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Research Article

Novel Concepts in Vague Graphs with Application in Hospital's Management System

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Many problems of practical interest can be modeled and solved by using vague graph (VG) algorithms. Vague graphs, belonging to the fuzzy graphs (FGs) family, have good capabilities when faced with problems that cannot be expressed by FGs. Hence, in this paper, we introduce the notion of (η, γ)- HMs of VGs and classify homomorphisms (HMs), weak isomorphisms (WIs), and coweak isomorphisms (CWIs) of VGs by (η, γ)- HMs. Hospitals are very important organizations whose existence is directly related to the general health of the community. Hence, since the management in each ward of the hospital is very important, we have tried to determine the most effective person in a hospital based on the performance of its staff.

1. Introduction

Graphs, from ancient times to the present day, have played a very important role in various fields, including computer science and social networks, so that with the help of the vertices and edges of a graph, the relationships between objects and elements in a social group can be easily introduced. But there are some phenomena in our lives that have a wide range of complexities that make it impossible for us to express certainty. These complexities and ambiguities were reduced with the introduction of FSs by Zadeh [1]. Since then, the theory of FSs has become a vigorous area of research in different disciplines including logic, topology, algebra, analysis, information theory, artificial intelligence, operations research, and neural networks and planning [2–6]. The FS focuses on the membership degree of an object in a particular set. But membership alone could not solve the complexities in different cases, so the need for a degree of membership was felt. To solve this problem, Gau and Buehrer [7] introduced false-membership degrees and defined a VS as the sum of degrees not greater than 1. The first definition of FGs was proposed by Kafmann [8] in 1993, from Zade's fuzzy relations [9, 10]. But Rosenfeld [11] introduced another elaborated definition including fuzzy vertex and fuzzy edges and several fuzzy analogs of graph theoretic concepts such as paths, cycles, and connectedness. Ramakrishna [12] introduced the concept of VGs and studied some of their properties. Akram et al. [13–16] defined the vague hypergraphs, Cayley-VGs, and regularity in vague intersection graphs and vague line graphs. Rashmanlou et al. [17] investigated categorical properties in intuitionistic fuzzy graphs. Bhattacharya [18] gave some remarks on FGs, and some operations of FGs were introduced by Mordeson and Peng [19]. The concepts of weak isomorphism, coweak isomorphism, and isomorphism between FGs were introduced by Bhutani in [2]. Khan et al. [20] studied vague relations. Talebi [21, 22] investigated Cayley-FGs and some results in bipolar fuzzy graphs. Borzooei [23] introduced domination in VGs. Ghorai and Pal studied some isomorphic properties of m-polar FGs [24]. Jiang et al. [25] defined vertex covering in cubic graphs. Krishna et al. [26] presented a new concept in cubic graphs. Rao et al. [27–29] investigated dominating set, equitable dominating set, and isolated vertex in VGs. Hoseini et al. [30] given maximal product of graphs under vague environment. Jan et al. [31] introduced some root-level modifications in interval-valued fuzzy graphs. Amanathulla et al. [32] defined new concepts of paths and interval graphs. Muhiuddin et al. [33, 34] presented the reinforcement number of a graph and new results in cubic graphs.

A VG is a generalized structure of an FG that provides more exactness, adaptability, and compatibility to a system when matched with systems run on FGs. Also, a VG is able to concentrate on determining the uncertainty coupled with the inconsistent and indeterminate information of any real-world problems, where FGs may not lead to adequate results. VGs have a wide range of applications in the field of psychological sciences as well as in the identification of individuals based on oncological behaviors. Thus, in this paper, we studied level graphs of VGs and investigated HMs, WIs, and CWIs of VGs by HMs of level graphs. Likewise, we characterized some VGs by their level graphs.

2. Preliminaries

In this section, we review some concepts of graph theory and VGs.

Definition 1. Let V be a finite nonempty set. A graph $G = (V, E)$ on V consist of a vertex set V and an edge set E, where an edge is an unordered pair of distinct nodes of G. We will use pq rather than $\{p, q\}$ to denote an edge. If pq is an edge, then we say that p and q are neighbor. A graph is called complete graph if each pair of nodes are neighbor.

Definition 2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. A mapping $h\colon V_1 \longrightarrow V_2$ is a homomorphism from G_1 to G_2 if $h(p)$ and $h(q)$ are neighbor whenever p and q are neighbor.

Definition 3. Two graphs G_1 and G_2 are isomorphic if there is a bijective mapping $\varphi\colon V_1 \longrightarrow V_2$ so that p and q are neighbor in G_1 if and only if $\varphi(p)$ and $\varphi(q)$ are neighbor in G_2, φ is named isomorphism from G_1 to G_2. An isomorphism from a graph G to itself is named an automorphism of G. The set of all automorphisms of G forms a group, which is named the automorphism group of G and shown by $\mathrm{Aut}(G)$.

Definition 4. A VS A is a pair (t_A, f_A) on set X where t_A and f_A are taken as real valued functions which can be defined on $V \longrightarrow [0,1]$ so that $t_A(p) + f_A(p) \leq 1$, $\forall p \in X$.

Definition 5. Let $A, B \in VS(V)$. We say that A is contained in B and write $A \subseteq B$, if for any $p \in V$,

$$\begin{aligned} t_A(p) &\leq t_B(p),\\ f_A(p) &\geq f_B(p). \end{aligned} \tag{1}$$

Let $K_* = \{(\eta, \gamma) | \eta, \gamma \in [0,1], \eta + \gamma \leq 1\}$. For any $(\eta_1, \gamma_1), (\eta_2, \gamma_2) \in L_*$, the orders $\leq$ and $<$ on K_* are defined as

$$\begin{aligned} &(\eta_1, \gamma_1) \leq (\eta_2, \gamma_2) \Leftrightarrow \eta_1 \leq \eta_2,\\ &\quad \gamma_1 \geq \alpha_2,\\ &(\eta_1, \gamma_1) < (\eta_2, \gamma_2) \Leftrightarrow (\eta_1, \gamma_1) \leq (\eta_2, \gamma_2),\\ &\quad \eta_1 < \eta_2,\\ &\text{or } \gamma_1 > \gamma_2. \end{aligned} \tag{2}$$

It is easy to see that, $(K_*, \leq)$ constitutes a complete lattice with maximum element $(1,0)$ and minimum element $(0,1)$.

Definition 6. Let $A \in VS(V)$. For each $(\eta, \gamma) \in K_*$, we define $A_{(\eta,\gamma)} = \{p \in V\colon t_A(p) \geq \eta, f_A(p) \leq \gamma\}$.

Then, $A_{(\eta,\gamma)}$ is named (η, γ)-level set of A. The set $\{p | p \in V, t_A(p) > 0 \text{ or } f_A(p) < 1\}$ is called the support A and is denoted by A^*.

Let V be a finite nonempty set. Denote by $\widetilde{V}^2$ the set of all 2-element subsets of V. A graph on V is a pair (V, E) where $E \subseteq \widetilde{V}^2$, V and E are named vertex set and edge set, respectively.

Definition 7. Let V be a finite nonempty set, $A \in VS(V)$ and $B \in VFS(\widetilde{V}^2)$. The triple $X = (V, A, B)$ is named a VG on V, if for each $(p, q) \in \widetilde{V}^2$,

$$\begin{aligned} t_B(p,q) &\leq t_A(p) \wedge t_A(q),\\ f_B(p,q) &\geq f_A(p) \vee f_A(q). \end{aligned} \tag{3}$$

If $X = (V, A, B)$ is a VG, then, it is easy to see that $X^* = (A^*, B^*)$ is a graph and it is called underlying graph of X. The set of all VG on V is denoted by $VG(V)$. For given $X = (V, A, B) \in VG(V)$, in this study suppose that $A^* = V$.

Definition 8. Let $X_1 = (V_1, A_1, B_1)$ and $X_2 = (V_2, A_2, B_2)$ be two VGs. Then,

(1) A mapping $\varphi\colon V_1 \longrightarrow V_2$ is a homomorphism from X_1 to X_2, if

(i) $t_{A_1}(p) \leq t_{A_2}(\varphi(p))$, $f_{A_1}(p) \geq f_{A_2}(\varphi(p))$, for all $p \in V_1$

(ii) $t_{B_1}(pq) \leq t_{B_2}(\varphi(p)\varphi(q))$, $f_{B_1}(pq) \geq f_{B_2}(\varphi(p)\varphi(q))$, for all $pq \in \widetilde{V}^2$

(2) A mapping $\varphi\colon V_1 \longrightarrow V_2$ is a weak isomorphism from X_1 to X_2, if φ is a BH from X_1 to X_2 and $t_{A_1}(p) = t_{A_2}(\varphi(p))$, $f_{A_1}(p) = f_{A_2}(\varphi(p))$, for all $p \in V_1$.

(3) A mapping $\varphi\colon V_1 \longrightarrow V_2$ is a coweak isomorphism from X_1 to X_2, if φ is a BH from X_1 to X_2 and $t_{B_1}(pq) = t_{B_2}(\varphi(p)\varphi(q))$, $f_{B_1}(pq) = f_{B_2}(\varphi(p)\varphi(q))$, for all $pq \in \widetilde{V}^2$.

(4) An isomorphism from X_1 to X_2 is a bijective mapping $\varphi\colon V_1 \longrightarrow V_2$ so that

(i) $t_{A_1}(p) = t_{A_2}(\varphi(p))$, $f_{A_1}(p) = f_{A_2}(\varphi(p))$, for all $p \in V_1$

(ii) $t_{B_1}(pq) = t_{B_2}(\varphi(p)\varphi(q))$, $f_{B_1}(pq) = f_{B_2}(\varphi(p)\varphi(q))$, for all $pq \in \widetilde{V}^2$

Definition 9. VG $X = (V, A, B)$ is called strong vague graph (SVG) if $t_B(pq) = t_A(p) \wedge t_A(q)$, $f_B(pq) = f_A(p) \vee f_A(q)$, for all $pq \in \widetilde{V}^2$, $(t_B(pq), f_B(pq)) \neq (0, 1)$ and is called complete vague graph (CVG), if $t_B(pq) = t_A(p) \wedge t_A(q)$, $f_B(pq) = f_A(p) \vee f_A(q)$, for all $pq \in \widetilde{V}^2$. A CVG $X = (V, A, B)$ with n nodes is denoted by $K_{n,A}$.

Definition 10. Suppose that $X = (V, A, B)$ and $Y = (V, A', B')$ be two VGs. Then, X is VSG of Y, if $A \subseteq A'$ and $B \subseteq B'$.

Definition 11. Let $X = (V, A, B)$ be VG and $W \subseteq V$. Then, the VG $Y = (W, A', B')$ so that $t_{A'}(p) = t_A(p)$, $f_{A'}(p) = f_A(p)$, for all $p \in W$, $t_{B'}(pq) = t_B(pq)$, $f_{B'}(pq) = f_B(pq)$, for all $pq \in \widetilde{W}^2$, is named the induced VSG by W and shown by $X[W]$.

Definition 12. A family $\Gamma = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of VSs on V is named a k-coloring of VG $X = (V, A, B)$ if

(i) $\vee \Gamma = A$.
(ii) $\lambda_i \wedge \lambda_j = 0$ for $1 \leq i, j \leq k$.
(iii) For each strong edge pq of X, $\min\{\lambda_i(p), \lambda_i(q)\} = 0$ for $1 \leq i \leq k$. We say that a graph is k-colorable if it can be colored with k colors.

All the basic notations are shown in Table 1.

Table 1: Some basic notations.

Notation	Meaning
FG	Fuzzy graph
VS	Vague set
FS	Fuzzy set
VG	Vague graph
CVG	Complete vague graph
SVG	Strong vague graph
BM	Bijective mapping
HM	Homomorphism
WI	Weak isomorphism
IH	Injective homomorphism
CWI	Coweak isomorphism
BH	Bijective homomorphism
SG	Subgraph
IV	Isolated vertex
CG	Complete graph
VSG	Vague subgraph

3. Homomorphisms and Isomorphisms of Vague Graphs

In this section, we discuss the homomorphism and isomorphism of VGs by the homomorphism of level graphs in VGs.

Theorem 1. *Let V be a finite nonempty set, $A \in VS(V)$ and $B \in VS(\widetilde{V}^2)$. Then, $X = (V, A, B) \in VFG(V)$ if and only if $X_{(\eta,\gamma)} = (A_{(\eta,\gamma)}, B_{(\eta,\gamma)})$ is a graph for all $(\eta, \gamma) \in L_*$, $A_{(\eta,\gamma)} \neq \varnothing$.*

Proof. Let $X = (V, A, B)$ be VG. For each $(\eta, \gamma) \in L_*$, $A_{(\eta,\gamma)} \neq \varnothing$, assume that $pq \in B_{(\eta,\gamma)}$. Then, $t_B(pq) \geq \eta$ and $f_B(pq) \leq \gamma$. Because X is VG,

$$\begin{aligned} \lambda \leq t_B(pq) \leq t_A(p) \wedge t_A(q), \\ \gamma \geq f_B(pq) \geq f_A(p) \vee f_A(q). \end{aligned} \tag{4}$$

It follows that $p, q \in A_{(\eta,\gamma)}$. Therefore, $(A_{(\eta,\gamma)}, B_{(\eta,\gamma)})$ is a graph.

Conversely, let $X_{(\eta,\gamma)} = (A_{(\eta,\gamma)}, B_{(\eta,\gamma)})$ is a graph, $\forall (\eta, \gamma) \in L_*$, $A_{(\eta,\gamma)} \neq \varnothing$. For each $pq \in \widetilde{V}^2$, let $t_B(pq) = \eta$, $f_B(pq) = \gamma$. Then, $pq \in B_{(\eta,\gamma)}$. Hence, $p, q \in A_{(\eta,\gamma)}$. Thus, $t_A(p) \geq \eta, t_A(q) \geq \eta, f_A(p) \leq \gamma$, and, $f_A(q) \leq \alpha$,. This implies that $t_A(p) \wedge t_A(q) \geq \eta = t_B(pq)$ and $f_A(p) \vee f_A(q) \leq \gamma = f_B(pq)$. Therefore, $X = (V, A, B)$ is VG. □

Definition 13. Let $X = (V, A, B)$ and $Y = (W, A', B')$ be two VGs, $h: V \longrightarrow W$ a mapping. For any $(\eta, \gamma) \in L_*$, $A_{(\eta,\gamma)} \neq \varnothing$, if h is a homomorphism from $X_{(\eta,\gamma)} = (A_{(\eta,\gamma)}, B_{(\eta,\gamma)})$ to $Y_{(\eta,\gamma)} = (A'_{(\eta,\gamma)}, B'_{(\eta,\gamma)})$, then, h is called (η, γ)– homomorphism mapping from X to Y.

Theorem 2. *Let $X = (V, A, B)$ and $Y = (W, A', B')$ be two VGs. Then, h: $X \longrightarrow Y$ is a homomorphism from X to Y if and only if h is (η, γ)- homomorphism from X to Y.*

Proof. Assume that h: $X \longrightarrow Y$ is a homomorphism from X to Y. Let, $A_{(\eta,\gamma)} \neq \varnothing$, $(\eta, \gamma) \in L_*$. If $p \in A_{(\eta,\gamma)}$, then

$$\begin{aligned} t_{A'}(h(p)) \geq t_A(p) \geq \eta, \\ f_{A'}(h(p)) \leq f_A(p) \leq \gamma. \end{aligned} \tag{5}$$

Hence, $h(p) \in A'_{(\eta,\gamma)}$ implying h is a mapping from $A_{(\eta,\gamma)}$ to $A'_{(\eta,\gamma)}$. For $p, q \in A_{(\eta,\gamma)}$, let $pq \in B_{(\eta,\gamma)}$. Then,

$$\begin{aligned} t_B(pq) \geq \eta, \\ f_B(pq) \leq \gamma. \end{aligned} \tag{6}$$

Hence,

$$\begin{aligned} t_{B'}(h(p)h(q)) \geq t_B(pq) \geq \eta, \\ f_{B'}(h(p)h(q)) \leq f_B(pq) \leq \gamma, \end{aligned} \tag{7}$$

which implies $h(p)h(q) \in B'_{(\eta,\gamma)}$. Therefore, h is a homomorphism from $X_{(\eta,\gamma)}$ to $Y_{(\eta,\gamma)}$.

Conversely, let h: $V \longrightarrow W$ be a (η, γ)- homomorphism from X to Y. For arbitrary element $p \in X$, let $t_A(p) = c$, $f_A(p) = d$. Then, $p \in A_{(c,d)}$, hence, $h(p) \in A'_{(c,d)}$, because h is a homomorphism from $(A_{(c,d)}, B_{(c,d)})$ to $(A'_{(c,d)}, B'_{(c,d)})$. It follows that

$$\begin{aligned} t_{A'}(h(p)) \geq c, \\ f_{A'}(h(p)) \leq d, \end{aligned} \tag{8}$$

that is,

$$t_{A_\prime}(h(p)) \geq t_A(p),$$
$$f_{A_\prime}(h(p)) \leq f_A(x). \tag{9}$$

Now for arbitraries $p, q \in V$, let $t_B(pq) = e$, $f_B(pq) = t$. Then,

$$e = t_B(pq) \leq t_A(p) \wedge t_A(q),$$
$$t = f_B(pq) \geq f_A p \vee f_A(q). \tag{10}$$

Hence, $p, q \in A_{(e,t)}$ and $pq \in B_{(e,t)}$. Because h is a homomorphism from $X_{(e,t)} = (A_{(e,t)}, B_{(e,t)})$ to $Y_{(e,t)} = (A_{(e,t)}^{\prime}, B_{(e,t)}^{\prime})$, we conclude that $h(p), h(q) \in A_{(e,t)}^{\prime}$ and $h(p)g(q) \in B_{(e,t)}^{\prime}$. Therefore,

$$t_{B_\prime}(h(p)h(q)) \geq e = t_B(pq),$$
$$f_{B_\prime}(h(p)h(q)) \leq t = f_B(pq). \tag{11}$$

□

Theorem 3. *Let $X = (V, A, B)$ and $Y = (W, A', B')$ be two VGs. Then, h: $V \longrightarrow W$ is a WI from X to Y if and only if h is a bijective (η, γ)- homomorphism from X to Y and*

$$t_A(p) = t_{A'}(h(p)),$$
$$f_A(p) = f_{A'}(h(p)), \tag{12}$$
$$\text{for all } p \in V.$$

Proof. Let h be a WI from X to Y. From the definition of homomorphism h is a bijective homomorphism from X to Y. By Theorem 2 h is a bijective (η, γ)- homomorphism from X to Y and also by the definition of WI we have

$$t_A(p) = t_{A'}(h(p)),$$
$$f_A(p) = f_{A'}(h(p)), \tag{13}$$
$$\text{for all } p \in V.$$

Conversely, from hypothesis, h: $A_{(0,1)} = V \longrightarrow A_{(0,1)}^{\prime} = W$ is a bijective mapping and

$$t_A(p) = t_{A'}(h(p)),$$
$$f_A(p) = f_{A'}(h(p)), \tag{14}$$
$$\text{for all } p \in V.$$

For $p, q \in V$, let $t_B(pq) = e$, $f_B(pq) = t$. Then,

$$e = t_B(pq) \leq t_A(p) \wedge t_A(q),$$
$$t = f_B(pq) \geq f_A(p) \vee f_A(q), \tag{15}$$

which implies $p, q \in A_{(e,t)}$ and $pq \in B_{(e,t)}$. Because h is a homomorphism from $(A_{(e,t)}, B_{(e,t)})$ to $(A_{(e,t)}^{\prime}, B_{(e,t)}^{\prime})$, we have $h(p), h(q) \in A_{(e,t)}^{\prime}$ and $h(p)h(q) \in B_{(e,t)}^{\prime}$. Hence,

$$t_{B'}(h(p)h(q)) \geq e = t_B(pq),$$
$$f_{B'}(h(p)h(q)) \leq t = f_B(pq), \tag{16}$$

which complete the proof. □

Theorem 4. *Let $X = (V, A, B)$ and $Y = (W, A', B')$ be two VGs. Then, h: $V \longrightarrow W$ is a CWI from X to Y if and only if h is a bijective (η, γ)- homomorphism from X to Y and*

$$t_B(pq) = t_{B'}(h(p)h(q)),$$
$$f_B(pq) = f_{B'}(h(p)h(q)), \tag{17}$$
$$\text{for all } pq \in \tilde{V}^2.$$

Proof. Let h: $V \longrightarrow W$ be a CWI from X to Y. Then, h is a bijective homomorphism from X to Y. By Theorem 2 h is a bijective (η, γ)- homomorphism from X to Y. Also by the definition of CWI

$$t_B(pq) = t_{B'}(h(p)h(q)),$$
$$f_B(pq) = f_{B'}(h(p)h(q)), \tag{18}$$
$$\text{for all } pq \in \tilde{V}^2.$$

Conversely, from hypothesis, we know that h: $A_{(0,1)} = V \longrightarrow A_{(0,1)}^{\prime} = W$ is a bijective mapping and

$$t_B(pq) = t_{B'}(h(p)h(q)),$$
$$f_B(pq) = f_{B'}(h(p)h(q)). \tag{19}$$

For arbitrary element $p \in V$, suppose that $t_A(p) = c$, $f_A(p) = d$. Then, we have $p \in A_{(c,d)}$. Now because h is a homomorphism from $(A_{(c,d)}, B_{(c,d)})$ to $(A_{(c,d)}^{\prime}, B_{(c,d)}^{\prime})$, $h(p) \in A_{(c,d)}^{\prime}$. Thus, $t_{A_\prime}(h(p)) \geq c = t_A(p)$, $f_{A_\prime}(h(p)) \leq d = f_A(p)$, which implies h is a CWI from X to Y. □

Corollary 1. *Let $X = (V, A, B) \in VG(V)$, $Y = (W, A', B') \in VG(W)$. If h: $V \longrightarrow W$ is a CWI from X to Y, then, h is an IH from $X_{(\eta,\gamma)}$ to $Y_{(\eta,\gamma)}$, $\forall (\eta, \gamma) \in K_*$, $A_{(\eta,\gamma)} \neq \varnothing$.*

From the following example, we conclude that the converse of Corollary 1 do not need to be true.

Example 1. Let $X = (V, A, B)$ and $Y = (W, A', B')$ be two VGs, as shown in Figure 1. Consider the mapping h: $V \longrightarrow W$, defined by $h(v_i) = w_i$, $1 \leq i \leq 5$. In view of the (η, γ)- level graphs of X and Y in Figure 1, if $A_{(\eta,\gamma)} \neq \varnothing$ then, h is an IH from $X_{(\eta,\gamma)}$ to $Y_{(\eta,\gamma)}$, but h is not a CWI.

Theorem 5. *Let $X = (V, A, B) \in VG(V)$, $Y = (W, A', B') \in VG(W)$, and h: $V \longrightarrow W$ be a mapping. For each $(\eta, \gamma) \in K_*$, $A_{(\eta,\gamma)} \neq \varnothing$, if h is an isomorphism from $X_{(\eta,\gamma)}$ to a SG of $Y_{(\eta,\gamma)}$, then, h is a CWI from X to an induced VSG of Y.*

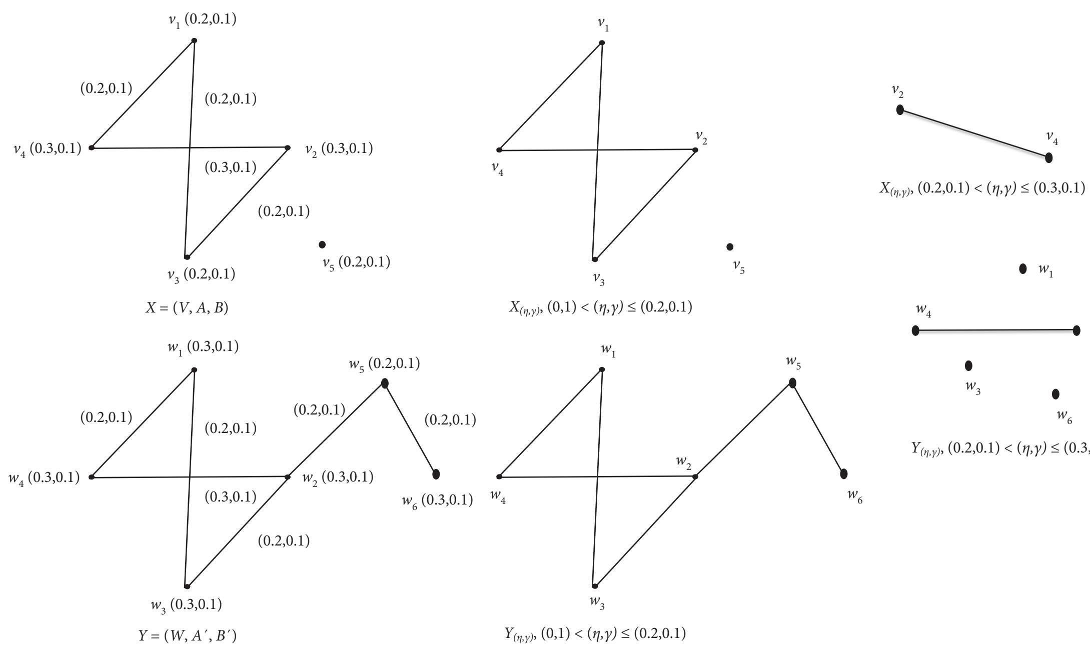

Figure 1: VGs X, Y and the mapping h: $V_i \longrightarrow W_i$ which is not a CWI.

Proof. The mapping h is an isomorphism from $X_{(0,1)} = (V, B_{(0,1)})$ to a SG $Y_{(0,1)} = (W, B_{(0,1)}')$, so h: $V \longrightarrow W$ is an IM. For arbitrary $p \in V$, suppose that $t_A(p) = \eta$, $f_A(p) = \gamma$. Then, $p \in A_{(\eta,\gamma)}$, and so $h(p) \in A_{(\eta,\gamma)}'$. Hence, $t_{A'}(h(p)) \geq \eta = t_A(p)$ and $f_{A'}(h(p)) \leq \gamma = f_A(p)$. For $p, q \in V$, let $t_B(pq) = \eta$ and $f_B(pq) = \gamma$. Then, $\eta \leq t_A(p)$, $\eta \leq t_A(q)$, $\gamma \geq f_A(p)$, $\gamma \geq f_A(q)$ and $pq \in B_{(\eta,\gamma)}$. Hence, $p, q \in A_{(\eta,\gamma)}$ and $pq \in B_{(\eta,\gamma)}$. Since h isomorphism from $X_{(\eta,\gamma)}$ to $Y_{(\eta,\gamma)}$, we get $h(p), h(q) \in A_{(\eta,\gamma)}'$ and $h(p)h(q) \in B_{(\eta,\gamma)}'$. Therefore,

$$\begin{aligned} t_{B'}(h(p)h(q)) &\geq \eta = t_B(pq), \\ f_{B'}(h(p)h(q)) &\leq \gamma = f_B(pq). \end{aligned} \tag{20}$$

Now, let $t_{B'}(h(p)h(q)) = k$, $f_{B'}(h(p)h(q)) = s$. Then, $h(p)h(q) \in A_{(k,s)}'$. Because h is injective and an isomorphism from $X_{(k,s)}$ to a SG of $Y_{(k,s)}$, we have $p, q \in A_{(k,s)}$ and $pq \in B_{(k,s)}$. Therefore,

$$\begin{aligned} t_B(pq) &\geq k = t_{B\prime}(h(p)h(q)), \\ f_B(pq) &\leq s = f_{B\prime}(h(p)h(q)). \end{aligned} \tag{21}$$

Now by (20) and (21), we conclude that

$$\begin{aligned} t_{B'}(h(p)h(q)) &= t_B(pq), \\ f_{B'}(h(p)h(q)) &= f_B(pq). \end{aligned} \tag{22}$$

□

Corollary 2. *Let $X = (V, A, B)$ and $Y = (W, A', B')$ be two VGs with $|V| = |W|$, and h: V <display> W a mapping. For $(\eta, \gamma) \in KL_*$, $A_{(\eta,\gamma)} \neq \varnothing$, if h is an isomorphism from $X_{(\eta,\gamma)}$ to a SG of $Y_{(\eta,\gamma)}$, then, h is a CWI from X to Y.*

Theorem 6. *Let $X = (V, A, B)$ and $Y = (W, A', B')$ be two VGs, h: $V \longrightarrow W$ be a bijective mapping. If for each $(\eta, \gamma) \in K_*$, h is an isomorphism from $X_{(\eta,\gamma)}$ to $Y_{(\eta,\gamma)}$, then, h is an isomorphism from X to Y.*

Proof. From hypothesis, h^{-1}: $W \longrightarrow V$ is a bijective mapping and an isomorphism from $Y_{(\eta,\gamma)}$ to $X_{(\eta,\gamma)}$. By Theorem 5 h is a CWI from X to Y and h^{-1} is a CWI from Y to X. Therefore, h is an isomorphism from X to Y. □

Corollary 3. *Let $X = (V, A, B)$ be VG and h: $V \longrightarrow V$ a bijective mapping. Then, h is an automorphism of X if and only if $h_{|A_{(\eta,\gamma)}}$ is an automorphism of $X_{(\eta,\gamma)}$, from an $(\eta, \gamma) \in K_*$, $A_{(\eta,\gamma)} \neq \varnothing$.*

Theorem 7. *Let $X = (V, A, B)$ be VG. Then, X is a CVG if and only if $X_{(\eta,\gamma)} = (A_{(\eta,\gamma)}, B_{(\eta,\gamma)})$ is a CG for $(\eta, \gamma) \in K_*$.*

Proof. If $X = (V, A, B)$ is a CVG and for $(\eta, \gamma) \in K_*$, $A_{(\eta,\gamma)} \neq \varnothing$, $p, q \in A_{(\eta,\gamma)}$, then, $t_A(p) \geq \eta$, $t_A(q) \geq \eta$, $f_A(p) \leq \gamma$, $f_A(q) \leq \gamma$, and so

$$\begin{aligned} t_B(pq) &= t_A(p) \wedge t_A(q) \geq \eta, \\ f_B(pq) &= f_A(p) \vee f_A(q) \leq \gamma. \end{aligned} \tag{23}$$

Hence, $pq \in B_{(\eta,\gamma)}$. It follows that $X_{(\eta,\gamma)}$ is a CG. Conversely, suppose that $X = (V, A, B)$ is not a CVG. Then, there are $p, q \in V$ so that $t_B(pq) < t_A(p) \wedge t_A(q)$ or $f_B(pq) > f_A(p) \vee f_A(q)$. Let $t_B(pq) < t_A(p) \wedge t_A(q)$, and $t_A(p) \wedge t_A(q) = \eta$, for $\eta \in (0, 1]$. Then, $t_A(p) \geq \eta$ and $t_A(q) \geq \eta$. Hence, $x, y \in A_{(\eta,\gamma)}$, for a $\gamma \in [0, 1]$, but

$pq \notin B_{(\eta,\gamma)}$. This implies that $X_{(\eta,\gamma)}$ is not a CG. For the case $f_B(pq) > f_A(p) \vee f_A(q)$, it follows similarly. □

Theorem 8. *Let $X = (V, A, B) \in VG(V)$. Then, $X_{(\eta,\gamma)}$ has not IV, for each $(\eta,\gamma) \in K_*$, $A_{(\eta,\gamma)} \neq \emptyset$ if and only if for each $p \in V$, $\exists q \in V$ so that $t_B(pq) = t_A(p)$, $f_B(pq) = f_A(p)$.*

Proof. Suppose that for each $(\eta,\gamma) \in K_*$, $A_{(\eta,\gamma)} \neq \emptyset$, graph $X_{(\eta,\gamma)}$ has not IV and there is a node $p \in V$ so that for each $q \in V$, $t_B(pq) < t_A(p)$ or $f_B(pq) > f_A(p)$. Let $t_B(pq) < t_A(p)$ and $t_A(p) = \eta$, $fA(p) = \gamma$, for $(\eta,\gamma) \in K_*$. Then, $p \in A_{(\eta,\gamma)}$ and for each $q \in V, q \neq p$, $pqinB_{(\eta,\gamma)}$. Therefore, p is an IV in the graph $X_{(\eta,\gamma)} = (A_{(\eta,\gamma)}, B_{(\eta,\gamma)})$, which is a contradiction.

Now suppose that for $(\eta,\gamma) \in K_*$, $A_{(\eta,\gamma)} \neq \emptyset$, node $p \in A_{(\eta,\gamma)}$ is an IV in $X_{(\eta,\gamma)}$. If $qinA_{(\eta,\gamma)}$, then, $t_B(pq) \leq t_A(q) < \eta \leq t_A(p)$ or $f_B(pq) \geq f_A(q) > \gamma \geq f_A(q)$, and if $q \in A_{(\eta,\gamma)}$, it is trivial that $pginB_{(\eta,\gamma)}$, hence, $t_B(pq) < \eta \leq t_A(p)$ or $f_B(pq) > \gamma \geq f_A(p)$. Therefore, for each $q \in V$, $t_B(pq) \neq t_A(p)$, $f_B(pq) \neq f_A(p)$.

Here, we describe the relationship between coloring graph and homomorphism of graph. □

Theorem 9. *A VG $X = (V, A, B)$ is r-colorable $\Leftrightarrow$ there exists a homomorphism from X to the $K_{r,A_{\prime}}$.*

Proof. Assume that X be r-colorable with r colors labeled $\Gamma = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. Let $V_i = \{v \in V | \lambda_i(v) \neq 0\}$. We define CVG $K_{r,A'}$ with vertices set $\{1, 2, \ldots, r\}$, so that the degree of membership vertex i is $t_{A'}(i) = \max\{t_A(v) \mid v \in V_i\}$ and the degree of non-membership vertex i is $f_{A'}(i) = \min\{f_A(v) | v \in V_i\}$. Now the mapping $h: X \longrightarrow K_{r,A'}$ defined by $h(v) = i$ is a graph homomorphism, because

$$\begin{aligned} t_A(v) &\leq \max\{t_A(w) | w \in V_i\} = t_{A'}(i) = t_{A'}(h(v)),\\ f_A(v) &\geq \min\{f_A(w) | w \in V_i\} = f_{A'}(i) = f_{A'}(h(v)). \end{aligned} \tag{24}$$

According to the definition of CVG, for $u \in V_i$ and $v \in V_j$ we have

$$\begin{aligned} t_B(uv) &\leq t_A(u) \wedge t_A(v) \leq t_{A'}(i) \wedge t_{A_{\prime}}(j) = t_{A'}(h(u)) \wedge t_{A'}(h(v)),\\ f_B(uv) &\geq f_A(u) \vee t_A(v) \leq t_{A'}(i) \vee t_{A'}(j) = t_{A'}(h(u)) \vee t_{A'}(h(v)). \end{aligned} \tag{25}$$

Then, $t_B(uv) \leq t_{B_{\prime}}(h(u)h(v))$, $f_B(uv) \geq f_{B'}(h(u)h(v))$, for all $uv \in \widetilde{V}^2$.

Conversely, let $g: X \longrightarrow K_{r,A'}$ be a homomorphism. For a given $k \in V(K_{r,A_{\prime}})$, define the set $h^{-1}(k) \subseteq V$ to be

$$h^{-1}(k) = \{x \in V | h(x) = k\}. \tag{26}$$

If $v \in h^{-1}(k)$, let $\lambda_k(v) = (t_{\lambda_k}(v), f_{\lambda_k}(v)) = (t_A(v), f_A(v))$, otherwise $\lambda_k(v) = 0$. Therefore, the VG X is r-colorable with coloring set $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. □

4. Application

Nowadays, the issue of coloring is very important in the theory of fuzzy graphs because it has many applications in controlling intercity traffic, coloring geographical maps, as well as finding areas with high population density. Therefore, in this section, we have tried to present an application of the coloring of vertices in a VG.

Example 2. Let $X = (V, A_1, B_1)$ be a VG (See Figure 2). We modeled a FG by considering countries A, B, C, D as vertices of graph. The membership and nonmembership value of the vertices are the good and bad activity of a country with respect technology so that are $(t_{A_1}(A), f_{A_1}(A)) = (0.1, 0.2)$, $(t_{A_1}(B), f_{A_1}(B)) = (0.4, 0.5)$, $(t_{A_1}(C), f_{A_1}(C)) = (0.2, 0.5)$, $(t_{A_1}(D), f_{A_1}(D)) = (0.2, 0.8)$, respectively. There is an edge if they share a boundary. Let AB, BC, AC, CD, and BD are edges of graph X. The membership and nonmembership value of the edges are the political relationship in a good and bad attitude such that $(t_{B_1}(AB), f_{B_1}(AB)) = (0.1, 0.5)$, $(t_{B_1}(BC), f_{B_1}(BC)) = (0.2, 0.8)$, $(t_{B_1}(AC), f_{B_1}(AC)) = (0.1, 0.5)$, $(t_{B_1}(CD), f_{B_1}(CD)) = (0.2, 0.8)$, $(t_{B_1}(BD), f_{B_1}(BD)) = (0.2, 0.8)$, respectively. We now want to see how many days we will need to hold a conference between these countries. Let S be a set of countries; $S = \{A, B, C, D\}$ and $P = \{AB\ BC\ AC\ CD\ BD\}$ Suppose that $S(p)$ be countries have boundary for $p \in P$. Now, form FG G with vertices set P, where $a, b \in P$ are neighbor if and only if $S(a) \cap S(b) \neq \emptyset$. For instance, $S(AB) = \{A, B\}$ and $S(BC) = \{C, B\}$. So $S(AB) \cap S(BC) = \{B\} \neq \emptyset$ and hence AB, BC are neighbor. By Theorem 9 there is a homomorphism from G to complete graph with $n = 3$. Then, 3 days are required to hold a conference between these countries, $\{\{AB, CD\}, \{BC\}, \{AC, BD\}\}$. The colored graph of the example 2 is shown in Figure 3.

In the next example, we want to identify the most effective employee of a hospital with the help of a vague influence digraph.

Example 3. Hospitals are very important organizations whose existence is directly related to the general health of the community. Researchers in each country examine factors that contribute to the success of strategic planning to improve the management status of these health organizations. The lives and health of many people are in the hands of health systems. From the safe delivery of a healthy baby to the respectful care of an elderly person, the health department has a vital and ongoing responsibility to individuals throughout their lives. The health industry has undergone many political, social, economic,

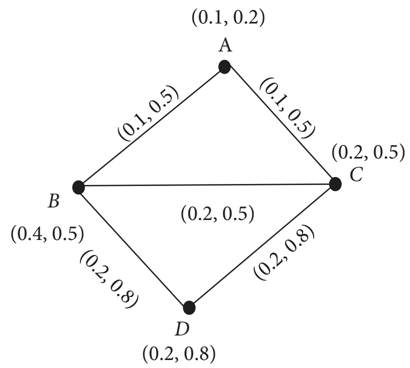

Figure 2: Vague graph $X = (V, A_1, B_1)$.

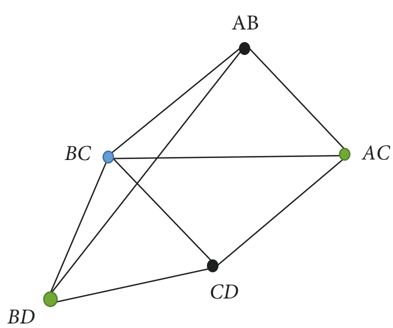

Figure 3: The colored graph of the example 2.

environmental, and technological changes since the early 1980s. These changes have created challenges for managers of healthcare organizations, especially hospitals that cannot be managed with operational plans. Thus, hospital managers have resorted to strategic planning since the 1980s to achieve excellence. Since the management in each ward of the hospital is very important, so in this section, we have tried to determine the most effective person in a hospital based on the performance of its staff. Therefore, we consider the vertices of the VIG as the heads of each ward of the hospital, and the edges of the graph as the degree of interaction and influence of each other. For this hospital, the set of staff is $F = \{\text{Taheri, Ameri, Talebi, Taleshi, Najafi, Kamali, Badri}\}$.

(i) Ameri has been working with Taleshi for 14 years and values his views on issues.

(ii) Taheri has been responsible for audiovisual affairs for a long time, and not only Ameri, but also Taleshi, are very satisfied with Taheri's performance.

(iii) In a hospital, the preservation of medical records is a very important task. Kamali is the most suitable person for this responsibility.

(iv) Talebi and Kamali have a long history of conflict.

(v) Talebi has an important role in the radiology department of the laboratory.

Table 2: Names of employees in a hospital and their services.

Name	Services
Taheri	Head of audiovisual department
Ameri	Environment health expert
Talebi	Head of radiology department
Taleshi	Network expert
Najafi	Medical equipment expert
Kamali	Medical records archive expert
Badri	Head of hospital

Table 3: The level of staff capability.

	Taheri	Ameri	Talebi	Taleshi	Najafi	Kamali	Badri
t_A	0.4	0.5	0.6	0.7	0.9	0.9	0.8
f_A	0.4	0.3	0.3	0.2	0.1	0.1	0.2

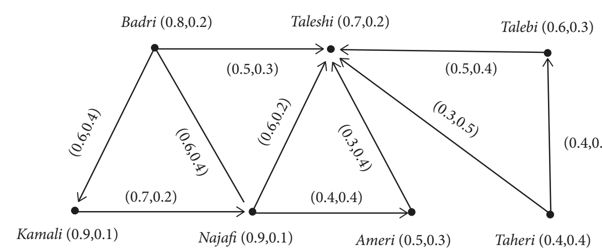

Figure 4: Vague influence digraph.

Given the abovementioned, we consider this a VIG. The vertices represent each of the hospital staff. Note that each staff member has the desired ability as well as shortcomings in the performance of their duties. Therefore, we use of VS to express the weight of the vertices. The true membership indicates the efficiency of the employee and the false membership shows the lack of management and shortcomings of each staff. But the edges describe the level of relationships and friendships between employees such that the true membership shows a friendly relationship between both employees and the false membership shows the degree of conflict between the two officials. Names of employees and levels of staff capability are shown in Tables 2 and 3. The adjacency matrix corresponding to Figure 4 is shown in Table 4.

Figure 4 shows that Najafi has 90% of the power needed to do the hospital work as the medical equipment expert, but does not have the 10% knowledge needed to be the boss. The directional edge of Taleshi–Ameri shows that there is 30% friendship among these two employees, and unfortunately, they have 40% conflict. Clearly, Badri has dominion over both Kamali and Najafi, and his dominance over both is 60%. It is clear that Badri is the most influential employee of the hospital because he controls both the head of the medical equipment and the medical records archive expert, who have 90% of the power in the hospital.

TABLE 4: Adjacency matrix corresponding to Figure 4.

	Taheri	Ameri	Talebi	Taleshi	Najafi	Kamali	Badri
Taheri	$(0, 0)$	$(0, 0)$	$(0.4, 0.5)$	$(0.3, 0.5)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Ameri	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0.3, 0.4)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Talebi	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0.5, 0.4)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Taleshi	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Najafi	$(0, 0)$	$(0.4, 0.4)$	$(0, 0)$	$(0.6, 0.2)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Kamali	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0.7, 0.2)$	$(0, 0)$	$(0, 0)$
Badri	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0.5, 0.3)$	$(0.6, 0.4)$	$(0.6, 0.4)$	$(0, 0)$

5. Conclusion

VGs have a wide range of applications in the field of psychological sciences as well as the identification of individuals based on oncological behaviors. With the help of VGs, the most efficient person in an organization can be identified according to the important factors that can be useful for an institution. Hence, in this paper, we introduced the notion of (η, γ)- homomorphism of VGs and classify HMs, WIs, and CWIs of VGs by (η, γ)- homomorphisms. We also investigated the level graphs of VGs to characterize some VGs. Finally, we presented two applications of VGs in coloring problem and also finding effective person in a hospital. In our future work, we will introduce new concepts of connectivity in VGs and investigate some of their properties. Also, we will study the new results of connected perfect dominating set, regular perfect dominating set, and independent perfect dominating set on VGs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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