

Research Article

Numerical Analysis of Iterative Fractional Partial Integro-Differential Equations

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Many nonlinear phenomena are modeled in terms of differential and integral equations. However, modeling nonlinear phenomena with fractional derivatives provides a better understanding of processes having memory effects. In this paper, we introduce an effective model of iterative fractional partial integro-differential equations (FPIDEs) with memory terms subject to initial conditions in a Banach space. The convergence, existence, uniqueness, and error analysis are introduced as new theorems. Moreover, an extension of the successive approximations method (SAM) is established to solve FPIDEs in sense of Caputo fractional derivative. Furthermore, new results of stability analysis of solution are also shown.

1. Introduction

Most of the physical phenomena are modeled in ordinary differential equations (ODEs) and partial differential equations (PDEs). During the last decades, it has been noted that modeling complex phenomena, using fractional derivatives, provides a good fit due to their nonlocal nature. Fractional derivatives are effective tools to formulate processes having memory effects. Furthermore, fractional PDEs, which are considered the generalization of PDEs with fractional-order derivatives, have been widely used in many areas of sciences and engineering, and they have been the topics of many workshops and conferences due to their essential uses applied in numerous diverse and widespread fields in applied sciences [1–7]. Furthermore, FPIDEs are applicable in sciences and engineering, and many works in FPIDEs have been introduced (see, for example, [3, 8–11]), while studying iterative FPIDEs is very rare and currently an active area of research due to their particular applications in neural networks. However, iterative FPIDEs are useful tools for modeling the memory properties of various materials and processes, with a

nonlinear relationship to time, such as anomalous diffusion, an elasticity theory, solids mechanic, and other applications [12–14]. The study of the theory of the iterative differential equations began with the work of Eder [15] where Eder worked on a solution of an iterative functional differential equation. Moreover, many studies on iterative differential equations have been conducted (see, for example, [16–18]).

In many physical systems described as models in terms of initial and boundary value problems, it is essential to develop techniques based on various types of successive approximations constructed explicitly in analytic forms. Several analytical and numerical methods for solving differential and integral equations are available in the literature. One of the powerful methods is the successive approximations method (SAM) which was introduced in 1891 by E. Picard, and it has been used to prove the existence and uniqueness of solutions of differential equations [19–22]. The SAM, which is also called the Picard iterative solutions method, has been increasingly applied to solve differential equations and integral equations [23, 24]. The SAM provides an approximate solution in a short series convergent with readily determinable terms [25].

The existence and uniqueness of solutions are proved with initial conditions for various types of iterative differential equations or iterative integro-differential in some works available in the literature, for example, the exact analytical solution for an iterative nonlinear differential equation was given in [26] where the authors studied a second-order nonlinear iterated differential equation, analytic solutions for an iterative differential equation were given in [27] where the authors studied an iterative functional differential equation, Yang and Zhang introduced solutions for iterative differential equations [28], and Zhang et al. [29] introduced the existence of wavefront solutions for an integral differential equation in a nonlinear nonlocal neuronal network. However, few works have been introduced for the stability analysis of solutions for iterative fractional integro-differential equations [30, 31].

This paper presents new analytical and numerical solutions of a new model called “iterative fractional partial integro-differential equation” of iterative Volterra-type equation. This model is solved by using the method of successive approximations. Moreover, the primary advances applied in this paper are very effective with applications of a Banach space and Gronwall–Bellman integral inequality in sense of Caputo derivative. The rest of the paper is organized as follows. Section 2 gives the preliminaries. Section 3 presents the description of the method of successive approximations, existence, uniqueness, convergence, and error analysis of the solution for the proposed model. Section 4 introduces solutions for two types of iterative FPIDEs. Numerical results and discussion are given in Section 5.

2. Preliminaries and Definitions

There are various definitions and theorems of fractional calculus available in the literature. This section presents some of these definitions and theorems that are needed in this paper and can be found in [32–36] and among other references cited therein.

Definition 1. Let $u(x, t): \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ and $n - 1 < \alpha < n \in \mathbb{N}$. The Riemann–Liouville integral of time fractional order α for a function u is defined by

$$\mathcal{I}_t^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(x, \tau) d\tau, \quad (1)$$

where Γ is the well-known gamma function.

Definition 2. Let $u(x, t): \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ and $n - 1 < \alpha < n \in \mathbb{N}$. The Riemann–Liouville time fractional partial derivative of order α for a function u is defined by

$${}^{\text{RL}}\mathcal{D}_t^\alpha u(x, t) = \frac{\partial^n}{\partial t^n} \int_0^t \frac{(t - \tau)^{n-\alpha-1}}{\Gamma(n - \alpha)} u(x, \tau) d\tau. \quad (2)$$

Definition 3. Let $u(x, t): \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ and $n - 1 < \alpha < n \in \mathbb{N}$; then, the Caputo derivative of time fractional order α for a function u is

$$\begin{cases} \mathcal{D}_t^\alpha u(x, t) = \int_0^t \frac{(t - \tau)^{n-\alpha-1}}{\Gamma(n - \alpha)} \frac{\partial^n u(x, \tau)}{\partial \tau^n} d\tau, \\ \mathcal{D}_t^\alpha u(x, t) = \frac{\partial^n u(x, t)}{\partial t^n}, \alpha = n \in \mathbb{N}. \end{cases} \quad (3)$$

Theorem 1 Let $u(x, t): \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ and $n - 1 < \alpha < n \in \mathbb{N}$. Then,

$$\begin{cases} \mathcal{F}_t^\alpha \mathcal{D}_t^\alpha u(x, t) = u(x, t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k u(x, 0^+)}{\partial t^k}, \\ \mathcal{D}_t^\alpha \mathcal{F}_t^\alpha u(x, t) = u(x, t). \end{cases} \quad (4)$$

Theorem 2 Let $\alpha, t \in \mathbb{R}, t > 0$, and $n - 1 < \alpha < n \in \mathbb{N}$. Then,

$$\begin{cases} \mathcal{D}_t^\alpha t^q = \frac{\Gamma(q + 1)}{\Gamma(q - \alpha + 1)} t^{q-\alpha}, \quad n \leq q, q \in \mathbb{R}, \\ \mathcal{D}_t^\alpha t^q = 0, \quad q \leq n - 1. \end{cases} \quad (5)$$

Lemma 1 (Gronwall–Bellman inequality). Let $u(x, t)$ be a nonnegative continuous function on $J \times J, J = [a, a + h], 0 < a, h \in \mathbb{R}$. If $u(x, t) \leq c + \int_a^t f(x, r)u(x, r)dr$ where f is an analytic function and c is a nonnegative constant, then $u(x, t) \leq c \exp(\int_a^t f(x, r)dr)$.

3. Description of the Numerical Scheme

In this section, we introduce an effective model of an iterative fractional partial integro-differential equation with memory term subject to initial value conditions of the following form:

$$\begin{cases} D_t^\alpha u(x, t) = f(x, t) + \int_0^t K(x, r)u(x, u(x, r))dr, \\ \frac{\partial^k u(x, 0)}{\partial t^k} = f_k(x), \quad k = 0, 1, 2, \dots, n - 1, (x, t) \in J \times J, J = [0, T], n - 1 < \alpha < n, \end{cases} \quad (6)$$

where D_t^α is the α -th Caputo fractional partial derivative, $K(x, t)$ is a bivariate kernel, $f(x, t)$ and $f_k(x)$ are known analytic functions, and $u(x, t)$ is the unknown function to be determined.

To find the solution for the iterative fractional partial integro-differential equation (6), we introduce an extension of the SAM as follows. We assume that (6) has an approximate solution given by

$$u_{n+1}(x, t) = u_0 + \int_0^t \frac{(x - \tau)^{\alpha-1}}{\Gamma(\alpha)} \cdot \left(f(x, \tau) + \int_0^\tau K(x, r)u_n(x, u_n(x, r))dr \right) d\tau, \tag{7}$$

for $n = 0, 1, 2, \dots$ where $u_0(x, t)$ is of class C^1 from $[0, T]$ to $[0, T]$ for $|u_0'(x, t)| \leq T$.

Our extension here is that all the components $u_n(x, t)$ are continuous where u_n can be given as a sum of successive differences in the following form:

$$u_n(x, t) = u_0(x, t) + \sum_{k=1}^n (u_k(x, t) - u_{k-1}(x, t)). \tag{8}$$

Next, if $(u_k(x, t) - u_{k-1}(x, t))$ converges, then $u_n(x, t)$ converges and the solution for (6) is given by

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t). \tag{9}$$

Lemma 2. Let a function $u \in C^1([0, T] \times [0, T])$ satisfy (6) on $[0, T] \times [0, T]$; then,

$$u(x, t) = u_0 + \int_0^t \frac{(x - \tau)^{\alpha-1}}{\Gamma(\alpha)} \cdot \left(f(x, \tau) + \int_0^\tau K(x, r)u(x, u(x, r))dr \right) d\tau. \tag{10}$$

3.1. Existence and Uniqueness. This section presents new results for existence and uniqueness of solution for the proposed model (6).

Theorem 3. Suppose that $|u_0 + T^\alpha(N + T^3k_T)/\Gamma(\alpha + 1)| \leq T$ and $0 < M < \Gamma(\alpha + 1)/T^{\alpha+1}k_T - 1 < 1$. Then, there is a unique solution for equation (6).

Proof. Let $B = C([0, T] \times [0, T])$ be a Banach space with a norm $\|u\| = \max_{(x,t) \in \Omega} |u(x, t)|$, $\Omega \subset J \times J$, $J = [0, T]$ and

$$S(\rho) = \left\{ u \in B : 0 \leq u \leq \rho, |u(x, t_1) - u(x, t_2)| \leq M|t_1 - t_2|, \forall t_1, t_2 \in J \right\}, \tag{11}$$

where $\rho = u_0 + T^\alpha(N + T^2k_T)/\Gamma(\alpha + 1)$ and $k_T = \sup\{|K(x, t)| : 0 \leq t \leq T\}$.

Before we apply the Banach contraction principle, we need to define an operator $P: B \rightarrow B$ as

$$P(u(x, t)) = u_0 + \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \cdot \left(f(x, \tau) + \int_0^\tau K(x, r)u(x, u(x, r))dr \right) d\tau. \tag{12}$$

From (12), we have

$$\begin{aligned} 0 \leq |P(u(x, t))| &= \left| u_0 + \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x, \tau) + \int_0^\tau K(x, r)u(x, u(x, r))dr \right) d\tau \right| \\ &\leq u_0 + \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(|f(x, \tau)| + \int_0^\tau |K(x, r)u(x, u(x, r))| dr \right) d\tau \\ &\leq u_0 + \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(|f(x, \tau)| + \int_0^\tau |K(x, r)||u(x, u(x, r))| dr \right) d\tau \leq u_0 + \frac{T^\alpha(N + T^3k_T)}{\Gamma(\alpha + 1)}. \end{aligned} \tag{13}$$

By similar argument, we obtain

$$\begin{aligned}
 |P(u(x, t_1)) - P(u(x, t_2))| &\leq \left| \int_0^{t_1} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x, \tau) + \int_0^\tau K(x, r)u(x, u(x, r))dr \right) d\tau - \int_0^{t_2} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
 &\quad \left. \cdot \left(f(x, \tau) + \int_0^\tau K(x, r)u(x, u(x, r))dr \right) d\tau \right| \\
 &\leq \int_{t_2}^{t_1} \left| \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right| \left(|f(x, \tau)| + \int_0^\tau |K(x, r)||u(x, u(x, r))|dr \right) d\tau \leq \frac{(N + k_T T^3)}{\Gamma(\alpha + 1)} |t_1 - t_2|^\alpha.
 \end{aligned}
 \tag{14}$$

This proves that P is a function from $S(\rho)$ to $S(\rho)$. Next, for $u, v \in S(\rho)$, we have

$$\begin{aligned}
 |P(u(x, t)) - P(v(x, t))| &\leq \int_0^t \frac{(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau (|K(x, r)||u(x, u(x, r)) - v(x, v(x, r))|)dr \right) d\tau \\
 &\leq k_T \int_0^t \frac{(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau (|u(x, u(x, r)) - u(x, v(x, r))| + |u(x, v(x, r)) - v(x, v(x, r))|)dr \right) d\tau \\
 &\leq k_T \int_0^t \frac{(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau (M(|u(x, r) - v(x, r)|) + |u(x, r) - v(x, r)|)dr \right) d\tau \\
 &\leq k_T \int_0^t \frac{(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau (M + 1)|u(x, r) - v(x, r)|dr \right) d\tau \\
 &\leq T k_T (M + 1) \|u - v\| \int_0^t \frac{(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \leq \frac{T^{\alpha+1} k_T (M + 1)}{\Gamma(\alpha + 1)} \|u - v\|.
 \end{aligned}
 \tag{15}$$

Therefore, we obtain

$$\| (P(u(x, t)) - P(v(x, t))) \| \leq \frac{T^{\alpha+1} k_T (M + 1)}{\Gamma(\alpha + 1)} \|u - v\|.
 \tag{16}$$

Since $M < \Gamma(\alpha + 1)/T^{\alpha+1} k_T - 1$ which implies that $T^{\alpha+1} k_T (M + 1)/\Gamma(\alpha + 1) < 1$, then by Banach principle, the

operator P has a unique fixed point. Therefore, equation (6) has a solution. \square

Theorem 4 (convergence).. *If the assumptions of Theorem 3 are proposed, then (7) converges.*

Proof. Define the sequence $S_k = \max_{(x,t) \in J \times J} |u_k(x, t) - u_{k-1}(x, t)|$. Then,

$$\begin{aligned}
 S_0 &= \max_{(x,t) \in J \times J} |u_0(x, t)|, \\
 S_1 &= \max_{(x,t) \in J \times J} |u_1(x, t)| \\
 &= \max_{(x,t) \in J \times J} \left| u_0 + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x, \tau) + \int_0^\tau K(x, r)u_0(x, u_0(x, r))dr \right) d\tau \right| \leq \left| u_0 + \frac{T^\alpha}{\Gamma(\alpha + 1)} (N + T^3 k_T) \right| < T.
 \end{aligned}
 \tag{17}$$

Since u_0 is a function from $[0, T]$ to $[0, T]$, we get $U_1 \leq u_0 + T^\alpha/\Gamma(\alpha + 1)(N + T^3 k_T) \leq T$:

$$\begin{aligned}
 S_2 &= \max_{(x,t) \in J \times J} |u_2(x,t) - u_1(x,t)| \\
 &= \max_{(x,t) \in J \times J} \left| u_0 + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x,\tau) + \int_0^\tau K(x,r)u_1(x,u_1(x,r))dr \right) d\tau - u_0 \right. \\
 &\quad \left. - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x,\tau) + \int_0^\tau K(x,r)u_0(x,u_0(x,r))dr \right) d\tau \right| \\
 &= \max_{(x,t) \in J \times J} \left| \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau K(x,r) (u_1(x,u_1(x,r)) - u_0(x,u_0(x,r))) dr \right) d\tau \right| \\
 &\leq \max_{(x,t) \in J \times J} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \|K(x,r) (u_1(x,u_1(x,r)) - u_0(x,u_0(x,r)))\| dr \right) d\tau \leq TS_1 \leq T^2,
 \end{aligned} \tag{18}$$

$$\begin{aligned}
 S_3 &= \max_{(x,t) \in J \times J} |u_3(x,t) - u_2(x,t)| \max_{(x,t) \in J \times J} \\
 &\quad \cdot \left| u_0 + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x,\tau) + \int_0^\tau K(x,r)u_2(x,u_2(x,r))dr \right) d\tau - u_0 \right. \\
 &\quad \left. - \int_0^t \frac{(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x,\tau) + \int_0^\tau K(x,r)u_1(x,u_1(x,r))dr \right) d\tau \right| \\
 &= \max_{(x,t) \in J \times J} \left| \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau K(x,r) (u_2(x,u_2(x,r)) - u_1(x,u_1(x,r))) dr \right) d\tau \right| \\
 &\leq \max_{(x,t) \in J \times J} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau \|K(x,r) (u_2(x,u_2(x,r)) - u_1(x,u_1(x,r)))\| dr \right) d\tau \leq TS_2 \leq T^3.
 \end{aligned} \tag{19}$$

By induction, we have $S_k \leq T^k$. Since $|u_0 + T^\alpha (N + T^3 k_T) / \Gamma(\alpha + 1)| \leq T$, we get $T < 1$ when $u_0 \geq 0$. Therefore, S_k goes to zero as k goes to infinity. For every subsequence $\{u_{k_j}\}$ of $\{S_k\}$, there exists a subsequence $\{s_{k_j}\}$ which uniformly converges and the limit must to be a solution of (6). Thus, $\{S_k\}$ uniformly goes to a unique solution of (6). \square

3.2. Error Analysis. In this section, we evaluate the maximum absolute error of the proposed method for the solution series (7) for (6).

Theorem 5. Suppose that the hypothesis of Theorem 3 holds. Let u_n and s_n be two solutions satisfying equation (6) for

$0 \leq x, t \leq T, M > 0$ with the initial approximations $u_n(x, t)$ and $s_n(x, t)$, respectively. Then, the maximum absolute error for a solution series (7) for (6) is estimated to be

$$\begin{aligned}
 &\max_{(x,t) \in J \times J} |u_n(x,t) - s_n(x,t)| \\
 &\leq \exp\left(\frac{k_T(M+1)T^{\alpha+1}}{\Gamma(\alpha+1)}\right) \max_{(x,t) \in J \times J} |u_0(x,t) - s_0(x,t)|.
 \end{aligned} \tag{20}$$

Proof. By using Theorem 3, we have

$$\begin{aligned}
 u_n(x,t) &= u_0(x,t) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x,\tau) + \int_0^\tau K(x,r)u_{n-1}(x,u_{n-1}(x,r))dr \right) d\tau, \\
 s_n(x,t) &= s_0(x,t) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(f(x,\tau) + \int_0^\tau K(x,r)s_{n-1}(x,s_{n-1}(x,r))dr \right) d\tau.
 \end{aligned} \tag{21}$$

Next, by using Theorem 4, we have

$$\begin{aligned}
 |u_n - s_n| &= \left| u_0(x, t) - s_0(x, t) + \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau K(x, r) (u_n(x, u_n(x, r)) - s_n(x, s_n(x, r))) dr \right) d\tau \right| \\
 &\leq |u_0(x, t) - s_0(x, t)| + \left| \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} k_T \left(\int_0^\tau (u_n(x, u_n(x, r)) - s_n(x, s_n(x, r))) dr \right) d\tau \right| \\
 &= |u_0(x, t) - s_0(x, t)| + k_T \left| \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau (u_n(x, u_n(x, r)) - u_n(x, s_n(x, r)) + u_n(x, s_n(x, r)) - s_n(x, s_n(x, r))) dr \right) d\tau \right| \\
 &\leq |u_0(x, t) - s_0(x, t)| + k_T \left| \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^\tau (M + 1) (u_n(x, r) - s_n(x, r)) dr \right) d\tau \right| \tag{22} \\
 &= |u_0(x, t) - s_0(x, t)| + k_T (M + 1) \left| \int_0^t \int_0^\tau \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} (u_n(x, r) - s_n(x, r)) dr d\tau \right| \\
 &= |u_0(x, t) - s_0(x, t)| + k_T (M + 1) \left| \int_0^t \int_0^\tau \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} (u_n(x, r) - s_n(x, r)) d\tau dr \right| \\
 &\leq |u_0(x, t) - s_0(x, t)| + \frac{k_T (M + 1) T^\alpha}{\Gamma(\alpha + 1)} \int_0^t |u_n(x, \tau) - s_n(x, \tau)| d\tau.
 \end{aligned}$$

By using Gronwall–Bellman inequality given by Lemma 1, we get

$$|u_n(x, t) - s_n(x, t)| \leq |u_0(x, t) - s_0(x, t)| \exp \left(\int_0^t \frac{k_T (M + 1) T^\alpha}{\Gamma(\alpha + 1)} dr \right) \leq |u_0(x, t) - s_0(x, t)| \exp \left(\frac{k_T (M + 1) T^{\alpha+1}}{\Gamma(\alpha + 1)} \right). \tag{23}$$

Thus, we obtain

$$\begin{aligned}
 \max_{(x,t) \in J \times J} |u_n(x, t) - s_n(x, t)| &\leq \exp \left(\frac{k_T (M + 1) T^{\alpha+1}}{\Gamma(\alpha + 1)} \right) \\
 &\quad \times \max_{(x,t) \in J \times J} |u_0(x, t) - s_0(x, t)|.
 \end{aligned} \tag{24}$$

This completes the proof of Theorem 5. □

3.3. Algorithms for Computer Implementations. In this section, we introduce algorithms for computing numerical results. Algorithm 1 computes the existence conditions given by Theorem 3.

Further, Algorithm 2 can be used to obtain particular approximate numerical solutions at particular values of the fractional order α .

4. Analytical Solutions for Iterative Volterra FPIDEs

This section introduces solutions for new examples of iterative FPIDEs. These examples are chosen since their solutions are not available in the literature or they have been solved previously some other well-known methods. for $0 \leq x, t \leq 0.75, 0 < \alpha < 1$.

Example 1. In this example, we solve the following iterative FPIDEs of Volterra type with initial value:

$$\begin{cases} D_t^\alpha u(x, t) = \cos\left(\frac{x}{2}\right) \int_0^t u(x, u(x, r)) dr, \\ u(x, 0) = \frac{\sin(x)}{2}, \end{cases} \tag{25}$$

Input: $T < 1, N = |f(x, t)|, k_T = K(x, t), u_0(x, 0) = f(x),$
 (1) **for** $0 \leq x \leq 1, 0 \leq t \leq 1, i - 1 < \alpha < i,$ **do**
 (2) special treatment of the first element of line $i;$
 (3) **for** $i = 1, 2, \dots, n,$ **do**
 (4) Compute $|u_0 + T^\alpha(N + T^3k_T)/\Gamma(\alpha + 1)|, \Gamma(\alpha + 1)/T^{\alpha+1}k_T - 1.$
Output: $|u_0 + T^\alpha(N + T^3k_T)/\Gamma(\alpha + 1)| < T, 0 < \Gamma(\alpha + 1)/T^{\alpha+1}k_T - 1 < 1.$

ALGORITHM 1: The computation of the existence conditions.

Input: $T < 1, N = |f(x, t)|, k_T = K(x, t), u_0(x, 0) = f(x),$
 (1) **for** $0 \leq x \leq 1, 0 \leq t \leq 1, i - 1 < \alpha < i,$ **do**
 (2) special treatment of the first element of line $i;$
 (3) **for** $i = 1, 2, \dots, n,$ **do**
 (4) Compute $|u_0 + T^\alpha(N + T^3k_T)/\Gamma(\alpha + 1)|, \Gamma(\alpha + 1)/T^{\alpha+1}k_T - 1;$
 (5) **if** $|u_0 + T^\alpha(N + T^3k_T)/\Gamma(\alpha + 1)| < T, \Gamma(\alpha + 1)/T^{\alpha+1}k_T - 1 < 1$ **then**
 (6) Compute $u_{i+1}(x, t) = u_0 + \int_0^t (x - \tau)^{\alpha-1}/\Gamma(\alpha) (f(x, \tau) + \int_0^\tau K(x, r)u_i(x, u_i(x, r))dr) d\tau.$
 (7) **Output:** $u_{i+1}(x, t) = u_0 + \int_0^t (x - \tau)^{\alpha-1}/\Gamma(\alpha) (f(x, \tau) + \int_0^\tau K(x, r)u_i(x, u_i(x, r))dr) d\tau.$

ALGORITHM 2: The computation of the numerical solutions.

Then, equation (25) is of form (6) with $T = 0.75, N = 0, k_T = 1$ which satisfies

$$\left| u_0 + \frac{T^\alpha(N + T^3k_T)}{\Gamma(\alpha + 1)} \right| = \left| \frac{\sin(x)}{2} + \frac{0.75^\alpha(0 + 0.75^3 \times \cos(x/2))}{\Gamma(\alpha + 1)} \right| < 0.75 = T, \tag{26}$$

where $0 < M < \Gamma(\alpha + 1)/T^{\alpha+1}k_T - 1 = \Gamma(\alpha + 1)/0.75^{\alpha+1} |\cos(x/2)| - 1 < 1$ for all $0 \leq x \leq 0.75$ and $0 < \alpha < 1.$

As the hypotheses of Theorem 3 are satisfied, a unique solution for equation (25) exists.

Next, by using Theorem 4 and assuming that $u_0(x, t) = u(x, 0) = \sin(x)/2,$ the first few iterative solutions are

$$\begin{aligned} u_1(x, t) &= u_1(x, 0) + \int_0^t \frac{(x - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\cos\left(\frac{x}{2}\right) \int_0^\tau u_0(x, u_0(x, r)) dr \right) d\tau, u_1(x, 0) = 0 \\ &= \frac{1}{2(\alpha^2 + \alpha)\Gamma(\alpha)} t^{\alpha+1} \sin(x) \cos\left(\frac{x}{2}\right), \end{aligned} \tag{27}$$

$$\begin{aligned} u_2(x, t) &= u_2(x, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t (x - \tau)^{\alpha-1} \left(\cos\left(\frac{x}{2}\right) \int_0^\tau u_1(x, u_1(x, r)) dr \right) d\tau, u_2(x, 0) = 0 \\ &= \frac{2^{-\alpha-2} \Gamma(\alpha + 2)^{-\alpha-1} \Gamma(\alpha(\alpha + 2) + 3) t^{(\alpha+1)(\alpha+2)} \csc(x) (\sin(x) \cos(x/2))^{\alpha+3}}{\times (\alpha(\alpha + 2) + 2) \Gamma(\alpha(\alpha + 3) + 3)}, \end{aligned} \tag{28}$$

$$\begin{aligned}
 u_3(x, t) &= u_2(x, 0) + \int_0^t \frac{(x - \tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\cos\left(\frac{x}{2}\right) \int_0^t u_2(x, u_2(x, r)) dr \right) d\tau, \quad u_3(x, 0) = 0, \\
 &= \frac{((\alpha(\alpha + 2) + 2)\Gamma(\alpha(\alpha + 3) + 3))^{-\alpha(\alpha+3)-3}}{(\alpha^2 + \alpha + 1)(\alpha(\alpha + 5) + 7)\Gamma(\alpha(\alpha(\alpha + 6) + 13) + 13) + 8)} \\
 &\quad \times \Gamma(\alpha(\alpha + 3)(\alpha(\alpha + 3) + 4) + 8) t^{\alpha(\alpha(\alpha+6)+13)+13+7} \\
 &\quad \times \left(2^{-\alpha-2} \Gamma(\alpha + 2)^{-\alpha-1} \Gamma(\alpha(\alpha + 2) + 3) \csc(x) \left(\sin(x) \cos\left(\frac{x}{2}\right) \right)^{\alpha+3} \right)^{\alpha(\alpha+3)+3}.
 \end{aligned}
 \tag{29}$$

Therefore, the approximate solution of (25) is obtained by $u(x, t) \approx \sum_{i=0}^3 u_i(x, t)$.

Example 2. In this example, we solve the following iterative FPIDEs of Volterra type with initial value:

$$\begin{cases} D_t^\alpha u(x, t) = \frac{\sin(x)}{3} + \int_0^t u(x, u(x, r)) dr, & 0 \leq x, t \leq 0.75, 0 < \alpha < 1, \\ u(x, 0) = 0. \end{cases}
 \tag{30}$$

Equation (30) is of form (8) with $T = 0.75, N = |\sin(x)/3|, k_T = 1$, which satisfies

where $0 < M < \Gamma(\alpha + 1)/T^{\alpha+1}k_T - 1 < 1$ for all $x \in [0, 0.75]$ and $0 < \alpha < 1$. As all the hypotheses of Theorem 3 are satisfied, a unique solution for (30) exists.

By using Theorem 4, we obtain a solution of (30) for different values of α . We assume that $u_0(x, t) = u(x, 0) = 0$ and by using Mathematica software, the first three iterative solutions are obtained as follows:

$$\begin{aligned}
 &\left| u_0 + \frac{T^\alpha(N + T^2k_T)}{\Gamma(\alpha + 1)} \right| \\
 &= \left| 0 + \frac{0.75^\alpha(\sin(x)/3 + 0.75^3)}{\Gamma(\alpha + 1)} \right| < 0.75 = T,
 \end{aligned}
 \tag{31}$$

$$\begin{aligned}
 u_1(x, t) &= \frac{1}{4\Gamma(\alpha)} \int_0^t (x - \tau)^{\alpha-1} \left(\sin(x)/3 + \int_0^t u_0(x, u_0(x, r)) dr \right) d\tau \\
 &= \frac{t^\alpha}{3\alpha\Gamma(\alpha)} \sin(x),
 \end{aligned}
 \tag{32}$$

$$\begin{aligned}
 u_2(x, t) &= \frac{1}{4\Gamma(\alpha)} \int_0^t (x - \tau)^{\alpha-1} \left(\frac{\sin(x)}{3} + \int_0^t u_1(x, u_1(x, r)) dr \right) d\tau \\
 &= \frac{3^{-\alpha-1} t^\alpha \sin(x) \left(2 \times 3^\alpha \Gamma(\alpha^2 + \alpha + 2) + \Gamma(\alpha^2 + 2) t^{\alpha^2+1} (\sin(x)/\Gamma(\alpha + 1))^\alpha \right)}{2\Gamma(\alpha + 1)\Gamma(\alpha^2 + \alpha + 2)},
 \end{aligned}
 \tag{33}$$

TABLE 1: Numerical values of the iterative solution when $q_1 = q_2 = 0.5, 1$ and $\alpha = \beta = 0.5$ for Example 2.

x	t	$u_0(x, t)$	$\alpha = 0.5$			$\alpha = 1$			$\alpha = 0.5$	$\alpha = 1$
			$u_1(x, t)$	$u_2(x, t)$	$u_3(x, t)$	$u_1(x, t)$	$u_2(x, t)$	$u_3(x, t)$	$\sum_{i=0}^3 u_i$	$\sum_{i=0}^3 u_i$
0.25	0.25	0.123702	0.011541	2.901×10^{-6}	9.774×10^{-29}	0.003836	3.732×10^{-9}	1.035×10^{-61}	0.135246	0.127538
	0.50	0.123702	0.032643	0.000039	1.892×10^{-23}	0.0153421	2.389×10^{-7}	1.138×10^{-49}	0.156384	0.139044
	0.75	0.123702	0.059970	0.000179	2.342×10^{-20}	0.034520	2.721×10^{-6}	1.258×10^{-42}	0.18385	158224
0.50	0.25	0.239713	0.021840	0.000014	1.700×10^{-25}	0.007258	2.470×10^{-8}	5.750×10^{-56}	0.261567	0.246971
	0.50	0.239713	0.061772	0.000188	3.291×10^{-20}	0.029033	1.581×10^{-6}	6.322×10^{-44}	0.301673	0.268747
	0.75	0.239713	0.113483	0.000859	4.073×10^{-17}	0.065323	0.000018	6.991×10^{-37}	0.354055	0.305054
0.75	0.25	0.340819	0.029821	0.000030	5.667×10^{-24}	0.003836	6.038×10^{-8}	3.002×10^{-53}	0.370669	0.350730
	0.50	0.340819	0.084346	0.000393	1.097×10^{-18}	0.039642	3.865×10^{-6}	3.301×10^{-41}	0.425558	0.380465
	0.75	0.340819	0.154953	0.001797	1.358×10^{-15}	0.089194	0.000044	3.649×10^{-34}	0.497569	0.430058

TABLE 2: Numerical values of the iterative solutions when $q_1 = q_2 = 0.5, 1$ and $\alpha = \beta = 0.5$ for Example 1.

x	t	$\alpha = 0.5$			$\alpha = 1$			$\alpha = 0.5$	$\alpha = 1$
		$u_1(x, t)$	$u_2(x, t)$	$u_3(x, t)$	$u_1(x, t)$	$u_2(x, t)$	$u_3(x, t)$	$\sum_{i=1}^3 u_i$	$\sum_{i=1}^3 u_i$
0.25	0.25	0.0465276	0.047411	0.047483	0.020617	0.020635	0.020644	0.141421	0.061895
	0.50	0.065800	0.068773	0.068501	0.041234	0.041376	0.041340	0.20307	0.123950
	0.75	0.080588	0.086632	0.085550	0.061851	0.062329	0.062090	0.25277	0.186270
0.50	0.25	0.090162	0.092546	0.092761	0.039952	0.040018	0.040052	0.275470	0.120023
	0.50	0.127509	0.135527	0.134859	0.079904	0.080436	0.080304	0.397895	0.240645
	0.75	0.156166	0.172468	0.169670	0.119856	0.121652	0.120756	0.498304	0.362264
0.75	0.25	0.128191	0.132233	0.132637	0.020617	0.056938	0.057005	0.393061	0.170746
	0.50	0.181290	0.194884	0.193864	0.113606	0.114682	0.114415	0.570037	0.342704
	0.75	0.222034	0.249672	0.245134	0.170410	0.174040	0.172229	0.716839	0.516678

$$\begin{aligned}
 u_3(x, t) &= \frac{1}{4\Gamma(\alpha)} \int_0^t (x-\tau)^{\alpha-1} \left(\frac{\sin(x)}{3} + \int_0^t u_2(x, u_2(x, r)) dr \right) d\tau \\
 &= \frac{\sin(x)}{12\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(\alpha^2+\alpha+2)} \left(\left(3^{-\alpha}\Gamma(\alpha^2+2)t^{\alpha+1} \left(\frac{\sin(x)}{\Gamma(\alpha+1)} \right)^{\alpha} \right. \right. \\
 &\quad \times \left(\frac{2^{-2\alpha-1}(2 \times 3^{\alpha}\Gamma(\alpha^2+\alpha+2) + 4^{-\alpha-1}\Gamma(\alpha^2+2)(\sin(x)/\Gamma(\alpha+1))^{\alpha})}{\Gamma(\alpha+1)\Gamma(\alpha^2+\alpha+2)} \right)^{\alpha^2+\alpha+1} \\
 &\quad \times \left(3^{-\alpha-1}\sin(x) \right)^{\alpha^2+\alpha+1} + \frac{4\Gamma(\alpha+2)\Gamma(\alpha^2+\alpha+2)t^{\alpha} + 2t\Gamma(\alpha^2+\alpha+2)}{\Gamma(\alpha+1)\Gamma(\alpha^2+\alpha+2)} \\
 &\quad \times \left(2^{-2\alpha-1}3^{-\alpha-1}t\sin(x) \left(2 \cdot 3^{\alpha}\Gamma(\alpha^2+\alpha+2) + 2^{-2(\alpha^2+1)}\Gamma(\alpha^2+2) \right. \right. \\
 &\quad \left. \left. \times \left(\frac{\sin(x)}{\Gamma(\alpha+1)} \right)^{\alpha} \right) \right)^{\alpha} \left. \right). \tag{34}
 \end{aligned}$$

Therefore, the third order term iterative solution (30) is $u(x, t) \approx \sum_{i=1}^3 u_i(x, t)$.

5. Numerical Solutions and Discussion

Table 1 presents numerical solutions for equation (25) through various values of x, t when $\alpha = 0.5, 1$. Table 2 includes numerical values of the iterative solutions for equation (30) by different values of x, t at $\alpha = 0.5, 1$. In Figures 1(a) and 1(b), we plot the graphs of first-order iterative solution for (25) using various values of x, t at $\alpha = 0.5, 1$ respectively. We plot the graph of second-order iterative solution for (25) by various values of x, t at $\alpha = 0.5, 1$ in Figures 2(a) and 2(b), respectively. The iterative solution

for Example 1 is graphically represented in Figures 3(a) and 3(b) through various points of x, t when $\alpha = 0.5, 1$, respectively. Figures 4(a) and 4(b) represent the graphs of solution through various values of t for a fixed value of $x = 0.75$ when $\alpha = 0.5, 1$, respectively, for Example 1. The first-order iterative solution for Example 2 is graphically represented in Figures 5(a) and 5(b) through various values of x, t when $\alpha = 0.5, 1$, respectively. The second order of iterative solution for Example 2 is graphically represented in Figures 6(a) and 6(b) by various values of x, t when $\alpha = 0.5, 1$, respectively. The third-order iterative solution for Example 2 is graphically represented in Figures 7(a) and 7(b) by various points of x, t at $\alpha = 0.5, 1$, respectively. The solution for Example 2 is graphically represented in

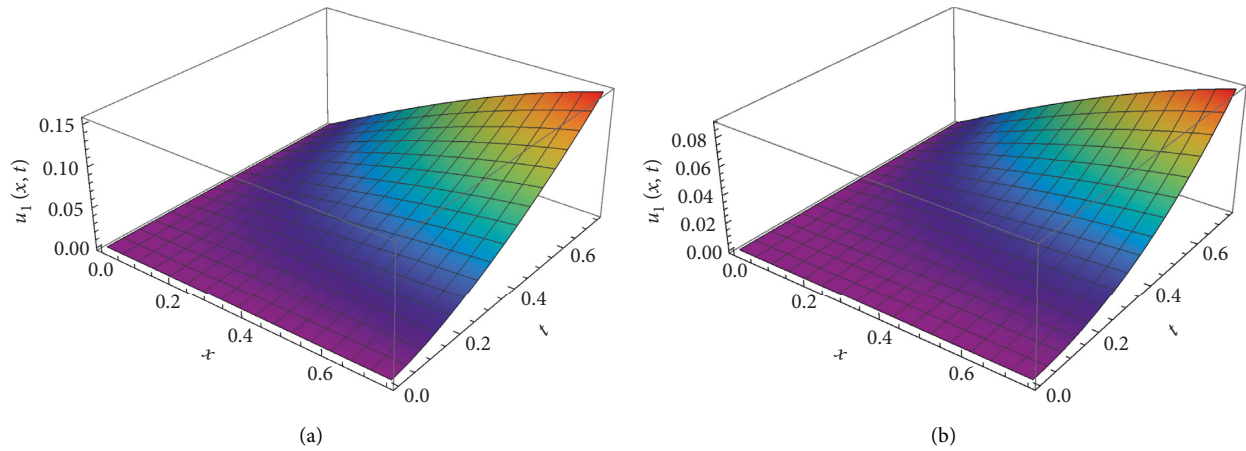


FIGURE 1: The graphs of the first-order iterative solution $u_1(x, t)$ for (25) through various values of x, t at $\alpha = 0.5, 1$, respectively. (a) The graph of first-order iterative solution for (25) through various values of x, t at $\alpha = 0.5$. (b) The graph of the first-order iterative solution for (25) through various values of x, t at $\alpha = 1$.

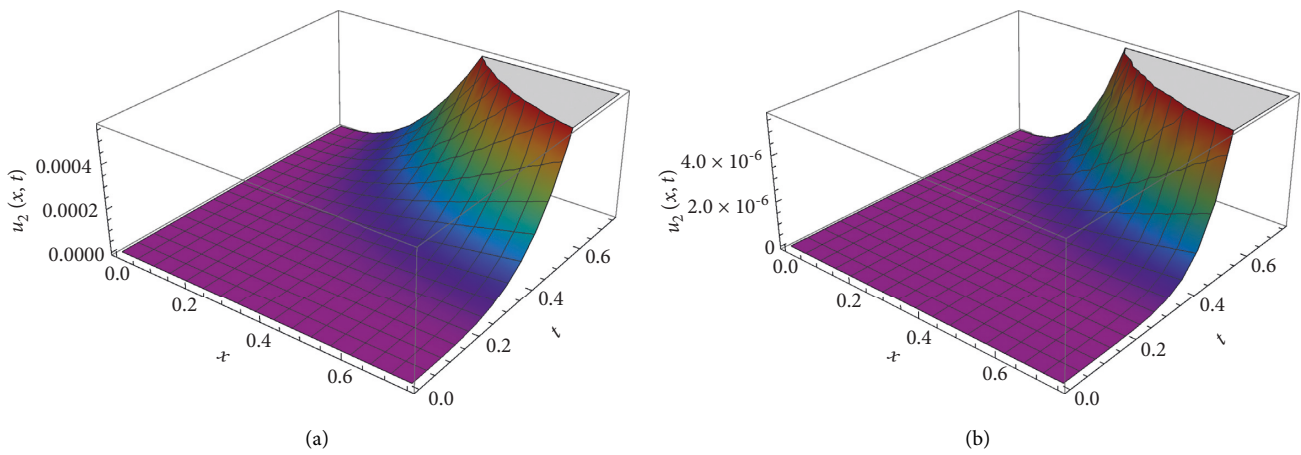


FIGURE 2: The graphs of the second-order iterative solution $u_2(x, t)$ for (25) through various values of x, t at $\alpha = 0.5, 1$, respectively. (a) The graph of the second-order iterative solution for (25) through various values of x, t at $\alpha = 0.5$. (b) The graph of the second-order iterative solution for (25) through various values of x, t at $\alpha = 1$.

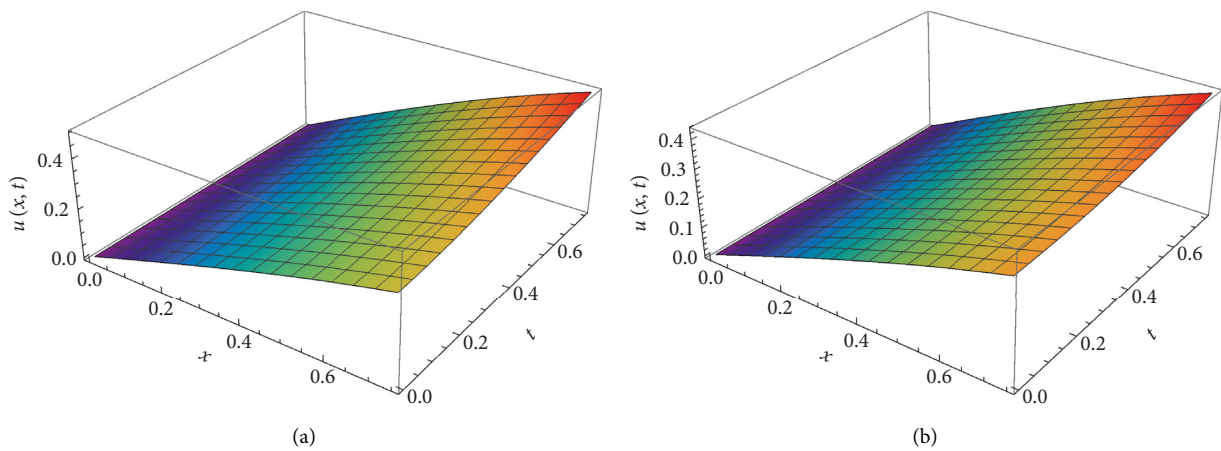


FIGURE 3: The graphs of the approximate iterative solution $u(x, t)$ for (25) through various values of x, t at $\alpha = 0.5, 1$, respectively. (a) The graph of the approximate iterative solution $u(x, t)$ for (25) through various values of x, t at $\alpha = 0.5$. (b) The graph of the approximate iterative solution $u(x, t)$ for (25) through various values of x, t at $\alpha = 1$.

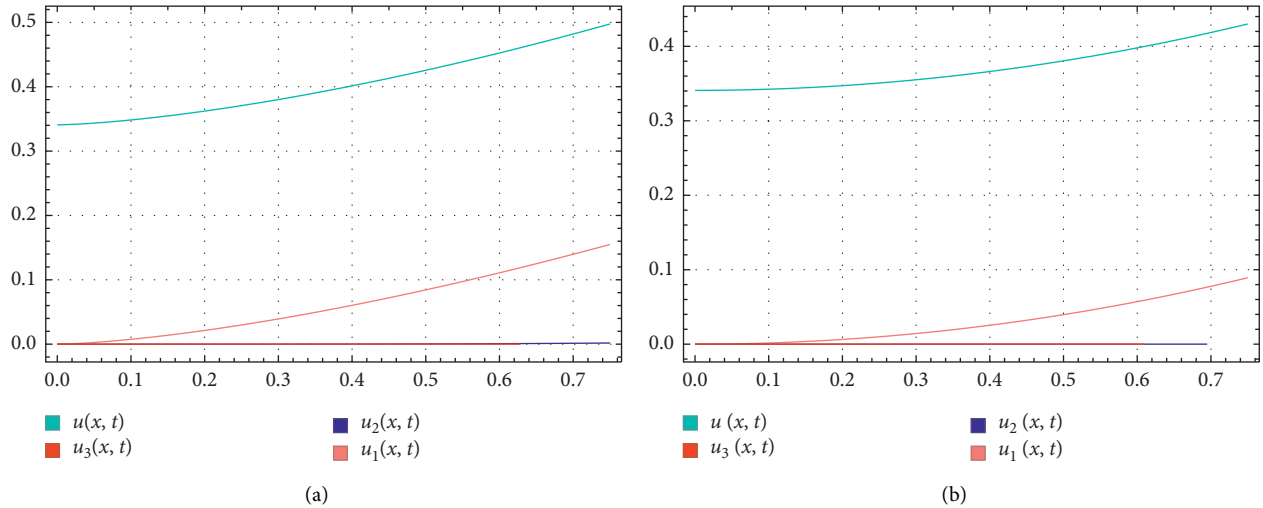


FIGURE 4: The graphical comparison of the iterative solutions for (25) through various values of t at $x = 0.75$ and $\alpha = 0.5, 1$, respectively. (a) The graphical comparison of the iterative solutions for (25) through various values of t at $x = 0.75$ and $\alpha = 0.5$. (b) The graphical comparison of the iterative solutions for (25) through various values of t at $x = 0.75$ and $\alpha = 1$.

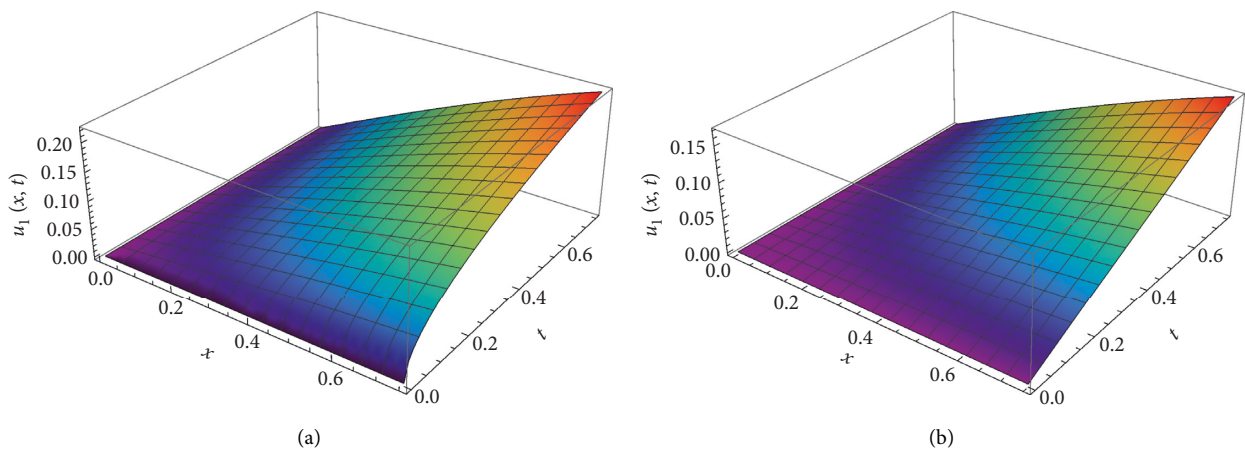


FIGURE 5: The graphs of the first-order iterative solution $u_1(x, t)$ for (30) through various values of x, t at $\alpha = 0.5, 1$, respectively. (a) The graph of first-order iterative solution $u_1(x, t)$ for (30) through various values of x, t at $\alpha = 0.5$. (b) The graph of the first-order iterative solution $u_1(x, t)$ for (30) through various values of x, t at $\alpha = 1$.

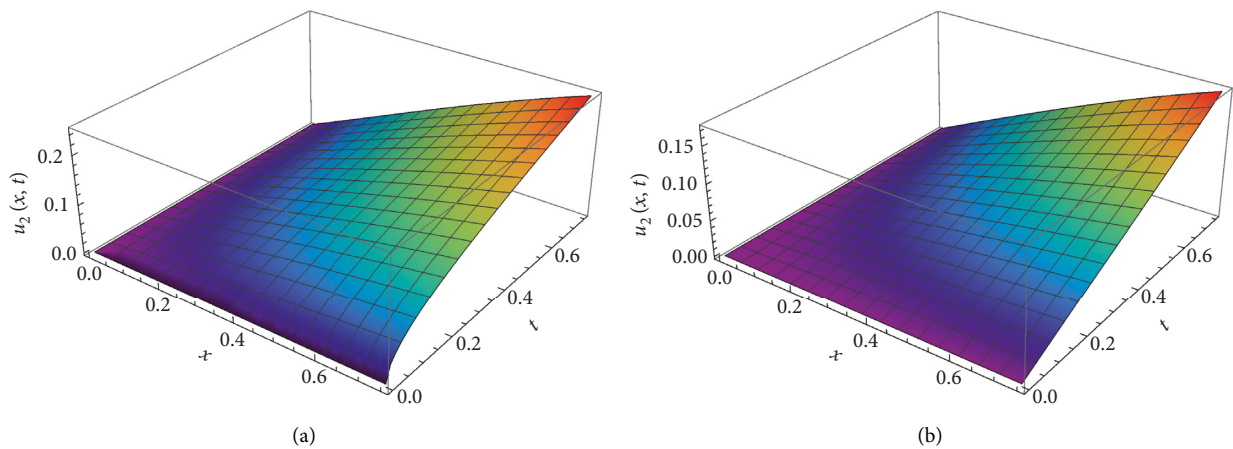


FIGURE 6: The graphs of second-order iterative solution $u_2(x, t)$ for (30) through various points x, t at $\alpha = 0.5, 1$, respectively. (a) The graph of second-order iterative solution $u_2(x, t)$ for (30) through various values of x, t at $\alpha = 0.5$. (b) The graph of second-order iterative solution $u_2(x, t)$ for (30) through various values of x, t at $\alpha = 1$.

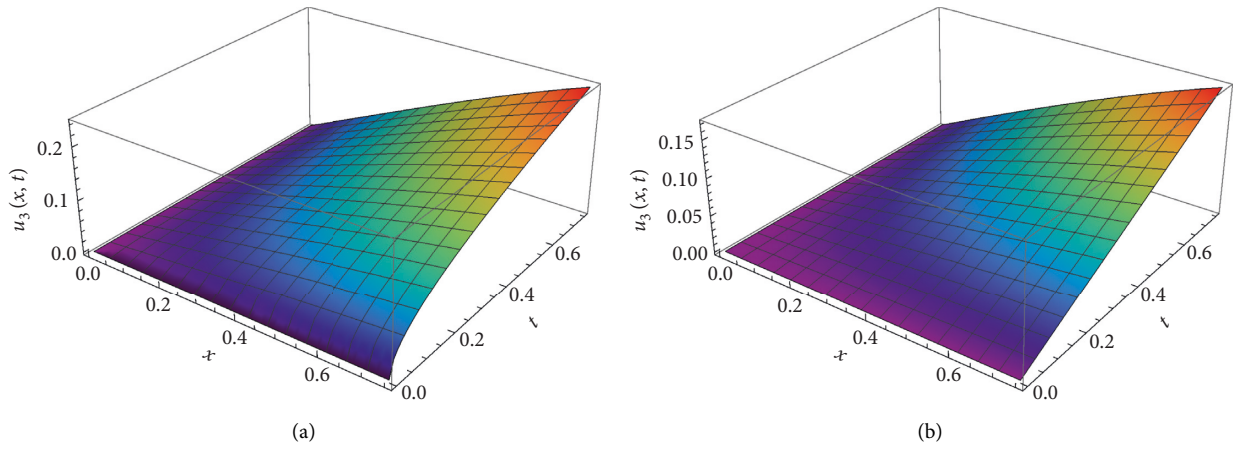


FIGURE 7: The graphs of the third-order iterative solution $u_3(x, t)$ for (30) through various values of x, t at $\alpha = 0.5, 1$, respectively. (a) The graph of third-order iterative solution $u_3(x, t)$ for (30) through various values of x, t at $\alpha = 0.5$. (b) The graph of the third-order iterative solution $u_3(x, t)$ for (30) through various values of x, t at $\alpha = 1$.

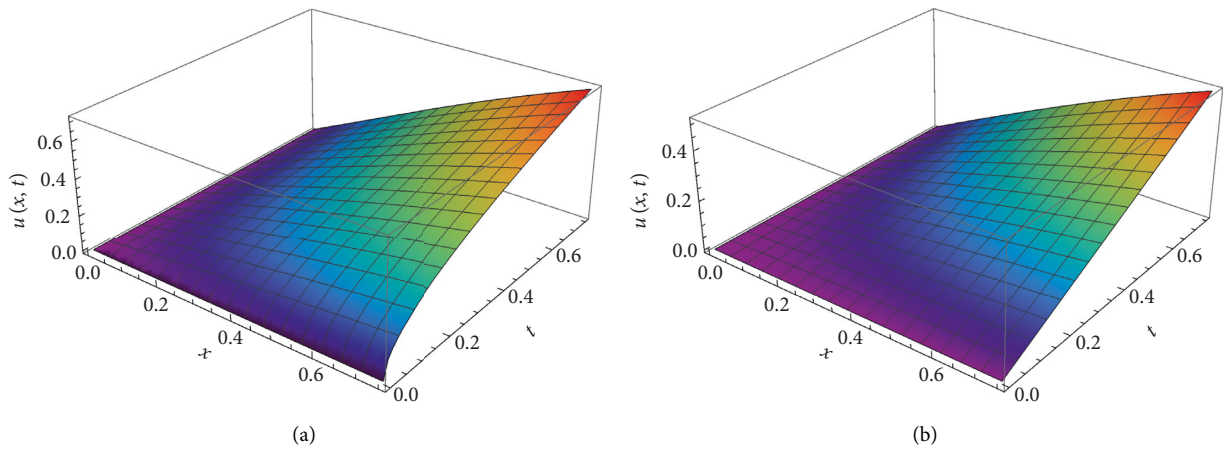


FIGURE 8: The graphs of the third-order approximate iterative solution $u(x, t)$ for (30) through various values of x, t at $\alpha = 0.5, 1$, respectively. (a) The graph of the approximate iterative solution $u(x, t)$ for (30) through various values of x, t at $\alpha = 0.5$. (b) The graph of the approximate iterative solution $u(x, t)$ for (30) through various values of x, t at $\alpha = 1$.

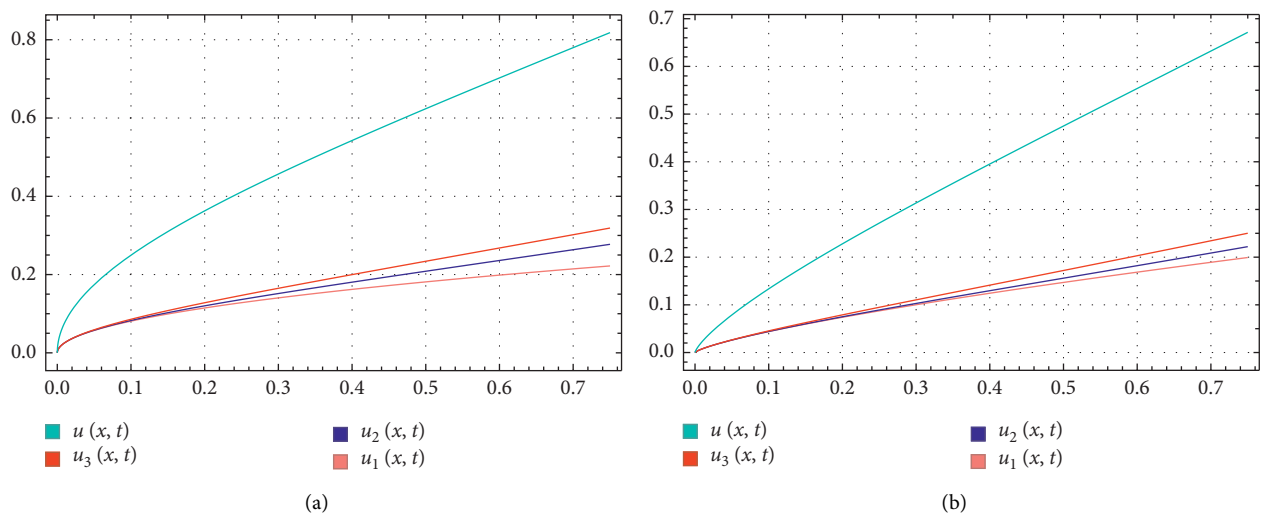


FIGURE 9: The graphical comparison of solutions for (30) through various values of t at $x = 0.75, \alpha = 0.5, 1$, respectively. (a) The graphical comparison of solutions for (30) through various values of t at $x = 0.75; \alpha = 0.5$. (b) The graphical comparison of solutions for (30) through various values of t at $x = 0.75; \alpha = 1$.

Figures 8(a) and 8(b) using various points of x, t when $\alpha = 0.5, 1$, respectively. In Figures 9(a) and 9(b), we plot the graphs of the solution through different values of t for a fixed value of $x = 0.75$ when $\alpha = 0.5, 1$, respectively, for Example 2.

6. Conclusion

In this paper, we introduced a model of FPIDEs. The proposed model is iterative with fractional derivative, which can be used in neural networks and help us to describe how the input data can be accessed. For instance, for subdiffusion in the porous media, fractional-order derivatives determine the decaying rate of the breakthrough curve for long-term observations. Moreover, new results on the local existence, uniqueness, and stability analysis of the solution for the proposed model were introduced. Furthermore, we extended the method of successive approximations to solve FPIDEs with memory terms subject to initial conditions in a Banach space. This extension derives good approximations and reliable techniques to handle iterative FPIDEs. New solutions for Volterra types of iterative FPIDEs were introduced. The numerical solutions were successfully obtained which confirm the presented results.

Data Availability

The datasets used or analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] M. Dehghan and M. Abbaszadeh, "Analysis of the element free Galerkin (EFG) method for solving fractional cable equation with Dirichlet boundary condition," *Applied Numerical Mathematics*, vol. 109, pp. 208–234, 2016.
- [2] H. Thabet, S. Kendre, and J. Peters, "Analytical solutions for nonlinear systems of conformable space-time fractional partial differential equations via generalized fractional differential transform," *Vietnam Journal of Mathematics*, vol. 47, no. 2, pp. 487–507, 2019.
- [3] H. Zhang, R. Ye, J. Cao, A. Ahmed, X. Li, and Y. Wan, "Lyapunov functional approach to stability analysis of riemann-liouville fractional neural networks with time-varying delays," *Asian Journal of Control*, vol. 20, no. 5, pp. 1938–1951, 2018.
- [4] H. Thabet and S. Kendre, "Modified least squares homotopy perturbation method for solving fractional partial differential equations," *Malaya Journal of Matematik*, vol. 06, no. 02, pp. 420–427, 2018.
- [5] M. Kaplan, A. R. Butt, H. Thabet, A. Akbulut, N. Raza, and D. Kumar, "An effective computational approach and sensitivity analysis to pseudo-parabolic-type equations," *Waves in Random and Complex Media*, pp. 1–15, 2021.
- [6] H. Thabet, S. Kendre, and J. Peters, "Advances in solving conformable nonlinear partial differential equations and new exact wave solutions for Oskolkov-type equations," *Mathematical Methods in the Applied Sciences*, vol. 45, no. 5, pp. 2658–2673, 2022.
- [7] H. Thabet, S. Kendre, D. Baleanu, and J. Peters, "Exact analytical solutions for nonlinear systems of conformable partial differential equations via an analytical approach," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 84, no. 1, pp. 109–120, 2022.
- [8] J. R. Loh, C. Phang, and A. Isah, "New operational matrix via genocchi polynomials for solving fredholm-volterra fractional integro-differential equations," *Advances in Mathematical Physics*, vol. 2017, pp. 1–12, 2017.
- [9] J. R. Loh and C. Phang, "A new numerical scheme for solving system of Volterra integro-differential equation," *Alexandria Engineering Journal*, vol. 57, no. 2, pp. 1117–1124, 2018.
- [10] J. R. Loh and C. Phang, "Numerical solution of fredholm fractional integro-differential equation with right-sided caputo's derivative using Bernoulli polynomials operational matrix of fractional derivative," *Mediterranean Journal of Mathematics*, vol. 16, no. 2, pp. 28–25, 2019.
- [11] J. R. Loh, C. Phang, and K. G. Tay, "New method for solving fractional partial integro-differential equations by combination of Laplace transform and resolvent kernel method," *Chinese Journal of Physics*, vol. 67, pp. 666–680, 2020.
- [12] F. Abdolrazaghi and A. Razani, "On the weak solutions of an overdetermined system of nonlinear fractional partial integro-differential equations," *Miskolc Mathematical Notes*, vol. 20, no. 1, pp. 3–6, 2019.
- [13] K. Al-Khaled, A. Darweesh, and M. H. Yousef, "Convergence of numerical schemes for the solution of partial integro-differential equations used in heat transfer," *Journal of Applied Mathematics and Computing*, vol. 61, no. 1-2, pp. 657–675, 2019.
- [14] H. Dehestani, Y. Ordokhani, and M. Razzaghi, "Pseudo-operational matrix method for the solution of variable-order fractional partial integro-differential equations," *Engineering with Computers*, vol. 37, no. 3, pp. 1791–1806, 2020.
- [15] E. Eder, "The functional differential equation $x'(t) = x(x(t))$," *Journal of Differential Equations*, vol. 54, no. 3, pp. 390–400, 1984.
- [16] M. I. Berenguer and D. Gámez, "Projected iterations of fixed-point type to solve nonlinear partial volterra integro-differential equations," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 6, pp. 4431–4442, 2020.
- [17] C. Bao, Y. Pu, and Y. Zhang, "Fractional-order deep back-propagation neural network," *Computational Intelligence and Neuroscience*, vol. 201810 pages, 2018.
- [18] G. Pang, L. Lu, and G. E. Karniadakis, "fPINNs: fractional physics-informed neural networks," *SIAM Journal on Scientific Computing*, vol. 41, no. 4, pp. A2603–A2626, 2019.
- [19] D. J. White, "Dynamic programming, Markov chains, and the method of successive approximations," *Journal of Mathematical Analysis and Applications*, vol. 6, no. 3, pp. 373–376, 1963.
- [20] R. E. Larson and A. J. Korsak, "A dynamic programming successive approximations technique with convergence proofs," *Automatica*, vol. 6, no. 2, pp. 245–252, 1970.
- [21] R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Academic Press, Cambridge, MA, USA, 1977.
- [22] Z. Aganovic and Z. Gajic, "Successive approximation procedure for the optimal control of bilinear systems," [*Proceedings 1992*] *IEEE International Conference on Systems Engineering*, vol. 39, no. 9, pp. 440–443, 1992.

- [23] A. M. A. El-Sayed, H. H. G. Hashem, and E. A. A. Ziada, "Picard and Adomian decomposition methods for a quadratic integral equation of fractional order," *Computational and Applied Mathematics*, vol. 33, no. 1, pp. 95–109, 2014.
- [24] A. M. A. El-Sayed, H. H. G. Hashem, and E. A. A. Ziada, "Picard and adomian methods for quadratic integral equation," *Computational and Applied Mathematics*, vol. 29, no. 3, pp. 447–463, 2010.
- [25] H. Jafari, "A comparison between the variational iteration method and the successive approximations method," *Applied Mathematics Letters*, vol. 32, no. 1, pp. 1–5, 2014.
- [26] H. Liu and W. Li, "The exact analytic solutions of a nonlinear differential iterative equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2466–2478, 2008.
- [27] X.-P. Wang and J.-G. Si, "Analytic solutions of an iterative functional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 2, pp. 490–498, 2001.
- [28] D. Yang and W. Zhang, "Solutions of equivariance for iterative differential equations," *Applied Mathematics Letters*, vol. 17, no. 7, pp. 759–765, 2004.
- [29] L. Zhang, L. Zhang, J. Yuan, and C. M. Khalique, "Existence of wave front solutions of an integral differential equation in nonlinear nonlocal neuronal network," *Abstract and Applied Analysis*, vol. 2014, no. 2, 2014.
- [30] R. W. Ibrahim, "Existence of iterative cauchy fractional differential equation," *Journal of Mathematics*, vol. 2013, no. 3, 7 pages, 2013.
- [31] M. Podisuk, "More on simple iterative ordinary differential equation," *Procedia - Social and Behavioral Sciences*, vol. 88, no. 3, pp. 187–195, 2013.
- [32] H. Thabet, S. Kendre, and D. Chalishajar, "New analytical technique for solving a system of nonlinear fractional partial differential equations," *Mathematics*, vol. 5, no. 4, p. 47, 2017.
- [33] H. Thabet and S. Kendre, *Elementary Course in Fractional Calculus*, LAP LAMBERT Academic Publishing, 2018.
- [34] H. Thabet and S. Kendre, "New modification of Adomian decomposition method for solving a system of nonlinear fractional partial differential equations," *International Journal of Advances in Applied Mathematics and Mechanics*, vol. 6, no. 3, pp. 1–13, 2019.
- [35] H. Thabet, S. Kendre, and J. Peters, "Travelling wave solutions for fractional Korteweg-de Vries equations via an approximate-analytical method," *AIMS Mathematics*, vol. 4, no. 4, pp. 1203–1222, 2019.
- [36] H. Thabet, S. Kendre, J. Peters, and M. Kaplan, "Solitary wave solutions and traveling wave solutions for systems of time-fractional nonlinear wave equations via an analytical approach," *Computational and Applied Mathematics*, vol. 39, no. 3, p. 144, 2020.