n\left(\xi\left\|z_n-z\right\|+\|f(z)-z\|\right)+\beta_n\left\|z_n-z\right\|+\delta_n\left\|z_n-z\right\|\\
&=\left(1-\gamma_n(1-\xi)\right)\left\|z_n-z\right\|+\gamma_n\|f(z)-z\|\\
&\le\max\left\{\left\|x_1-z\right\|,\frac{\|f(z)-z\|}{1-\xi}\right\}.
\end{aligned}\tag{58}$$

By induction, we obtain $\|z_n-z\|\le\max\{\|x_1-z\|,(\|f(z)-z\|/1-\xi)\}$, $\forall n\in\mathbb{N}$. It follows that $\{z_n\}$ is bounded so are $\{v_n\}$ and $\{c_n^i\}$, $\forall i=1,2,\ldots,\overline{N}$.

Step 3. Show that $\lim_{n\longrightarrow\infty}\|z_{n+1}-z_n\|=0$.

By the definition of z_n, we obtain

$$\begin{aligned}
&\left\|z_{n+1}-z_n\right\|\\
&\le\gamma_n\left\|f\left(v_n\right)-f\left(v_{n-1}\right)\right\|+\left|\gamma_n-\gamma_{n-1}\right|\left\|f\left(v_{n-1}\right)\right\|\\
&\quad+\beta_n\left\|z_n-z_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|z_{n-1}\right\|\\
&\quad+\delta_n\left\|S_nv_n-S_nv_{n-1}\right\|+\delta_n\left\|S_nv_{n-1}-S_{n-1}v_{n-1}\right\|\\
&\quad+\left|\delta_n-\delta_{n-1}\right|\left\|S_{n-1}v_{n-1}\right\|\\
&\le\gamma_n\xi\left\|v_n-v_{n-1}\right\|+\left|\gamma_n-\gamma_{n-1}\right|\left\|f\left(v_{n-1}\right)\right\|\\
&\quad+\beta_n\left\|z_n-z_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|z_{n-1}\right\|\\
&\quad+\delta_n\left\|v_n-v_{n-1}\right\|\\
&\quad+\delta_n\left\|S_nv_{n-1}-S_{n-1}v_{n-1}\right\|+\left|\delta_n-\delta_{n-1}\right|\left\|S_{n-1}v_{n-1}\right\|\\
&\le\gamma_n\xi\left[\sum_{i=1}^{\overline{N}}a_n^i\left\|c_n^i-c_{n-1}^i\right\|+\sum_{i=1}^{\overline{N}}\left|a_n^i-a_{n-1}^i\right|\left\|c_{n-1}^i\right\|\right]+\left|\gamma_n-\gamma_{n-1}\right|\left\|f\left(v_{n-1}\right)\right\|\\
&\quad+\beta_n\left\|z_n-z_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|z_{n-1}\right\|\\
&\quad+\delta_n\left[\sum_{i=1}^{\overline{N}}a_n^i\left\|c_n^i-c_{n-1}^i\right\|+\sum_{i=1}^{\overline{N}}\left|a_n^i-a_{n-1}^i\right|\left\|c_{n-1}^i\right\|\right]\\
&\quad+\delta_n\left\|S_nv_{n-1}-S_{n-1}v_{n-1}\right\|+\left|\delta_n-\delta_{n-1}\right|\left\|S_{n-1}v_{n-1}\right\|.
\end{aligned}\tag{59}$$

By using the same method of proof in Step 3 in [17], we obtain

$$\begin{aligned}\|c_n^i - c_{n-1}^i\| &\le \|z_n - z_{n-1}\| + |r_n^i - r_{n-1}^i|\|Az_{n-1}\| \\ &\quad + \frac{1}{\varepsilon}|r_n^i - r_{n-1}^i|\|c_n^i - (I - r_n^i A)z_n\|.\end{aligned} \tag{60}$$

Substitute (60) into (59), we get

$$\begin{aligned}&\|z_{n+1} - z_n\| \\ &\le \gamma_n \xi \left[\sum_{i=1}^{\overline{N}} a_n^i \left(\|z_n - z_{n-1}\| + |r_n^i - r_{n-1}^i|\|Az_{n-1}\| + \frac{1}{\varepsilon}|r_n^i - r_{n-1}^i|\|c_n^i - (I - r_n^i A)z_n\|\right) + \sum_{i=1}^{\overline{N}}|a_n^i - a_{n-1}^i|\|c_{n-1}^i\|\right] + |\gamma_n - \gamma_{n-1}|\|f(v_{n-1})\| \\ &\quad + \beta_n\|z_n - z_{n-1}\| + |\beta_n - \beta_{n-1}|\|z_{n-1}\| \\ &\quad + \delta_n\left[\sum_{i=1}^{\overline{N}} a_n^i \left(\|z_n - z_{n-1}\| + |r_n^i - r_{n-1}^i|\|Az_{n-1}\| + \frac{1}{\varepsilon}|r_n^i - r_{n-1}^i|\|c_n^i - (I - r_n^i A)z_n\|\right) + \sum_{i=1}^{\overline{N}}|a_n^i - a_{n-1}^i|\|c_{n-1}^i\|\right] \\ &\quad + \delta_n\|S_n v_{n-1} - S_{n-1}v_{n-1}\| + |\delta_n - \delta_{n-1}|\|S_{n-1}v_{n-1}\| \\ &\le (1 - \gamma_n(1 - \xi))\|z_n - z_{n-1}\| + 2\sum_{i=1}^{\overline{N}} a_n^i |r_n^i - r_{n-1}^i|\|Az_{n-1}\| \\ &\quad + \frac{2}{\varepsilon}\sum_{i=1}^{\overline{N}} a_n^i |r_n^i - r_{n-1}^i|\|c_n^i - (I - r_n^i A)z_n\| + 2\sum_{i=1}^{\overline{N}}|a_n^i - a_{n-1}^i|\|c_{n-1}^i\| \\ &\quad + |\gamma_n - \gamma_{n-1}|\|f(v_{n-1})\| + |\beta_n - \beta_{n-1}|\|z_{n-1}\| + \|S_n v_{n-1} - S_{n-1}v_{n-1}\| \\ &\quad + |\delta_n - \delta_{n-1}|\|S_{n-1}v_{n-1}\|.\end{aligned} \tag{61}$$

Using the conditions (i), (v), Lemmas 1 and 3 (ii), we obtain

$$\lim_{n\longrightarrow\infty}\|z_{n+1} - z_n\| = 0. \tag{62}$$

Step 4. Claim that $\lim_{n\longrightarrow\infty}\|c_n^i - z_n\| = \lim_{n\longrightarrow\infty}\|S_n v_n - v_n\| = 0$, $\forall i = 1, 2, \ldots, \overline{N}$.

By following the same method as in Step 4 of [17], we deduce that

$$\|c_n^i - z\|^2 \le \|z_n - z\|^2 - \|z_n - c_n^i\|^2 + 2r_n^i\|z_n - c_n^i\|\|Az_n - Az\|, \tag{63}$$

and also

$$\lim_{n\longrightarrow\infty}\|Az_n - Az\| = 0. \tag{64}$$

From the definition of z_n and (63), we derive that

$$\begin{aligned}&\|z_{n+1} - z\|^2 \\ &\le \gamma_n\|f(v_n) - z\|^2 + \beta_n\|z_n - z\|^2 + \delta_n\|S_n v_n - z\|^2 \\ &\le \gamma_n\|f(v_n) - z\|^2 + \beta_n\|z_n - z\|^2 + \delta_n\sum_{i=1}^{\overline{N}} a_n^i\|c_n^i - z\|^2 \\ &\le \gamma_n\|f(v_n) - z\|^2 + \beta_n\|z_n - z\|^2 + \delta_n\sum_{i=1}^{\overline{N}} a_n^i\left(\|z_n - z\|^2 - \|z_n - c_n^i\|^2 + 2r_n^i\|z_n - c_n^i\|\|Az_n - Az\|\right)\end{aligned}$$

$$\leq \gamma_n \|f(v_n) - z\|^2 + \|z_n - z\|^2 - \delta_n \sum_{i=1}^{\overline{N}} a_n^i \|z_n - c_n^i\|^2 + 2\delta_n \sum_{i=1}^{\overline{N}} a_n^i r_n^i \|z_n - c_n^i\| \|Az_n - Az\|. \tag{65}$$

which implies that

$$\begin{aligned}\delta_n \sum_{i=1}^{\overline{N}} a_n^i \|z_n - c_n^i\|^2 &\leq \|z_n - z\|^2 - \|z_{n+1} - z\|^2 + \gamma_n \|f(v_n) - z\|^2 + 2\delta_n \sum_{i=1}^{\overline{N}} a_n^i r_n^i \|z_n - c_n^i\| \|Az_n - Az\| \\ &\leq (\|z_n - z\| + \|z_{n+1} - z\|)\|z_{n+1} - z_n\| + \gamma_n \|f(v_n) - z\|^2 + 2\delta_n \sum_{i=1}^{\overline{N}} a_n^i r_n^i \|z_n - c_n^i\| \|Az_n - Az\|.\end{aligned} \tag{66}$$

From (62), (64) and the conditions (i), (ii), (iii), we get

$$\lim_{n\longrightarrow\infty} \|z_n - c_n^i\| = 0, \quad \text{for all } i = 1, 2, \ldots, \overline{N}. \tag{67}$$

Consider

$$\|v_n - z_n\| = \left\| \sum_{i=1}^{\overline{N}} a_n^i c_n^i - z_n \right\| \leq \sum_{i=1}^{\overline{N}} a_n^i \|c_n^i - z_n\|. \tag{68}$$

Then, by (67), this follows that

$$\lim_{n\longrightarrow\infty} \|v_n - z_n\| = 0. \tag{69}$$

Since

$$\|z_{n+1} - v_n\| \leq \|z_{n+1} - z_n\| + \|z_n - v_n\|, \tag{70}$$

then, from (62) and (69), we obtain

$$\lim_{n\longrightarrow\infty} \|z_{n+1} - v_n\| = 0. \tag{71}$$

By the definition of z_n, we obtain

$$\begin{aligned} z_{n+1} - v_n &= \gamma_n (f(v_n) - v_n) + \beta_n \sum_{i=1}^{\overline{N}} a_n^i (c_n^i - v_n) + \delta_n (S_n v_n - v_n) \\ &= \gamma_n (f(v_n) - v_n) + \beta_n \sum_{i=1}^{\overline{N}} a_n^i ((c_n^i - z_n) + (z_n - v_n)) + \delta_n (S_n v_n - v_n). \end{aligned} \tag{72}$$

From (67), (69) and (71) and the conditions (i) and (ii), we can conclude that

$$\lim_{n\longrightarrow\infty} \|S_n v_n - v_n\| = 0. \tag{73}$$

Step 5. Prove that $\{z_n\}, \{p_n^i\}$ and $\{r_n^i\}$ are Cauchy sequences, for each $i = 1, 2, \ldots, \overline{N}$.

Let $a \in (0, 1)$, by (62), there exists $N \in \mathbb{N}$ such that

$$\|z_{n+1} - z_n\| < a^n, \quad \forall n \geq N. \tag{74}$$

Therefore, for any $n \geq N \in \mathbb{N}$ and $p \in \mathbb{N}$, we derive that

$$\|z_{n+p} - z_n\| \leq \sum_{k=n}^{n+p-1} \|z_{k+1} - z_k\| \leq \sum_{k=n}^{n+p-1} a^k < \sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}. \tag{75}$$

Since $a \in (0, 1)$, we get $\lim_{n\longrightarrow\infty} a^n = 0$. From (79), taking $n \longrightarrow \infty$, we obtain $\{z_n\}$ is a Cauchy sequence in a Hilbert space H. Let $\lim_{n\longrightarrow\infty} z_n = z^*$. Since $V_i: C \longrightarrow CB(H)$ be $\mathscr{H}$-Lipschitz continuous on H with coefficients μ_i, for every $i = 1, 2, \ldots, \overline{N}$, and (51), we have

$$\begin{aligned} \|p_n^i - p_{n+1}^i\| &\leq \left(1 + \frac{1}{n}\right) \mathscr{H}\left(V_i(I - r_n^i A) z_n, V_i(I - r_{n+1}^i A) z_{n+1}\right) \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \left\| (I - r_n^i A) z_n - (I - r_{n+1}^i A) z_{n+1} \right\| \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \Big(\left\| (I - r_n^i A) z_n - (I - r_n^i A) z_{n+1} \right\| + \left\| (I - r_n^i A) z_{n+1} - (I - r_{n+1}^i A) z_{n+1} \right\| \Big) \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \left(\|z_n - z_{n+1}\| + |r_{n+1}^i - r_n^i| \|A z_{n+1}\| \right) \\ &\leq \left(1 + \frac{1}{n}\right) \mu_i \left(\|z_n - z_{n+1}\| + |r_{n+1}^i - r_n^i| M \right), \end{aligned} \tag{76}$$

where $M = \max_{n\in\mathbb{N}}\{\|Az_n\|\}$. From (62), (76) and the condition (vi), we obtain

$$\lim_{n\longrightarrow\infty}\left\|p_n^i - p_{n+1}^i\right\| = 0, \quad \text{for every } i = 1, 2, \ldots, \overline{N}. \tag{77}$$

It is obvious that $\{p_n^i\}$ and $\{r_n^i\}$ are Cauchy sequences in a Hilbert space H, for all $i = 1, 2, \ldots, \overline{N}$. So we let $\lim_{n\longrightarrow\infty} p_n^i = p_i^*$, and $\lim_{n\longrightarrow\infty} r_n^i = r_i^*$ for every $i = 1, 2, \ldots, \overline{N}$.

Next, claim that $p_i^* \in V_i(I - r_i^* A)z^*$, for each $i = 1, 2, \ldots, \overline{N}$.

Because $p_n^i \in V_i(I - r_n^i A)z_n$, we obtain

$$\begin{aligned}
&d\left(p_n^i, V_i(I - r_i^* A)z^*\right)\\
&\le \max\left\{d\left(p_n^i, V_i(I - r_i^* A)z^*\right), \sup_{\widetilde{p}_i\in V_i(I-r_i^*A)z^*} d\left(V_i\left(I - r_n^i A\right)z_n, \widetilde{p}_i\right)\right\}\\
&\le \max\left\{\sup_{\widehat{p}_i\in V_i(I-r_n^iA)z_n} d\left(\widehat{p}_i, V_i(I - r_i^* A)z^*\right), \sup_{\widetilde{p}_i\in V_i(I-r_i^*A)z^*} d\left(V_i\left(I - r_n^i A\right)z_n, \widetilde{p}_i\right)\right\}\\
&= \mathscr{H}\left(V_i\left(I - r_n^i A\right)z_n, V_i(I - r_i^* A)z^*\right), \quad \text{for each } i = 1, 2, \ldots, \overline{N}.
\end{aligned} \tag{78}$$

Since

$$\begin{aligned}
&d\left(p_i^*, V_i(I - r_i^* A)z^*\right)\\
&\le \left\|p_i^* - p_n^i\right\| + d\left(p_n^i, V_i(I - r_i^* A)z^*\right)\\
&\le \left\|p_i^* - p_n^i\right\| + \mathscr{H}\left(V_i\left(I - r_n^i A\right)z_n, S_i(I - r_i^* A)z^*\right)\\
&\le \left\|p_i^* - p_n^i\right\| + \mu_i\left\|\left(I - r_n^i A\right)z_n - (I - r_i^* A)z^*\right\|\\
&= \left\|p_i^* - p_n^i\right\| + \mu_i\left\|(z_n - z^*) - \left(r_n^i A z_n - r_i^* A z^*\right)\right\|.
\end{aligned} \tag{79}$$

taking $n \longrightarrow \infty$, we have

$$d\left(p_i^*, V_i(I - r_i^* A)z^*\right) = 0, \tag{80}$$

which implies that

$$p_i^* \in V_i(I - r_i^* A)z^*, \quad \text{for all } i = 1, 2, \ldots, \overline{N}. \tag{81}$$

Step 6. Finally, show that $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$ converge strongly to $z^* \in \Omega$, for every $i = 1, 2, \ldots, \overline{N}$.

Without loss of generality, we can assume that $z_{n_k} \rightharpoonup z^*$ as $k \longrightarrow \infty$. By (69), we easily obtain that $v_{n_k} \rightharpoonup z^*$ as $k \longrightarrow \infty$.

For each $j = 1, 2, \ldots, N$, $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [b_1, b_2] \subset [0, 1]$, without loss of generality, we may assume that

$$\alpha_i^{n_k,j} \longrightarrow \alpha_i^j \in (0,1) \text{as } k \longrightarrow \infty, \quad \text{for every } i = 1, 2, 3 \text{ and } j = 1, 2, \ldots, N. \tag{82}$$

Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$, $W_1, W_2, \ldots, W_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. By Lemma 2, we have S is nonexpansive and $\mathrm{Fix}(S) = \cap_{i=1}^N \mathrm{Fix}(T_i) \cap \cap_{i=1}^N \mathrm{Fix}(W_i)$.

From Lemma 3 (i), we obtain

$$\lim_{k\longrightarrow\infty}\left\|S_{n_k} v_{n_k} - S v_{n_k}\right\| = 0. \tag{83}$$

Since

$$\left\|v_{n_k} - S v_{n_k}\right\| \le \left\|v_{n_k} - S_{n_k} v_{n_k}\right\| + \left\|S_{n_k} v_{n_k} - S v_{n_k}\right\|, \tag{84}$$

by (73) and (83), we have

$$\lim_{k\longrightarrow\infty}\left\|v_{n_k} - S v_{n_k}\right\| = 0. \tag{85}$$

Since $v_{n_k} \rightharpoonup z^*$ as $n \longrightarrow \infty$, (85) and $I - S$ is demiclosed, we deduce that

$$z^* \in \mathrm{Fix}(S) = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \cap \bigcap_{i=1}^N \mathrm{Fix}(W_i). \tag{86}$$

Finally, we show that $z^* \in \cap_{i=1}^{\overline{N}} (GEP)_s(\Psi_i, \phi, A)$.

Since $z_{n_k} \longrightarrow z^*$ as $k \longrightarrow \infty$ and (67), we get

$$c_{n_k}^i \longrightarrow z^* \text{as } k \longrightarrow \infty, \quad \text{for all } i = 1, 2, \ldots, \overline{N}. \tag{87}$$

From (51), we have

$$\begin{aligned}
&\Psi_i\left(p_{n_k}^i, c_{n_k}^i, y\right) + \phi(y) - \phi\left(c_{n_k}^i\right)\\
&\quad + \frac{1}{r_{n_k}^i}\langle c_{n_k}^i - z_{n_k}, y - c_{n_k}^i\rangle + \langle A z_{n_k}, y - c_{n_k}^i\rangle \ge 0.
\end{aligned} \tag{88}$$

for every $y \in C$ and $i = 1, 2, \ldots, \overline{N}$. From (67), (87), the condition $(H1)$ and the lower semicontinuity of ϕ, we deduce that

$$\Psi_i\left(p_i^*, z^*, y\right) + \phi(y) - \phi(z^*) + \langle A z^*, y - z^*\rangle \ge 0. \tag{89}$$

for every $y \in C$ and $i = 1, 2, \ldots, \overline{N}$, which follows by (81) that

$$z^* \in (GEP)_s(\Psi_i, \phi, A), \quad \text{for every } i = 1, 2, \ldots, \overline{N}. \tag{90}$$

It follows that

$$z^* \in \bigcap_{i=1}^{\overline{N}} (GEP)_s(\Psi_i, \phi, A). \tag{91}$$

From (86) and (91), we have

$$z^* \in \Omega. \tag{92}$$

Therefore, we obtain the sequence $\{z_n\}$ converges strongly to $z^* \in \Omega$. Moreover, from (67) and (69), we have $\{v_n\}$ and $\{c_n^i\}$ converge strongly to $z^* \in \Omega$, for every $i = 1, 2, \ldots, \overline{N}$. This completes the proof.

The following corollary is a direct consequence of Theorem 2. Therefore, the proof is omitted.

Corollary 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of η_i-contractive mappings and $\{W_i\}_{i=1}^N$ be a finite family of L_i-Lipschitzian mappings of C into itself, respectively, with $\eta_i L_i \leq 1$, for all $i = 1, 2, \ldots, N$. Assume that $\Omega := \cap_{i=1}^N Fix(T_i) \cap \cap_{i=1}^N Fix(W_i) \cap \cap_{i=1}^{\overline{N}} (GEP)_s(\Psi_i, \phi) \neq \emptyset$. For each $i = 1, 2, \ldots, \overline{N}$, $V_i: H \longrightarrow CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ_i, $\Psi_i: H \times C \times C \longrightarrow \mathbb{R}$ be equilibrium-like function satisfying (H1)–(H3). Let $\phi: C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. For every $n \in \mathbb{N}$, let S_n be the S-mapping generated by $W_1, W_2, \ldots, W_n$, $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [b_1, b_2] \subset [0, 1]$, for all $j = 1, 2, \ldots, N$. For every $i = 1, 2, \ldots, \overline{N}$, let $\{z_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in V_i(x_1)$, there exists sequences $\{p_n^i\} \in H$ and $\{z_n\}$, $\{c_n^i\} \subseteq C$ such that*

$$\begin{cases} p_n^i \in V_i(z_n), \left\| p_n^i - p_{n+1}^i \right\| \leq \left(1 + \dfrac{1}{n}\right) \mathcal{H}(V_i(z_n), V_i(z_{n+1})), \\ \Psi_i\left(p_n^i, c_n^i, y\right) + \phi(y) - \phi\left(c_n^i\right) + \dfrac{1}{r_n^i} \langle c_n^i - z_n, y - c_n^i \rangle \geq 0, \quad \forall y \in C, \\ v_n = \sum_{i=1}^{\overline{N}} a_n^i c_n^i, \\ z_{n+1} = \gamma_n f(v_n) + \beta_n z_n + \delta_n S_n v_n, \forall n \geq 1, \end{cases} \tag{93}$$

where $f: C \longrightarrow C$ is a contraction mapping with a constant ξ and $\{\gamma_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\gamma_n + \beta_n + \delta_n = 1$, $\forall n \geq 1$. Suppose the following statement are true:

(i) $\lim_{n \longrightarrow \infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $0 < \tau \leq \beta_n, \delta_n \leq \upsilon < 1$;

(iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, for each $i = 1, 2, \ldots, \overline{N} - 1$ and $0 < \eta \leq a_n^{\overline{N}} \leq 1$ with $\sum_{n=1}^{\overline{N}} a_n^i = 1$;

(iv) $0 < \varepsilon \leq r_n^i \leq \omega < 2\lambda$, for every $n \in \mathbb{N}$ and $i = 1, 2, \ldots, \overline{N}$;

(v) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty$, $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty$, $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$, for each $i = 1, 2, \ldots, \overline{N}$ and $j = 1, 2, \ldots, N$;

(vi) For each $i = 1, 2, \ldots, \overline{N}$, there exists $\rho_i > 0$ such that

$$\begin{aligned} &\Psi_i\left(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)\right) + \Psi_i\left(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)\right) \\ &\quad \leq -\rho_i \left\| T_{r_1^i}(x_1) - T_{r_2^i}(x_2) \right\|^2. \end{aligned} \tag{94}$$

for every $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in C \times C$ and $w_j^i \in V_i(x_j)$, for $j = 1, 2$, where $\Theta_i = \{r_n^i: n \geq 1\}$. Then $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$ converge strongly to $z^* \in \Omega$, for all $i = 1, 2, \ldots, \overline{N}$.

In 2014, Suwannaut and Kangtunyakarn [17] introduced the viscosity approximation method for the modified generalized equilibrium problem and a finite family of strictly pseudo-contractive mappings in Hilbert spaces. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mappings with $\mathcal{F}\colon = \cap_{i=1}^{N} F(T_i) \cap \cap_{i=1}^{N} (MGEP)_s(\Psi_i, \phi, A) \neq \varnothing$. For $i = 1, 2, \ldots, \overline{N}$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in V_i(I - r_1^i A)x_1$, there exists sequences $\{p_n^i\} \in H$ and $\{z_n\}$, $\{c_n^i\} \subseteq C$ such that

$$\begin{cases} p_n^i \in V_i\left(I - r_n^i A\right)z_n, \left\|p_n^i - p_{n+1}^i\right\| \leq \left(1 + \frac{1}{n}\right)\mathcal{H}\left(V_i\left(I - r_n^i A\right)z_n, V_i\left(I - r_{n+1}^i A\right)z_{n+1}\right), \\ \Psi_i\left(p_n^i, c_n^i, y\right) + \phi(y) - \phi\left(c_n^i\right) + \frac{1}{r_n^i}\langle c_n^i - z_n, y - c_n^i\rangle + \langle Az_n, y - c_n^i\rangle \geq 0, \quad \forall y \in C, \\ v_n = \sum_{i=1}^{\overline{N}} a_n^i c_n^i, \\ z_{n+1} = \gamma_n f(v_n) + \beta_n z_n + \delta_n K_n v_n, \forall n \geq 1, \end{cases} \tag{95}$$

where K_n is a K-mapping generated by a finite family of strictly pseudo-contractive mappings and real numbers. Then, under some control conditions, the sequences $\{z_n\}$ and $\{c_n^i\}$ converge strongly to $q = P_{\mathcal{F}} f(q)$, for all $i = 1, 2, \ldots, \overline{N}$.

Remark 1. The iterative method (51) is a modification and extension of the iteration (95) as follows:

(1) S-mapping can be reduced to K-mapping. Therefore, K-mapping is a special case of S-mapping.

(2) In this research, a finite family of Lipschitzian mappings is considered instead of using a finite family of strictly pseudo-contractive mappings.

4. Numerical Examples

In this section, we give numerical examples to support our main theorem.

Example 1. Let the mappings $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$, $A\colon \mathbb{R} \longrightarrow \mathbb{R}$, $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} \varphi u &= u^2, \\ Au &= \frac{u}{2}, \\ fu &= \frac{u}{4}, \quad \text{for all } u \in \mathbb{R}. \end{aligned} \tag{96}$$

For $i = 1, 2, \ldots, N$, let $T_i\colon \mathbb{R} \longrightarrow \mathbb{R}$ and $W_i\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} T_i u &= \frac{u}{500i}, \\ W_i u &= 2iu, \quad \text{for every } u \in \mathbb{R}. \end{aligned} \tag{97}$$

For $i = 1, 2, \ldots, \overline{N}$, let $V_i\colon \mathbb{R} \longrightarrow \mathbb{R}$, $\Phi_i\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} V_i u &= \frac{u}{8}, \\ \Phi_i(w, u, v) &= iw(v - u), \quad \text{for each } w, u, v \in \mathbb{R}. \end{aligned} \tag{98}$$

Let $\gamma_n = (1/6n)$, $\beta_n = ((3n-2)/6n)$, $\delta_n = ((3n+1)/6n)$, $r_n = ((5n+2)/(7n+9))$ and $a_n^1 = ((2n+1)/(5n+3))$, $a_n^2 = ((3n+2)/(5n+3))$ for each $n \in \mathbb{N}$. For every $j = 1, 2, \ldots, N$ and let $\alpha_1^j = (3/(5j^2+3))$, $\alpha_2^j = (3j^2/(5j^2+3))$, $\alpha_3^j = (2j^2/(5j^2+3))$. Then, the sequences $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$ converge strongly to 0, for each $i = 1, 2, \ldots, \overline{N}$.

Solution. It is clear that the sequences $\{\gamma_n\}, \{\beta_n\}, \{\delta_n\}$, and $\{r_n\}$ satisfy all the conditions of Theorem 2. It is easy to show that T_i is a contractive mapping with coefficient $\eta_i = (1/5i)$ and W_i is $3i$-Lipschitzian mapping. Since $\alpha_1^j = (3/(5j^2+3))$, $\alpha_2^j = (3j^2/(5j^2+3))$, $\alpha_3^j = (2j^2/(5j^2+3))$, then $\sigma_j = ((3/(5j^2+3)), (3j^2/(5j^2+3)), (2j^2/(5j^2+3)))$ for $j = 1, 2, \ldots, N$. Since S_n is S-mapping generated by $T_1, T_2, \ldots, T_n, W_1, W_2, \ldots, W_n$ and $\sigma_1, \sigma_2, \ldots, \sigma_n$, we obtain

$$\begin{aligned} U_{n,0} z_n &= z_n \\ U_{n,1} z_n &= \frac{1}{5}\left(\frac{9}{8}U_{n,0} + \left(\frac{3}{8}\right)U_{n,0} + \frac{2}{8}\right)z_n \\ U_{n,2} z_n &= \frac{1}{25}\left(\frac{18}{23}U_{n,1} + \left(\frac{12}{23}\right)U_{n,1} + \frac{8}{23}\right)z_n \\ &\vdots \end{aligned}$$

$$\begin{aligned} U_{n,k}z_n &= \frac{1}{5k}\left(\frac{9k}{5k^2+3}U_{n,k-1} + \left(\frac{3k^2}{5k^2+3}\right)U_{n,k-1} + \frac{2k^2}{5k^2+3}\right)z_n \\ &\vdots \\ U_{n,N-1}z_n &= \frac{1}{5(N-1)}\left(\frac{9(N-1)}{5(N-1)^2+3}U_{n,N-2} + \left(\frac{3(N-1)^2}{5(N-1)^2+3}\right)U_{n,N-2} + \frac{2(N-1)^2}{5(N-1)^2+3}\right)z_n \\ S_nz_n = U_{n,N}z_n &= \frac{1}{5N}\left(\frac{9N}{5(N)^2+3}U_{n,N-1} + \left(\frac{3N^2}{5N^2+3}\right)U_{n,N-1} + \frac{2N^2}{5N^2+3}\right)z_n. \end{aligned} \tag{99}$$

From the definition of T_n and W_n, we deduce that

$$\{0\} = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(W_i). \tag{100}$$

For $i = 1, 2, \ldots, \overline{N}$, we have that Φ_i, V_i, φ, A satisfy all conditions in Theorem 2 and

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(W_i) \cap \bigcap_{i=1}^{\overline{N}} (MGEP)_s(\Phi_i, \varphi, A) = \{0\}. \tag{101}$$

Therefore

$$\begin{aligned} 0 &\le \Phi_i\left(p_n^i, c_n^i, y\right) + \varphi(y) - \varphi\left(c_n^i\right) + \frac{1}{r_n^i}\langle y - c_n^i, c_n^i - z_n\rangle + \langle Az_n, y - c_n^i\rangle \\ &= ip_n^i\left(y - c_n^i\right) + y^2 - \left(c_n^i\right)^2 + \frac{1}{r_n^i}\left(y - c_n^i\right)\left(c_n^i - z_n\right) + \frac{z_n}{2}\left(y - c_n^i\right) \\ &\Leftrightarrow \\ 0 &\le r_n^i\left(ip_n^i\left(y - c_n^i\right) + y^2 - \left(c_n^i\right)^2 + \frac{1}{r_n^i}\left(y - c_n^i\right)\left(c_n^i - z_n\right) + \frac{z_n}{2}\left(c_n^i - z_n\right)\right) \\ &= -\left(c_n^i\right)^2 - r_n^i\left(c_n^i\right)^2 - ir_n^ic_n^ip_n^i + c_n^iz_n - \frac{1}{2}r_n^ic_n^iz_n + c_n^iy + ir_n^ip_n^iy - z_ny \\ &\quad + \frac{1}{2}r_n^iz_ny + r_n^iy^2. \end{aligned} \tag{102}$$

Let $G(y) = -(c_n^i)^2 - r_n^i\ (c_n^i)^2 - ir_n^ic_n^ip_n^i + c_n^iz_n - (1/2)r_n^ic_n^iz_n + c_n^iy + ir_n^ip_n^iy - z_ny + (1/2)r_n^iz_ny + r_n^iy^2$, where $G(y)$ is a quadratic function of y with coefficients $a = r_n^i$, $b = c_n^i + ir_n^ip_n^i - z_n + (1/2)r_n^iz_n$, and $c = -(c_n^i)^2 - r_n^i(c_n^i)^2 - ir_n^ic_n^ip_n^i + c_n^iz_n - (1/2)r_n^ic_n^iz_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= \left(c_n^i + ir_n^ip_n^i - z_n + \frac{1}{2}r_n^iz_n\right)^2 \\ &\quad - 4\left(r_n^i\right)\left(-\left(c_n^i\right)^2 - r_n^i\left(c_n^i\right)^2 - ir_n^ic_n^ip_n^i + c_n^iz_n - \frac{1}{2}r_n^ic_n^iz_n\right) \\ &= \left(c_n^i\right)^2 + 4r_n^i\left(c_n^i\right)^2 + 4\left(r_n^i\right)^2\left(c_n^i\right)^2 + 2ir_n^ic_n^ip_n^i + 4i\left(r_n^i\right)^2c_n^ip_n^i + i^2\left(r_n^i\right)^2\left(p_n^i\right)^2 \end{aligned}$$

$$-2c_n^i z_n - 3r_n^i c_n^i z_n + 2\left(r_n^i\right)^2 c_n^i z_n - 2ir_n^i p_n^i z_n + i\left(r_n^i\right)^2 p_n^i z_n + z_n^2 - r_n^i z_n^2$$
$$+\frac{1}{4}\left(\left(r_n^i\right)^2 z_n^2\right) \quad (103)$$
$$=\frac{1}{4}\left(2c_n^i + 4r_n^i c_n^i + 2ir_n^i p_n^i - 2z_n + r_n^i z_n\right)^2.$$

From (102), we have $G(y) \geq 0$, for every $y \in \mathbb{R}$. If $G(y)$ has most one solution in $\mathbb{R}$, thus we have $\Delta \leq 0$. This deduces that

$$c_n^i = \frac{-2ir_n^i p_n^i + 2z_n - r_n^i z_n}{2\left(1 + 2r_n^i\right)}. \quad (104)$$

For each $n \in \mathbb{N}$, we rewrite (51) as follows:

$$\begin{cases} p_n^i = V_i\left(I - r_n^i A\right) z_n \\ c_n^i = \dfrac{-2ir_n^i p_n^i + 2z_n - r_n^i z_n}{2\left(1 + 2r_n^i\right)}, \\ v_n = \sum_{i=1}^{\overline{N}} a_n^i c_n^i, \\ z_{n+1} = \gamma_n f(v_n) + \beta_n z_n + \delta_n S_n v_n, \quad \forall n \in \mathbb{N} \text{ and } i = 1, 2, \ldots, \overline{N}. \end{cases} \quad (105)$$

From Theorem 2, the sequences $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$ generated by (105) converge strongly to 0, for every $i = 1, 2, \ldots, \overline{N}$.

The numerical values of all sequences $\{z_n\}$, $\{v_n\}$ and $\{c_n^1\}$, $\{c_n^2\}$ are shown in Table 1 and Figure 1, where $\overline{N} = 2$ and $n = N = 200$.

Next, we will give an numerical example for the iterative method (95) in the work of Suwannaut and Kangtunyakarn [17].

Example 2. For $i = 1, 2, \ldots, N$, let the mapping $T_i\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$T_i u = -\frac{5}{3}u, \quad \forall u \in \mathbb{R}, \quad (106)$$

and

$$\lambda_i^n = \frac{2n}{250n + i}, \quad \forall n \in \mathbb{N}. \quad (107)$$

Let all parameters and mappings be defined the same as mentioned in Example 1. It is obvious that T_i is 1/4-strictly pseudo-contractive mapping, for each $i = 1, 2, \ldots, N$ and all parameters and mappings satisfy all conditions of Theorem 2 in [17]. Thus, we get

$$\{0\} = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i) \cap \bigcap_{i=1}^{\overline{N}} (MGEP)_s(\Phi_i, \varphi, A). \quad (108)$$

The numerical values of all sequences $\{z_n\}$, $\{v_n\}$ and $\{c_n^1\}$, $\{c_n^2\}$ are shown in Table 2 and Figure 1, where $\overline{N} = 2$ and $n = N = 200$.

Remark 2. From the above numerical results, we can conclude that

(i) Table 1 shows that the sequences $\{c_n^1\}$, $\{c_n^2\}$, $\{v_n\}$ and $\{z_n\}$ converge to 0, where $\{0\} = \cap_{i=1}^{N} \operatorname{Fix}(T_i) \cap \cap_{i=1}^{N} \operatorname{Fix}(W_i) \cap \cap_{i=1}^{N} (MGEP)_s(\Phi, \varphi, A)$ and the convergence of all sequences can be guaranteed by Theorem 2.

(ii) Table 2 shows that the sequences $\{c_n^1\}$, $\{c_n^2\}$, $\{v_n\}$ and $\{z_n\}$ converge to 0, where $\{0\} = \cap_{i=1}^{N} \operatorname{Fix}(T_i) \cap \cap_{i=1}^{N} (MGEP)_s(\Phi, \varphi, A)$ and the convergence of all sequences can be guaranteed by Theorem 2 in [17].

(iii) From Tables 1 and 2, we have that the iterative method (51) converges faster than the iterative method (95).

Similar to Example 1, we give another example for Theorem 2. Moreover, we use this numerical example to approximate the value of π.

Example 3. Let the mappings $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}$, $A\colon \mathbb{R} \longrightarrow \mathbb{R}$, $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\phi u = (u - \pi)^2,$$
$$Au = \frac{u - \pi}{2}, \quad (109)$$
$$fu = \frac{u}{4}, \quad \text{for all } u \in \mathbb{R}.$$

For every $i = 1, 2, \ldots, N$, let $T_i\colon \mathbb{R} \longrightarrow \mathbb{R}$ and $W_i\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$T_i u = \frac{u}{3i + 1} + \frac{3i\pi}{3i + 1},$$
$$W_i u = (i + 1)u - i\pi, \quad \text{for all } u \in \mathbb{R}. \quad (110)$$

For $i = 1, 2, \ldots, \overline{N}$, let $V_i\colon \mathbb{R} \longrightarrow \mathbb{R}$, $\Psi_i\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$V_i u = \frac{x - \pi}{8},$$
$$\Psi_i(w, u, v) = iw(v - u), \quad \text{for each } w, u, v \in \mathbb{R}. \quad (111)$$

Let all parameter sequences be defined as in Example 1. Then, the sequences $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$ converge strongly to π, for each $i = 1, 2, \ldots, \overline{N}$.

Solution. It is clear that the sequences $\{\gamma_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{r_n\}$ satisfy all the conditions of Theorem 2. It is obvious that T_i is a contractive mapping with coefficient

Table 1: The values of all sequences for the iterative method (51).

n	c_n^1	c_n^2	v_n	z_n
1	0.226562	0.192633	0.205356	1.000000
2	0.021764	−0.002027	0.007123	0.175224
3	−0.007967	−0.030669	−0.021841	0.058556
4	−0.017253	−0.039651	−0.030887	0.022469
5	−0.020720	−0.042995	−0.034244	0.009040
⋮	⋮	⋮	⋮	⋮
100	−0.022991	−0.044964	−0.036184	−0.000031
⋮	⋮	⋮	⋮	⋮
196	−0.022985	−0.044950	−0.036168	−0.000016
197	−0.022985	−0.044949	−0.036168	−0.000016
198	−0.022985	−0.044949	−0.036168	−0.000015
199	−0.022985	−0.044949	−0.036168	−0.000015
200	−0.022985	−0.044949	−0.036168	−0.000015

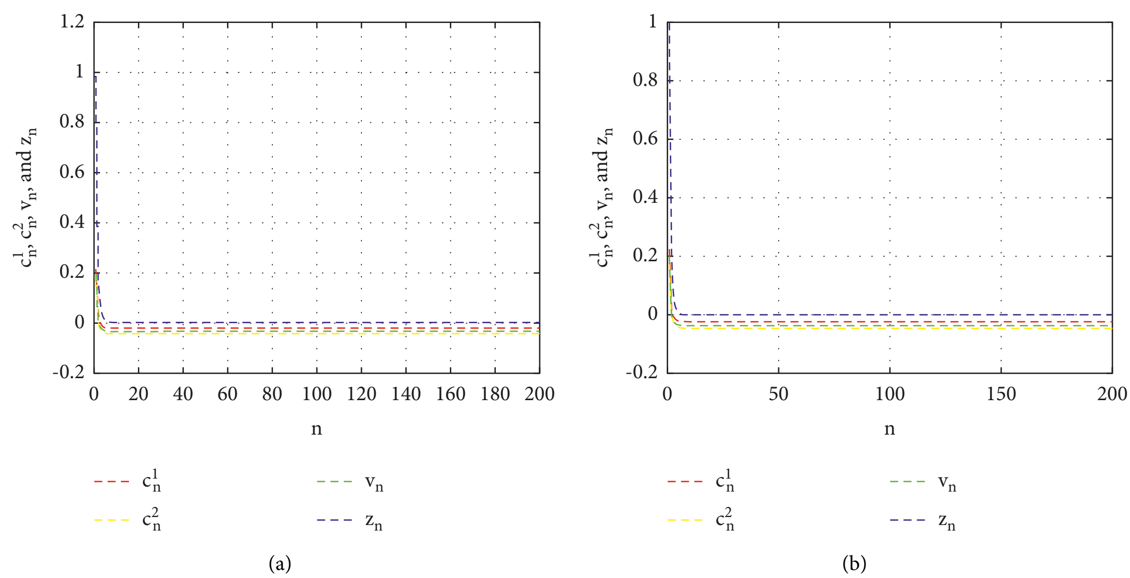

Figure 1: The convergence of all sequences for the iterative method (51) and (95). (a) The iterative method (51) (b) The iterative method (95).

Table 2: The values of all sequences for the iterative method (95).

n	c_n^1	c_n^2	v_n	z_n
1	0.226562	0.192633	0.205356	1.000000
2	0.023261	−0.000569	0.008596	0.181057
3	−0.007420	−0.030133	−0.021300	0.060666
4	−0.017105	−0.039506	−0.030740	0.023036
5	−0.020743	−0.043018	−0.034267	0.008950
⋮	⋮	⋮	⋮	⋮
100	−0.023122	−0.045094	−0.036314	−0.000524
⋮	⋮	⋮	⋮	⋮
196	−0.023114	−0.045079	−0.036298	−0.000505
197	−0.023114	−0.045079	−0.036298	−0.000505
198	−0.023114	−0.045079	−0.036298	−0.000505
199	−0.023114	−0.045079	−0.036297	−0.000505
200	−0.023114	−0.045079	−0.036297	−0.000505

$\eta_i = (1/(3i+1))$ and W_i is $i+1$-Lipschitzian mapping. From the definition of T_n and W_n, we obtain

$$\{\pi\} = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(W_i). \tag{112}$$

For each $i = 1, 2, \ldots, \overline{N}$, it is obvious that Ψ_i, V_i, ϕ, A satisfy all conditions in Theorem 2 and $\pi \in \cap_{i=1}^{\overline{N}} (MGEP)_s (\Psi_i, \phi, A)$. Then we have

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(W_i) \cap \bigcap_{i=1}^{\overline{N}} (MGEP)_s (\Psi_i, \phi, A) = \{\pi\}. \tag{113}$$

Then we deduce that

$$\begin{aligned}
0 &\leq \Psi_i\left(p_n^i, u_n^i, y\right) + \phi(y) - \phi\left(u_n^i\right) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - z_n\rangle + \langle Az_n, y - u_n^i\rangle \\
&= ip_n^i\left(y - u_n^i\right) + (y-\pi)^2 - \left(u_n^i - \pi\right)^2 + \frac{1}{r_n^i}\left(y - u_n^i\right)\left(u_n^i - z_n\right) \\
&\quad + \frac{(z_n - \pi)}{2}\left(y - u_n^i\right) \\
&\Leftrightarrow \\
&= \frac{5\pi r_n^i u_n^i}{2} - \left(u_n^i\right)^2 - r_n^i\left(u_n^i\right)^2 - ir_n^i u_n^i p_n^i + u_n^i z_n - \frac{1}{2} r_n^i u_n^i z_n - \frac{5\pi r_n^i y}{2} + u_n^i y \\
&\quad + ir_n^i p_n^i y - z_n y + \frac{1}{2} r_n^i z_n y + r_n^i y^2.
\end{aligned} \tag{114}$$

Let $G(y) = (5\pi r_n^i u_n^i/2) - (u_n^i)^2 - r_n^i (u_n^i)^2 - ir_n^i u_n^i p_n^i + u_n^i z_n - (1/2) r_n^i u_n^i z_n - (5\pi r_n^i y/2) + u_n^i y + ir_n^i p_n^i y - z_n y + (1/2) r_n^i z_n y + r_n^i y^2$, where $G(y)$ is a quadratic function of y with coefficients $a = r_n^i$, $b = -(5\pi r_n^i/2) + u_n^i y + ir_n^i p_n^i - z_n + (1/2) r_n^i z_n$, and $c = (5\pi r_n^i u_n^i/2) - (u_n^i)^2 - r_n (u_n^i)^2 - ir_n^i u_n^i p_n^i + u_n^i z_n - (1/2) r_n^i u_n^i z_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= \left(-\frac{5\pi r_n^i}{2} + u_n^i y + ir_n^i p_n^i - z_n + \frac{1}{2} r_n^i z_n\right)^2 \\
&\quad - 4\left(r_n^i\right)\left(\frac{5\pi r_n^i u_n^i}{2} - \left(u_n^i\right)^2 - r_n^i\left(u_n^i\right)^2 - ir_n^i u_n^i p_n^i + u_n^i z_n - \frac{1}{2} r_n^i u_n^i z_n\right) \\
&= \frac{25\pi^2\left(r_n^i\right)^2}{4} - 5\pi r_n^i u_n^i - 10\pi\left(r_n^i\right)^2 u_n^i + \left(u_n^i\right)^2 + 4r_n^i\left(u_n^i\right)^2 + 4\left(r_n^i\right)^2\left(u_n^i\right)^2 \\
&\quad + 5i\pi\left(r_n^i\right)^2 p_n^i + 2ir_n^i u_n^i p_n^i + 4i\left(r_n^i\right)^2 u_n^i p_n^i + i^2\left(r_n^i\right)^2\left(p_n^i\right)^2 + 5\pi r_n^i z_n \\
&\quad - \frac{5}{2}\pi\left(r_n^i\right)^2 z_n - 2u_n^i z_n - 3r_n^i u_n^i z_n + 2\left(r_n^i\right)^2 u_n^i z_n - 2ir_n^i p_n^i z_n + i\left(r_n^i\right)^2 p_n^i z_n \\
&\quad + z_n^2 - r_n^i z_n^2 + \frac{1}{4}\left(r_n^i\right)^2 z_n^2. \\
&= \frac{1}{5}\left(5\pi r_n^i - 2u_n^i - 4r_n^i u_n^i - 2ir_n^i p_n^i + 2z_n - r_n^i z_n\right)^2.
\end{aligned} \tag{115}$$

Table 3: The values of $\{c_n^1\}$, $\{c_n^2\}$, $\{v_n\}$ and $\{z_n\}$ with $x_1 = 3$, $\overline{N} = 2$ and $n = N = 20000$.

n	c_n^1	c_n^2	v_n	z_n
1	3.109513	3.114317	3.112516	3.000000
2	3.037897	3.043914	3.041600	2.724918
3	3.057601	3.062415	3.060543	2.804982
4	3.076533	3.080691	3.079064	2.879347
5	3.090648	3.094446	3.092954	2.934169
⋮	⋮	⋮	⋮	⋮
250	3.144346	3.147455	3.146212	3.139700
⋮	⋮	⋮	⋮	⋮
1000	3.144638	3.147747	3.146504	3.140807
⋮	⋮	⋮	⋮	⋮
10000	3.144825	3.147934	3.146691	3.141514
⋮	⋮	⋮	⋮	⋮
19998	3.144836	3.147945	3.146701	3.141553
19999	3.144836	3.147945	3.146701	3.141553
20000	3.144836	3.147945	3.146701	3.141553

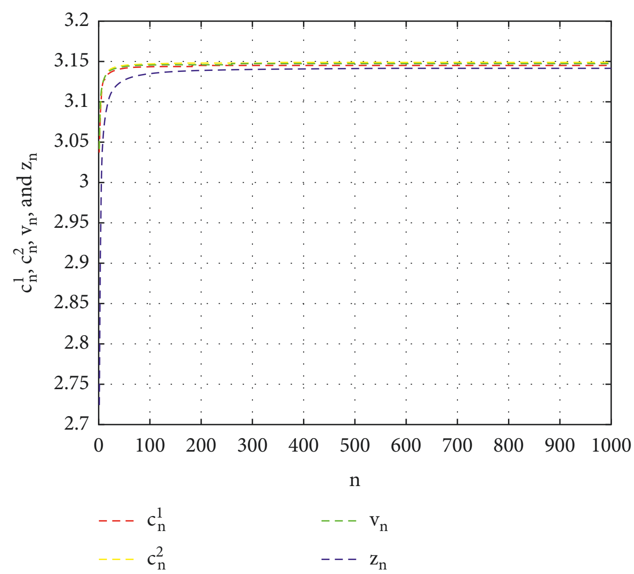

Figure 2: The comparison of sequences $\{c_n^1\}$, $\{c_n^2\}$, $\{v_n\}$ and $\{z_n\}$ with $x_1 = 3$, $\overline{N} = 2$ and $n = N = 1000$.

From (114), we have $G(y) \geq 0$, for every $y \in \mathbb{R}$. If $G(y)$ has most one solution in $\mathbb{R}$, thus we get $\Delta \leq 0$. This yields that

$$c_n^i = \frac{5\pi r_n - 2ir_n^i p_n^i + 2z_n - r_n^i z_n}{2\left(1 + 2r_n^i\right)}. \tag{116}$$

For each $n \in \mathbb{N}$, (51) becomes

$$\begin{cases} p_n^i = V_i\left(I - r_n^i A\right)z_n \\ c_n^i = \dfrac{5\pi r_n - 2ir_n^i p_n^i + 2z_n - r_n^i z_n}{2\left(1 + 2r_n^i\right)}, \\ v_n = \displaystyle\sum_{i=1}^{\overline{N}} a_n^i c_n^i, \\ z_{n+1} = \gamma_n f(v_n) + \beta_n z_n + \delta_n S_n v_n, \quad \forall n \in \mathbb{N} \text{ and } i = 1, 2, \ldots, \overline{N}. \end{cases} \tag{117}$$

From Theorem 2, $\{z_n\}$, $\{v_n\}$ and $\{c_n^i\}$ generated by (117) converge strongly to π, for every $i = 1, 2, \ldots, \overline{N}$.

The numerical values of all sequences $\{z_n\}$, $\{v_n\}$ and $\{c_n^1\}$, $\{c_n^2\}$ are shown in Table 3 and Figure 2, where $\overline{N} = 2$ and $n = N = 20000$.

Remark 3.

(i) From Table 3 and Figure 2, the sequences $\{c_n^1\}$, $\{c_n^2\}$, $\{v_n\}$ and $\{z_n\}$ converge to π, where $\{\pi\} = \cap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \cap_{i=1}^{N} \mathrm{Fix}(W_i) \cap \cap_{i=1}^{N} (MGEP)_s(\Psi, \phi, A)$.

(ii) The convergence of $\{c_n^1\}$, $\{c_n^2\}$, $\{v_n\}$ and $\{z_n\}$ can be guaranteed by Theorem 2.

(iii) Using this as an example, Theorem 2 can be used to approximate the value of π.

5. Conclusion

In this research, we study and analyze the viscosity iterative method for approximating a common solution of the modified generalized equilibrium problems and a common fixed point of a finite family of Lipchitzian mappings. It can be seen as an improvement and modification of some existing algorithms for solving an equilibrium problem and a fixed point problem of Lipchitzian mappings and some related mappings. Some previous research works, for example, [6, 7, 16, 17, 25] can be considered as special cases of Theorem 2. Moreover, some numerical examples for our main theorem are provided.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The two authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Acknowledgments

This research is supported by Lampang Rajabhat University and the Research Administration Division of King Mongkut's Institute of Technology Ladkrabang. No funding to declare.

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