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Research Article

A New Efficient Method for Solving System of Weakly Singular Fractional Integro-Differential Equations by Shifted Sixth-Kind Chebyshev Polynomials

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In this paper, a new approach for solving the system of fractional integro-differential equation with weakly singular kernels is introduced. The method is based on a class of symmetric orthogonal polynomials called shifted sixth-kind Chebyshev polynomials. First, the operational matrices are constructed, and after that, the method is described. This method reduces a system of weakly singular fractional integro-differential equations (WSFIDEs) by the collocation method into a system of algebraic equations. Thereupon, an upper error bound for the proposed method is determined. Finally, some numerical examples are prepared to test the accuracy and efficiency of the presented method.

1. Introduction

The study of fractional calculus has applications and popularity in various and wide fields of biology, physics, and fluid mechanics. Fractional calculus is actually integration and differentiation of arbitrary orders [1–4]. In various problems of physics and engineering, the fractional differential equations have been proved to be valuable tools in modeling of many phenomena [5, 6]. As we know, many mathematical models of real phenomena (arising in engineering and physics) are described as linear or nonlinear systems. It is worth mentioning that with the development of fractional calculus, the behavior of many systems can be described using the fractional differential and fractional integro-differential system [7, 8]. In recent years, systems of the fractional differential and integral equations are the subjects of extensive study due to their frequent appearance in many engineering and scientific disciplines [9–11]. However, most of the fractional-order equations and integral equations do not have analytic solutions or are hard to find. So, it is essential to find numerical methods to get approximate or exact solutions of a system of integro-differential equations. So far, researchers have utilized diverse numerical methods for solving a system of fractional integro-differential equations. In [12], the homotopy perturbation method was proposed for solving linear and nonlinear systems of fractional integro-differential equations. Heydari et al. have used the Chebyshev wavelet method for solving a class of systems of nonlinear fractional singular Volterra integro-differential equation in [13]. In the next year, for the first time, the hybrid functions composed of the Block-pulse functions and Bernoulli polynomials were applied for problems with fractional-order differential equations in [14]. Also, a novel technique based on iterative refinement was presented to analytically approximate a system of linear fractional integro-differential equations [15]. In 2018, Hesameddini and Shahbazi developed the concept of [14] and used it to solve the FDIE system in [16]. Also, Xie and Yi presented the simple and fast method based on the Block-pulse function to solve a nonlinear system of fractional

Volterra-Fredholm integro-differential equations in the same year [17]. In [18], the authors implemented the new Jacobi operational matrices to reduce the complexity of calculations to solve WSFDIEs. Next, the Haar wavelet method was employed to solve a coupled system of FIDEs, and the Muntz-Legendre wavelets were introduced to solve FIDVFEs in [19, 20]. Also, the other authors applied the Chebyshev Pseudo spectral method for solving fractional-order nonlinear system of Volterra integro-differential equations and a least square collocation Chebyshev technique for solving a system of linear fractional integro-differential equations [21, 22]. In this paper, we consider the following system of weakly singular integro-differential equations:

$$\begin{aligned}\mathscr{D}^{\nu_i}u_i(x) &= \mathscr{F}_i(x, u_1(x), u_2(x), \cdots, u_m(x)) \\ &\quad + \sum_{j=1}^{m}\theta_{ij}\int_0^x \frac{K_{ij}(x,z)G_{ij}(z,u_j(z))}{(x-z)^{\alpha_{ij}}}\,dz \\ &\quad + f_i(x), i = 1, \cdots, m, u_i^{(k)}(0^+) = u_{0i}^{(k)},\ k = 0, 1, \\ &\quad \cdots, r-1,\ r-1 < \nu_i \le r,\end{aligned} \tag{1}$$

where $u_i(x) \in C^r[0,1]$, $i = 1, \cdots, m$, are the unknown functions, $\mathscr{F}_i : [0,1] \times (C^r[0,1])^m \longrightarrow \mathbb{R}$, $G_{ij} : [0,1] \times C^r[0,1] \longrightarrow \mathbb{R}$, and $K_{ij}(x,z)$: $[0,1]^2 \longrightarrow \mathbb{R}$ are continuous operators and functions that satisfy Lipschitz conditions, and $\mathscr{D}^{\nu_i}$ is the Caputo fractional derivative operator where $\nu_i \in (0,1]$. The parameters $\theta_{ij}, \alpha_{ij} \in \mathbb{R}$ such that $|\theta_{ij}| \le 1$, $0 < \alpha_{ij} < 1$, and $i, j = 1, 2, \cdots, m$. Moreover, $f_i(x)$, $i = 1, \cdots, m$, are known and sufficiently smooth functions.

As usual, a way for solving functional equations is to express the solution as a linear combination of the so-called basis functions. In most researches, various polynomials such as the Legendre, Chebyshev, Taylor, Hermit, and Bernstein are used as basis functions. Among all of them, the Chebyshev polynomials are the most important in the analysis and numerical analysis. Chebyshev polynomials are orthogonal on the interval $[-1, 1]$ and have good properties that are used widely in the approximation of the functions. For this reason, many studies are done based on the different kinds of Chebyshev polynomials. In [23], Masjed-Jamei introduced two half-trigonometric orthogonal Chebyshev polynomials, and he named them as the Chebyshev polynomials of the fifth and sixth kinds. The basic formulas and properties of this class of polynomials are displayed in [24, 25]. Up to now, many researchers have used various kinds of Chebyshev polynomials for the fractional-order differential and integro-differential equations (see [26, 27]). However, there are only a few works that have used the sixth-kind Chebyshev polynomials. The main aim of this work is to introduce these polynomials as a new basis function for solving WSFDIEs. In the current paper, we apply the orthogonal shifted sixth-kind Chebyshev polynomials together with the collocation method for solving a system of weakly singular integro-differential equations with fractional derivatives that, to the best of our knowledge, is proposed here for the first time. For solving these equations, we derive the fractional operational matrices of fractional and integer orders and the product operational matrix, as well. Also, we introduce an operational matrix to approximate the integral term that has the weakly singular kernel in Equation (1). As far as we can tell, this operational matrix is presented for the first time. By substituting appropriate approximations in Equation (1), the original equations convert into algebraic equations that each of the equations of algebraic systems is collocated at $N+1$ roots of the $(N+1)$ th shifted sixth-kind Chebyshev polynomials (SSKCPs). By solving these algebraic systems, the approximate solutions of the original system are obtained. Although the calculation of the operational matrices may be complicated, we show that the obtained results are equal to other methods or are even more accurate. Implementing these matrices leads to a decrease in the number of required computations, and therefore, the computation time will be reduced.

The rest of the paper is organized as follows. In section 2, some essential preliminaries are mentioned briefly. Section 3 is devoted to constructing the operational matrices of SSKCPs. The proposed numerical procedure is described in Section 4. The error analysis of the proposed method is discussed in Section 5. Some numerical applications are indicated in Section 6, and conclusions are presented in Section 7.

2. Preliminaries and Notations

In this section, we recall some definitions and properties of fractional integral and derivative operators which will be used later. After that, some necessary definitions and fundamental properties of the shifted sixth-kind Chebyshev polynomials are reviewed briefly.

2.1. Some Essentials of the Fractional Calculus

Definition 1. The Riemann-Liouville fractional integral operator J^α of order α is given by [2]

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1} f(z)\,dz, \alpha > 0, x > 0. \tag{2}$$

Definition 2. Let $\alpha \in \mathbb{R}$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$, and $f(x)$ be a real-valued continuous function defined on $[0,\infty)$. Then, the Caputo fractional derivative of order $\alpha > 0$ is defined by [2]

$$\begin{cases} 0\mathscr{D}_x^\alpha f(x) = \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x \frac{f^n(z)}{(x-z)^{\alpha+1-n}}\,dz, \\ f^{(n)}(z),\ \alpha = n, \end{cases} \tag{3}$$

where $\Gamma(x)$ is the Gamma function as

$$\begin{gathered}\Gamma(x) = \int_0^\infty e^{-z}z^{x-1}\,dz, \mathrm{Re}\ (z) > 0, \\ \Gamma(x+1) = x\Gamma(x), \\ B(u,v) = \int_0^1 z^{u-1}(1-z)^{v-1}\,dz = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \mathrm{Re}\ (u) > 0, \mathrm{Re}\ (v) > 0.\end{gathered} \tag{4}$$

The last integral is often called the Beta integral. For $\alpha_1, \alpha_2 > 0$, the Riemann-Liouville integral and Caputo fractional derivative operators satisfy the following properties:

(1) $J^{\alpha_1}(J^{\alpha_2}f(x)) = J^{\alpha_2}(J^{\alpha_1}f(x)) = J^{\alpha_1+\alpha_2}f(x)$

(2) $J^{\alpha}(\lambda_1 f(x) + \lambda_2 g(x)) = \lambda_1 J^{\alpha}f(x) + \lambda_2 J^{\alpha}g(x)$

(3) $J^{\alpha}(\mathscr{D}^{\alpha}f(x)) = f(x) - \sum_{i=0}^{n-1} f^i(0)(x^i/i!), n-1 < \alpha \le n, x > 0$

(4) $\mathscr{D}^{\alpha}x^{\gamma} = \begin{cases} 0, & \alpha > \gamma, \\ (\Gamma(\gamma+1)/\Gamma(\gamma-\alpha+1))x^{\gamma-\alpha}, & \text{otherwise} \end{cases}$

(5) $J^{\alpha}x^{\nu} = ((\Gamma(\nu+1))/(\Gamma(\nu+\alpha+1)))x^{\nu+\alpha}, \nu > -1$

2.2. Shifted Sixth-Kind Chebyshev Polynomials and Their Properties (SSKCPs)

Definition 3. The sixth-kind Chebyshev polynomials are orthogonal functions on the interval $[-1, 1]$ and can be determined with the following recursive formula [23, 24]:

$$\begin{aligned} S_j(x) &= xS_{j-1}(x) - \frac{j(j+1) + (-1)^j(2j+1) + 1}{4j(j+1)} S_{j-2}(x), j \ge 2, \\ S_0(x) &= 1, \\ S_1(x) &= x. \end{aligned} \tag{5}$$

Definition 4. The shifted sixth-kind Chebyshev polynomials on $[0, 1]$ is defined by [23, 24]

$$S_j^*(x) = S_j(2x-1), j = 0, 1, 2, \cdots. \tag{6}$$

These polynomials have the following explicit analytic form:

$$S_j^*(x) = \sum_{k=0}^{j} \rho_{kj}x^k, \tag{7}$$

where

$$\rho_{kj} = \begin{cases} \dfrac{2^{2k-j}}{(2k+1)!} \displaystyle\sum_{i=\lfloor (k+1)/2 \rfloor}^{j/2} \dfrac{(-1)^{(j/2)+i+k}(2i+k+1)!}{(2i-k)!}, & j \text{ even}, \\ \dfrac{2^{2k-j+1}}{(2k+1)!(j+1)} \displaystyle\sum_{i=\lfloor k/2 \rfloor}^{j-1/2} \dfrac{(-1)^{((j+1)/2)+i+k}(i+1)(2i+k+2)!}{(2i-k+1)!}, & j \text{ odd}. \end{cases} \tag{8}$$

Moreover, the shifted polynomials $S_j^*(x)$ are orthogonal on $[0, 1]$ with respect to the weight function $V(x) = (2x-1)^2 \sqrt{x - x^2}$ in the sense that

$$\int_0^1 S_i^*(x)S_j^*(x)V(x)dx = \lambda_i\delta_{ij}, \tag{9}$$

$$\lambda_i = \begin{cases} \dfrac{\pi}{2^{2i+5}}, & i \text{ even}, \\ \dfrac{\pi(i+3)}{2^{2i+5}(i+1)}, & i \text{ odd}. \end{cases} \tag{10}$$

Now, let $h(x) \in L^2[0, 1]$; then, $h(x)$ can be approximated in terms of $S_j^*(x)$ as

$$h(x) \approx \sum_{j=0}^{N} \varrho_j S_j^*(x) = F^T S(x) = S^T(x)F, \tag{11}$$

where

$$S(x) = [S_0^*(x), S_1^*(x), \cdots, S_N^*(x)]^T, F = [\varrho_0, \varrho_1, \cdots, \varrho_N]^T, \tag{12}$$

where the coefficients ϱ_j are given by

$$\varrho_j = \frac{1}{\lambda_j} \int_0^1 h(x)S_j^*(x)V(x)dx, \tag{13}$$

and λ_j is defined in Equation (10). Similarly, any continuous two-variable function, $\mathscr{F}(x, z)$, defined on $[0, 1] \times [0, 1]$ can be approximated by means of the double-shifted sixth-kind Chebyshev polynomials as

$$\mathscr{F}(x, z) \approx \sum_{j=0}^{N}\sum_{i=0}^{N} \mathscr{F}ij S_i^*(x)S_j^*(z) = S^T(x)\mathscr{F}S(z), \tag{14}$$

where $\mathscr{F}$ is a $(N+1) \times (N+1)$ matrix, and its entries are given by

$$\mathscr{F}_{ij} = \frac{1}{\lambda_i\lambda_j} \int_0^1\int_0^1 \mathscr{F}(x, z)S_i^*(x)S_j^*(z)V(x)V(z)dxdz. \tag{15}$$

3. Operational Matrices of SSKCPs

In this section, the formulas of operational matrices with the fractional order will be derived for the sixth-kind Chebyshev polynomials. The following are the required lemmas.

Lemma 5. *If $r \ge l$, $l \in \mathbb{N}$, then we have*

$$\int_0^l x^r S_l^*(x)V(x)dx = \sum_{m=0}^{l} \frac{\rho_{ml}\sqrt{\pi}\Gamma(r+m+(3/2))}{2\Gamma(r+m+5)} \left(m^2 + m + r^2 + 2rm + 3 + r\right). \tag{16}$$

Proof. From the properties of the orthogonal polynomials, if $r \le l$, we have

$$\int_0^1 x^r S_l^*(x)V(x)\,dx = 0. \tag{17}$$

Hence, we suppose $r \ge l$. The lemma can be easily proved by the integration of the analytic form of SSKCPs in Equation (7). □

Theorem 6. *Let $S(x)$ be the SSKCP vector given by Equation (12) and $\mu \in \mathbb{R}$; then,*

$$J^{\mu}S(x) \approx \mathscr{P}^{(\mu)}S(x), \tag{18}$$

where $\mathscr{P}^{(\mu)}$ is the $(N+1)\times(N+1)$ operational matrix of the fractional integration of the order μ in the Riemann-Liouville sense which is defined by

$$\mathscr{P}^{(\mu)} = \begin{bmatrix} \tilde{a}_{00} & \tilde{a}_{01} & \cdots & \tilde{a}_{0N} \\ \tilde{a}_{10} & \tilde{a}_{11} & \cdots & \tilde{a}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{N0} & \tilde{a}_{N1} & \cdots & \tilde{a}_{NN} \end{bmatrix}, \tag{19}$$

$$\tilde{a}_{ij} = \sum_{l=0}^{i} \varpi_{ijl}, i = 0 \cdots N, j = 0 \cdots N. \tag{20}$$

ϖ_{ijl} are given by

$$\varpi_{ijl} = \rho_{li}\frac{\Gamma(l+1)}{\lambda_j\Gamma(l+\mu+1)}\sum_{k=0}^{j}\rho_{kj}\frac{\sqrt{\pi}\Gamma(k+\mu+l+(3/2))}{2\Gamma(k+\mu+l+5)} \times \left((k+\mu)^2+(2l+1)(k+\mu)+l(l+1)+3\right), i = 0\cdots N, j = 0\cdots N. \tag{21}$$

Proof. By applying the Riemann-Liouville integral operator to the SSKCPs' analytic form, we have

$$J^{\mu}(S_i^*(x)) = \sum_{l=0}^{i}\rho_{li}\frac{\Gamma(l+1)}{\Gamma(l+\mu+1)}x^{\mu+l}. \tag{22}$$

Now, we can express $x^{\mu+l}$ in terms of the shifted sixth-kind Chebyshev polynomials as follows:

$$x^{\mu+l} \approx \sum_{j=0}^{N}\tilde{C}_{lj}S_j^*(x), \tag{23}$$

where the coefficients $\tilde{C}_{lj}$ are given by

$$\tilde{C}_{lj} = \frac{1}{\lambda_j}\int_0^1 x^{\mu+l}S_j^*(x)V(x)dx. \tag{24}$$

According to Lemma 5, we can rewrite Equation (22) as

$$J^{\mu}S_j^{(*)}(x) \approx \sum_{j=0}^{N}\left\{\sum_{l=0}^{i}\rho_{li}\frac{\Gamma(l+1)}{\lambda_j\Gamma(l+\mu+1)}\times\sum_{k=0}^{j}\rho_{kj}\frac{\sqrt{\pi}\Gamma(k+\mu+l+(3/2))}{2\Gamma(k+\mu+l+5)}\left((k+\mu)^2+(2l+1)(k+\mu)+l(l+1)+3\right)\right\}S_j^*(x) = \sum_{j=0}^{N}\tilde{a}_{ij}S_j^*(x), \tag{25}$$

where $\tilde{a}_{ij}$ is given in Equation (20). Rewriting the last relation in the vector form gives

$$J^{\mu}S_j^*(x) = [\tilde{a}_{i0}, \tilde{a}_{i1}, \cdots, \tilde{a}_{iN}]S_j^*(x), i = 0, 1, \cdots, N. \tag{26}$$

This leads to the desired result. □

In the following, some useful and applicable lemmas are presented to get the Chebyshev operational matrix of product.

Lemma 7. *If $S_j^*(x)$ and $S_i^*(x)$ are jth and ith shifted sixth-kind Chebyshev polynomials, respectively, then we can write the product of $S_j^*(x)$ and $S_i^*(x)$ as*

$$Q_{i+j}(x) = S_i^*(x)S_j^*(x) = \sum_{k=0}^{i+j}\chi_k^{(i,j)}x^k. \tag{27}$$

Proof. See [18]. □

Lemma 8. *If $S_i^*(x)$, $S_j^*(x)$, and $S_k^*(x)$ are ith, jth, and kth shifted sixth-kind Chebyshev polynomials, then*

$$\begin{aligned} d_{ijk} &= \int_0^1 S_i^*(x)S_j^*(x)S_k^*(x)V(x)dx \\ &= \sum_{r=0}^{j+k}\sum_{l=0}^{i}\frac{\sqrt{\pi}\rho_{li}\chi_k^{(i,j)}\Gamma(r+l+(3/2))}{2\Gamma(r+l+5)}((l+r)(l+r+1)+3), \end{aligned} \tag{28}$$

where $\chi_k^{(i,j)}$ is obtained by Lemma 7.

Proof. According to Lemma 7, we can write

$$Q_{j+k}(x) = S_j^*(x)S_k^*(x) = \sum_{r=0}^{j+k}\chi_r^{(j,k)}x^r. \tag{29}$$

Then,

$$d_{ijk} = \int_0^1 S_i^*(x)\sum_{r=0}^{j+k}\chi_r^{(j,k)}x^rV(x)\,dx = \sum_{r=0}^{j+k}\chi_r^{(j,k)}\int_0^1 S_i^*(x)x^rV(x)\,dx. \tag{30}$$

The value of the last integral is obtained by Lemma 5. □

Assuming that E is a $(N+1)\times 1$ vector, we have

$$S(x)S^T(x)E \simeq \tilde{E}\,S(x), \tag{31}$$

where $\tilde{E}$ is a $(N+1)\times(N+1)$ matrix called the product operational matrix. The next theorem presents a general form for entries of the matrix $\tilde{E}$.

Theorem 9. *The entries of the matrix $\tilde{E}$ in Equation (31) are as follows:*

$$\tilde{E}_{jk} = \frac{1}{\lambda_k}\sum_{i=0}^{N} E_i\, d_{ijk}, j, k = 0, 1, \cdots, N, \tag{32}$$

where d_{ijk} is obtained by Lemma 8 and E_i is the element of the vector E.

Proof. See [18]. □

In the following, we get an approximation for the integral part with the singular kernel in Equation (1). Before that, we present a theorem.

Theorem 10. *The following relation is determined for $0<\kappa<1$:*

$$\int_0^x \frac{z^r}{(x-z)^\kappa}\,dz = \frac{\Gamma(r+1)\Gamma(1-\kappa)}{\Gamma(r-\kappa+2)}x^{r-\kappa+1}, r = 0, 1, 2, \cdots. \tag{33}$$

Proof. By performing Equation (33) and the substitution of $z=\xi x$ into Equation (33) and then using the definition of the Beta function, we obtain

$$x^{r-\kappa+1}\int_0^1 (1-\xi)^{-\kappa}\xi^r\,d\xi = \frac{\Gamma(1-\kappa)\Gamma(r+1)}{\Gamma(r-\kappa+2)}x^{r-\kappa+1}, r = 0, 1, 2, \cdots. \tag{34}$$

□

Now, we present an approximation for the integral part with a weakly singular kernel. For this purpose, see the following theorem.

Theorem 11. *Suppose that $u(x)$ is a continuous function on the interval $[0,1]$ and $\kappa\in(0,1)$ and $u(x)\approx S^T(x)F = F^T S(x)$ where $S(x)$ and F are defined by Equation (12), then we have*

$$\int_0^x \frac{u(z)}{(x-z)^\kappa}\,dz \approx F^T \mathfrak{J}^{(\kappa)}S(x), \tag{35}$$

where $\mathfrak{J}^{(\kappa)}$ is a $(N+1)\times(N+1)$ matrix as follows:

$$\mathfrak{J}^{(\kappa)} = \begin{bmatrix} \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \sigma_{10} & \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N0} & \tilde{a}_{N1} & \cdots & \sigma_{NN} \end{bmatrix}, \tag{36}$$

and its entries are determined as follows:

$$\mathfrak{J}^{(\kappa)}_{ij} = \sigma_{ij} = \sum_{l=0}^{i} \rho_{li}\frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)}\bar{C}_{j(l-\kappa+1)}, i, j = 0, 1, \cdots, N, \tag{37}$$

where the quantities ρ_{li} and $\bar{C}_{j(l-\kappa+1)}$ are introduced by relation (8) and Lemma 5.

Proof. By the definition of vector $S(x)$ and Lemma 5, we can write

$$S^T(x) = [S_0^*(x), S_1^*(x), \cdots, S_N^*(x)] = \left[\sum_{l=0}^{0}\rho_{l0}x^l, \cdots, \sum_{l=0}^{N}\rho_{lN}x^l\right]. \tag{38}$$

By applying Theorem 10, we have

$$\int_0^x \frac{S^T(z)}{(x-z)^\kappa}\,dz = \left[\sum_{l=0}^{0}\int_0^x \rho_{l0}\frac{z^l}{(x-z)^\kappa}\,dz, \cdots, \sum_{l=0}^{N}\rho_{lN}\int_0^x \frac{z^l}{(x-z)^\kappa}\,dz\right] = \left[\sum_{l=0}^{0}\rho_{l0}\frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)}x^{l-\kappa+1}, \cdots, \sum_{l=0}^{N}\rho_{lN}\frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)}x^{l-\kappa+1}\right]. \tag{39}$$

Now, we approximate $x^{l-\kappa+1}$ in terms of SSKCPs as follows:

$$\begin{aligned} x^{l-\kappa+1} &\approx \sum_{j=0}^{N}\bar{C}_{j(l-\kappa+1)}S_j^*(x), \\ \bar{C}_{j(l-\kappa+1)} &= \frac{1}{\lambda_j}\int_0^1 x^{l-\kappa+1}S_j^*(x)V(x)\,dx, j = 0, 1, \cdots, N, \end{aligned} \tag{40}$$

where $\bar{C}_{j(l-\kappa+1)}$ is obtained by applying Lemma 5. Thus,

$$\begin{aligned} &\sum_{l=0}^{i}\rho_{li}\frac{\Gamma(l+1)\Gamma(1-\kappa)}{\Gamma(l-\kappa+2)}x^{l-\kappa+1} \\ &\approx \sum_{j=0}^{N}\left\{\sum_{l=0}^{i}\frac{\rho_{li}\Gamma(l+1)\Gamma(1-\kappa)\bar{C}_{j(l-\kappa+1)}}{\Gamma(l-\kappa+2)}\right\}S_j^*(x) \\ &= \sum_{j=0}^{N}\sigma_{ij}S_j^*(x), i = 0, 1, \cdots, N. \end{aligned} \tag{41}$$

Equation (39) is then represented as follows:

$$\int_0^x \frac{S^T(z)}{(z-x)^\kappa}\,dz \approx \begin{bmatrix} \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \sigma_{10} & \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N0} & \tilde{a}_{N1} & \cdots & \sigma_{NN} \end{bmatrix} \begin{bmatrix} S_0^*(x) \\ S_1^*(x) \\ \vdots \\ S_N^*(x) \end{bmatrix} = \mathfrak{F}^{(\kappa)} S(x). \tag{42}$$

□

4. Numerical Procedure

We consider the system of fractional integro-differential equation with weakly singular kernels described in Equation (1). To solve the system, we approximate the function $\mathscr{D}^{\nu_i}u$ in a matrix form

$$\mathscr{D}^{\nu_i}u_i(x) \approx S^T(x)F_i,\ F_i = \left[\varrho_0^i, \varrho_1^i, \cdots, \varrho_N^i\right]^T,\ i = 1, 2, \cdots, m. \tag{43}$$

Then, according to the initial conditions of the problem, we can approximate the known function $\mathscr{H}_i(x) = \sum_{k=0}^{r-1}(u_{0i}^{(k)} x^k/(\Gamma(k+1))),\ k = \lceil \nu_i \rceil$ as follows:

$$\mathscr{H}_i(x) \approx S^T(x)\mathscr{C}_i,\ i = 1, 2, \cdots, m. \tag{44}$$

Using Equations (43) and (44), we compute an approximate for $u_i(x)$:

$$u_i(x) \approx S^T(x)\mathscr{P}^{(\nu_i)}F_i + S^T(x)\mathscr{C}_i = S^T(x)U_i,\ i = 1, 2, \cdots, m, \tag{45}$$

where $\mathscr{P}^{(\nu_i)}$ is the integral operational matrix presented in Theorem 6. We have the following approximations for the rest of the system:

$$\begin{gathered} \mathscr{F}_i(x, u_1(x), u_2(x), \cdots, u_m(x)) \approx \mathscr{X}_i^T S(x), \\ G_{ij}\left(x, u_j(x)\right) \approx S^T(x)\mathscr{V}_{ij}, \quad i, j = 1, 2, \cdots, m, \\ K_{ij}(x, z) \approx S^T(x)K_{ij}S(z). \end{gathered} \tag{46}$$

Using Theorems 9 and 11, we have

$$\begin{aligned} &\int_0^x \frac{K_{ij}(x,z)G_{ij}\left(z, y_j(z)\right)}{(x-z)^{\alpha_{ij}}}\,dz \\ &\approx \int_0^x \frac{S^T(x)K_{ij}S(z)S^T(z)\mathscr{V}_{ij}}{(x-z)^{\alpha_{ij}}}\,dz \\ &\approx S^T(x)K_{ij}\tilde{\mathscr{V}}_{ij}\int_0^x \frac{S(z)}{(x-z)^{\alpha_{ij}}}\,dz \approx S^T(x)K_{ij}\tilde{\mathscr{V}}_{ij}\mathfrak{F}^{(\alpha_{ij})}S(x). \end{aligned} \tag{47}$$

Now, by substituting Equations (43)-(47) into Equation (1), we obtain

$$\begin{aligned} S^T(x)F_i - \mathscr{X}_i^T S(x) - \sum_{j=0}^{m}\theta_{ij}S^T(x)K_{ij}\tilde{\mathscr{V}}_{ij}\mathfrak{F}^{(\alpha_{ij})}S(x) \\ - f_i(x) \approx 0,\ i = 1, 2, \cdots, m. \end{aligned} \tag{48}$$

Each equation of algebraic system (48) is collocated at $N+1$ roots of the $(N+1)$th shifted sixth-kind Chebyshev polynomials. Thus, an algebraic system, including $m(N+1)$ equations, is acquired. By solving the resultant algebraic system, we can obtain an approximation for unknown vectors $F_i,\ i = 1, 2, \cdots, m$, and by substituting the vector F_i into Equation (45), we obtain an approximation for $u_i(x),\ i = 1, 2, \cdots, m$.

5. Error Analysis

In this section, we prove some theorems. Then, we obtain an upper error bound for the approximation error. For this aim, we need the following norms:

$$\begin{gathered} \|f\|_{L^2(I)} = \left(\int_I |f(x)|^2 V(x)\,dx\right)^{1/2}, \\ \|X\|_1 = \sum_{i=0}^{n}|x_i|, \end{gathered} \tag{49}$$

where $f \in L^2(I)$ is a square integrable function on the interval $I = [0, 1]$ and $X = [x_0, x_1, \cdots, x_n]^T$ is a vector.

Theorem 12. *Suppose that $Y_N(x) = \sum_{i=0}^{N}\mathscr{E}_i S_i^*(x)$ is an approximation in SSKCPs to the continuous function $Y(x)$ on the interval $[0, 1]$. Then, the coefficients $\mathscr{E}_i$, for $i = 0, 1, \cdots, N$, are bounded as*

$$|\mathscr{E}_i| \le \frac{\mathscr{M}_Y}{\lambda_i}\sum_{m=0}^{i}\rho_{mi}\frac{\sqrt{\pi}\Gamma(m+(3/2))}{2\Gamma(m+5)}\left(m^2+m+3\right), \tag{50}$$

where $\mathscr{M}_Y$ denotes the maximum value of $|Y(x)|$ on the interval $[0, 1]$.

Proof. Using Equations (7) and (9) for $i = 0, 1, \cdots, N$, we have

$$\begin{aligned} \mathscr{E}_i &= \frac{1}{\lambda_i}\int_0^1 Y(x)S_i^*(x)V(x)\,dx = \frac{1}{\lambda_i}\int_0^1 Y(x)\sum_{m=0}^{i}\rho_{mi}x^m V(x)\,dx \\ &= \frac{1}{\lambda_i}\sum_{m=0}^{i}\rho_{mi}\int_0^1 Y(x)x^m V(x)\,dx. \end{aligned} \tag{51}$$

Since $Y(x)$ is a continuous function on the interval $[0, 1]$, so it is bounded and there is a constant $\mathscr{M}_Y$ such that

$$\forall x \in [0, 1],\ |Y(x)| \le \mathscr{M}_Y. \tag{52}$$

Using Equations (51) and (52), inequality (50) is deduced. □

Theorem 13. *Suppose that $Y(x)$ is a continuous function and $Y_N(x)$ is an approximation to $Y(x)$ in terms of SSKCPs. Then, the error bound can be achieved as follows:*

$$\|Y(x) - Y_N(x)\|_{L^2} \leq \left(\sum_{i=N+1}^{\infty} \Omega_i \right)^{1/2} = \Omega_N, \quad (53)$$

where

$$\Omega_i = \frac{\mathcal{M}_Y^2}{\lambda_i} \sum_{m=0}^{i} \left(\rho_{mi} \frac{\sqrt{\pi}\Gamma(m + (3/2))}{2\Gamma(m+5)} \left(m^2 + m + 3\right) \right)^2. \quad (54)$$

Proof. Assume $Y(x)$ is an arbitrary function. So, $Y(x)$ and $Y_N(x)$ have the following forms using SSKCP series:

$$\begin{aligned} Y(x) &= \sum_{i=0}^{\infty} \mathcal{E}_i S_i^*(x), \\ Y_N(x) &= \sum_{i=0}^{N} \mathcal{E}_i S_i^*(x), \end{aligned} \quad (55)$$

so,

$$Y(x) - Y_N(x) = \sum_{i=N+1}^{\infty} \mathcal{E}_i S_i^*(x). \quad (56)$$

Using Equations (9) and (56) and Theorem 12, we have

$$\begin{aligned} \|Y(x) - Y_N(x)\|_{L^2}^2 &= \int_0^1 |Y(x) - Y_N(x)|^2 V(x)\, dx \\ &= \int_0^1 \left(\sum_{i=N+1}^{\infty} \mathcal{E}_i S_i^*(x) \right)^2 V(x)\, dx \\ &= \int_0^1 \sum_{j=N+1}^{\infty} \sum_{i=N+1}^{\infty} \mathcal{E}_i \mathcal{E}_j S_i^*(x) S_j^*(x) V(x)\, dx \\ &= \sum_{i=N+1}^{\infty} \mathcal{E}_i^2 \lambda_i \leq \sum_{i=N+1}^{\infty} \Omega_i. \end{aligned} \quad (57)$$

□

Theorem 14. *Suppose that the continuous two-variable function $\mathcal{H}(x, y)$ is approximated on the interval $[0, 1] \times [0, 1]$ in terms of SSKCPs as $\mathcal{H}_N(x, y) = \sum_{i=0}^{N} \sum_{j=0}^{N} \mathcal{H}_{ij} S_i^*(x) S_j^*(y)$; then, the coefficients $\mathcal{H}_{ij}$, for $i, j = 0, 1, \cdots, N$, can be bounded as follows:*

$$\begin{aligned} |\mathcal{H}_{ij}| \leq \frac{\mathcal{M}_{\mathcal{H}} \pi}{4\lambda_i \lambda_j} &\sum_{m=0}^{i} \frac{\rho_{mi} \Gamma(m + (3/2))}{\Gamma(m+5)} \left(m^2 + m + 3\right) \\ &\sum_{r=0}^{j} \frac{\rho_{rj} \Gamma(r + (3/2))}{\Gamma(r+5)} \left(r^2 + r + 3\right), \end{aligned} \quad (58)$$

where $\mathcal{M}_{\mathcal{H}}$ denotes the maximum value of $|\mathcal{H}(x, y)|$ on the interval $[0, 1] \times [0, 1]$.

Proof. Using Equations (7) and (15), we have

$$\begin{aligned} \mathcal{H}_{ij} &= \frac{1}{\lambda_i \lambda_j} \int_0^1 \int_0^1 \mathcal{H}(x, y) S_i^*(x) S_j^*(y) V(x) V(y)\, dx dy \\ &= \frac{1}{\lambda_i \lambda_j} \int_0^1 \sum_{m=0}^{i} \rho_{mi} x^m V(x) \left(\int_0^1 \mathcal{H}(x, y) \sum_{r=0}^{j} \rho_{rj} y^r V(y) dy \right) dx \\ &= \frac{1}{\lambda_i \lambda_j} \sum_{m=0}^{i} \rho_{mi} \sum_{r=0}^{j} \rho_{rj} \int_0^1 \int_0^1 x^m \mathcal{H}(x, y) y^r V(x) V(y) dx dy. \end{aligned} \quad (59)$$

Since $\mathcal{H}(x, y)$ is a bounded and continuous function on the interval $[0, 1] \times [0, 1]$, so there is a constant $\mathcal{M}_{\mathcal{H}}$ such that

$$\forall (x, y) \in [0, 1] \times [0, 1], |\mathcal{H}(x, y)| \leq \mathcal{M}_{\mathcal{H}}. \quad (60)$$

Using Equations (59) and (60), Theorem 14 is proved. □

Theorem 15. *Suppose that $\mathcal{H}(x, y)$ is a continuous function with two variables and $\mathcal{H}_N(x, y)$ is the approximation to $\mathcal{H}(x, y)$ using SSKCPs. Then, the error bound can be obtained as*

$$\begin{aligned} \|\mathcal{H}(x, y) - \mathcal{H}_N(x, y)\|_{L^2} \leq &\left(\sum_{i=0}^{N} \sum_{j=N+1}^{\infty} \varsigma_{ij}^2 \lambda_i \lambda_j \right)^{1/2} \\ &+ \left(\sum_{i=N+1}^{\infty} \sum_{j=0}^{\infty} \varsigma_{ij}^2 \lambda_i \lambda_j \right)^{1/2} = \Lambda_{\mathcal{H}}, \end{aligned} \quad (61)$$

where

$$\begin{aligned} \varsigma_{ij} = \frac{\mathcal{M}_{\mathcal{H}} \pi}{4\lambda_i \lambda_j} &\sum_{m=0}^{i} \rho_{mi} \frac{\Gamma(m + (3/2))}{\Gamma(m+5)} \left(m^2 + m + 3\right) \\ &\sum_{r=0}^{j} \rho_{rj} \frac{\Gamma(r + (3/2))}{\Gamma(r+5)} \left(r^2 + r + 3\right). \end{aligned} \quad (62)$$

Proof. Suppose that $\mathcal{H}(x, y)$ is an arbitrary function. SSKCP series of $\mathcal{H}(x, y)$ and its approximation in terms of SSKCPs have the following form:

$$\mathscr{H}(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathscr{H}_{ij}S_i^*(x)S_j^*(y),\mathscr{H}_N(x,y)=\sum_{i=0}^{N}\sum_{j=0}^{N}\mathscr{H}_{ij}S_i^*(x)S_j^*(y). \tag{63}$$

□

Thus,

$$\mathscr{H}(x,y)-\mathscr{H}_N(x,y)=\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathscr{H}_{ij}S_i^*(x)S_j^*(y)+\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathscr{H}_{ij}S_i^*(x)S_j^*(y). \tag{64}$$

Using Equations (9) and (64) and Theorem 14, we conclude

$$\begin{aligned}\|\mathscr{H}(x,y)(x)-\mathscr{H}_N(x,y)\|_{L^2}&\le\left\|\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathscr{H}_{ij}S_i^*(x)S_j^*(y)\right\|_{L^2}+\left\|\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathscr{H}_{ij}S_i^*(x)S_j^*(y)\right\|_{L^2}=\left(\int_0^1\int_0^1\left(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathscr{H}_{ij}S_i^*(x)S_j^*(y)\right)^2V(x)V(y)dydx\right)^{1/2}\\&+\left(\int_0^1\int_0^1\left(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathscr{H}_{ij}S_i^*(x)S_j^*(y)\right)^2V(x)V(y)dydx\right)^{1/2}=\left(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\mathscr{H}_{ij}^2\lambda_i\lambda_j\right)^{1/2}\\&+\left(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\mathscr{H}_{ij}^2\lambda_i\lambda_j\right)^{1/2}\le\left(\sum_{i=0}^{N}\sum_{j=N+1}^{\infty}\varsigma_{ij}^2\lambda_i\lambda_j\right)^{1/2}+\left(\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\varsigma_{ij}^2\lambda_i\lambda_j\right)^{1/2}.\end{aligned} \tag{65}$$

In the following theorem, we obtain an upper error bound of the proposed method. First, suppose that for the $u_i(x), y_i(x)\in C^r[0,1], i=1,2,\cdots,m$, there exist positive constants ξ_r^i, $\eta_i^j>0$ such that the following Lipschitz conditions hold.

(1) $\|\mathscr{F}_i(x,u_1(x),u_2(x),\cdots,u_m(x))-\mathscr{F}_i(x,y_1(x),y_2(x),\cdots,y_m(x))\|_{L^2}\le\xi_1^i\|u_1(x)-y_1(x)\|_{L^2}+\xi_2^i\|u_2(x)-y_2(x)\|_{L^2}+\cdots+\xi_m^i\|u_m(x)-y_m(x)\|_{L^2}, i=1,2,\cdots,m$

(2) $\|G_{ij}(z,u_j(z))-G_{ij}(z,y_j(z))\|_{L^2}\le\eta_i^j\|u_j(z)-y_j(z)\|_{L^2}, i,j=1,2,\cdots,m$

Theorem 16. *Suppose that $\tilde{U}(x)=(\tilde{u}_1(x),\tilde{u}_2(x),\cdots,\tilde{u}_m(x))$ is a set of approximate solutions obtained from the SSKCP collocation method, $U(x)=(u_1(x),u_2(x),\cdots,u_m(x))$ is the set of exact solutions of system (1), $E=(\|u_1(x)-\tilde{u}_1(x)\|_{L^2},\cdots,\|u_m(x)-\tilde{u}_m(x)\|_{L^2})$ is the error vector of approximate solutions, and also $r_i(x), i=1,2,\cdots,m$ denote the residual functions associated to the approximate solutions that are named perturbation terms. Assume that Hypotheses (1) and (2) are satisfied; then, a bound for the method error can be achieved as*

$$\|E\|_1\le\frac{\Gamma_N^*+\sum_{i=1}^m\sum_{j=1}^m\gamma_j^i}{1-\Delta_N^*}, 0<\Delta_N^*<1. \tag{66}$$

Proof. First, we apply the Riemann-Liouville integral operator on Equation (1) and obtain the following equation:

$$\begin{aligned}u_i(x)&=g_i(x)+\frac{1}{\Gamma(\nu_i)}\int_0^x(x-z)^{\nu_i-1}\mathscr{F}_i(z,u_1(z),u_2(z),\cdots,u_m(z))\,dz\\&+\sum_{j=1}^m\theta_{ij}\frac{\Gamma(1-\alpha_{ij})}{\Gamma(\nu_i-\alpha_{ij}+1)}\int_0^x(x-z)^{\nu_i-\alpha_{ij}}K_{ij}(x,z)G_{ij}(z,u_j(z))\,dz,\end{aligned} \tag{67}$$

where

$$g_i(x)=\sum_{k=0}^{r-1}\frac{u_{0i}^{(k)}}{\Gamma(k+1)}x^k+\frac{1}{\Gamma(\nu_i)}\int_0^x(x-z)^{\nu_i-1}f_i(z)\,dz. \tag{68}$$

We can write the approximate equation of Equation (67) as follows:

$$\begin{aligned}\tilde{u}_i(x)&=g_i(x)+\frac{1}{\Gamma(\nu_i)}\int_0^x(x-z)^{\nu_i-1}\mathscr{F}_i(z,\tilde{u}_1(z),\tilde{u}_2(z),\cdots,\tilde{u}_m(z))\,dz\\&+\sum_{j=1}^m\theta_{ij}\frac{\Gamma(1-\alpha_{ij})}{\Gamma(\nu_i-\alpha_{ij}+1)}\int_0^x(x-z)^{\nu_i-\alpha_{ij}}\tilde{K}_{ij}(x,z)G_{ij}(z,\tilde{u}_j(z))\,dz\\&+r_i(x),\end{aligned} \tag{69}$$

where $r_i(x)$ is the perturbation term. We subtract Equation (69) from Equation (67) and obtain the following result:

$$\begin{aligned}r_i(x)&=u_i(x)-\tilde{u}_i(x)+\frac{1}{\Gamma(\nu_i)}\int_0^x(x-z)^{\nu_i-1}(\mathscr{F}_i(z,u_1(z),u_2(z),\cdots,u_m(z))-\mathscr{F}_i(z,\tilde{u}_1(z),\tilde{u}_2(z),\cdots,\tilde{u}_m(z)))\,dz\\&+\sum_{j=1}^m\theta_{ij}\frac{\Gamma(1-\alpha_{ij})}{\Gamma(\nu_i-\alpha_{ij}+1)}\int_0^x(x-z)^{\nu_i-\alpha_{ij}}\left(K_{ij}(x,z)G_{ij}(z,u_j(z))-\tilde{K}_{ij}(x,z)G_{ij}(z,\tilde{u}_j(z))\right)dz.\end{aligned} \tag{70}$$

First, we obtain a bound for the perturbation term, so by taking the L^2-norm on Equation (70), we get the following inequality:

$$\|r_i(x)\|_{L^2} \leq \|u_i(x) - \tilde{u}_i(x)\|_{L^2} + \frac{1}{\Gamma(\nu_i)}\left\|\int_0^x (x-z)^{\nu_i-1}(\mathscr{F}_i(z, u_1(z), \cdots, u_m(z)) - \mathscr{F}_i(z, \tilde{u}_1(z), \cdots, \tilde{u}_m(z)))\, dz\right\|_{L^2} + \sum_{j=1}^{m}\theta_{ij}\frac{\Gamma(1-\alpha_{ij})}{\Gamma(\nu_i - \alpha_{ij} + 1)} \times \left\|\int_0^x (x-z)^{\nu_i-\alpha_{ij}}\left(K_{ij}(x,z)G_{ij}(z, u_j(z)) - \tilde{K}_{ij}(x,z)G_{ij}(z, \tilde{u}_j(z))\right) dz\right\|_{L^2}. \tag{71}$$

Using Hypothesis 1 and Theorem 13, we have

$$\frac{1}{\Gamma(\nu_i)}\left\|\int_0^x (x-z)^{\nu_i-1}(\mathscr{F}_i(z, u_1(z), u_2(z), \cdots, u_m(z)) - \mathscr{F}_i(z, \tilde{u}_1(z), \tilde{u}_2(z), \cdots, \tilde{u}_m(z)))dz\right\|_{L^2} \leq \Psi_{\nu_i}\sum_{j=1}^{m}\xi_j^i\left\|u_j(z) - \tilde{u}_j(z)\right\|_{L^2} \leq \Psi_{\nu_i}\sum_{j=1}^{m}\xi_j^i\Omega_N^j, \tag{72}$$

where

$$\Psi_{\nu_i} = \frac{1}{\Gamma(\nu_i)}\left(\frac{4\Gamma(7/2)\Gamma(2\nu_i - (1/2))}{\Gamma(2\nu_i + 3)} - \frac{4\Gamma(5/2)\Gamma(2\nu_i - (1/2))}{\Gamma(2\nu_i + 2)} + \frac{\Gamma(3/2)\Gamma(2\nu_i - (1/2))}{\Gamma(2\nu_i + 1)}\right)^{1/2}. \tag{73}$$

Noting that $K_{ij}(x,z)$ and $G_{ij}(z, u_j(z))$ are continuous and known functions, thus, there are constants $M_{K_{ij}}$ and $N_{G_{ij}}$ such that

$$\left\|K_{ij}(x,z)\right\|_{L^2} \leq M_{K_{ij}}, \left\|G_{ij}\left(z, u_j(z)\right)\right\|_{L^2} \leq N_{G_{ij}}. \tag{74}$$

From Hypothesis 2, Theorem 13, Theorem 15, and Equation (74), the following inequality is obtained:

$$\sum_{j=1}^{m}\theta_{ij}\frac{\Gamma(1-\alpha_{ij})}{\Gamma(\nu_i - \alpha_{ij} + 1)}\left\|\int_0^x (x-z)^{\nu_i-\alpha_{ij}}\left(K_{ij}(x,z)G_{ij}(z, u_j(z)) - \tilde{K}_{ij}(x,z)G_{ij}(z, \tilde{u}_j(z))\right)dz\right\|_{L^2} \leq \sum_{j=1}^{m}\theta_{ij}\Psi_{\nu_i-\alpha_{ij}}\left(M_{K_{ij}}\eta_j^i\left\|u_j(z) - \tilde{u}_j(z)\right\|_{L^2} + N_{G_{ij}}\Lambda_{K_{ij}}\right) \leq \sum_{j=1}^{m}\theta_{ij}\Psi_{\nu_i-\alpha_{ij}}\left(M_{K_{ij}}\eta_j^i\Omega_N^j + N_{G_{ij}}\Lambda_{K_{ij}}\right), \tag{75}$$

where

$$\Psi_{\nu_i-\alpha_{ij}} = \frac{\Gamma(1-\alpha_{ij})}{\Gamma(\nu_i - \alpha_{ij} + 1)} \times \left(\frac{4\Gamma(7/2)\Gamma(2(\nu_i - \alpha_{ij}) + (3/2))}{\Gamma(2(\nu_i - \alpha_{ij}) + 5)} - \frac{4\Gamma(5/2)\Gamma(2(\nu_i - \alpha_{ij}) + (3/2))}{\Gamma(2(\nu_i - \alpha_{ij}) + 4)} + \frac{\Gamma(3/2)\Gamma(2(\nu_i - \alpha_{ij}) + (3/2))}{\Gamma(2(\nu_i - \alpha_{ij}) + 3)}\right)^{1/2}. \tag{76}$$

Here, Δ_j^i and γ_j^i can be introduced as below:

$$\Delta_j^i = \Psi_{\nu_i}\xi_j^i + \Psi_{\nu_i-\alpha_{ij}}\theta_{ij}M_{K_{ij}}\eta_j^i, \quad (77)$$

$$\gamma_j^i = \Psi_{\nu_i-\alpha_{ij}}\theta_{ij}N_{G_{ij}}\Lambda_{K_{ij}}. \quad (78)$$

From Equations (72)-(78), we can get the following upper bound for $r_i(x)$:

$$\|r_i(x)\|_{L^2} \le \Omega_N^i + \sum_{j=1}^m \Delta_j^i\Omega_N^j + \sum_{j=1}^m \gamma_j^i, \, i = 1, 2, \cdots, m. \quad (79)$$

By adding the above m inequalities, we have

$$\sum_{i=1}^m \|r_i(x)\|_{L^2} \le \sum_{i=1}^m \Omega_N^i + \sum_{i=1}^m\sum_{j=1}^m \Delta_j^i\Omega_N^j + \sum_{i=1}^m\sum_{j=1}^m \gamma_j^i = \Gamma_N^*. \quad (80)$$

We define the vector R as

$$R = [\|r_1\|_{L^2}, \|r_2\|_{L^2}, \cdots, \|r_{\mathrm{m}}\|_{L^2}]^T. \quad (81)$$

Thus, we have

$$\|R\|_1 = \sum_{i=1}^m \|r_{\mathrm{i}}\|_{L^2} \le \Gamma_N^*. \quad (82)$$

Again, we consider Equation (70). So we have

$$\begin{aligned}\|u_i(x) - \tilde{u}_i(x)\|_{L^2} &\le \|r_i(x)\|_{L^2} + \sum_{j=1}^m \Delta_j^i\|u_i(x) - \tilde{u}_i(x)\|_{L^2} \\ &\quad + \sum_{j=1}^m \gamma_j^i, \, i = 1, 2, \cdots, m.\end{aligned} \quad (83)$$

Adding the above m inequalities leads to the following inequality:

$$\begin{aligned}&\sum_{i=1}^m \|u_i(x) - \tilde{u}_i(x)\|_{L^2} \\ &\le \sum_{i=1}^m \|r_i(x)\|_{L^2} + \sum_{i=1}^m\sum_{j=1}^m \Delta_j^i\|u_i(x) - \tilde{u}_i(x)\|_{L^2} + \sum_{i=1}^m\sum_{j=1}^m \gamma_j^i \\ &= \|R\|_1 + \left(\Delta_1^1\|u_1(x) - \tilde{u}_1(x)\|_{L^2} + \cdots + \Delta_m^1\|u_m(x) - \tilde{u}_m(x)\|_{L^2}\right) \\ &\quad + \left(\Delta_1^2\|u_1(x) - \tilde{u}_1(x)\|_{L^2} + \cdots + \Delta_m^2\|u_m(x) - \tilde{u}_m(x)\|_{L^2}\right) \\ &\quad + \cdots + \left(\Delta_1^m\|u_1(x) - \tilde{u}_1(x)\|_{L^2} + \cdots + \Delta_m^m\|u_m(x) - \tilde{u}_m(x)\|_{L^2}\right) \\ &\quad + \sum_{i=1}^m\sum_{j=1}^m \gamma_j^i \\ &= \|R\|_1 + \left(\Delta_1^1 + \Delta_1^2 + \cdots + \Delta_1^m\right)\|u_1(x) - \tilde{u}_1(x)\|_{L^2} \\ &\quad + \cdots + \left(\Delta_m^1 + \Delta_m^2 + \cdots + \Delta_m^m\right)\|u_m(x) - \tilde{u}_m(x)\|_{L^2} + \sum_{i=1}^m\sum_{j=1}^m \gamma_j^i.\end{aligned} \quad (84)$$

By defining the following quantities

$$\begin{aligned}\Delta_N^* &= \mathrm{Max}\left\{\sum_{i=1}^m \Delta_j^i, j = 1, \cdots, m\right\}, \\ E &= \left[\|u_1(x) - \tilde{u}_1(x)\|_{L^2}, \|u_2(x) - \tilde{u}_2(x)\|_{L^2}, \cdots, \|u_m(x) - \tilde{u}_m(x)\|_{L^2}\right]^T,\end{aligned} \quad (85)$$

we obtain the following upper bound for the method error:

$$\begin{aligned}\sum_{i=1}^m \|u_i(x) - \tilde{u}_i(x)\|_{L^2} &\le \|R\|_1 + \Delta_N^*\sum_{i=1}^m \|u_i(x) - \tilde{u}_i(x)\|_{L^2} + \sum_{i=1}^m\sum_{j=1}^m \gamma_j^i, \\ \|E\|_1 &\le \frac{\Gamma_N^* + \sum_{i=1}^m\sum_{j=1}^m \gamma_j^i}{1 - \Delta_N^*}, 0 < \Delta_N^* < 1.\end{aligned} \quad (86)$$

□

6. Numerical Applications

In this section, three linear and nonlinear examples are presented to illustrate the practical implementation of our numerical method. Also, the comparison of results obtained from the proposed method with those of the other methods is shown. All calculations are done with mathematical software Maple 18.

Example 1. Let us consider the following linear system of WSFIDE:

$$\begin{cases}\mathscr{D}^{\nu_1}u_1(x) + u_1(x) + \int_0^x \frac{u_1(z)}{(x-z)^{1/2}}\,dz - \frac{1}{2}\int_0^x \frac{xu_2(z)}{(x-z)^{1/2}}\,dz = f_1(x), \\ \mathscr{D}^{\nu_2}u_2(x) + u_2(x) + \frac{1}{3}\int_0^x \frac{x^2zu_1(z)}{(x-z)^{1/2}}\,dz + \frac{1}{3}\int_0^x \frac{u_2(z)}{(x-z)^{1/2}}\,dz = f_2(x),\end{cases} \quad (87)$$

where

$$\begin{aligned}f_1(x) &= 2x + x^2 + \frac{2}{5}x^{5/2}, \\ f_2(x) &= 1 + x + \frac{32}{105}x^{11/2} + \frac{4}{9}x^{3/2},\end{aligned} \quad (88)$$

and $0 < \nu_i \le 1, i = 1, 2,$ and the initial conditions are $u_1(0) = u_2(0) = 0$. The exact solutions are $u_1(x) = x^2$ and $u_2(x) = x$ if $\nu_1 = \nu_2 = 1$. According to the procedure presented in Section 4, we reach the following approximations:

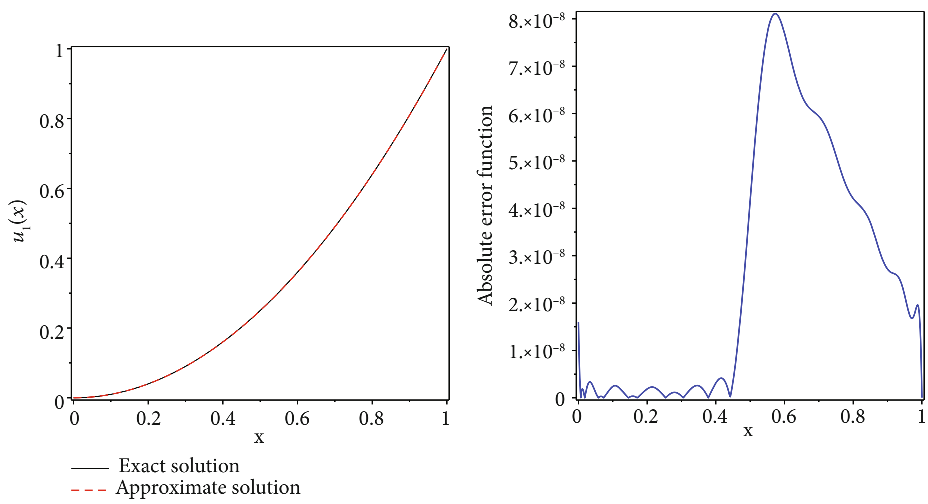

FIGURE 1: The graphs of the exact and approximate solutions and the absolute error function of $\tilde{u}_1(x)$ for $N = 20$ and $\nu_1 = \nu_2 = 1$ of Example 1.

$$\begin{gathered}
\mathscr{D}^{\nu_1} u_1(x) \approx S^T(x)\mathrm{F}_1, \quad u_1(x) \approx S^T(x)\mathscr{P}^{(\nu_1)T}\mathrm{F}_1 = S^T(x)U_1, \\
\mathscr{D}^{\nu_2} u_2(x) \approx S^T(x)\mathrm{F}_2, \quad u_2(x) \approx S^T(x)\mathscr{P}^{(\nu_2)T}\mathrm{F}_2 = S^T(x)U_2, \\
x \approx S^T(x)K_{12}S(z), x^2 z \approx S^T(x)K_{21}S(z), \\
\int_0^x \frac{u_1(z)}{(x-z)^{1/2}}\,dz \approx \int_0^x \frac{U_1^T S(z)}{(x-z)^{1/2}}\,dz = U_1^T \int_0^x \frac{S(z)}{(x-z)^{1/2}}\,dz \approx U_1^T \mathfrak{F}^{(1/2)} S(x), \\
\int_0^x \frac{x u_2(z)}{(x-z)^{1/2}}\,dz \approx \int_0^x \frac{S^T(x)K_{12}S(z)S^T(z)U_2}{(x-z)^{1/2}}\,dz = S^T(x)K_{12}\int_0^x \frac{S(z)S^T(z)U_2}{(x-z)^{1/2}}\,dz \approx S^T(x)K_{12}\tilde{U}_2\int_0^x \frac{S(z)}{(x-z)^{1/2}}\,dz \approx S^T(x)K_{12}\tilde{U}_2\mathfrak{F}^{(1/2)}S(x), \\
\int_0^x \frac{x^2 z u_1(z)}{(x-z)^{1/2}}\,dz \approx \int_0^x \frac{S^T(x)K_{21}S(z)S^T(z)U_1}{(x-z)^{1/2}}\,dz = S^T(x)K_{21}\int_0^x \frac{S(z)S^T(z)U_1}{(x-z)^{1/2}}\,dz = S^T(x)K_{21}\tilde{U}_1\int_0^x \frac{S(z)}{(x-z)^{1/2}}\,dz = S^T(x)K_{21}\tilde{U}_1\mathfrak{F}^{(1/2)}S(x), \\
\int_0^x \frac{u_2(z)}{(x-z)^{1/2}}\,dz \approx \int_0^x \frac{U_2^T S(z)}{(x-z)^{1/2}}\,dz \approx U_2^T \int_0^x \frac{S(z)}{(x-z)^{1/2}}\,dz \approx U_2^T \mathfrak{F}^{(1/2)} S(x),
\end{gathered} \tag{89}$$

where $\tilde{U}_1$ and $\tilde{U}_2$ are the operational matrices of the product, corresponding to the vectors U_1 and U_2, respectively. Setting above approximations in system (87) leads to the following linear algebraic system.

$$\begin{cases} S^T(x)\mathrm{F}_1 + S^T(x)U_1 + U_1^T\mathfrak{F}^{(1/2)}S(x) - \dfrac{1}{2}S^T(x)K_{12}\tilde{U}_2\mathfrak{F}^{(1/2)}S(x) - f_1(x) \approx 0, \\ S^T(x)\mathrm{F}_2 + S^T(x)U_2 + \dfrac{1}{3}S^T(x)K_{21}\tilde{U}_1\mathfrak{F}^{(1/2)}S(x) + \dfrac{1}{3}U_2^T\mathfrak{F}^{(1/2)}S(x) - f_2(x) \approx 0. \end{cases} \tag{90}$$

Figures 1 and 2 show a comparison between the exact and numerical solutions and absolute error functions for $\nu_1 = \nu_2 = 1$ and $N = 20$. The maximum absolute errors are listed in Table 1 in versus of N. It can be seen that our proposed method shows good consistency between the numerical results and analytic solutions and also this method can achieve a higher convergence result when N increases. Figure 3 shows the behavior of the numerical solutions for $N = 20$ and $\nu_i = 0.8, 0.9, 1$, $i = 1, 2$. As seen from Figure 3, as $\nu_i \longrightarrow 1$, the approximate solutions are $\tilde{u}_i(x) \longrightarrow u_i(x)$ for $i = 1, 2$.

Here, we calculate a numerical error bound for the first example. This result could confirm the correctness of the analytical error bound. This validation could be accomplished similarly for the other cases, but the calculations

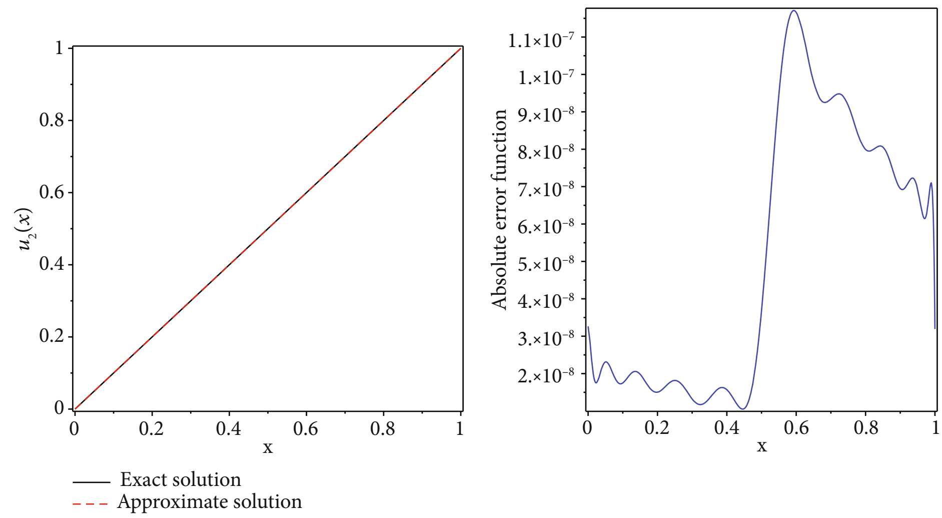

FIGURE 2: The graphs of the exact and approximate solutions and the absolute error function of $\tilde{u}_2(x)$ for $N = 20$ and $\nu_1 = \nu_2 = 1$ of Example 1.

are long.

$$\begin{gathered}
\|u_1(x) - \tilde{u}_1(x)\|_{L^2} = 8.93650018 \times 10^{-9}, \\
\|u_2(x) - \tilde{u}_2(x)\|_{L^2} = 1.81941246 \times 10^{-8}, \\
E = \left[8.93650018 \times 10^{-9}, 1.81941246 \times 10^{-8}\right]^T, \\
\|E\|_1 = 2.71305687 \times 10^{-8}, \\
\xi_1^1 = 0.4, \\
\xi_2^1 = 0.3, \\
\xi_1^2 = 0.8, \\
\xi_2^2 = 0.62, \\
\eta_1^1 = \eta_1^2 = \eta_2^1 = \eta_2^2 = 1, \\
\Psi_{\nu_1} = \Psi_{\nu_2} = 0.31332853, \\
\Psi_{\nu_1-\alpha_{11}} = \Psi_{\nu_1-\alpha_{12}} = 0.31332834, \\
\Psi_{\nu_2-\alpha_{21}} = \Psi_{\nu_2-\alpha_{22}} = 0.31332834, \\
\Delta_1^1 = 0.43865995, \\
\Delta_1^2 = 0.25392268, \\
\Delta_2^1 = 0.12405832, \\
\Delta_2^2 = 0.29870654, \\
\Delta_N^* = \mathrm{Max}\left\{\sum_{i=1}^{2} \Delta_j^i, j = 1, 2\right\} = 0.69258263, \\
\Lambda_{K_{11}} = \Lambda_{K_{12}} = \Lambda_{K_{21}} = \Lambda_{K_{22}} = 14.31641021, \\
\gamma_1^1 = 0.72969033, \\
\gamma_2^1 = -0.43034788, \\
\gamma_1^2 = 0.24323011, \\
\gamma_2^2 = 0.28689859, \\
\Gamma_N^* = 2.71305877 \times 10^{-8}, \\
\frac{\Gamma_N^* + \sum_{i=1}^m \sum_{j=1}^m \gamma_j^i}{1 - \Delta_N^*} = 2.69819226, \\
\|E\|_1 = 2.71305687 \times 10^{-8} \leq 2.69819226.
\end{gathered} \tag{91}$$

TABLE 1: Maximum absolute errors for different values of N for Example 1.

N	$u_1(x)$	$u_2(x)$	CPU time
3	2.7867×10^{-5}	3.0371×10^{-5}	3.010
8	3.7450×10^{-6}	4.7250×10^{-6}	8.642
11	2.3054×10^{-7}	7.7804×10^{-7}	18.330
16	2.1369×10^{-7}	3.0206×10^{-7}	68.048
20	8.0541×10^{-8}	1.1680×10^{-7}	166.843

Example 2. In this example, consider the following system of linear WSFIDEs [13, 18]:

$$\begin{aligned}
&\mathscr{D}^{\nu_1} u_1(x) - u_3(x) - \int_0^x \frac{xzu_1(z)}{(x-z)^{1/2}}\,dz - \int_0^x \frac{u_2(z)}{(x-z)^{1/2}}\,dz = f_1(x), \\
&\mathscr{D}^{\nu_2} u_2(x) - u_1(x) - \int_0^x \frac{u_2(z)}{(x-z)^{1/3}}\,dz - \int_0^x \frac{u_3(z)}{(x-z)^{1/3}}\,dz = f_2(x), \\
&\mathscr{D}^{\nu_3} u_3(x) - u_3(x) - \int_0^x \frac{u_1(z)}{(x-z)^{1/4}}\,dz - \int_0^x \frac{x^2 z u_2(z)}{(x-z)^{1/4}}\,dz = f_3(x),
\end{aligned} \tag{92}$$

where $0 \leq x \leq 1$ and

$$\begin{aligned}
f_1(x) &= 2x - 1 - x^3 + \frac{16}{15}x^{7/2} - \frac{16}{15}x^{5/2} - \frac{32}{35}x^{9/2}, \\
f_2(x) &= 2x - x(x-1) - \frac{27}{40}x^{8/3} - \frac{243}{440}x^{11/3}, \\
f_3(x) &= 3x^2 - x^3 + \frac{16}{21}x^{7/4} - \frac{128}{231}x^{11/4} - \frac{512}{1155}x^{23/4},
\end{aligned} \tag{93}$$

and $0 < \nu_i \leq 1, i = 1, 2, 3,$ the initial conditions are $u_1(0) = u_2(0) = u_3(0) = 0$. The exact solutions are $u_1(x) = x(x-1)$,

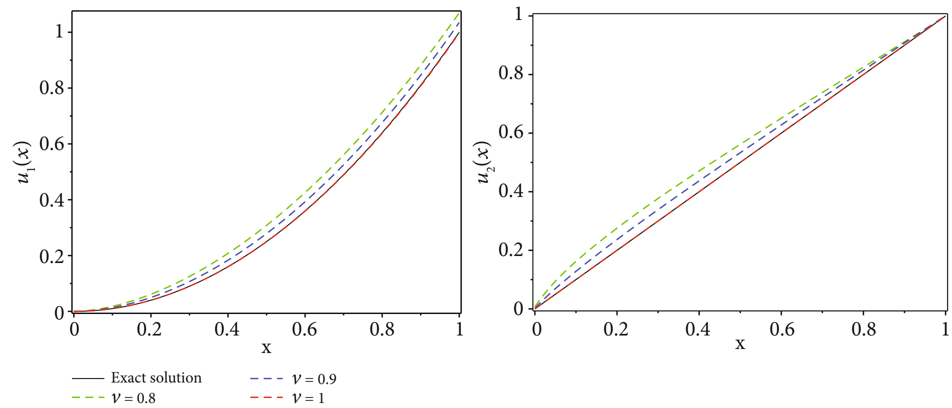

Figure 3: Some illustrations for the exact and approximate solutions using different values of ν_i: $\nu_i = 0.8, 0.9, 1$, $i = 1, 2$ in Example 1.

Table 2: Maximum absolute errors for different values of N for Example 2.

N	$u_1(x)$	$u_2(x)$	$u_3(x)$	CPU time
4	8.1514×10^{-5}	6.7763×10^{-5}	1.0743×10^{-4}	5.382
9	2.3334×10^{-6}	1.8107×10^{-6}	3.3112×10^{-6}	17.457
12	6.8859×10^{-7}	5.4537×10^{-7}	1.1140×10^{-6}	36.941
17	1.2296×10^{-7}	9.5306×10^{-8}	2.0253×10^{-7}	115.316
20	5.9278×10^{-8}	4.6454×10^{-8}	1.0049×10^{-7}	216.638

Table 3: Maximum absolute errors obtained by SSKCP collocation and Jacobi collocation methods for $N = 9$, $\alpha = -1/2$, and $\beta = 1/2$ in Example 2.

Method	$u_1(x)$	$u_2(x)$	$u_3(x)$
SSKCP collocation	2.3334×10^{-6}	1.8107×10^{-6}	3.3112×10^{-6}
Jacobi collocation [18]	2.0034×10^{-6}	1.8376×10^{-6}	2.0763×10^{-6}

$u_2(x) = x^2, u_3(x) = x^3$ if $\nu_1 = \nu_2 = \nu_3 = 1$. By assuming that $\mathscr{D}^{\nu_1} u_1(x) \approx S^T(x)F_1$, $\mathscr{D}^{\nu_2} u_2(x) \approx S^T(x)F_2$, and $\mathscr{D}^{\nu_3} u_3(x) \approx S^T(x)F_3$ and using proper operational matrices, the following approximations are obtained:

$$\begin{gathered}
u_1(x) \approx S^T(x)\mathscr{P}^{(\nu_1)T} F_1 = S^T(x)U_1,\\
u_2(x) \approx S^T(x)\mathscr{P}^{(\nu_2)T} F_2 = S^T(x)U_2,\\
u_3(x) \approx S^T(x)\mathscr{P}^{(\nu_3)T} F_3 = S^T(x)U_3,\\
xz \approx S^T(x)K_{11}S(z), x^2 z \approx S^T(x)K_{32}S(z),\\
\int_0^x \frac{xzu_1(z)}{(x-z)^{1/2}}\,dz + \int_0^x \frac{u_2(z)}{(x-z)^{1/2}}\,dz \approx S^T(x)K_{11}\tilde{U}_1\mathfrak{F}^{(1/2)}S(x) + U_2^T\mathfrak{F}^{(1/2)}S(x),\\
\int_0^x \frac{u_2(z)}{(x-z)^{1/3}}\,dz + \int_0^x \frac{u_3(z)}{(x-z)^{1/3}}\,dz \approx U_2^T\mathfrak{F}^{(1/3)}S(x) + U_3^T\mathfrak{F}^{(1/3)}S(x),\\
\int_0^x \frac{u_1(z)}{(x-z)^{1/4}}\,dz + \int_0^x \frac{x^2 z u_2(z)}{(x-z)^{1/4}}\,dz \approx U_1^T\mathfrak{F}^{(1/4)}S(x) + S^T(x)K_{32}\tilde{U}_2\mathfrak{F}^{(1/4)}S(x),
\end{gathered} \tag{94}$$

where $\tilde{U}_1$ and $\tilde{U}_2$ are the operational matrices of product, corresponding to the vectors U_1 and U_2, respectively. By substituting the above approximations into Equation (92), we achieve the following algebraic system

$$\begin{gathered}
S^T(x)\mathrm{F}_1 - S^T(x)U_3 - S^T(x)K_{11}\tilde{U}_1\mathfrak{F}^{(1/2)}S(x) - U_2^T\mathfrak{F}^{(1/2)}S(x) - f_1(x) \approx 0,\\
S^T(x)\mathrm{F}_2 - S^T(x)U_1 - U_2^T\mathfrak{F}^{(1/3)}S(x) - U_3^T\mathfrak{F}^{(1/3)}S(x) - f_2(x) \approx 0,\\
S^T(x)\mathrm{F}_3 - S^T(x)U_3 - U_1^T\mathfrak{F}^{(1/4)}S(x) - S^T(x)K_{32}\tilde{U}_2\mathfrak{F}^{(1/4)}S(x) - f_3(x) \approx 0.
\end{gathered} \tag{95}$$

Solving the above system by the collocation method for $N = 9$, we can determine the unknown vectors F_i, $i = 1, 2, 3$. Table 2 displays the maximum absolute errors for various N. The data in this table show that the numerical solutions get close to the analytical solutions with the increase of values of N. Table 3 shows a comparison between the SSKCP collocation and Jacobi collocation methods. This table shows that the results are approximately the same as reported by [18]. Figures 4–6 display a graphical comparison between approximate solutions and exact solutions and also absolute error functions for $N = 9$. Figure 7 shows the behavior of the numerical solutions for $N = 9$ and $\nu_i = 0.85, 0.90, 0.95, 1$, $i = 1, 2, 3$. From Figure 7, it can be seen that as $\nu_i \longrightarrow 1$, the approximate solutions are $\tilde{u}_i(x) \longrightarrow u_i(x)$, $i = 1, 2, 3$.

Example 3. Consider the following nonlinear system of WSFIDEs [13, 18]:

$$\begin{gathered}
\mathscr{D}^{3/4}u_1(x) - u_2(x) - \int_0^x \frac{u_1(z)}{(x-z)^{1/2}}\,dz - \int_0^x \frac{u_2^2(z)}{(x-z)^{1/2}}\,dz = f_1(x),\\
\mathscr{D}^{1/2}u_2(x) - u_1^3(x) - \int_0^x \frac{u_1(z)}{(x-z)^{2/3}}\,dz - \int_0^x \frac{u_2(z)}{(x-z)^{2/3}}\,dz = f_2(x),
\end{gathered} \tag{96}$$

where $0 \le x \le 1$ and

$$\begin{gathered}
f_1(x) = \frac{2}{\Gamma(9/4)}x^{5/4} - x^3 - \frac{\Gamma(3)\Gamma(1/2)}{\Gamma(7/2)}x^{5/2} - \frac{\Gamma(7)\Gamma(1/2)}{\Gamma(5/2)}x^{13/2},\\
f_2(x) = \frac{\Gamma(4)}{\Gamma(7/2)}x^{5/2} - x^6 - \frac{\Gamma(3)\Gamma(1/3)}{\Gamma(10/3)}x^{7/3} - \frac{\Gamma(4)\Gamma(1/3)}{\Gamma(13/3)}x^{10/3}.
\end{gathered} \tag{97}$$

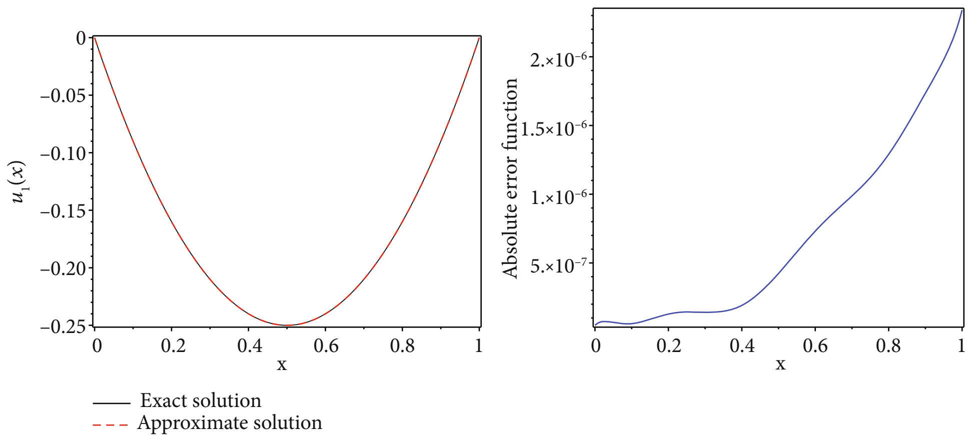

FIGURE 4: The graphs of the exact and approximate solutions and the absolute error function of $\tilde{u}_1(x)$ for $N = 9$ and $\nu_1 = \nu_2 = \nu_3 = 1$ of Example 2.

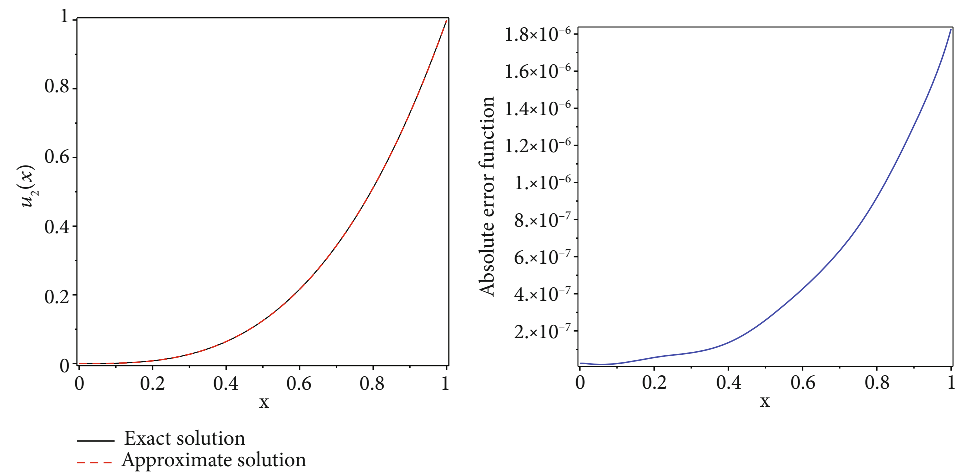

FIGURE 5: The graphs of the exact and approximate solutions and the absolute error function of $\tilde{u}_2(x)$ for $N = 9$ and $\nu_1 = \nu_2 = \nu_3 = 1$ of Example 2.

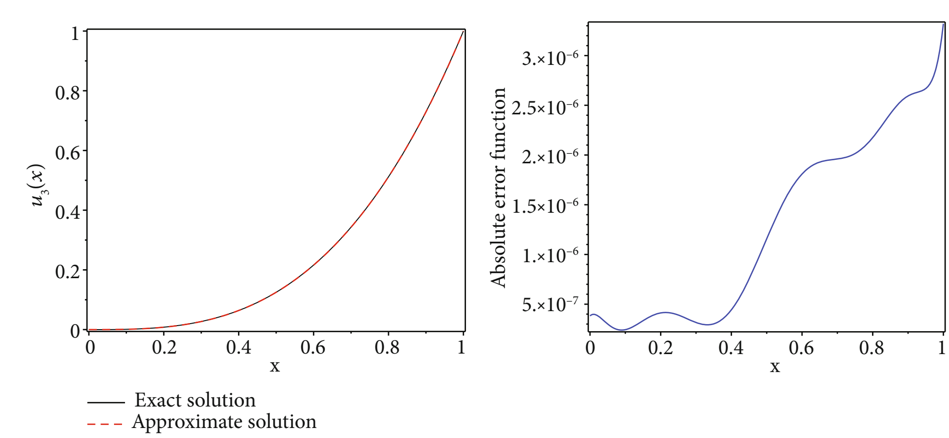

FIGURE 6: The graphs of the exact and approximate solutions and the absolute error function of $\tilde{u}_3(x)$ for $N = 9$ and $\nu_1 = \nu_2 = \nu_3 = 1$ of Example 2.

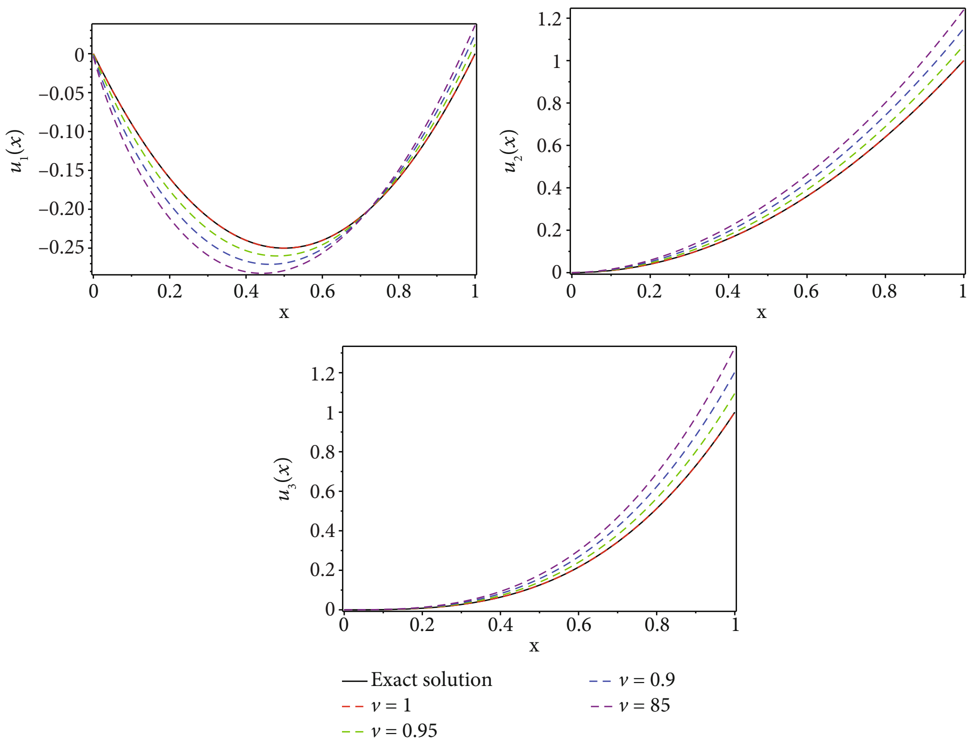

FIGURE 7: Some illustrations for the exact and approximate solutions using different values of ν_i: $\nu_i = 0.85, 0.90, 0.95, 1$, $i = 1, 2, 3$ for $N = 9$ in Example 2.

TABLE 4: Comparison between absolute errors of SSKCP collocation and Jacobi collocation methods at equally spaced points for $N = 12$ in Example 3.

x_i	SSKCP collocation method		Jacobi collocation method [18]	
	Error (u_1)	Error (u_2)	Error (u_1)	Error (u_2)
0.0	1.7182×10^{-5}	9.8329×10^{-7}	9.2840×10^{-6}	6.2830×10^{-7}
0.1	5.5657×10^{-6}	3.3268×10^{-6}	1.5202×10^{-6}	1.6472×10^{-7}
0.2	5.5935×10^{-6}	6.9555×10^{-6}	7.0504×10^{-7}	9.8923×10^{-8}
0.3	8.1095×10^{-6}	1.2777×10^{-5}	8.7678×10^{-7}	2.9284×10^{-7}
0.4	1.1355×10^{-5}	2.2448×10^{-5}	4.8193×10^{-7}	4.5120×10^{-7}
0.5	2.0566×10^{-5}	3.9797×10^{-5}	8.8367×10^{-7}	7.7271×10^{-7}
0.6	3.2387×10^{-5}	7.0527×10^{-5}	3.7500×10^{-7}	1.3807×10^{-6}
0.7	5.7063×10^{-5}	1.2877×10^{-4}	1.0664×10^{-6}	2.4661×10^{-6}
0.8	1.1121×10^{-4}	2.5191×10^{-4}	2.2537×10^{-6}	4.8902×10^{-6}
0.9	2.3722×10^{-4}	5.4688×10^{-4}	4.7116×10^{-6}	1.0475×10^{-5}
1.0	5.7984×10^{-4}	1.3758×10^{-3}	1.2704×10^{-5}	2.6197×10^{-5}

TABLE 5: Maximum absolute errors obtained by SSKCP collocation method for different values of N in Example 3.

N	Error (u_1)	Error (u_2)	CPU time
8	2.8334×10^{-3}	6.7948×10^{-3}	5.772
12	5.7819×10^{-4}	1.3597×10^{-3}	23.587
16	1.8841×10^{-4}	4.4719×10^{-4}	97.048
20	7.8006×10^{-5}	1.8536×10^{-4}	164.409
25	2.9931×10^{-5}	6.1196×10^{-5}	483.681

TABLE 6: Values of absolute errors at equally spaced points for $N = 25$ in Example 3.

x_i	Error (u_1)	Error (u_2)
0.0	1.3659×10^{-6}	3.2865×10^{-8}
0.1	2.6535×10^{-7}	1.7914×10^{-7}
0.2	3.1410×10^{-7}	3.8266×10^{-7}
0.3	4.3165×10^{-7}	7.0524×10^{-7}
0.4	6.3587×10^{-7}	1.2387×10^{-6}
0.5	9.6524×10^{-7}	2.1315×10^{-6}
0.6	1.5052×10^{-6}	3.6310×10^{-6}
0.7	2.8039×10^{-6}	6.5226×10^{-6}
0.8	5.5119×10^{-6}	1.2576×10^{-6}
0.9	1.1848×10^{-5}	2.7085×10^{-5}
1.0	2.8779×10^{-5}	6.9756×10^{-5}

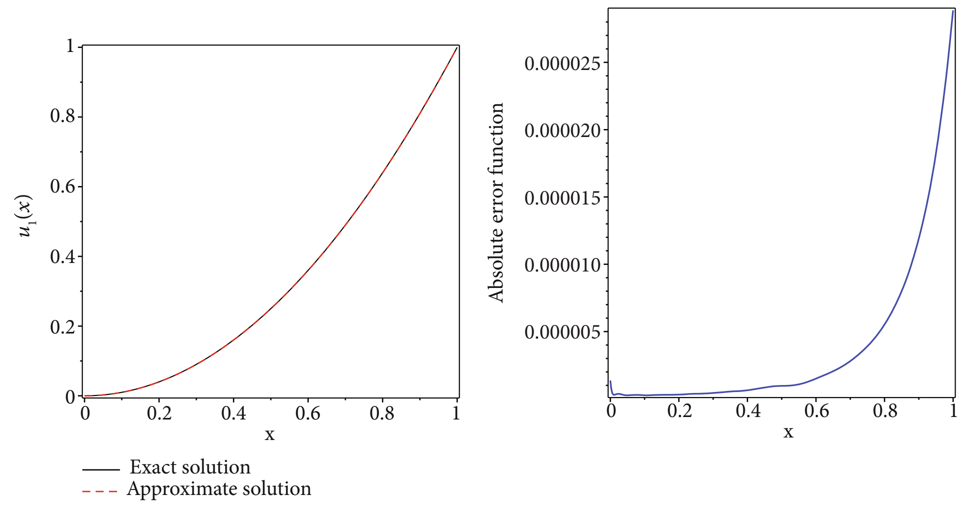

Figure 8: The graphs of the exact and approximate solutions and the absolute error function of $\tilde{u}_1(x)$ for $N = 25$ of Example 3.

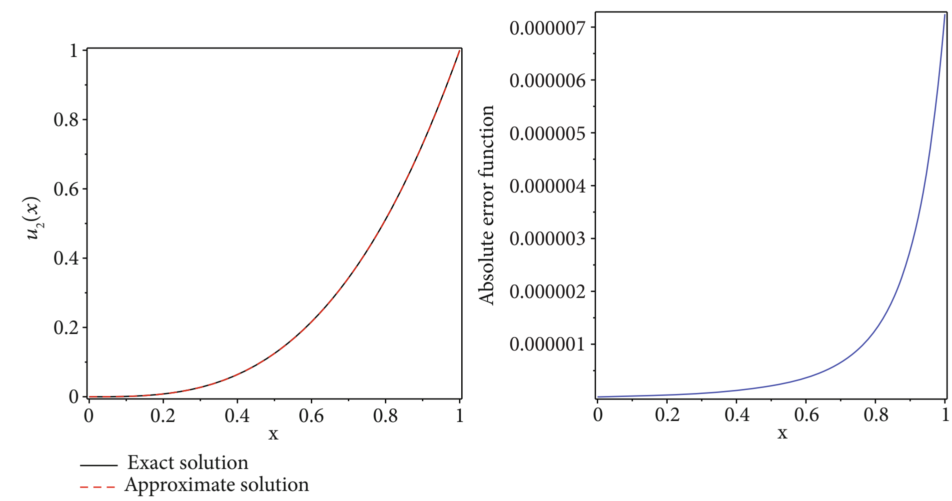

Figure 9: The graphs of the exact and approximate solutions and the absolute error function of $\tilde{u}_2(x)$ for $N = 25$ of Example 3.

The initial conditions are $u_1(0) = u_2(0) = 0$ and the exact solutions for this example are $u_1(x) = x^2$ and $u_2(x) = x^3$. We use the following approximations and convert the system (96) into an algebraic system, which will be described below:

$$\begin{aligned}
&\mathscr{D}^{3/4}u_1(x) \approx S^T(x)\mathrm{F}_1, \quad u_1(x) \approx S^T(x)\mathscr{P}^{(3/4)T}\mathrm{F}_1 = S^T(x)U_1,\\
&\mathscr{D}^{1/2}u_2(x) \approx S^T(x)\mathrm{F}_2, \quad u_2(x) \approx S^T(x)\mathscr{P}^{(1/2)T}\mathrm{F}_2 = S^T(x)U_2,\\
&u_1^3(x) \approx S^T(x)U_3, \quad U_3 \approx \left(\tilde{U}_1^2\right)^T U_1,\\
&u_2^2(x) \approx S^T(x)U_4, \qquad U_4 \approx \tilde{U}_2^T U_2,\\
&\int_0^x \frac{u_1(z)}{(x-z)^{1/2}}\,dz + \int_0^x \frac{u_2^2(z)}{(x-z)^{1/2}}\,dz \approx U_1^T\mathfrak{F}^{(1/2)}S(x) + U_4^T\mathfrak{F}^{(1/2)}S(x),\\
&\int_0^x \frac{u_1(z)}{(x-z)^{2/3}}\,dz + \int_0^x \frac{u_2(z)}{(x-z)^{2/3}}\,dz \approx U_1^T\mathfrak{F}^{(2/3)}S(x) + U_2^T\mathfrak{F}^{(2/3)}S(x),
\end{aligned} \tag{98}$$

where $\tilde{U}_1$ and $\tilde{U}_2$ are operational matrices of the product, corresponding to the vectors U_1 and U_2, respectively. So we have

$$\begin{cases} S^T(x)\mathrm{F}_1 - S^T(x)U_2 - U_1^T\mathfrak{F}^{(1/2)}S(x) - U_4^T\mathfrak{F}^{(1/2)}S(x) - f_1(x) \approx 0,\\ S^T(x)\mathrm{F}_2 - S^T(x)U_3 - U_1^T\mathfrak{F}^{(2/3)}S(x) - U_2^T\mathfrak{F}^{(2/3)}S(x) - f_2(x) \approx 0. \end{cases} \tag{99}$$

Table 4 shows a comparison between the absolute errors of the SSKCP collocation method and the Jacobi collocation method at equally spaced points $x_i = 0.1i, i = 0, 1, \cdots, 10$ for $N = 12$, which shows that the absolute errors of the Jacobi collocation method are less than the presented method, but with increasing N, the errors of the proposed method decrease. Maximum absolute errors for different values of N and numerical results for $N = 25$ reported in Tables 5 and 6 confirm that the results are close to those reported by [18]. Figures 8 and 9 show the comparison between the numerical results and the exact solutions and also the absolute error functions of $u_1(x), u_2(x)$ for $N = 25$, respectively.

7. Conclusion

In this paper, the shifted sixth-kind Chebyshev polynomials together with the collocation method were used to solve a class of the system of fractional integro-differential equations with weakly singular kernels. For this purpose, the integral and product operational matrices were calculated, and using the obtained approximations, the original system of equations was transformed into a corresponding linear and nonlinear system of algebraic equations that are easier for solving. Choosing an appropriate value of N, each of algebraic equations was collocated in the roots of $S^*_{N+1}(x)$, and finally, for obtaining the unknown vectors $F_i, i = 1, 2, \cdots, m$, an algebraic system involving $m(N+1)$ algebraic equations was solved. To eliminate the singularity of the kernels of the equations under study, an operational matrix was derived. Also, an error bound was determined for the proposed method. To show the ability and efficiency of the proposed method, three examples were presented and the maximum absolute errors were calculated for different N, and graphs of the absolute error functions and numerical solutions were plotted and numerical results showed a good agreement between the approximate and exact solutions. When the order of the fractional derivative ν was uncertain, the numerical solutions for the various values of ν, $0 < \nu \leq 1$, were approached to the exact solutions as $\nu \longrightarrow 1$. The comparison of the proposed method with the Jacobi collocation method [18] showed good implementation of SSKCP collocation method for solving a system of linear fractional integro-differential equations with weakly singular kernels and for a system of nonlinear WSFIDEs; the error decreased with increasing N. CPU times were computed for all examples. According to the numerical results which were obtained in the relevant instances and compared with exact solutions and those obtained from the Jacobi collocation method, it can be concluded that the SSKCP collocation method is very helpful to look for approximate solutions of a system of WSFIDEs. This method can be applied to the linear and nonlinear systems of fractional-order Volterra-Fredholm integro-differential equations with weakly singular kernels, but additional operational matrices are required.

Data Availability

All results have been obtained by conducting the numerical procedure, and the ideas can be shared for the researchers.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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