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Research Article

A Computational Model for q-Bernstein Quasi-Minimal Bézier Surface

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A computational model is presented to find the q-Bernstein quasi-minimal Bézier surfaces as the extremal of Dirichlet functional, and the Bézier surfaces are used quite frequently in the literature of computer science for computer graphics and the related disciplines. The recent work [1–5] on q-Bernstein–Bézier surfaces leads the way to the new generalizations of q-Bernstein polynomial Bézier surfaces for the related *Plateau–Bézier problem*. The q-Bernstein polynomial-based *Plateau–Bézier problem* is the minimal area surface amongst all the q-Bernstein polynomial-based Bézier surfaces, spanned by the prescribed boundary. Instead of usual area functional that depends on square root of its integrand, we choose the Dirichlet functional. Related Euler–Lagrange equation is a partial differential equation, for which solutions are known for a few special cases to obtain the corresponding minimal surface. Instead of solving the partial differential equation, we can find the optimal conditions for which the surface is the extremal of the Dirichlet functional. We workout the minimal Bézier surface based on the q-Bernstein polynomials as the extremal of Dirichlet functional by determining the vanishing condition for the gradient of the Dirichlet functional for prescribed boundary. The vanishing condition is reduced to a system of algebraic constraints, which can then be solved for unknown control points in terms of known boundary control points. The resulting Bézier surface is q-Bernstein–Bézier minimal surface.

1. Introduction

We observe the tendency of nature to be minimum in constructing the path predicting models of different objects around us. For instance, the principles of flexibility involves the optimization of processes based on the observation that a business model should change and grow with time. Other examples are the optimal designs in solid and fluid mechanics, electromagnet of an electromagnetic particle, and the mathematical models for gravity theory are the phenomena in nature that access to physical systems for optimization. These phenomena are analysed in the optimization theory, which include the calculus of variations, game theory, decision theory, linear programming, control theory, network analysis, and Markov chains.

One of the active research areas in the optimization theory [1] is calculus of variations in which we try to find a best suitable function, which could be a curve or a surface subject to certain constraint usually expressed in the integral form called the functional, which is in fact a critical point of that functional, a maxima or minima of the functional under consideration based on the mathematical analysis of minimization principals on multidimensional function spaces. The constraints can be in the form of integrals (functionals based on unknown function of one or more parameters), differential constraints (they arise for instance in case of rolling wheels, flying aircrafts, in situations arising from dynamics of any mechanical system, and so on) or the algebraic structure in constraints. This work is related to finding the surfaces subject to constraints in the form of certain integral called the functional, and one of such functionals is Dirichlet functional with appropriate boundary conditions. The Dirichlet functional being the integral of non-negative quantity is itself non-negative. The variational problem of solving the Dirichlet functional for the minimal vector function $x(s,t)$ subject to certain boundary conditions is the Euler–Lagrange partial differential equation $\nabla^2 x(s,t) = 0$, which is Laplace equation in this case. Such a vector function $x(s,t)$ has minimal Dirichlet energy and is said to be harmonic. This can be done by finding the extremal of area functional of a surface $x(s,t)$, but it involves square root of the integrand, and instead, we can use variational approach for finding such a surface as the extremal of Dirichlet functional and we obtain a quasi-minimal surface. A minimal surface is a surface for which mean curvature of the surface vanishes for all possible parameterizations of the same surface.

In curve theory, the familiar examples are that of shortest distance between two points in a plane, which is a straight line in case when no constrains are involved; however, in case of given constraints, the shortest path or the distance between two points is usually called a geodesic on a surface, and the related ancient well-known problem is the brachistochrone problem. In optics, one of the related problems is the Fermat's principle (1657), to find the path in least possible time. Fermat explained that the laws of geometric optics, lens design such as reflection, refraction, focusing, and aberrations could be explained from the geometric and analytical properties of this principle. The natural generalization of the minimal curve or geodesic problem is the minimal surface problem [2, 3]. It consists of searching the surface $x \subset \mathbb{R}^3$ with possible bounded area spanned by a closed contour Γ. This essentially means setting the surface area functional $\mathscr{A} = \int_{\mathscr{R}} dx$ as an objective of extremization functional over all possible $x \subset \mathbb{R}^3$ with prescribed contour Γ to achieve a minimal area surface. Minimal surface problem is referred to as *Plateau–Bézier problem* [4, 5] in the honour of physicist of Belgian Joeseph Plateau, who established that minimal surfaces can be associated with soap films spanned by closed wire frames in 1849. He explained further through his experiments that a minimal surface can be achieved in the form of a thin soap film spanned by a wire frame by immersing it into soapy water and displacing it back cautiously. The soap film itself is a surface of minimal area, and the wire frame in shape of closed contour Γ serves as a spanning curve.

The soap films and bubbles have been a source of fascination from aesthetical, physical, and mathematical point of view that is why it is an active field of research for several hundred years, finding its applications in different disciplines of science. The earliest significant work is that of Euler, who in 1744, while searching for rotational surfaces of minimal area, proved that a minimal surface is planer if and only if its Gaussian curvature is identically equal to zero and a minimal surface is always locally saddled-shaped. Later in 1762, Lagrange considered the minimal surface problem by deriving associated Euler–Lagrange equations, which is a quadratic partial differential equation ($p\ de$). The solutions of such partial differential equations are the functions that minimize a given functional. However, he was not able to find out a general solution of the equation other than a plane. In 1776, Meusnier illustrated the condition for minimality for a specific form of a regular surface, known as Monge's patch. The outcome of the minimal condition for the Monge's patch is a quasilinear, quadratic partial differential equation derived earlier by Lagrange. He also showed that catenoid and helicoid satisfy Euler–Lagrange equations too and proved that mean curvature vanishing surfaces are in fact minimal surfaces. In 1830, Heinrich Scherk used the Euler–Lagrange equation to find the nontrivial examples of complete minimal surfaces. The first golden era of minimal surface theory began in mid-19$^{\text{th}}$ century after the discovery made by J. Plateau. In 1867, Schwarz [6] found the solution of *Plateau–Bézier problem* for a general quadrilateral by an appeal to methods in complex analysis. Specifically, he discovered CLP (crossed layers of parallels) surface Schwarz T for tetragonal, Schwarz D for diamond, Schwarz H for hexagonal, and Schwarz P for primitive surfaces. Weierstrass and Enneper [4] developed representation formulas for minimal surface in conformal parametrization as a pair of holomorphic and meromorphic functions. Weierstrass and Enneper parametric representation of minimal surfaces serves as bridge between the geometry and the complex analysis [7]. Later on, achievements of Jesse Douglas [8], Rado [9], Tonelli [10], Courant [7, 11], Wang [12] (minimal surfaces based modelling of atoms and molecules), and others contributed many revolutionary results in the subsequent years in minimal surfaces spanned by given boundary curves. General solution of *Plateau–Bézier problem* remained challenging for it involves extremization of area functional that is highly nonlinear in its integrand. In 1931, American mathematician J. Douglas [8] and, in 1933, Hungarian Tibor Radó [9] showed the existence of a minimal surface spanned by a closed curve by exchanging the area integral by rather a simpler integral the so-called Douglas-Dirichlet integral. Revolutionary achievements in subsequent years by Courant [11], Max. Shiffman [13], Morse [14], Tompkins [14], Robert-Osserman [15], and others are worth mentioning that contributed many significant results to theory of minimal surfaces. A detailed account of related results of measure theory and differential geometry for *Plateau–Bézier problem* can be seen in reference [16]. Struwe [17] points out the parametric minimal surfaces and the developments in *Plateau–Bézier problem* with a concise treatment of constant mean curvature surfaces. Harrison [18] considered *Plateau–Bézier problem* as *test problem* for differential chains Theory.

The properties of minimal surfaces provide a support for shape modelling and shape fairing techniques. Minimal surface theory is an area of contemporary research and is of great importance in engineering design, computer-aided geometric designs (CAGD), architectural design, and biology that includes foams, domes, and cell membranes, and so on. In 2004, Monterde [19] used tensor product Bézier surfaces to find an approximate minimal surface giving a continuous surface spanned by a given boundary curve, called the *Plateau–Bézier problem*. They extremized the Dirichlet functional by finding the gradient of the functional with respect to the control points. Monterde and Ugail [20], in 2006, introduced a different functional, namely, the general bi-quadratic functional, which could be reduced to a functional introduced by Farin and Hansford [21], standard biharmonic functional introduced by Schneider and Kobbelt [22], or Bloor and Wilson's modified biharmonic functional [23]. In 2011, Monterde [24] constructed triangular Bézier patches by introducing a variational approach, and in 2015, Monterde and Ugail [25] found the quasi-harmonic surface as the extremal of quasi-harmonic functional in an attempt to solve the related *Plateau–Bézier problem*. Farin and Handford [26] obtained combinatorial relations between q-Genocchi numbers and polynomials with weight α and β and integral representation of weighted q-Bernstein polynomials. Bernstein polynomial is an important class of polynomials related to special polynomials like Bernoulli polynomials, Euler polynomials, and so on, in the theory of analytical numbers, and they find their applications in designing smooth curves and surfaces called Bézier curves and Bézier surfaces. They play a significant role for constructing the surfaces of different shapes and desired characteristics that depend on the choice of Bernstein polynomials, namely, the classical Bernstein polynomials, shifted knots Bernstein polynomials, and q-Bernstein polynomials, generally called the modified Bernstein polynomials. An extension of q-Bernstein polynomials in surface theory is (p,q)-Bernstein polynomials, basic properties, and generating functions for Bernstein polynomial-related results with the help of (p,q)-calculus, which can be seen in references [27, 28]. Araci [29] and Jang et al. [30] study the q-analogue of Euler numbers and polynomials naturally arising from the p-adic fermionic integrals on Z_p and investigate some properties for these numbers and polynomials. We use one of these Bernstein polynomials, recently introduced by Kim [31] to formulate the related Bézier surfaces for extreme values of Dirichlet functional resulting in a quasi-minimal surface.

The minimal surfaces occur naturally in the developing fields, namely, CAGD, (computer-aided geometric design), CG (computer graphics), and CAD (computer-aided design) in mathematical. The minimal surfaces are known through the variation of the respective area functional. The area functional involves square-root in its integrand, and it is not always possible to find the solution of the PDE as the outcome of the vanishing condition of its gradient for a surface to be a minimal surface. However, the variation of other functionals is more useful for some specific desired features of a surface based on the restriction or the constraint. These constraints are usually an integral to be minimized, for instance, an energy integral, for example, the Dirichlet functional, rms of mean curvature, quasi-harmonic functional, and biharmonic functional. These variational techniques can be applied to a famous class of surfaces known as Bézier surfaces, which have their own importance for their useful properties and applications in CAGD, CG, and CAD. There are two main categories of Bézier patches, namely, the rectangular and triangular Bézier patches. For a given Bézier control net of points P_{jk}, a rectangular Bézier surface is defined as

$$x(s,t) = \sum_{j=0}^{n}\sum_{k=0}^{m} \mathscr{B}_j^n(s)\mathscr{B}_k^m(t)P_{jk}, \tag{1}$$

where $\mathscr{B}_j^n(s)$ are the Bernstein polynomials. The rectangular Bézier surfaces (1) are based on the univariate Bernstein polynomials:

$$\mathscr{B}_j^n(s) = \binom{n}{j} s^j(1-s)^{n-j}, \tag{2}$$

which named after the Sergei Natanovich Bernstein [32], where

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} \tag{3}$$

are the usual binomial coefficients. The triangular Bézier patches $S(s,t,r) = \sum_{j+k+l=n} P_{jkl}B^n_{j,k,l}(s,t,r)$ are expressed in terms of bivariate Bernstein polynomials $B^n_{j,k,l}(s,t,r) = \binom{n}{j,k,l} s^j t^k r^l$. One of the problems in CAGD, for example, is to construct a surface of least area amongst all the surfaces spanned by the prescribed boundary, called the *Plateau–Bézier problem*. A surface of least area can be determined as one of the critical points of the area functional by finding constraints on the interior control points as the outcome of vanishing condition for the gradient of area functional; however, the area functional involves square root in its integrand and makes it difficult to find the surface as one of its critical points. The area functional can be replaced by some other suitable energy functional having similar properties as that of area functional or may be some other desired feature of the surface. It is then possible to find the gradient of this chosen energy functional (constraint functional in the form of an integral). The vanishing condition of the gradient of the functional gives linear algebraic constraints on the unknown control points in terms of known boundary control points. One of the widely used class of surfaces is known as the Bézier surfaces based on Bernstein polynomials as has been done by Monterde [19,20], and he found the quasi-minimal Bézier surface as the extremal of Dirichlet functional, other related works for ansatz method, and for the vanishing condition of certain energy functionals for the Bézier surfaces based on modified Bernstein polynomials and Coons patch one can see Ahmad et al. [33–38].

As mentioned above, one of the widely used restrictions is to find out the minimal Bézier surface as the extremal of various energy functionals by the vanishing condition of gradient of such a functional. We choose one of such restrictions, namely, the Bézier surface with q-Bernstein polynomials, and find the corresponding minimal surface as the extremal of Dirichlet functional by finding the vanishing condition of gradient of this Dirichlet functional for Bézier surface with q-Bernstein polynomials, which gives us constraints on the interior control points in terms of boundary control points for which the new surface is minimal. It is to be remarked that the minimizers of a certain chosen functional are possibly the candidate functions for a quasi-minimal surface obtained from the zero-functional gradient of the surface $x(s,t)$, which is equivalent to finding the extremal points of a function of several variables in calculus. The functional gradient equated to zero gives us linear constraints on the unknown interior control points depending on the known boundary control points. The surface spanned by these new interior control points with prescribed border is minimal, which we refer to as the q-Bernstein quasi-minimal Bézier surface. The applications of these surfaces are in building and material sciences, and formation of a surface in CAGD, for example, in modelling, a problem related to protest investigation and mechanics of cell material. Bézier models can assist in designing the computational-based prediction models, for instance, for human-engineered representations for their possible applications for machine learning capabilities [39–42].

Our aim is to solve Plateau q-Bernstein–Bézier problem, which is to find the q-Bernstein–Bézier surface as one of the critical points of Dirichlet functional instead of area functional, and the resulting surface has minimal area from amongst all the possible q-Bernstein–Bézier surfaces constructed from the given prescribed boundary control points. The q-Bernstein–Bézier surface can be written as follows:

$$x(s,t) = \sum_{j=0}^{m}\sum_{k=0}^{n} \mathcal{Q}_{j,q}^{m}(s)\mathcal{Q}_{k,q}^{n}(t)P_{jk}, \tag{4}$$

where $\mathcal{Q}_j^n(s)_q$ are the q-Bernstein polynomials introduced by Kim [31] in 2011. The Kim's q-Bernstein polynomials are different from the Philips version of q-Bernstein polynomials (1997) [26] that depend on q-integers, and later in 2003, Oruc and Phillips obtained related Bézier curves [43] and some interesting properties. We intend to find the Kim's q-Bernstein quasi-minimal Bézier surface as the extremal of the Dirichlet functional.

$$D(x(s,t)) = \frac{1}{2}\int_{\mathcal{R}}\left(\|x_s\|^2 + \|x_t\|^2\right)du\ dv, \tag{5}$$

for Kim's q-Bernstein–Bézier surface (equation (4)). The vanishing condition of a functional gradient generates a system of algebraic conditions on the unknown inner control points as boundary control points.

The rest of the article is organized as follows: in the forthcoming Section 2, we give few definitions of differential geometry-related quantities for curves and surface, the minimal surfaces, q-Bernstein polynomials, and Kim type [31] q-Bernstein–Bézier surfaces, specific classes of surfaces, their basic construction schemes, properties of the general Bézier curves with kim operator [31], the generalized rational Bézier surfaces, and the q-Bernstein–Bézier surfaces. In Section 3, we develop a technique to find the quasi-minimal surfaces corresponding to q-Bernstein–Bézier surfaces. In the same section, we have included the illustrative examples for the bi-quadratic and bi-cubic q-Bernstein–Bézier surfaces as the application of the technique developed. Finally, Section 4 includes the final remarks and the future prospects of the work.

2. Preliminaries

In this section, we give few geometric quantities and basics of curve and surface theory, q-Bernstein polynomials, classical Bézier and q-Bézier curves and surfaces, derivatives, integrals, mean, and Gaussian curvature of q-Bézier surfaces.

Definition 1. The area functional of a surface.

A surface $x(s,t)$ of minimal area is a surface that locally minimizes its area. This is equivalent to having zero mean curvature:

$$H = \frac{1}{2}\frac{Eg - 2Ff + Ge}{EG - F^2}. \tag{6}$$

For every parametrization, which is direct consequence of minimizing the area functional,

$$\begin{aligned} A(\mathcal{P}) &= \int_{\mathcal{R}} \|x_s \wedge x_t\| \mathrm{d}s\mathrm{d}t \\ &= \int_{\mathcal{R}} \left(EG - F^2\right)^{1/2} \mathrm{d}s\mathrm{d}t, \end{aligned} \tag{7}$$

where $E = \langle x_s, x_s\rangle, F = \langle x_s, x_t\rangle$, and $G = \langle x_t, x_t\rangle$ are coefficients of the quadratic form $I(s,t) = \langle dx(s,t), dx(s,t)\rangle$, for $dx(s,t) = x_s ds + x_t dt$, a $1-1$ linear mapping of vectors $(ds\ ,dt)$ onto $dx(s,t) = x_s ds + x_t dt$, which lies in the tangent plane. The quadratic form $I(s,t)$, called first fundamental form, is usually written as $ds^2 = E(s,t)ds^2 + 2F(s,t)ds\ dt + G(s,t)dt^2$ for the surface $x(s,t)$.

Definition 2. Bernstein polynomials, Bézier curves, and surfaces.

Blending functions are used to define the curves and surfaces in parameterized form, and they effect the type and the shape of curves and surfaces. A set of points used to produce a curve or a surface is called a set of control points for that curve or the surface. If we denote blending functions by $f_j(s)$ and the control net of points by P_j, then the parameterized form of the curve for $0 \le s \le 1$ is $x(s) = \sum_{j=0}^{n} f_j(s)P_j$. Similarly for a net of control points P_{jk} for $j,k = 0,\dots,n$ and the blending functions $f_{jk}(s,t)$, the parameterized form of the surface is $x(s,t) = \sum_{j,k=0}^{n} f_{jk}(s,t)P_{jk}$, where $0 \le s,t \le 1$. For a given continuous function f on the interval $[0,1]$, the Bernstein polynomial is expressed as

$$B_n(f;u) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)B_j^n(u) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)\binom{n}{j}u^j(1-u)^{n-k}, \tag{8}$$

where n is a positive integer. For each function f, the equation (8) results in a sequence of Bernstein polynomials. The continuity of the function f assures the uniform convergence of Bernstein polynomials to the function f, and this means that on the interval $[0,1]$, $\lim_{n\longrightarrow\infty} B_n(f;u) = f$. It can be seen from (8) that for $u = 0$, $B_n(f;0) = f(0)$, and for

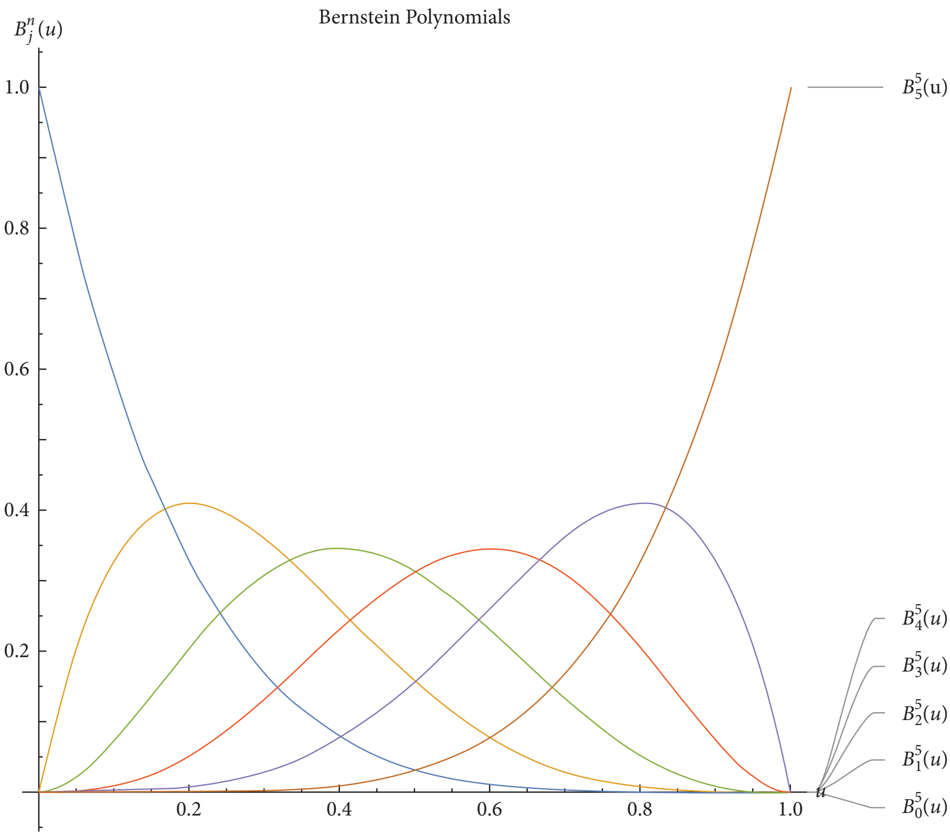

FIGURE 1: Polynomials in Bernstein form, $\mathscr{B}_0^5(u), \mathscr{B}_1^5(u), \mathscr{B}_2^5(u), \mathscr{B}_3^5(u), \mathscr{B}_4^5(u), \mathscr{B}_5^5(u)$, are shown.

$u = 1$, $B_n(f;1) = f(1)$. For $f = 1$, (8) reduces to $B_n(f;u) = B_n(1;u) = 1$, whereas for $f = u$, (8) reduces to $B_n(f;u) = B_n(u;u) = u$, and for $f = u^2$, it is $B_n(f;u) = B_n(u^2;u) = (n-1)u^2/n + u/n$ and so on. Bernstein [32] introduced the so-called the Bernstein polynomials, a linear combination of functions $\mathscr{B}_j^n(u)$, $\mathscr{B}_n(u) = \sum j = 0n\beta_j\mathscr{B}_j^n(u)$, also called polynomials in Bernstein form, with β_k as the Bernstein coefficients or Bézier coefficients $\mathscr{B}_k^n(u)$ (equations (2) and (3)), and the Bernstein operator of order n for $k, n \in z\,(0 \le k \le n)$. For instance, the Bernstein polynomials of degree $n = 5$, namely, $\mathscr{B}_0^5(u), \mathscr{B}_1^5(u), \mathscr{B}_2^5(u), \mathscr{B}_3^5(u), \mathscr{B}_4^5(u)$, and $\mathscr{B}_5^5(u)$, are shown in Figure 1.

A Bézier curve is a parametric curve, which is used in computer graphics and related fields [2,19]. The Bézier curve depends on Bernstein polynomials, which are called the blending functions or the basis of Bézier curve with a set of $(n+1)$ control points (also called Bézier points) denoted by $\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n$. A Bézier curve of degree n is given in the form $x(s) = \sum_{j=0}^n \mathscr{B}_j^n(s) P_j$ for the Bernstein polynomials $\mathscr{B}_j^n(s)$ (equation (2)) of degree n for $u \in [0,1]$. Bézier surfaces $\mathbf{x}(s,t)$ (equation (1)) are the higher dimension generalization of Bézier curves for a given set of $n+1, m+1$ control points $\{P_{jk}\}_{j,k=0}^{m,n}$ for the blending functions $\mathscr{B}_j^n(s)\mathscr{B}_k^m(t) = \mathscr{B}_{j,k}^{n,m}(s,t)$: $\mathscr{R}^2 \longrightarrow \mathscr{R}$, where $\mathscr{B}_j^n(s)$ and $\mathscr{B}_k^m(t)$ are Bernstein basis functions given by the equation (2) for $0 \le s, t \le 1$ (Figure 2).

Definition 3. q-Bernstein–Bézier curves and surfaces.

A q-Bernstein–Bézier surface (4) is a Bézier surface based on q-Bernstein polynomials, where q-Bernstein polynomials $\mathscr{Q}_{j,q}^m(s)$, the function of curve parameter s, serve as the blending functions for the Bézier surface, taken in the form,

$$\mathscr{Q}_{j,q}^m(s) = \binom{m}{j} [s]_q^j [1-s]_{q^{-1}}^{-j+m}, \quad [s]_q^j = \left(\frac{1-q^s}{1-q}\right)^j, \tag{9}$$

where $\binom{m}{j}$ are the usual binomial coefficients, $[s]_q^j$ is the Kim's q-Bernstein operator (for detail, see Equation (1.7) of reference [31]) for $0 \le q \le 1$. Note that $\lim_{q\longrightarrow 1} [s]_q = \lim_{q\longrightarrow 1} 1 - q^s/1-q = s$ and $\lim_{q\longrightarrow 1} [1-s]_{q^{-1}} = \lim_{q\longrightarrow 1} 1 - (q^{-1})^{1-s}/\ 1-q^{-1} = 1-s$, so that the q-Bernstein polynomials $\mathscr{Q}_{j,q}^m(s)$ reduce to the classical Bernstein polynomials $\mathscr{B}_k^n(s)$. In particular, the q-Bernstein polynomials of degrees $n = 1, 2, 3$ are shown in Figure 2. Plugging the q-Bernstein polynomials given by equation (9) for the surface parameters s, t in the equation (4), the Bézier surface with q-Bernstein polynomials may be rewritten as

$$x(s,t) = \sum_{j,k=0}^{m,n} \binom{m}{j}\binom{n}{k} [s]_q^j [t]_q^k [1-s]_{q^{-1}}^{m-j} [1-t]_{q^{-1}}^{n-k} P_{jk}. \tag{10}$$

The bi-quadratic q-Bernstein–Bézier surface and the bi-cubic q-Bernstein–Bézier surface obtained from the equation (10) for $q = 0.2$ for the prescribed are shown in Figure 3.

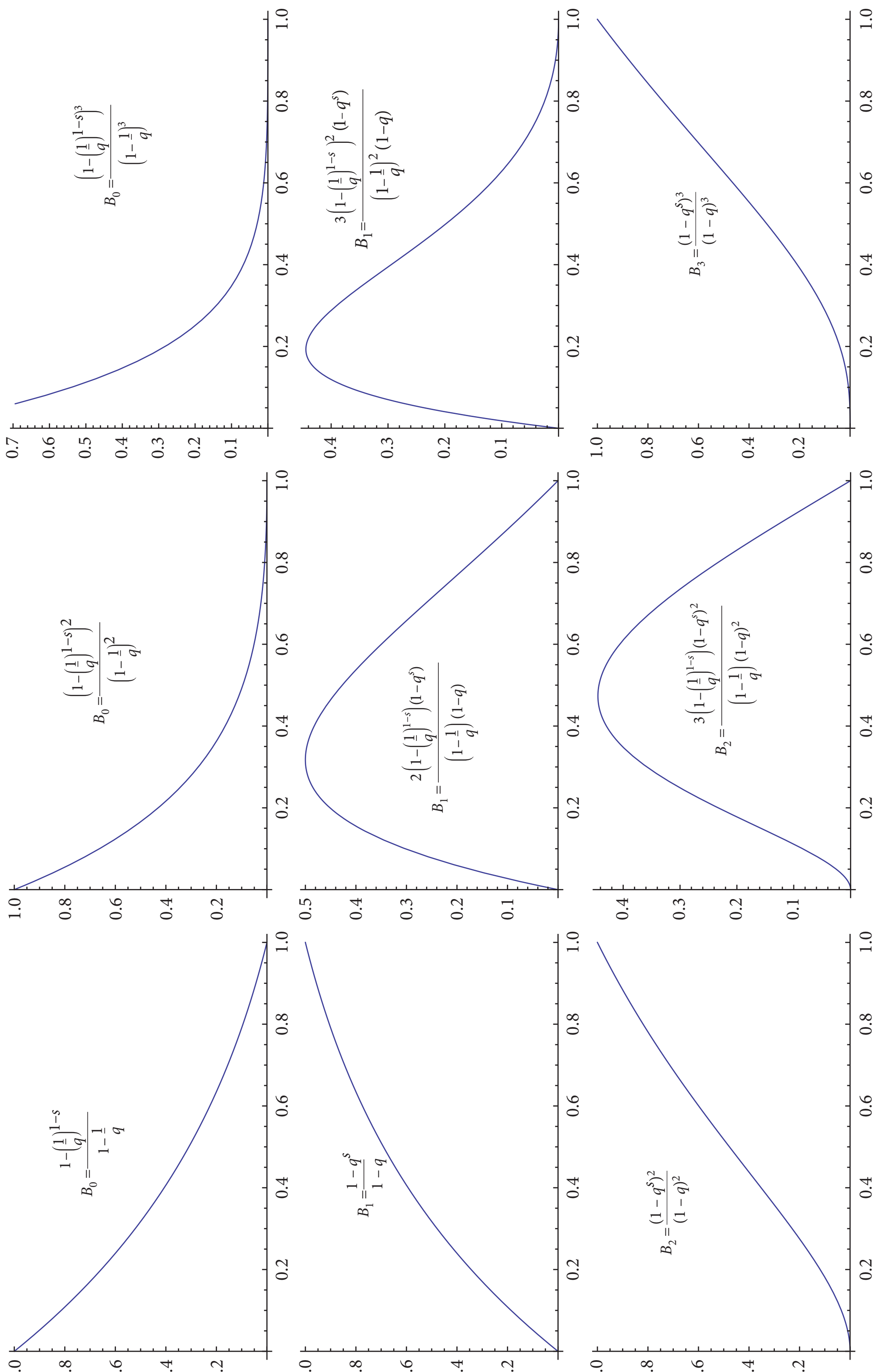

Figure 2: q-Bernstein polynomials of various degrees are shown.

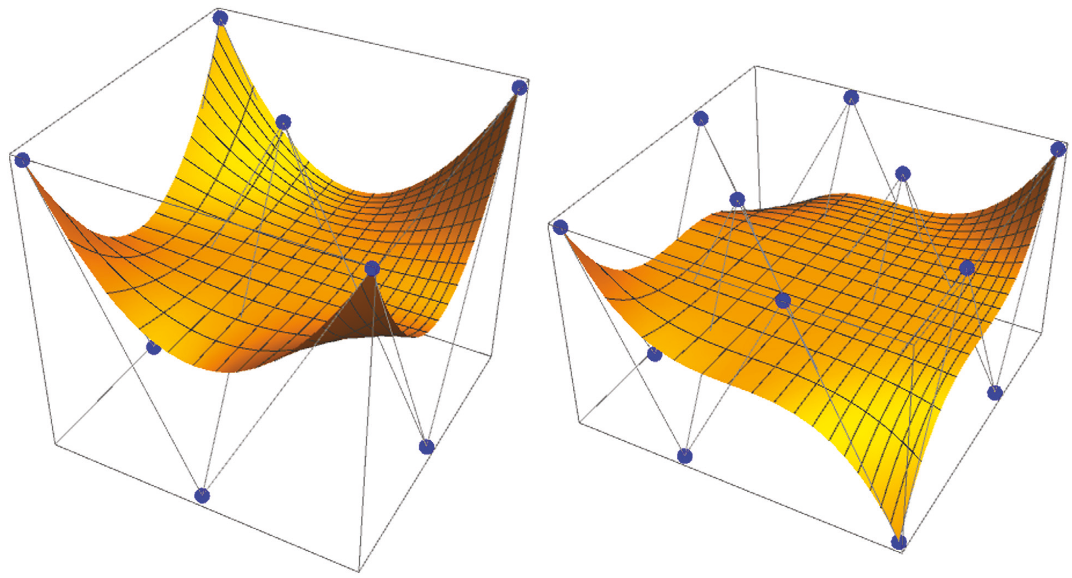

FIGURE 3: Bi-quadratic and bi-cubic q-Bernstein–Bézier surfaces for the prescribed border.

Definition 4. Partial derivatives of q-Bernstein–Bézier surface with respect to the control points.

The partial derivatives $(\partial \mathbf{x}_s/\partial x^a_{pq}) = (\partial/\partial s)(\sum_{j,k=0}^{m,n} \mathcal{Q}^m_{j,q}(s)\mathcal{Q}^n_{k,q}(t)\partial P_{jk}/\partial x^a_{pq})$ of tangent vectors $\mathbf{x}_s$ to the coordinate curves on the Bézier surface with q-Bernstein functions based on the Kim operator [31] with respect to the components $x^a_{pq}(p = 0, 1, 2, \ldots, m, q = 0, 1, 2, \ldots n, a = 1, 2, 3)$ of control points P_{pq}, where $P_{jk} = (x_{jk}, y_{jk}, z_{jk}) = (x^1_{jk}, x^2_{jk}, x^3_{jk})$ are the control points as mentioned above, $\partial P_{jk}/\partial x^a_{pq}$ is one of the standard basis vectors e^a for $p = j, q = k$, and otherwise, for $p \neq j$ or $q \neq k$, $\partial P_{jk}/\partial x^a_{pq}$ is zero vector, which can be written as

$$\frac{\partial x_s}{\partial x^a_{jk}} = \frac{\partial}{\partial s}\left(\mathcal{Q}^m_{j,q}(s)\mathcal{Q}^n_{k,q}(t)e^a\right). \tag{11}$$

The q-Bernstein polynomial [31], for $m \geq j$, $\mathcal{Q}^m_{j,q}(s)$, is defined as

$$\mathcal{Q}^m_{j,q}(s) = \binom{m}{j}[s]^j_q[1-s]^{m-j}_{q^{-1}}, \tag{12}$$

and $\mathcal{Q}^m_{j,q}(s) = 0$; otherwise, where $\binom{m}{j}$ is the usual binomial coefficient and

$$[s]^j_q = \left(\frac{1-q^s}{1-q}\right)^j, [1-s]^{(m-j)}_{q^{-1}} = \left(\frac{1-(q^{-1})^{(1-s)}}{1-(q^{-1})}\right)^{(m-j)}, \tag{13}$$

the partial derivative of q-Bernstein polynomial $\mathcal{Q}^m_j(s)_q$ with respect to s is given by

$$\frac{\partial}{\partial s}\mathcal{Q}^m_{j,q}(s) = \binom{m}{j}\left(-j[s]^{j-1}_q[1-s]^{m-j}_{q^{-1}} + (m-j)[s]^j_q[1-s]^{m-j-1}_{q^{-1}}\right)\left(\frac{q^s \log q}{1-q}\right). \tag{14}$$

Definition 5. Integral of q-Bernstein polynomial

The q-Bernstein polynomials (9) for the Kim operator $[s]^j_q$ can be written as

$$\mathcal{Q}^m_{j,q}(s) = \binom{m}{j}\left(\frac{1-q^s}{1-q}\right)^j\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j}. \tag{15}$$

The product of two q-Bernstein polynomials with same parameter s is given by

$$\mathcal{Q}^m_{j,q}(s)\mathcal{Q}^n_{k,q}(s) = \binom{m}{j}\binom{n}{k}\left(\frac{1-q^s}{1-q}\right)^j\left(\frac{1-q^s}{1-q}\right)^k\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{n-k}, \tag{16}$$

and it ((16) can be written as

$$\mathcal{Q}_{j,q}^{m}(s)\mathcal{Q}_{k,q}^{n}(s)=\binom{m}{j}\binom{n}{k}\left(\frac{1-q^{s}}{1-q}\right)^{j+k}\cdot\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{(m+n)-(j+k)}. \tag{17}$$

However, the $(j+k)$-th term of q-Bernstein polynomial of degree $(m+n)$ is given by

$$\mathcal{Q}_{j+k,q}^{m+n}(s)=\binom{m+n}{j+k}\left(\frac{1-q^{s}}{1-q}\right)^{j+k}\cdot\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{(m+n)-(j+k)}. \tag{18}$$

Thus, the product of two q-Bernstein polynomials (17) can be written in the form

$$\mathcal{Q}_{j,q}^{m}(s)\mathcal{Q}_{k,q}^{n}(s)=\binom{m}{j}\binom{n}{k}\binom{m+n}{j+k}^{-1}\mathcal{Q}_{j+k,q}^{m+n}(s). \tag{19}$$

For the integral of q-Bernstein polynomial based on Kim operator and the integral of product of two q-Bernstein polynomials having the same parameter s, we state the result below.

Theorem 1. *The integral of $q^{2s}\mathcal{Q}_{j,q}^{m}(s)$, where $\mathcal{Q}_{j,q}^{m}(s)$ is the q-Bernstein polynomial for $0\le i\le m-j$ and $0\le p\le i+j$, with respect to the parameter s, is given by*

$$\int_0^1 q^{2s}\mathcal{Q}_{j,q}^{m}(s)ds=\sum_{i=0}^{m-j}\sum_{p=0}^{i+j}(-1)^{i+p}\binom{m}{j}\binom{m-j}{i}\binom{i+j}{p}\cdot\frac{(1-q)^{-i-j}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)}, \tag{20}$$

where

$$\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}[s]_q^j[1-s]_{q^{-1}}^{m-j},\text{and }\lim_{p\longrightarrow 0}\frac{q^p-1}{\log\left(q^p\right)}=1. \tag{21}$$

Proof. In order to find out the above integral, we shall use the following result that helps us to write the product of $[s]_q^j$ and $[1-s]_{q^{-1}}^{m-j}$ in the following form [26]:

$$q^{2s}[s]_q^j[1-s]_{q^{-1}}^{m-j}=q^{2s}\sum_{i=0}^{m-j}(-1)^i\binom{m-j}{i}[s]_q^{i+j}, \tag{22}$$

which can then be used to establish the given result. Let us write the integral of $[s]_q^j$ as follows:

$$\int_0^1 q^{2s}[s]_q^j ds=\int_0^1 q^{2s}\left(\frac{1-q^s}{1-q}\right)^j ds=(1-q)^{-j}I_j, \tag{23}$$

where

$$I_j=\int_0^1 q^{2s}(1-q^s)^j ds\ . \tag{24}$$

The binomial expansion of $(1-q^s)^j$ is given by

$$\begin{aligned}q^{2s}(1-q^s)^j&=q^{2s}\sum_{k=0}^{j}(-1)^p\binom{j}{p}(q^s)^p\\&=\sum_{p=0}^{j}(-1)^p\binom{j}{p}\left(q^{p+2}\right)^s,\end{aligned} \tag{25}$$

which enable us to write the integral (24) in the form

$$I_j=\sum_{p=0}^{j}(-1)^p\binom{j}{p}\int_0^1\left(q^{p+2}\right)^s ds, \tag{26}$$

and thus, the integral (26) reduces to

$$\begin{aligned}I_j&=\sum_{p=0}^{j}(-1)^p\binom{j}{p}\left|\frac{(q^{p+2})^s}{\log(q^{p+2})}\right|_0^1\\&=\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{q^{p+2}-1}{\log\left(q^{p+2}\right)},\end{aligned} \tag{27}$$

where

$$I_0=1. \tag{28}$$

Plugging the (27) in (23), we get

$$\int_0^1[s]_q^j ds=(1-q)^{-j}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{q^{p+2}-1}{\log\left(q^{p+2}\right)}, \tag{29}$$

and thus,

$$\int_0^1[s]_q^{i+j}ds=(1-q)^{-i-j}\sum_{p=0}^{i+j}(-1)^p\binom{i+j}{p}\frac{q^{p+2}-1}{\log\left(q^{p+2}\right)}. \tag{30}$$

The integral of (22)

$$\int_0^1[s]_q^j[1-s]_{q^{-1}}^{m-j}ds=\sum_{i=0}^{m-j}(-1)^i\binom{m-j}{i}\int_0^1[s]_q^{i+j}ds \tag{31}$$

along with (30) is given by

$$\int_0^1[s]_q^j[1-s]_{q^{-1}}^{m-j}ds=\sum_{i=0}^{m-j}(-1)^i\binom{m-j}{i}\cdot(1-q)^{-i-j}\sum_{p=0}^{i+j}(-1)^p\binom{i+j}{p}\cdot\frac{q^{p+2}-1}{\log\left(q^{p+2}\right)}, \tag{32}$$

which reduces to

$$\int_0^1 [s]_q^j [1-s]_{q^{-1}}^{m-j} ds = \sum_{l=0}^{m-j}\sum_{p=0}^{i+j} (-1)^{i+p} (1-q)^{-i-j} \binom{m-j}{i} \cdot \binom{i+j}{p} \frac{q^{p+2}-1}{\log\left(q^{p+2}\right)}, \tag{33}$$

where

$$\lim_{p\longrightarrow 0} \frac{q^p-1}{\log\left(q^p\right)} = 1. \tag{34}$$

Hence, the integral of q-Bernstein polynomial is given by (20). The results in the corollaries below are given for later use in Section 3. □

Corollary 1. *The integral of q-Bernstein polynomial with respect to the parameter s when m is replaced by $2m-2$ and j by $j+l-1$ is given by*

$$\int_0^1 q^{2s} \mathcal{Q}_{j+l-1,q}^{2m-2}(s) ds = \sum_{i=0}^{2m-j-2}\sum_{p=0}^{i+j+l-1} (-1)^{i+p} \binom{2m-2}{j} \cdot \binom{2m-j-l-1}{i}\binom{i+j+l-1}{p} \cdot \frac{(1-q)^{-i-j-l+1}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)}. \tag{35}$$

Corollary 2. *The integral of q-Bernstein polynomial with respect to the parameter s when m replaced by $2m-2$ and j by $j+l$ is given by*

$$\int_0^1 q^{2s} \mathcal{Q}_{j+l,q}^{2m-2}(s) ds = \sum_{i=0}^{2m-j+l-2}\sum_{p=0}^{i+j+l} (-1)^{i+p} \binom{2m-2}{j+l} \cdot \binom{2m-j+l-2}{i}\binom{i+j+l}{p} \cdot \frac{(1-q)^{-i-j-l}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)}. \tag{36}$$

Corollary 3. *The integral of q-Bernstein polynomial with respect to the parameter s when m replaced by $2m$ and j by $j+l$ is given by*

$$\int_0^1 q^{2s} \mathcal{Q}_{j+l,q}^{2m}(s) ds = \sum_{i=0}^{2m-j+l}\sum_{p=0}^{i+j+l} (-1)^{i+p} \binom{2m}{j+l} \cdot \binom{2m-j+l}{i}\binom{i+j+l}{p} \cdot \frac{(1-q)^{-i-j+l}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)}. \tag{37}$$

Corollary 4. *The integral of q-Bernstein polynomial with respect to the parameter t when $m=2n-2$, $j=k+r-1$ is given by*

$$\int_0^1 q^{2t} \mathcal{Q}_{k+r-1,q}^{2n-2}(t) dt = \sum_{i=0}^{2n-k-r-1}\sum_{p=0}^{i+k+r-1} \left((-1)^{i+p} \binom{2n-2}{k+r-1}\binom{2n-k-r-1}{i} \times \binom{i+k+r-1}{p} \frac{(1-q)^{-i-k-r+1}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)} \right). \tag{38}$$

Corollary 5. *The integral of q-Bernstein polynomial with respect to the parameter t when m is replaced by $2n-2$ and j by $r+k$ is given by*

$$\int_0^1 q^{2t} \mathcal{Q}_{k+r,q}^{2n-2}(t) dt = \sum_{i=0}^{2n-2-k+r}\sum_{p=0}^{i+k+r} (-1)^{i+p} \binom{2n-2}{k+r}\binom{2n-2-k+r}{i}\binom{i+k+r}{p} \frac{(1-q)^{-i-k+r}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)}. \tag{39}$$

Corollary 6. *The integral of q-Bernstein polynomial with respect to the parameter t when $m=2n$, $j=r+k$ is given by*

$$\int_0^1 q^{2t}\mathcal{Q}^{2n}_{k+r,q}(t)dt = \sum_{i=0}^{2n-k+r}\sum_{p=0}^{i+k+r}(-1)^{i+p}\begin{pmatrix}2n\\ k+r\end{pmatrix}\begin{pmatrix}2n-k+r\\ i\end{pmatrix}\begin{pmatrix}i+k+r\\ p\end{pmatrix}\frac{(1-q)^{-i-k+r}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)}. \tag{40}$$

Definition 6. Mean and Gaussian curvature of q-Bernstein–Bézier surface.

For the mean and Gaussian curvature of the q-Bernstein–Bézier surface, (10), we find the fundamental coefficients E, F, G and e, f, g of the q-Bernstein surface with Kim operator $x_q(s,t)$, which requires computation of the partial derivatives of the surface $x_q(s,t)$ with respect to the surface parameters s, t. Note that the partial derivatives of the surface with respect to its surface parameters appear as terms involving the derivatives of q-Bernstein polynomials. The fundamental coefficients of the q-Bernstein–Bézier surface (10) help us to compute the mean curvature and the Gaussian curvature of the surface. The mean curvature (6) of the surface is

$$H = \sum_{j,k=0}^{m,n}\begin{pmatrix}\left(\frac{jk(m)-(j)q^{(1-t)}}{1-q^{(1-t)}} - \frac{j^2k^4q^{(t)}}{1-q^{(t)}}\right)\left(\frac{m^2(m)-(j)q^{(1-t)}}{1-q^{(1-t)}} - \frac{jm^2q^{(t)}}{1-q^{(t)}}\right)\\ \left(\frac{jk(m)-(j)q^{(1-s)}}{1-q^{(1-s)}} - \frac{j^2k^4q^{(s)}}{1-q^{(s)}}\right)\left(\frac{m^2(m)-(j)q^{(1-s)}}{1-q^{(1-s)}} - \frac{jm^2q^{(s)}}{1-q^{(s)}}\right)\\ \left(\frac{1-q^{(1-s)}}{1-q}\right)^{m-j}\left(\frac{1-q^{(1-t)}}{1-q}\right)^{m-j}\left(\frac{1-q^{(s)}}{1-q}\right)^{j}\left(\frac{1-q^{(t)}}{1-q}\right)^{j})\end{pmatrix}\frac{\log(q)}{1-q}P_{jk}. \tag{41}$$

The Gaussian curvature $\mathcal{K} = \mathcal{L}\mathcal{N} - \mathcal{M}^2/EG - F^2$ of the surface is given by the following expression:

$$\mathcal{K} = \sum_{j,k=0}^{m,n}\begin{pmatrix}\left(\frac{jk(m-j)q^{(1-t)}}{1-q^{(1-t)}} - \frac{j^2k^4q^{(t)}}{1-q^{(t)}}\right)^2\left(\frac{m^2(m-j)q^{(1-t)}}{1-q^{(1-t)}} - \frac{jm^2q^{(t)}}{1-q^{(t)}}\right)^2\\ \left(\frac{jk(m-j)q^{(1-s)}}{1-q^{(1-s)}} - \frac{j^2k^4q^{(s)}}{1-q^{(s)}}\right)^2\left(\frac{m^2(m-j)q^{(1-s)}}{1-q^{(1-s)}} - \frac{jm^2q^{(s)}}{1-q^{(s)}}\right)^2\\ \left(\frac{1-q^{(1-s)}}{1-q}\right)^{m-j}\left(\frac{1-q^{(1-t)}}{1-q}\right)^{m-j}\left(\frac{1-q^{(s)}}{1-q}\right)^{j}\left(\frac{1-q^{(t)}}{1-q}\right)^{j}\end{pmatrix}\frac{\log(q)}{1-q}P_{jk}. \tag{42}$$

We have discussed restricted class of surfaces, namely, the Bézier surfaces with q-Bernstein polynomials, the basic construction scheme of these surfaces, their properties based on kim operator [31], mean curvature, and Gaussian curvature of the surfaces. In the section below, we come up with a scheme for finding the quasi-minimal q-Bernstein–Bézier surface as the extremal of Dirichlet functional.

3. Quasi-Minimal q-Bernstein–Bézier Surfaces as the Extremal of Dirichlet Functional

This section is devoted to the problem of finding the quasi-minimal q-Bernstein–Bézier surface as the extremal of Dirichlet functional by determining the constraints on the interior control points of a given q-Bernstein–Bézier surface for the prescribed border. For this, we shall find the gradient of the Dirichlet functional (5) for the q-Bernstein–Bézier surface (10) with the prescribed border in terms of its boundary points and equate it to zero that results in a system of linear algebraic constraints. These algebraic constraints can be solved for interior control points in terms of boundary control points for a specific mesh structure. When these new interior control points are plugged in the q-Bernstein–Bézier surface, the emerging surface is a quasi-minimal surface. For illustration of the given scheme, we have included bi-quadratic and bi-cubic quasi-minimal q-Bernstein–Bézier surfaces in the section immediately after exploring the scheme.

Following proposition enables us to write the linear algebraic constraints satisfied by the interior control points of q-Bernstein–Bézier surface meshed with the boundary control points as the outcome of vanishing condition of gradient of the Dirichlet functional for the q-Bernstein–Bézier surface. The interior control points worked out from these linear algebraic constraints together with known boundary control points help us to construct the desired quasi-minimal q-Bernstein–Bézier surface. The proposition is given below.

Proposition 1. *A control net $\mathscr{P} = \{P_{jk}\}_{j,k=0}^{n,m}$ of q-Bernstein–Bézier surface of prescribed border is the extremal of Dirichlet functional if the control points $\{P_{jk}\}_{j,k=0}^{n,m}$ (for $0 < q < 1$) satisfy the following constraint equation:*

$$\left(\frac{m\log q}{1-q}\right)^2\sum_{i,j=0}^{m-1,n}\left(\xi_{j-1,l}^{m-1,m-1}\delta_{j+l-1}^{2m-2}(s)-\xi_{j,l}^{m-1,m-1}\delta_{j+l}^{2m-2}(s)\right)\xi_{k,r}^{n,n}\delta_{k+r}^{2n}(t)\Delta^{10}P_{ij}+\left(\frac{n\log q}{1-q}\right)^2 \sum_{i,j=0}^{m,n-1}\left(\xi_{k-1,r}^{n-1,n-1}\delta_{k+r-1}^{2n-2}(t)-\xi_{r,k}^{n-1,n-1}\delta_{r+k}^{2n-2}(t)\right)\xi_{l,j}^{m,m}\delta_{l+j}^{2m}(s)\Delta^{01}P_{lr}=0, \tag{43}$$

where

$$\delta_{j,q}^{m}(s)=\sum_{i=0}^{m-j}\sum_{p=0}^{j+l}(-1)^{i+p}\binom{m}{j}\binom{m-j}{i}\binom{j+i}{p}\cdot\frac{(1-q)^{-j-i}\left(q^{p+2}-1\right)}{\log\left(q^{p+2}\right)}. \tag{44}$$

Proof. Let $\mathscr{P} = \{P_{jk}\}_{j,k=0}^{n,m}$ be the control net of a q-Bernstein–Bézier surface with the corresponding patch defined in equation (4). The Dirichlet functional [44], equation (5), can be written in the following convenient form:

$$\mathscr{D}(\mathscr{P})=\frac{1}{2}\int_{\mathscr{R}}\left(\langle x_s,x_s\rangle+\langle x_t,x_t\rangle\right)\mathrm{d}s\mathrm{d}t. \tag{45}$$

Let us calculate the gradient of Dirichlet functional with respect to the coordinates (x_{jk}^a) of the control point $P_{jk} = (x_{jk}^1, x_{jk}^2, x_{jk}^3)$. For $a \in \{1,2,3\}$, $j = 1,2,\ldots,m-1$ and $k = 1,2,\ldots,n-1$, the gradient of the Dirichlet functional (45) can be written as

$$\frac{\partial D(\mathscr{P})}{\partial x_{jk}^a}=\int_{\mathscr{R}}\left(\langle\frac{\partial \mathbf{x}_s}{\partial x_{jk}^a},\mathbf{x}_s\rangle+\langle\frac{\partial \mathbf{x}_t}{\partial x_{jk}^a},\mathbf{x}_t\rangle\right)\mathrm{d}s\mathrm{d}t. \tag{46}$$

We need to find the partial derivatives $(\partial \mathbf{x}_s/\partial x_{jk}^a)$ and $(\partial \mathbf{x}_t/\partial x_{jk}^a)$ of the (46) for the q-Bernstein–Bézier surface (4). For the q-Bernstein–Bézier surface given by the equation (4),

$$\frac{\partial \mathbf{x}_s}{\partial x_{pq}^a}=\frac{\partial}{\partial s}\left(\sum_{j,k=o}^{n,m}\mathscr{Q}_{j,q}^m(s)\mathscr{Q}_{k,q}^n(t)\frac{\partial P_{jk}}{\partial x_{pq}^a}\right), \tag{47}$$

where $P_{jk} = (x_{jk}, y_{jk}, z_{jk}) = (x_{jk}^1, x_{jk}^2, x_{jk}^3) \equiv (x_{jk}^a)$, $(a = 1,2,3)$, and $\partial P_{jk}/\partial x_{pq}^a$ reduces to one of the standard basis vectors e^a for $p = j, q = k$; otherwise, for $p \neq j$ or $q \neq k$, $(\partial P_{jk}/\partial x_{pq}^a)$ is zero vector, and the equation (47) reduces to

$$\frac{\partial \mathbf{x}_s}{\partial x_{jk}^a}=\left(\frac{\partial}{\partial s}\mathscr{Q}_{j,q}^m(s)\right)\mathscr{Q}_{k,q}^n(t)e^a. \tag{48}$$

The q-Bernstein polynomials $\mathscr{Q}_{j,q}^m(s)$ (Kim [31]) are given by the following expression:

$$\mathscr{Q}_{j,q}^m(s)=\binom{m}{j}[s]_q^j[1-s]_{q^{-1}}^{m-j}, \tag{49}$$

where $\binom{m}{j}$ is the usual binomial coefficients,

$$[s]_q^j=\left(\frac{1-q^s}{1-q}\right)^j. \tag{50}$$

By virtue of the equation (50) for $[s]_q^j$, the q-Bernstein polynomial $\mathscr{Q}_{j,q}^m(s)$ (49) can be written in the form

$$\mathscr{Q}_{j,q}^m(s)=\binom{m}{j}\left(\frac{1-q^s}{1-q}\right)^j\left(\frac{1-\left(q^{-1}\right)^{1-s}}{1-q^{-1}}\right)^{m-j}, \tag{51}$$

the partial derivative of $\mathscr{Q}_{j,q}^m(s)$ (51) with respect to one of (the surface) parameter s,

$$\frac{\partial}{\partial s}\mathscr{Q}_{j,q}^m(s)=\binom{m}{j}\left(\left(\frac{\partial}{\partial s}\left(\frac{1-q^s}{1-q}\right)^j\right)\left(\frac{1-\left(q^{-1}\right)^{1-s}}{1-q^{-1}}\right)^{m-j}+\left(\frac{1-q^s}{1-q}\right)^j\frac{\partial}{\partial s}\left(\frac{1-\left(q^{-1}\right)^{1-s}}{1-q^{-1}}\right)^{m-j}\right), \tag{52}$$

for

$$\frac{\partial}{\partial s}\left(\frac{1-q^s}{1-q}\right)^j=-j\frac{q^s}{1-q}\left(\frac{1-q^s}{1-q}\right)^{j-1}\log q, \tag{53}$$

and

$$\frac{\partial}{\partial s}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j} = \frac{(m-j)(q^{-1})^{1-s}}{1-q^{-1}}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j-1}\log(q^{-1}), \tag{54}$$

can be written as

$$\begin{aligned}\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s) = \binom{m}{j}\Bigg(\Bigg(&-\frac{jq^{s}}{1-q}\left(\frac{1-q^{s}}{1-q}\right)^{j-1}\log q\Bigg)\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j} + \left(\frac{1-q^{s}}{1-q}\right)^{j} \\ &\times\left((n-j)\frac{(q^{-1})^{1-s}}{1-q^{-1}}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{n-j-1}\log(q^{-1})\right)^{m-j}\Bigg),\end{aligned} \tag{55}$$

which is then reduced to the form (using the property that $\log(q^{-1}) = -\log q$).

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s) = \binom{m}{j}\left(-\frac{jq^{s}}{1-q}\left(\frac{1-q^{s}}{1-q}\right)^{j-1}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j}\log q - \frac{(m-j)(q^{-1})^{1-s}}{1-q^{-1}}\left(\frac{1-q^{s}}{1-q}\right)^{j}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j-1}\log q\right). \tag{56}$$

Taking out the common factors from the above expression, we get

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s) = \binom{m}{j}\left(\frac{1-q^{s}}{1-q}\right)^{j}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j}\left(-\frac{jq^{s}}{1-q}\left(\frac{1-q^{s}}{1-q}\right)^{-1} - (m-j)\times\frac{(q^{-1})^{1-s}}{1-q^{-1}}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{-1}\right)\log q, \tag{57}$$

which can be further simplified in the form

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s) = \binom{m}{j}\left(\frac{1-q^{s}}{1-q}\right)^{j}\left(\frac{1-(q^{-1})^{1-s}}{1-q^{-1}}\right)^{m-j}\left(-\frac{jq^{s}}{1-q^{s}} - \frac{(m-j)(q^{-1})^{1-s}}{1-(q^{-1})^{1-s}}\right)\log q. \tag{58}$$

By virtue of the equation (50), last equation (21) can be written as

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}[s]_q^{j}[1-s]_{q^{-1}}^{m-j}\cdot\left(-\frac{jq^{s}}{1-q^{s}}-\frac{(m-j)\left(q^{-1}\right)^{1-s}}{1-\left(q^{-1}\right)^{1-s}}\right)\log q. \tag{59}$$

Note that for $-jq^s/1-q^s-(m-j)(q^{-1})^{1-s}/1-(q^{-1})^{1-s}=-j[s]_q^{-1}\times q^s/1-q-(m-j)[1-s]_{q^{-1}}^{-1}\times (q^{-1})^{1-s}/1-q^{-1}$ of the above (59) can be reduced to the following form:

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}[s]_q^{j}[1-s]_{q^{-1}}^{m-j}\left(-j[s]_q^{-1}\times\frac{q^{s}}{1-q}-(m-j)[1-s]_{q^{-1}}^{-1}\times\frac{\left(q^{-1}\right)^{1-s}}{1-q^{-1}}\right)\log q, \tag{60}$$

in which $(q^{-1})^{1-s}/1-q^{-1}=q^{s-1}/1-q^{-1}=q^sq^{-1}/1-\ q^{-1}=q^sq^{-1}/q^{-1}(q-1)=q^s/q-1$, and thus, we can be written in the equation (60) as

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}[s]_q^{j}[1-s]_{q^{-1}}^{m-j}\left(-j[s]_q^{-1}\times\frac{q^{s}}{1-q}-(m-j)[1-s]_{q^{-1}}^{-1}\times\frac{q^{s}}{q-1}\right)\log q, \tag{61}$$

and taking out the common factor from (61) yields

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}[s]_q^{j}[1-s]_{q^{-1}}^{m-j}\left(-j[s]_q^{-1}+(m-j)[1-s]_{q^{-1}}^{-1}\right)\frac{q^{s}}{1-q}\log q. \tag{62}$$

Combining the terms $[s]_q^j[1-s]_{q^{-1}}^{m-j}$ with the terms inside the parentheses of above (62), which is

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}\left(-j[s]_q^{j}[s]_q^{-1}[1-s]_{q^{-1}}^{m-j}+(m-j)[s]_q^{j}[1-s]_{q^{-1}}^{m-j}[1-s]_{q^{-1}}^{-1}\right)\frac{q^{s}}{1-q}\log q, \tag{63}$$

equation (63) can be written in the following form:

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}\left(-j[s]_q^{j-1}[1-s]_{q^{-1}}^{m-j}+(m-j)[s]_q^{j}[1-s]_{q^{-1}}^{m-j-1}\right)\left(\frac{q^{s}\log q}{1-q}\right). \tag{64}$$

Using the (49), we note that

$$\mathcal{Q}_{j-1,q}^{m-1}(s)=\binom{m-1}{j-1}[s]_q^{j-1}[1-s]_{q^{-1}}^{(m-1)-(j-1)}, \tag{65}$$

or

$$\binom{m-1}{j-1}^{-1}\mathcal{Q}_{j-1,q}^{m-1}(s)=[s]_q^{j-1}[1-s]_{q^{-1}}^{m-j}. \tag{66}$$

and in the similar way, we write

$$\binom{m-1}{j}^{-1}\mathcal{Q}_{j,q}^{m-1}(s)=[s]_q^{j}[1-s]_{q^{-1}}^{m-1-j}. \tag{67}$$

Plugging the (66 and 67) in the (64) allows us to write

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=\binom{m}{j}\left(-j\binom{m-1}{j-1}^{-1}\mathcal{Q}_{j-1,q}^{m-1}(s)+(m-j)\binom{m-1}{j}^{-1}\mathcal{Q}_{j,q}^{m-1}(s)\right)\left(\frac{q^{s}\log q}{1-q}\right). \tag{68}$$

It is to be noted that in the above (68), the binomial expressions $j\binom{m}{j}\binom{m-1}{j-1}^{-1}=m$ and $(m-j)\binom{m}{j}\binom{m-1}{j}^{-1}=m$, and thus, the (68) for partial derivative of q-Bernstein polynomial can be written in lower degree polynomials given by

$$\frac{\partial}{\partial s}\mathcal{Q}_{j,q}^{m}(s)=m\left(\mathcal{Q}_{j,q}^{m-1}(s)-\mathcal{Q}_{j-1,q}^{m-1}(s)\right)\left(\frac{q^{s}\log q}{1-q}\right). \tag{69}$$

Substituting (69) in equation (48), we obtain the relation for the partial derivative of the q-Bernstein–Bézier surface with respect to the surface parameter s as follows:

$$\frac{\partial \mathbf{x}_s}{\partial x_{jk}^{a}}=m\left(\frac{q^{s}\log q}{1-q}\right)\left(\mathcal{Q}_{j,q}^{m-1}(s)-\mathcal{Q}_{j-1,q}^{m-1}(s)\right)\mathcal{Q}_{k,q}^{n}(t)e^{a}. \tag{70}$$

Adopting the same procedure as done for the derivation of (70), we can find the partial derivative $(\partial x_t/\partial x_{jk}^{a})_q$ that, with respect to other surface parameter t,

$$\frac{\partial x_t}{\partial x_{jk}^{a}}=\frac{\partial}{\partial x_{jk}^{a}}\left(\frac{\partial x}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial x_{jk}^{a}}\right), \tag{71}$$

reduces to

$$\frac{\partial \mathbf{x}_t}{\partial x_{jk}^{a}}=\frac{\partial}{\partial t}\left(\mathcal{Q}_{j,q}^{m}(s)\mathcal{Q}_{k,q}^{n}(t)e^{a}\right)=\mathcal{Q}_{j,q}^{m}(s)\left(\frac{\partial}{\partial t}\mathcal{Q}_{k,q}^{n}(t)\right)e^{a}, \tag{72}$$

and we can write the partial derivative $\partial/\partial t\mathcal{Q}_{k}^{n}(t)$ in the similar way as we worked out (69). Thus, the equation (70) can be written by appropriately replacing s, m, and j by t, n, and k, respectively, in the above (70) to obtain

$$\frac{\partial x_t}{\partial x_{jk}^{a}}=n\left(\frac{q^{t}\log q}{1-q}\right)\mathcal{Q}_{j,q}^{m}(s)\left(\mathcal{Q}_{k,q}^{n-1}(t)-\mathcal{Q}_{k-1,q}^{n-1}(t)\right)e^{a}. \tag{73}$$

Substituting equations (70) and (73) in equation (46) to get

$$\frac{\partial D(\mathcal{P})}{\partial x_{jk}^{a}}=\int_{\mathcal{R}}\left(m\left(\frac{\log q}{1-q}\right)q^{s}\left(\mathcal{Q}_{j,q}^{m-1}(s)-\mathcal{Q}_{j-1,q}^{m-1}(s)\right)\mathcal{Q}_{k,q}^{n}(t)\langle e^{a},x_s\rangle+n\left(\frac{\log q}{1-q}\right)q^{t}\mathcal{Q}_{j,q}^{m}(s)\left(\mathcal{Q}_{k}^{n-1}(t)_q-\mathcal{Q}_{k-1}^{n-1}(t)_q\right)\langle e^{a},x_t\rangle\right)\mathrm{d}s\mathrm{d}t. \tag{74}$$

In above equation (74), we need now the partial derivatives $x_s(s,t)_q$ and $x_t(s,t)_q$ of q-Bernstein–Bézier surface $x(s,t)_q$. For this purpose, we use the following expression for the q-Bernstein–Bézier surface $x(s,t)_q$,

$$x(s,t)_q=\sum_{l=0}^{m}\sum_{r=0}^{n}\mathcal{Q}_{l,q}^{m}(s)\mathcal{Q}_{r,q}^{n}(t)P_{lr}, \tag{75}$$

and then find its partial derivative with respect to the surface parameter s and t. (This can be established by finding directly the partial derivative of $\mathcal{Q}_{k}^{n}(s)$.) The partial derivative of q-Bernstein–Bézier surface $x(s,t)_q$ with respect to its parameter s is

$$x_s(s,t)_q=\sum_{l=0}^{m}\sum_{r=0}^{n}\left(\frac{\partial}{\partial s}\mathcal{Q}_{l,q}^{m}(s)\right)\mathcal{Q}_{r,q}^{n}(t)P_{lr}. \tag{76}$$

Plugging the value of $\partial/\partial s\mathcal{Q}_{l,q}^{m}(s)$ from (69) in above (76) to obtain

$$x_s(s,t)_q=m\left(\frac{q^{s}\log q}{1-q}\right)\sum_{l=0}^{m-1}\sum_{r=0}^{n}\left(\mathcal{Q}_{l,q}^{m-1}(s)-\mathcal{Q}_{l-1,q}^{m-1}(s)\right)\cdot\mathcal{Q}_{r}^{n}(t)P_{lr}, \tag{77}$$

which can be written as

$$x_s(s,t)_q=-m\left(\frac{q^{s}\log q}{1-q}\right)\sum_{l=0}^{m-1}\sum_{r=0}^{n}\mathcal{Q}_{l,q}^{m-1}(s)\mathcal{Q}_{r,q}^{n}(t)\Delta^{10}P_{lr}. \tag{78}$$

By the symmetry of the resulting (78), we can write the partial derivative of q-Bernstein–Bézier surface $x(s,t)_q$ with respect to surface parameter t as

$$x_t(s,t)_q=-n\left(\frac{q^{t}\log q}{1-q}\right)\sum_{l=0}^{m}\sum_{r=0}^{n-1}\mathcal{Q}_{l,q}^{m}(s)\mathcal{Q}_{r,q}^{n-1}(t)\Delta^{01}P_{lr}. \tag{79}$$

Substituting (78) and (79) in (74), we get

$$\frac{\partial D(\mathscr{P})}{\partial x_{jk}^{a}} = \int_{\mathscr{R}} \left(\begin{array}{l} \left(\dfrac{m \log q}{1-q}\right)^{2} q^{2s}\left(\mathcal{Q}_{j-1,q}^{m-1}(s) - \mathcal{Q}_{j,q}^{m-1}(s)\right)\mathcal{Q}_{k,q}^{n}(t)\langle e^{a}, \displaystyle\sum_{l=0}^{m-1}\sum_{r=0}^{n}\mathcal{Q}_{l,q}^{m-1}(s)\mathcal{Q}_{r,q}^{n}(t)\Delta^{10}P_{lr}\rangle \\ +\left(\dfrac{n \log q}{1-q}\right)^{2} q^{2t}\mathcal{Q}_{j,q}^{m}(s)\left(\mathcal{Q}_{k-1,q}^{n-1}(t) - \mathcal{Q}_{k,q}^{n-1}(t)\right)\langle e^{a}, \displaystyle\sum_{l=0}^{m}\sum_{r=0}^{n-1}\mathcal{Q}_{l,q}^{m}(s)\mathcal{Q}_{r,q}^{n-1}(t)\Delta^{01}P_{lr}\rangle \end{array} \right) \mathrm{d}s\mathrm{d}t, \tag{80}$$

which can be written as

$$\frac{\partial D(\mathscr{P})}{\partial x_{jk}^{a}} = \int_{\mathscr{R}} \left(\frac{\log q}{1-q}\right)^{2} \left(\begin{array}{l} m^{2}q^{2s}\displaystyle\sum_{l=0}^{m-1}\sum_{r=0}^{n}\left(\mathcal{Q}_{j-1,q}^{m-1}(s)\mathcal{Q}_{l,q}^{m-1}(s) - \mathcal{Q}_{j,q}^{m-1}(s)\mathcal{Q}_{l,q}^{m-1}(s)\right)\mathcal{Q}_{k,q}^{n}(t)\mathcal{Q}_{r,q}^{n}(t)\langle e^{a}, \Delta^{10}P_{lr}\rangle \\ +n^{2}q^{2t}\displaystyle\sum_{l=0}^{m}\sum_{r=0}^{n-1}\mathcal{Q}_{j,q}^{m}(s)\mathcal{Q}_{l,q}^{n-1}(s)\left(\mathcal{Q}_{k-1,q}^{n-1}(t)\mathcal{Q}_{r,q}^{n-1}(t) - \mathcal{Q}_{k,q}^{n-1}(t)\mathcal{Q}_{r,q}^{n-1}(t)\right)\langle e^{a}, \Delta^{01}P_{lr}\rangle \end{array} \right) \mathrm{d}s\mathrm{d}t. \tag{81}$$

The equation (81) can be written as

$$\frac{\partial D(\mathscr{P})}{\partial x_{jk}^{a}} = R_{jk} + S_{jk}, \tag{82}$$

where

$$R_{jk} = \left(\frac{m \log q}{1-q}\right)^{2}\int_{\mathscr{R}} M_{jk}(s,t)ds\, dt, \tag{83}$$

and

$$S_{jk} = \left(\frac{n \log q}{1-q}\right)^{2}\int_{\mathscr{R}} N_{jk}(s,t)ds\, dt, \tag{84}$$

in which $M_{jk}(s,t)$ represents

$$M_{jk}(s,t) = q^{2s}\sum_{l=0}^{m-1}\sum_{r=0}^{n}\left(\mathcal{Q}_{j-1,q}^{m-1}(s)\mathcal{Q}_{l,q}^{m-1}(s) - \mathcal{Q}_{j,q}^{m-1}(s)\mathcal{Q}_{l,q}^{m-1}(s)\right)\mathcal{Q}_{k,q}^{n}(t)\mathcal{Q}_{r,q}^{n}(t)\langle e^{a}, \Delta^{10}P_{lr}\rangle, \tag{85}$$

and $N_{jk}(s,t)$ stands for

$$N_{jk}(s,t) = q^{2t}\sum_{l=0}^{m}\sum_{r=0}^{n-1}\left(\mathcal{Q}_{k-1,q}^{n-1}(t)\mathcal{Q}_{r,q}^{n-1}(t) - \mathcal{Q}_{k,q}^{n-1}(t)_{q}\mathcal{Q}_{r,q}^{n-1}(t)\right)\mathcal{Q}_{j,q}^{m}(s)\mathcal{Q}_{l,q}^{m}(s)\langle e^{a}, \Delta^{01}P_{lr}\rangle. \tag{86}$$

Equations (85) and (86) can be written as

$$M_{jk}(s,t) = q^{2s}\sum_{l=0}^{m-1}\sum_{r=0}^{n}\left(\frac{\binom{m-1}{j-1}\binom{m-1}{l}}{\binom{2m-2}{j+l-1}}\mathcal{Q}_{j+l,q}^{2m-2}(s) - \frac{\binom{m-1}{j}\binom{m-1}{l}}{\binom{2m-2}{l+j}}\mathcal{Q}_{l+j,q}^{2m-2}(t)\right)\frac{\binom{n}{k}\binom{n}{r}}{\binom{2n}{k+r}}\mathcal{Q}_{k+r,q}^{2n}(t)\langle e^{a}, \Delta^{10}P_{lr}\rangle, \tag{87}$$

and

$$N_{jk}(s,t) = q^{2t}\sum_{l=0}^{m}\sum_{r=0}^{n-1}\left(\frac{\binom{n-1}{k-1}\binom{n-1}{r}}{\binom{2n-2}{k+r-1}}\mathcal{Q}_{k+r-1,q}^{2n-2}(t) - \frac{\binom{n-1}{k}\binom{n-1}{r}}{\binom{2n-2}{r+k}}\mathcal{Q}_{r+k,q}^{2n-2}(t)\right)\frac{\binom{m}{l}\binom{m}{j}}{\binom{2m}{l+j}}\mathcal{Q}_{l+j,q}^{2m}(s)\langle e^{a}, \Delta^{01}P_{lr}\rangle. \tag{88}$$

The equations (87) and (88) can be rewritten as

$$M_{jk}(s,t) = q^{2s}\sum_{l=0}^{m-1}\sum_{r=0}^{n}\left(\frac{\binom{m-1}{j-1}\binom{m-1}{l}}{\binom{2m-2}{j+l-1}}\mathcal{Q}_{j+l,q}^{2m-2}(s) - \frac{\binom{m-1}{j}\binom{m-1}{l}}{\binom{2m-2}{l+j}}\mathcal{Q}_{l+j,q}^{2m-2}(t)\right)\frac{\binom{n}{k}\binom{n}{r}}{\binom{2n}{k+r}}\mathcal{Q}_{k+r,q}^{2n}(t)\Delta^{10}P_{lr}, \tag{89}$$

and

$$N_{jk}(s,t) = q^{2t}\sum_{l=0}^{m}\sum_{r=0}^{n-1}\left(\frac{\binom{n-1}{r}\binom{n-1}{k}}{\binom{2n-2}{k+r}}\mathcal{Q}_{k+r,q}^{2n-2}(t) - \frac{\binom{n-1}{k-1}\binom{n-1}{r}}{\binom{2n-2}{r+k-1}}\mathcal{Q}_{r+k-1,q}^{2n-2}(t)\right)\frac{\binom{m}{l}\binom{m}{j}}{\binom{2m}{l+j}}\mathcal{Q}_{l+j,q}^{2m}(s)\Delta^{01}P_{lr}. \tag{90}$$

Now, we find the integrals of $M_{jk}(s,t)$ (89) and that of $N_{jk}(s,t)$ (90),

$$\int_R M_{jk}(s,t)\mathrm{d}s\mathrm{d}t = \sum_{l=0}^{m-1}\sum_{r=0}^{n}\left(\xi_{j-1,l}^{m-1,m-1}\delta_{j+l-1}^{2m-2}(s) - \xi_{j,l}^{m-1,m-1}\delta_{j+l}^{2m-2}(s)\right)\xi_{k,r}^{n,n}\delta_{k+r}^{2n}(t)\Delta^{10}P_{lr}, \tag{91}$$

and

$$\int_R N_{jk}(s,t)\mathrm{d}s\mathrm{d}t = \sum_{l=0}^{m}\sum_{r=0}^{n-1}\left(\xi_{k-1,r}^{n-1,n-1}\delta_{k+r-1}^{2n-2}(t) - \xi_{r,k}^{n-1,n-1}\delta_{r+k}^{2n-2}(s)\right)\xi_{l,j}^{m,m}\delta_{l+j}^{2m}(s)\Delta^{01}P_{lr}, \tag{92}$$

where

$$\xi_{j,l}^{m,m} = \frac{\binom{m}{j}\binom{m}{l}}{\binom{2m}{j+l}},$$
$$\delta_{j+l}^{2m}(s) = \int_R q^{2s} Q_{j+l}^{2m}(s)\mathrm{d}s. \tag{93}$$

Plugging the (83) and (84) in (91) and (92), respectively, it gives

$$R_{jk} = \left(\frac{m \log q}{1-q}\right)^2 \sum_{i,j=0}^{m-1,n} \left(\xi_{j-1,l}^{m-1,m-1}\delta_{j+l-1}^{2m-2}(s) - \xi_{j,l}^{m-1,m-1}\delta_{j+l}^{2m-2}(s)\right)\xi_{k,r}^{n,n}\delta_{k+r}^{2n}(t)\Delta^{10}P_{lr}, \tag{94}$$

and

$$S_{jk} = \left(\frac{n \log q}{1-q}\right)^2 \sum_{i,j=0}^{m,n-1} \left(\xi_{k-1,r}^{n-1,n-1}\delta_{k+r-1}^{2n-2}(t) - \xi_{r,k}^{n-1,n-1}\delta_{r+k}^{2n-2}(t)\right)\xi_{l,j}^{m,m}\delta_{l+j}^{2m}(s)\Delta^{01}P_{lr}. \tag{95}$$

Substituting the value of R_{jk} (94) and the value of S_{jk} (95) in (81), we find that

$$\begin{aligned}\frac{\partial D(\mathscr{P})}{\partial x_{jk}^{a}} &= \left(\frac{m \log q}{1-q}\right)^2 \sum_{i,j=0}^{m-1,n} \left(\xi_{j-1,l}^{m-1,m-1}\delta_{j+l-1}^{2m-2}(s) - \xi_{j,l}^{m-1,m-1}\delta_{j+l}^{2m-2}(s)\right)\xi_{k,r}^{n,n}\delta_{k+r}^{2n}(t)\Delta^{10}P_{ij} \\ &+ \left(\frac{n \log q}{1-q}\right)^2 \sum_{i,j=0}^{m,n-1} \left(\xi_{k-1,r}^{n-1,n-1}\delta_{k+r-1}^{2n-2}(t) - \xi_{r,k}^{n-1,n-1}\delta_{r+k}^{2n-2}(t)\right)\xi_{l,j}^{m,m}\delta_{l+j}^{2m}(s)\Delta^{01}P_{lr}.\end{aligned} \tag{96}$$

The vanishing condition of gradient of Dirichlet function is $\partial D(\mathscr{P})/\partial x_{jk}^{a} = 0$ for the q- Bernstein–Bézier surface. Thus, the above (96) yields the equation (43). ☐

Corollary 7. *A control net, $\mathscr{P} = \{P_{jk}\}_{j,k=0}^{n,m}$, is an extremal of the Dirichlet functional with the prescribed border for $m = n$ if the following constraint on the interior control points is satisfied.*

$$\begin{aligned}&\left(\frac{m \log q}{1-q}\right)^2 \left(\sum_{i,j=0}^{m-1,m} \left(\xi_{j-1,l}^{m-1,m-1}\delta_{j+l-1}^{2m-2}(s) - \xi_{j,l}^{m-1,m-1}\delta_{j+l}^{2m-2}(s)\right)\xi_{k,r}^{m,m}\delta_{k+r}^{2m}(t) \times \Delta^{10}P_{lr}\right. \\ &\left. + \sum_{i,j=0}^{m,m-1} \left(\xi_{k-1,r}^{m-1,m-1}\delta_{k+r-1}^{2m-2}(t) - \xi_{r,k}^{m-1,m-1}\delta_{r+k}^{2m-2}(t)\right)\xi_{l,j}^{m,m}\delta_{l+j}^{2m}(s)\Delta^{01}P_{lr}\right) = 0.\end{aligned} \tag{97}$$

Corollary 8. *A bi-quadratic quasi-minimal q-Bernstein–Bézier surface as the extremal of Dirichlet functional satisfies the following constraint, which explicitly gives P_{11}, the only interior point in this case as the linear combination of known boundary control points (by plugging $n = m = 2$ and $j = k = 1$ in the equation (97)).*

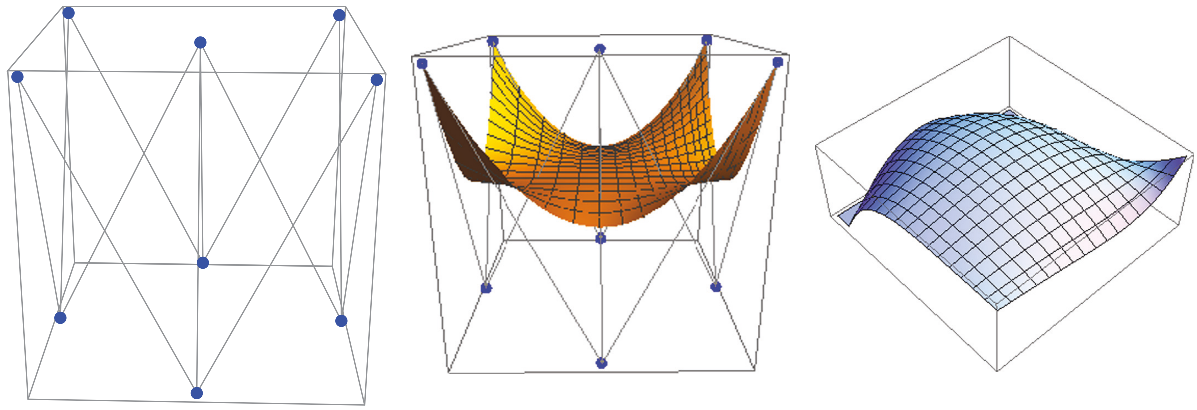

Figure 4: Prescribed border, the quasi-minimal bi-quadratic q-Bernstein–Bézier surface, and the mean curvature function.

$$\begin{aligned} &q^{2t}\left(\Delta^{01}P_{00}+\Delta^{01}P_{01}+\Delta^{01}P_{02}+\Delta^{01}P_{10}+\Delta^{01}P_{11}+\Delta^{01}P_{12}+\Delta^{01}P_{21}+\Delta^{01}P_{22}\right)\left(q^3-15q^2+27q-37\right)\\ &\quad-2q^{2s}\left(\Delta^{10}P_{00}+\Delta^{10}P_{01}+\Delta^{10}P_{02}+\Delta^{10}P_{10}+\Delta^{10}P_{11}+\Delta^{10}P_{12}+\Delta^{10}P_{21}+\Delta^{10}P_{22}\right)(1-q)=0. \end{aligned} \tag{98}$$

Corollary 9. *The shift operator properties that $\Delta^{01}P_{ij}=P_{ij+1}-P_{ij}$ and $\Delta^{10}P_{ij}=P_{i+1j}-P_{ij}$, for $q=0.2$, enable us to reduce the equation (98) for the bi-quadratic quasi-minimal q-Bernstein–Bézier surface as the extremal of Dirichlet functional as follows:*

$$\begin{aligned} P_{11}=\frac{1}{113}&(-29P_{00}+29P_{01}-29P_{02}+42P_{10}+86P_{12}\\ &-29P_{20}+0.1P_{21}-2P_{22}). \end{aligned} \tag{99}$$

Figure 4 represents the prescribed control net (first figure of Figure 4), related bi-quadratic quasi-minimal q-Bernstein–Bézier surface (for $q=0.2$, $m=2$, and $n=2$ given by second figure of Figure 4) and the mean curvature function (third figures of Figure 4) for the schematic illustration of the quasi-minimal q-Bernstein–Bézier surfaces described in Proposition 1, together with the unknown interior point P_{11} obtained from the vanishing condition for the gradient of Dirichlet functional (99). Corollary 10. A bi-quadratic quasi-minimal q-Bernstein–Bézier surface as the extremal of Dirichlet functional is obtained by taking $n=m=2$; in this case, we have only the choice that $j=1$ and $k=1$ for P_{jk}, which gives us P_{11} in terms of known boundary control points. In particular, for $q=1$, the above expression (98) reduces to equation (100) which is standard result obtained for P_{11} for bi-quadratic quasi-minimal Bézier surface by Monterde [19], and its mean curvature function of surface parameters is shown in Figure 5.

$$P_{11}=\left(2P_{00}-3P_{01}-5P_{12}+9P_{02}+P_{10}+2P_{02}+3P_{21}+P_{22}\right). \tag{100}$$

Corollary 11. *A bicubic q-Bernstein–Bézier surface is an extremal of the Dirichlet functional with prescribed border for $n=m=3$ and $j=k=1,2$. For example, for $q=0.2$, the constraint equation (43) gives us four constraints on the interior control points $P_{11},P_{12},P_{21},P_{22}$ that depend on the boundary control points, which are*

$$\begin{aligned} P_{11}&=-11.48P_{00}-11.48P_{01}-11.48P_{02}-P_{10}-P_{13}-0.50P_{20}-0.50P_{23}+7.96P_{31}+7.97P_{32}+7.97P_{33},\\ P_{12}&=4.86P_{01}+4.86P_{02}+4.86P_{03}+0.81P_{13}+P_{20}+P_{21}+P_{22}+P_{23}-2.16P_{30}-2.16P_{31}-2.16P_{32},\\ P_{21}&=4.86P_{01}+4.86P_{02}+4.86P_{03}+0.81P_{13}+P_{20}+P_{21}+P_{22}+P_{23}-2.16P_{30}-2.16P_{31}-2.16P_{32},\\ P_{22}&=4.10P_{00}+4.32P_{01}+4.32P_{02}-0.33P_{10}-0.33P_{13}+P_{20}+P_{23}+1.99P_{30}+1.99P_{31}+1.99P_{32}+1.99P_{33} \end{aligned}\quad. \tag{101}$$

As an instance of a bi-cubic quasi-minimal q-Bernstein–Bézier surface for $q=0.2$, $m=3$, and $n=3$, Figure 6 serves as an illustrative example of the Proposition 1 for the quasi-minimal q-Bernstein–Bézier surface. In Figure 6, the boundary control points of the bicubic quasi-minimal q-Bernstein–Bézier surface are shown in its first figure, whereas the bicubic quasi-minimal q-Bernstein–Bézier surface and its mean curvature are shown in second and third figures along with the unknown interior control points P_{11}, P_{12}, P_{21}, and P_{22}, worked out from the vanishing condition of gradient of the Dirichlet functional given by equation (43) as control net of the surface. We conclude the

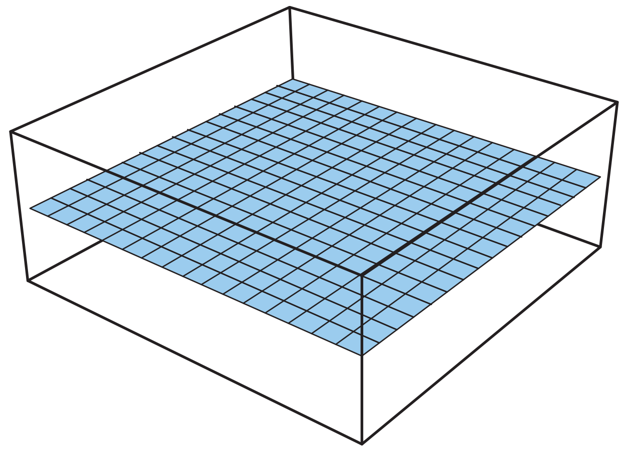

Figure 5: The mean curvature function of the bi-quadratic quasi-minimal q-Bernstein–Bézier surface.

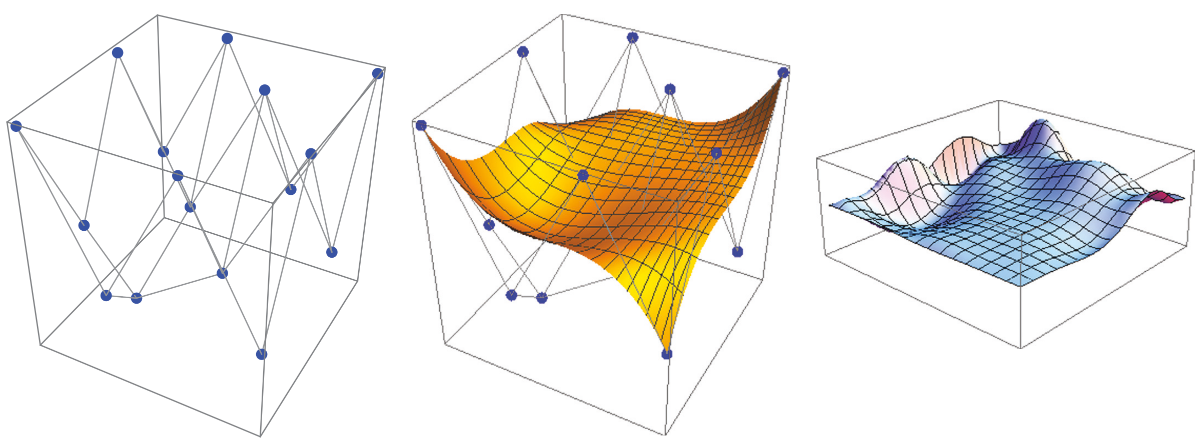

Figure 6: Prescribed border, the bi-cubic quasi-minimal q-Bernstein–Bézier surface, and the mean curvature function.

section with the remark that the problem of finding the quasi-minimal q-Bernstein–Bézier surface as the extremal of Dirichlet functional is thus reduced to solving the algebraic constraints for the interior control points in terms of boundary control points as the outcome of vanishing condition of gradient of Dirichlet functional. For schematic illustration, bi-quadratic and bicubic quasi-minimal q-Bernstein–Bézier surfaces are given as representative examples.

4. Conclusion

The Bézier surfaces and minimal surfaces arising as the extremal of certain energy functional appear quite frequently in the mathematical models of surface formation in computer science for computer-aided geometric design (CAGD), computer graphics, and other disciplines of mathematics. A minimal surface is defined as a surface of minimal area that has vanishing mean curvature everywhere on the surface. One of the widely used restrictions is to find out the Bézier surface of minimal area as the extremal of various energy integrals by the vanishing condition of gradient of such a functional. A class of surfaces, namely, the q-Bernstein–Bézier surfaces, have been discussed and the corresponding quasi-minimal surfaces as the extremal of Dirichlet functional are achieved by finding the linear algebraic constraints on the interior control points of the prescribed border (boundary control points) as the outcome of the vanishing condition for this functional gradient of Dirichlet functional. The unknown interior control points computed through this scheme along with given boundary control points can be used to plot the minimal surfaces. We call the quasi-minimal surface determined as the extremal of Dirichlet functional as the quasi-minimal q-Bernstein–Bézier surface. We have included the representative examples for the illustration of the scheme for bi-quadratic and bi-cubic q-Bernstein–Bézier surfaces for $q = 0.2$. The work can be extended not only for the surfaces as the extremal of other functionals but also for the surfaces spanned by various polynomials as well as for Stancu polynomials, Hermite polynomials, Bernoulli polynomials, and others.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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