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Research Article

Direct Algorithm for Bernstein Enclosure Boundary of Polynomials

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Multivariate polynomials of finite degree can be expanded into Bernstein form over a given simplex domain. The minimum and maximum Bernstein control points optimize the polynomial curve over the same domain. In this paper, we address methods for computing these control points in the simplicial case of maximum degree L. To this end, we provide arithmetic operations and properties for obtaining a fast computational method of Bernstein coefficients. Furthermore, we give an algorithm for direct determination of the minimum and maximum Bernstein coefficients (enclosure boundary) in the simplicial multivariate case. Subsequently, the implicit form, monotonicity, and dominance cases are investigated.

1. Introduction

The enclosure of Bernstein function F estimates the range of polynomials over a given simplex. In order to determine the enclosure boundary, all Bernstein coefficients of degree L are needed in the traditional approach and their number is large for Bernstein functions with moderately many variables. The problem of minimizing and bounding polynomials for global optimization problems was considered in [1]. The use of Bernstein expansion for a given power form polynomial over a simplex (triangles) was considered in [2–6]. Applications of a similar approach on shape designs and geometric representations in computer-aided geometric design were generalized in [7]. Furthermore, a computational method for reaction diffusion model was addressed in [8]. In [9], the authors published results in degree elevation and subdivision of the underlying simplex of Bernstein basis for solving global optimization and system problems. In [10, 11], the tensorial Bernstein case over boxes was addressed for computing the enclosure range of a given (multivariate) rational polynomial function, which is slow. Generally speaking, minimizing and maximizing of Bernstein coefficients provide bounds for the range of its polynomial function F over any given simplex, whereas the complexity of computing these coefficients is high. In this paper, we simplify the computational of Bernstein coefficients in high degree over a multidimensional simplex. Moreover, we provide a fast method for direct determination of the Bernstein enclosure boundary depending on the indices. This method covers the monotonicity of indices, tolerance case, and implicit Bernstein coefficients. Since our results are mathematically proven, we only give simple examples to be followed by readers.

This paper is organized as follows. In the next section, we briefly recall the simplical polynomial Bernstein form. In Section 3, we present the Bernstein expansion and properties over a simplex. The main results about the optimization and fast computation of Bernstein control points are given in Section 4. Conclusions are given in Section 5.

2. Background of Bernstein Expansion

We present the background and fundamental notations of Bernstein basis over a nondegenerate simplex. From the literature, we give the following definition of simplices.

Definition 1. Let $\sigma_0, \ldots, \sigma_n$ be $n+1$ points of $\mathbb{R}^n$. The list $v = [\sigma_0, \ldots, \sigma_n]$ defines a simplex of vertices $\sigma_0, \ldots, \sigma_n$. The convex hull of the vertex points $\sigma_0, \ldots, \sigma_n$ of v is the set of $\mathbb{R}^n$ defined as $|v|$. The largest edge of $|v|$ defines the diameter of v.

Throughout this work, the points $\sigma_0, \ldots, \sigma_n$ are affinely independent in which case the simplex v is nondegenerate. For simplicity, we consider the standard simplex $\Delta = [e_0, e_1, \ldots, e_n]$, where e_0 is the zero vector in $\mathbb{R}^n$ and e_i is the i^{th} vector of the canonical basis of $\mathbb{R}^n$, $i \in \{1, \ldots, n\}$. This is because any simplex v in $\mathbb{R}^n$ can be linearly transformed to Δ. We also recall that $x \in \mathbb{R}^n$ can be formulated as an affine combination of $\sigma_0, \ldots, \sigma_n$ with the barycentric coordinates $\chi_0, \ldots, \chi_n$. If $x = (x_1, \ldots, x_n) \in \Delta$, then $(\chi_1, \ldots, \chi_n) = (x_1, \ldots, x_n)$ and $\chi_0 = 1 - \sum_{i=1}^{n} x_i$. For every multi-index $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ and $\chi = (\chi_0, \ldots, \chi_n) \in \mathbb{R}^{n+1}$, we write $|\alpha| = \alpha_0 + \cdots + \alpha_n$ and $\chi^\alpha := \prod_{i=0}^{n} \chi_i^{\alpha_i}$. Let the entry-wise relation $\leq$ be given, then for $\alpha, \beta \in \mathbb{N}^{n+1}$ with $\beta \leq \alpha$, we define

$$\binom{\alpha}{\beta} := \prod_{i=0}^{n} \binom{\alpha_i}{\beta_i}. \tag{1}$$

If $L \in \mathbb{N}$ is the degree of any polynomial function such that $|\alpha| = L$, then we use the notation $\binom{L}{\alpha} = L!/(\alpha_0! \cdots \alpha_n!)$.

Definition 2. The Bernstein basis of maximum degree L over v is defined as $(S_\alpha^L)_{|\alpha|=L}$, where

$$S_\alpha^L = \binom{L}{\alpha} \chi^\alpha. \tag{2}$$

Note that Bernstein basis takes nonnegative values on v and $\sum_{|\alpha|=L} S_\alpha^L = 1$. Let F be a power form polynomial of degree l,

$$F(x) = \sum_{|\beta| \leq l} c_\beta x^\beta. \tag{3}$$

Since the Bernstein expansion forms a basis of the vector space $\mathbb{R}_n[X]$ of polynomials of maximum degree L, see Proposition 1.6 in [7], then $F(x)$ can be expanded as $(l \leq L)$

$$F(x) = \sum_{|\alpha|=L} C_\alpha(F, L, v) S_\alpha^L(x), \tag{4}$$

where $C_\alpha(F, L, v)$ denote the simplicial coefficients of Bernstein polynomial F of degree L over v.

Remark 1. Let $\Delta = v$, the grid points of degree L associated to Δ are the points

$$\sigma_\alpha(L, \Delta) = \frac{\alpha_0 e_0 + \cdots + \alpha_n e_n}{L} \in \mathbb{R}^n \ (|\alpha| = L), \tag{5}$$

where the associated control points to F are

$$(\sigma_\alpha(L, \Delta), C_\alpha(F, L, \Delta)) \in \mathbb{R}^{n+1} \ (|\alpha| = L). \tag{6}$$

Proposition 1 (see Proposition 2.7 [9]). *For $F \in \mathbb{R}_n[X]$, the following properties hold:*

(1) *Interpolation at the vertices is as follows:*

$$C_{Le_i} = F(\sigma_i), \quad i = 0, \ldots, n; \tag{7}$$

(2) *Convex hull of control points: the graph of F over v is convexly optimized by the convex hull of its associated control points, see Figure 1.*

(3) *Enclosure bound property: the following bound for F holds:*

$$\min_{|\alpha|=L} C_\alpha(F, L, v) \leq F(x) \leq \max_{|\alpha|=L} C_\alpha(F, L, v), \quad x \in v. \tag{8}$$

Theorem 1 (see Theorem 3.3 in [12]). *Let $F(x)$ be a polynomial in Bernstein basis of degree L. Then, its power form is*

$$F(x) = \sum_{|\alpha|=L} c_\alpha x^\alpha, \tag{9}$$

where

$$c_\alpha = \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta} \binom{L}{\alpha} \binom{\alpha}{\beta} C_\beta^{(L)}, \quad |\alpha| = L. \tag{10}$$

3. Simplification of Bernstein

First, we provide the method of affine transformation of v upon the standard simplex Δ, where the barycentric coordinates χ can be transformed to Cartesian coordinates. Subsequently, we simply express power form polynomials into Bernstein form over simplices. Let $v = (\sigma_0, \ldots, \sigma_n)$ and $\sigma_j = (x_1^j, \ldots, x_n^j)$ for all $j \in \{0, \ldots, n\}$. Therefore, $x = (x_1, \ldots, x_n) = \chi_1(\sigma_1 - \sigma_0) + \cdots + \chi_n(\sigma_n - \sigma_0) + \sigma_0$ and $x_1 = (\chi_1(x_1^1 - x_1^0) + \cdots + \chi_n(x_1^n - x_1^0) + x_1^0), \ldots, x_n = (\chi_1(x_n^1 - x_n^0) + \cdots + \chi_n(x_n^n - x_n^0) + x_n^0)$. Then, we have

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1^1 - x_1^0 & \ldots & x_1^n - x_1^0 \\ \vdots & & \vdots \\ x_n^1 - x_n^0 & \ldots & x_n^n - x_n^0 \end{pmatrix} \cdot \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} + \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}, \tag{11}$$

from which we have

$$\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} = \begin{pmatrix} x_1^1 - x_1^0 & \ldots & x_1^n - x_1^0 \\ \vdots & & \vdots \\ x_n^1 - x_n^0 & \ldots & x_n^n - x_n^0 \end{pmatrix}^{-1} \cdot \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix} \right). \tag{12}$$

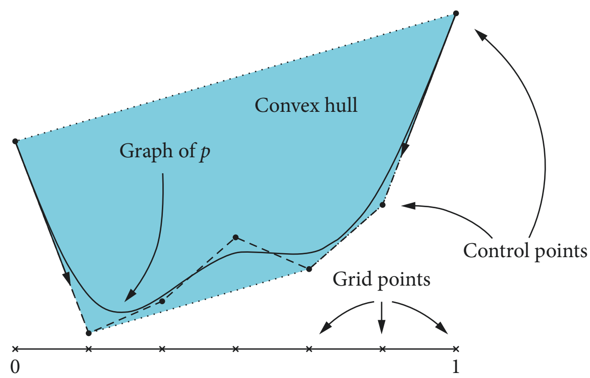

FIGURE 1: The curve of a univariate polynomial p, the colored convex hull of control points optimizes the polynomial curve, and the minimum Bernstein value approximates the minimum range.

In the following, we simplify the method of expanding power form polynomials in Bernstein form over a simplex.

Notation 1. Consider $\alpha = (\widehat{\alpha}, \alpha_0) \in \mathbb{N}^{n+1}$ with $\chi \in \mathbb{R}^{n+1}$, and from (3), we get $0 \le \beta_1 \le l =: l_1, 0 \le \beta_2 \le l - \beta_1 =: l_2, \ldots, 0 \le \beta_n \le (l - (\beta_1 + \cdots + \beta_{n-1})) =: l_n$. Any polynomial power form $F(x)$ in (3) of degree $\widehat{l} = (l_1, \ldots, l_n)$ can be expressed on Δ as

$$F(x) = \sum_{\widehat{\beta} \le \widehat{l}} c_{\widehat{\beta}} x^{\widehat{\beta}}. \tag{13}$$

From Notation 1, we can write $\sum_{|\widehat{\beta}| \le l} = \sum_{\beta_1 \le l_1} \cdots \sum_{\beta_n \le l_n}$. Additionally $(|\widehat{\alpha}| + \alpha_0 = L)$,

$$\begin{aligned} \binom{L}{\widehat{\alpha}, \alpha_0} &= \binom{L}{\alpha_1} \cdot \binom{L - \alpha_1}{\alpha_2} \cdots \binom{L - (\alpha_1 + \cdots + \alpha_{n-1})}{\alpha_n} \cdot \binom{L - (\alpha_1 + \cdots + \alpha_n)}{\alpha_0} \\ &=: \binom{L_1}{\alpha_1} \cdots \binom{L_n}{\alpha_n} \cdot \binom{L_0}{\alpha_0} =: \binom{\widehat{L}}{\widehat{\alpha}, \alpha_0}. \end{aligned} \tag{14}$$

Proposition 2. *For $(\widehat{\alpha}, \alpha_0) \in \mathbb{N}^{n+1}$ and $\widehat{L} \in \mathbb{N}^{n+1}$, the simplicial Bernstein expansion of maximum degree L can be given as*

$$F(x) = \sum_{(\widehat{\alpha}, \alpha_0) \le \widehat{L}} C^{(\widehat{L})}_{(\widehat{\alpha}, \alpha_0)}(F) S^{(\widehat{L})}_{(\widehat{\alpha}, \alpha_0)}(x), \tag{15}$$

where

$$S^{(\widehat{L})}_{(\widehat{\alpha}, \alpha_0)}(x) = \binom{\widehat{L}}{\widehat{\alpha}, \alpha_0} x^{\widehat{\alpha}} (1 - |x|)^{\alpha_0}, \tag{16}$$

$$C^{(\widehat{L})}_{(\widehat{\alpha}, \alpha_0)}(F) = \sum_{\widehat{\beta} \le \widehat{\alpha}} \frac{\binom{\widehat{\alpha}}{\widehat{\beta}}}{\binom{\widehat{L}}{\widehat{\beta}, \beta_0}} c_{\widehat{\beta}}, \quad 0 \le (\widehat{\alpha}, \alpha_0) \le \widehat{L}. \tag{17}$$

Proof. Let F be a polynomial power form of maximum degree l. For $\beta_0 = L - |\widehat{\beta}|$, we have $(L \ge l)$

$$\begin{aligned} F(x) &= \sum_{|\widehat{\beta}| \le l} c_{\widehat{\beta}} x^{\widehat{\beta}} \\ &= \sum_{\beta_1 \le l_1} \cdots \sum_{\beta_n \le l_n} c_{\widehat{\beta}} x_1^{\beta_1} \cdots x_n^{\beta_n} (|x| + 1 - |x|)^{L - |\widehat{\beta}|} \\ &= \sum_{\beta_1 \le l_1} \cdots \sum_{\beta_n \le l_n} c_{\widehat{\beta}} x_1^{\beta_1} \cdots x_n^{\beta_n} \sum_{|\widehat{\gamma}| \le L - |\widehat{\beta}|} \binom{L - |\widehat{\beta}|}{\widehat{\gamma}} \\ &\quad x^{\widehat{\gamma}} (1 - |x|)^{L - |\widehat{\beta}| - |\widehat{\gamma}|} \\ &= \sum_{\widehat{\beta} \le \widehat{l}} \sum_{|\widehat{\gamma}| \le L - |\widehat{\beta}|} a_{\widehat{\beta}} \binom{L - |\widehat{\beta}|}{\widehat{\gamma}} x^{\widehat{\gamma} + \widehat{\beta}} (1 - |x|)^{L - |\widehat{\beta}| - |\widehat{\gamma}|} \\ &= \sum_{\widehat{\beta} \le \widehat{l}} \sum_{|\widehat{\alpha}| \le L} c_{\widehat{\beta}} \binom{L - |\widehat{\beta}|}{\widehat{\alpha} - \widehat{\beta}} x^{\widehat{\alpha}} (1 - |x|)^{L - |\widehat{\alpha}|} \end{aligned}$$

$$= \sum_{\widehat{\beta} \le \widehat{l}} \sum_{|\alpha|=L} c_{\widehat{\beta}} \frac{\binom{\widehat{\alpha}}{\widehat{\beta}} \binom{L}{\widehat{\alpha}, \alpha_0}}{\binom{L}{\widehat{\beta}}} x^{\widehat{\alpha}} (1 - |x|)^{\alpha_0}$$

$$= \sum_{(\widehat{\alpha}, \alpha_0) \le \widehat{L}} \sum_{\widehat{\beta} \le \widehat{\alpha}} c_{\widehat{\beta}} \frac{\binom{\widehat{\alpha}}{\widehat{\beta}}}{\binom{\widehat{L}}{\widehat{\beta}, \beta_0}} \binom{\widehat{L}}{\widehat{\alpha}, \alpha_0} x^{\widehat{\alpha}} (1 - |x|)^{\alpha_0}. \tag{18}$$

□

4. Main Results

The number of Bernstein coefficients of n-variable polynomial $F(x)$ of maximum degree L is equal to $D := \binom{L+n}{L}$. We aim to store and represent the minimum and maximum Bernstein coefficients in a fast determination method.

In the following proposition, we show the linear combination property of Bernstein coefficients.

Proposition 3. *Consider F be in the Bernstein basis over a given simplex. The coefficients of Bernstein of maximum degree L obtain a linear combination of coefficients of lower degree l.*

Proof. Let the Bernstein form F of maximum degree l be given on Δ,

$$F(x) = \sum_{|\widehat{\alpha}|+\alpha_0=l} C_{(\widehat{\alpha}, \alpha_0)}(F, l, \Delta) S^{(l)}_{(\widehat{\alpha}, \alpha_0)}. \tag{19}$$

We deduce from the Bernstein basis that

$$S^{(l)}_{(\widehat{\alpha}, \alpha_0)} = \binom{l}{\widehat{\alpha}, \alpha_0} x^{\widehat{\alpha}} (1 - |x|)^{l - |\widehat{\alpha}|}$$
$$= \binom{l}{\widehat{\alpha}, \alpha_0} x^{\widehat{\alpha}} (1 - |x|)^{\alpha_0} (|x| + 1 - |x|)^{L-l}, \quad l \le L$$
$$= \binom{l}{\widehat{\alpha}, \alpha_0} x^{\widehat{\alpha}} (1 - |x|)^{\alpha_0} \sum_{|\widehat{\gamma}|+\gamma_0=L-l} \binom{L-l}{\widehat{\gamma}, \gamma_0}$$
$$x^{\widehat{\gamma}} (1 - |x|)^{\gamma_0}$$
$$= \sum_{|\widehat{\gamma}|+\gamma_0=L-l} \binom{l}{\widehat{\alpha}, \alpha_0} \binom{L-l}{\widehat{\gamma}, \gamma_0} x^{\widehat{\alpha}+\widehat{\gamma}} (1 - |x|)^{\alpha_0+\gamma_0}$$
$$(\widehat{\alpha} + \widehat{\gamma} =: \widehat{\kappa}, \alpha_0 + \gamma_0 =: \kappa_0)$$
$$= \sum_{|\widehat{\kappa}|+\tau_0=L} \binom{l}{\widehat{\alpha}, \alpha_0} \binom{L-l}{\widehat{\kappa} - \widehat{\alpha}, \kappa_0 - \alpha_0} x^{\widehat{\kappa}} (1 - |x|)^{\kappa_0}$$
$$= \sum_{|\widehat{\kappa}|+\kappa_0=L} \frac{\binom{l}{\widehat{\alpha}, \alpha_0} \binom{L-l}{\widehat{\kappa} - \widehat{\alpha}, \kappa_0 - \alpha_0} \binom{L}{\widehat{\kappa}, \kappa_0}}{\binom{L}{\widehat{\kappa}, \kappa_0}} x^{\widehat{\kappa}} (1 - |x|)^{\kappa_0}$$
$$= \sum_{|\widehat{\kappa}|+\kappa_0=L} \frac{\binom{l}{\widehat{\alpha}, \alpha_0} \binom{L-l}{\widehat{\kappa} - \widehat{\alpha}, \kappa_0 - \alpha_0}}{\binom{L}{\widehat{\kappa}, \kappa_0}} S^{L}_{(\widehat{\kappa}, \kappa_0)}, \tag{20}$$

from which we have

$$C_{(\widehat{\kappa}, \kappa_0)}(F, L, \Delta) = \sum_{|\widehat{\alpha}|+\alpha_0=l} \frac{\binom{l}{\widehat{\alpha}, \alpha_0} \binom{L-l}{\widehat{\kappa} - \widehat{\alpha}, \kappa_0 - \alpha_0}}{\binom{L}{\widehat{\kappa}, \kappa_0}} \tag{21}$$

$$C_{(\widehat{\alpha}, \alpha_0)}(F, l, \Delta).$$

□

Remark 2. Consider $F(x)$ be a given polynomial of degree L over a simplex. The Bernstein coefficients of $(\partial F / \partial x_\mu)(x)$ can be calculated by taking linear combinations of Bernstein coefficients of F of degree $L-1$, i.e., for all $|\alpha| = L - 1$,

$$C_\alpha(F_i', L-1, \Delta) = L\left(C_{\alpha+e_i} - C_\alpha\right). \tag{22}$$

Example 1. Let $F(x_1, x_2) := 2x_1 x_2^2 - 0.6x_1^2 + x_1 + 0.5$ be given over the standard simplex Δ. The computed Bernstein coefficients of $(\partial F / \partial x_1)(x)$ are as follows:

$$C_{\widehat{\alpha}}(F, l, \Delta) = \begin{pmatrix} 1 & 1 & 3 \\ 0.4 & 0.4 & \\ -0.2 & & \end{pmatrix}. \tag{23}$$

For determining the enclosure boundary in only one dimensional polynomial of degree 2, we need to compute 6 coefficients.

Remark 3. The minimum value of the enclosure bound optimizes the minimum range of its original polynomial (see Figure 2). For positivity analysis of polynomial systems, if the minimum enclosure is positive, then we certify the global positivity of its polynomial over the given domain.

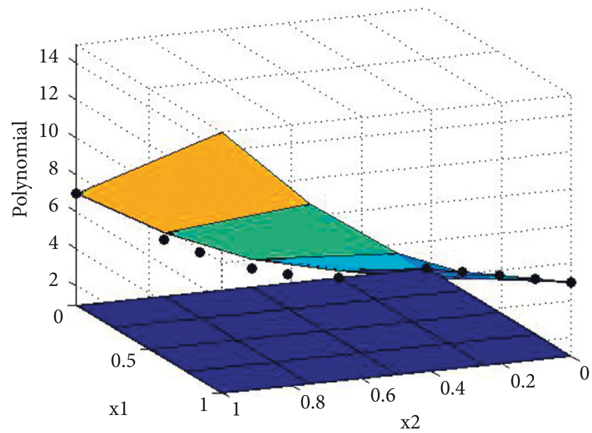

FIGURE 2: The surface of a bi-variate polynomial over the standard simplex, the coefficients of Bernstein marked by black points, and the minimum enclosure bound (blue surface) bounds its polynomial surface over the whole domain.

4.1. The Implicit Bernstein Form. In this subsection, we provide a method of computing the implicit Bernstein coefficients over Δ. Let

$$F(x) = F(x)^{(1)} + \cdots + F(x)^{(t)}, \tag{24}$$

be a given polynomial power form comprising the monomials $F(x)^{(r)}$, $r = 1, \ldots, t$. Let $\widehat{\beta}^{(r)} \in \mathbb{N}^n$, $r \in \{1, \ldots, t\}$, then F can be written as

$$F(x) = c_{\widehat{\beta}^{(1)}} x^{\widehat{\beta}^{(1)}} + \cdots + c_{\widehat{\beta}^{(t)}} c^{\widehat{\beta}^{(t)}}, \quad \text{with} \left|\widehat{\beta}^{(r)}\right| = l, \tag{25}$$

for some $r \in \{1, \ldots, t\}$.

For avoiding the constant terms, we assume that $|\widehat{\beta}^{(r)}| > 0$ for all $r = 1, \ldots, t$.

If F consists t terms of monomials, then each Bernstein coefficient of degree L can be added to the corresponding Bernstein coefficient of the next term:

$$C_{(\widehat{\alpha},\alpha_0)}(F, L, \Delta) = \sum_{r=1}^{t} C^{(r)}_{\left(\widehat{\alpha}^{(r)},\alpha_0^{(r)}\right)}(F, L, \Delta), \quad \left|\widehat{\alpha}^{(r)}\right| + \alpha_0^{(r)} = L, \tag{26}$$

where $C^{(r)}_{(\widehat{\alpha}^{(r)},\alpha_0^{(r)})}(F, L, \Delta)$ are the coefficients of the r th term of Bernstein F.

Remark 4. Consider $F(x)$ and $P(x)$ be in Bernstein form of the same degree L. Then, we have

$$F(x) + P(x) = \sum_{|\alpha|=L} \left(C_\alpha(F, L, \Delta) + C_\alpha(P, L, \Delta)\right) S_\alpha(x). \tag{27}$$

Example 2. Let $F(x_1, x_2) = 2x_1x_2 - 0.1x_1^3 + 5$ of degree $l = 3$ be given over the standard simplex $\Delta = [e_0, e_1, e_2]$. Adding Bernstein coefficients to the corresponding coefficients of each term gives the total number of Bernstein coefficients for $F(x)$:

$$\begin{aligned} C_{\widehat{\alpha}}(F, l, \Delta) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & \\ 0 & \frac{2}{3} & & \\ 0 & & & \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & & \\ -0.1 & & & \end{pmatrix} + \begin{pmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & \\ 5 & 5 & & \\ 5 & & & \end{pmatrix} \\ &= \begin{pmatrix} 5 & 5 & 5 & 5 \\ 5 & \frac{16}{3} & \frac{17}{3} & \\ 5 & \frac{17}{3} & & \\ 4.5 & & & \end{pmatrix}. \end{aligned} \tag{28}$$

The minimum and maximum values of $C_{\widehat{\alpha}}(F, l, \Delta)$ approximate F, and the number of computed coefficients is $D = 10$.

Remark 5. Let $F(x)$ be of degree L_F and $p(x)$ be of degree L_p with $L_p \leq L_F$. Then, we have

$$F + p = \sum_{|\alpha| = L_F} \left(C_\alpha (F, L_F, \Delta) + \sum_{|\beta| = \min\{L_p, |\alpha|\}} \frac{\binom{L_p}{\beta} \cdot \binom{L_F - L_p}{\alpha - \beta}}{\binom{L_F}{\alpha}} C_\beta (p, L_p, \Delta) \right) S_\alpha^{(L_F)}. \tag{29}$$

4.2. Monotonicity of Monomial Coefficients. In this subsection, we show the monotonicity of Bernstein coefficients for high dimensional monomials over a simplex. We assume $t = 1$ and $\widehat{F}(x)$ is an $n-$ variable monomial.

Lemma 1. *Let $\widehat{F}(x) = c_{\widehat{\beta}} x^{\widehat{\beta}}$, $x \in \mathbb{R}^n$, be a monomial and $(e_0, \ldots, e_n)$ denote the canonical basis of $\mathbb{R}^{n+1}$. Then, the Bernstein coefficients $C_{(\widehat{\alpha}, \alpha_0)}(\widehat{F}, L, \Delta)$ are monotone with respect to $\widehat{\alpha}$, i.e.,*

$$\begin{aligned} C_{\widehat{\alpha}}(\widehat{F}, L, \Delta) &\leq C_{\widehat{\alpha}+e_i}(\widehat{F}, L, \Delta), \quad \text{if } c_{\widehat{\beta}} > 0, \\ C_{\widehat{\alpha}}(\widehat{F}, L, \Delta) &\geq C_{\widehat{\alpha}+e_i}(\widehat{F}, L, \Delta), \quad \text{if } c_{\widehat{\beta}} < 0. \end{aligned} \tag{30}$$

Proof. Let $\widehat{F}(x) = c_{\widehat{\beta}} x^{\widehat{\beta}}$, $0 < |\widehat{\beta}| \leq l$, be a multivariate power form monomial of degree l on Δ. We can express $\widehat{F}(x)$ in the monomial Bernstein form of degree $L \geq l$. Assume without loss the generality that $c_{\widehat{\beta}} > 0$. Then, we conclude that

$$C_{\widehat{\alpha}}(\widehat{F}, L, \Delta) = \frac{\binom{\widehat{\alpha}}{\widehat{\beta}}}{\binom{L}{\widehat{\beta}}} c_{\widehat{\beta}} \leq \frac{\binom{\widehat{\alpha} + e_i}{\widehat{\beta}}}{\binom{L}{\widehat{\beta}}} c_{\widehat{\beta}} = C_{\widehat{\alpha}+e_i}(\widehat{F}, L, \Delta), \tag{31}$$

for. $i \in \{0, \ldots, n\}$. □

4.3. Polynomials under Dominance. In this subsection, we consider polynomials (of two terms) $F(x) = F(x)^{(+)} + F(x)^{(-)}$ of orders $\widehat{\beta}^{(1)}, \widehat{\beta}^{(2)}$, respectively, and $c_{\widehat{\beta}^{(1)}} > 0, c_{\widehat{\beta}^{(2)}} < 0$. Assume for simplicity that $l = L$. We provide a direct determination of $\max C_\alpha(F)$ and $\min C_\alpha(F)$ that occur at some $\widehat{a}^+, \widehat{a}^* \in \mathbb{N}^n$, respectively.

Proposition 4. *If $\forall |\widehat{\alpha}| < l$ and $|\widehat{\gamma}| = l, \widehat{\gamma} \in \mathbb{N}^n$,*

$$\begin{aligned} &C_{\widehat{\alpha}}\left(F^{(-)}, l, \Delta\right) + b_{e_0}\left(F^{(+)}, l, \Delta\right) \\ &\quad > C_{\widehat{\alpha}+e_i}\left(F^{(-)}, l, \Delta\right) + C_{\widehat{\gamma}}\left(F^{(+)}, l, \Delta\right), \end{aligned} \tag{32}$$

then $\widehat{\alpha}^+ = e_0$ and $\widehat{\alpha}^* = \widehat{\alpha}$ for some $\widehat{\alpha}$ obtains $|\widehat{\alpha}| = l$ with $\alpha_0 = 0$.

If $\forall |\widehat{\alpha}| < l$ and $|\widehat{\gamma}| = l$

$$\begin{aligned} &C_{\widehat{\gamma}}\left(F^{(-)}, l, \Delta\right) + C_{\widehat{\alpha}+e_i}\left(F^{(+)}, l, \Delta\right) \\ &\quad > C_{e_0}\left(F^{(-)}, l, \Delta\right) + C_{\widehat{\alpha}}\left(F^{(+)}, l, \Delta\right), \end{aligned} \tag{33}$$

then $\widehat{\alpha}^* = e_0$ and $\widehat{\alpha}^+ = \widehat{\alpha}$ for some $\widehat{\alpha}$ obtains $|\widehat{\alpha}| = l$ with $\alpha_0 = 0$.

Proof. We give the proof of (32) and the proof of (33) is analogous. For all $|\widehat{\alpha}| < l$ and $|\widehat{\gamma}| = l$, we have

$$\begin{aligned} C_{\widehat{\alpha}+e_i}(F, l, \Delta) &= C_{\widehat{\alpha}+e_i}\left(F^{(-)}, l, \Delta\right) + C_{\widehat{\alpha}+e_i}\left(F^{(+)}, l, \Delta\right) \\ &\leq C_{\widehat{\alpha}+e_i}\left(F^{(-)}, l, \Delta\right) + C_{\widehat{\gamma}}\left(F^{(+)}, l, \Delta\right) \\ &< C_{\widehat{\alpha}}\left(F^{(-)}, l, \Delta\right) + C_{e_0}\left(F^{(+)}, l, \Delta\right) \\ &\leq C_{\widehat{\alpha}}\left(F^{(-)}, l, \Delta\right) + C_{\widehat{\alpha}}\left(F^{(+)}, l, \Delta\right) = C_{\widehat{\alpha}}(F, l, \Delta), \end{aligned} \tag{34}$$

where $C_{\widehat{\alpha}}(F, l, \Delta)$ are decreasing with respect to $\widehat{\alpha}$ and the proof follows. □

4.4. Monotonicity of Polynomial Coefficients. The determination of enclosure boundary for multivariate polynomials in Bernstein form required D number of coefficients. The minimum Bernstein coefficient of multivariate polynomials approximates the minimum range over the same domain (see Figure 2). We dramatically reduce the search space of coefficients by obtaining the monotonicity of Bernstein.

Remark 6. Consider $F(x)$ and $P(x)$ be polynomials in Bernstein form of degree L_F and L_P, respectively. From [13], we have

$$F(x)\cdot P(x)=\sum_{|\gamma|=L_F+L_P}\left(\sum_{|\beta|=\min\{L_F,|\gamma|\}}\frac{\binom{L_F}{\beta}\binom{L_P}{\gamma-\beta}}{\binom{L_F+L_P}{\gamma}}.\right.$$
$$\left.C_\beta(F,L_F)C_{\gamma-\beta}(P,L_P)\right)S_\gamma^{(L_F+L_P)}(x)$$
$$=\sum_{|\gamma|=L_F+L_P}C_\gamma(F\cdot P)S_\gamma^{(L_F+L_P)}(x). \tag{35}$$

Example 3. Let $F(x_1,x_2)=2x_1x_2-0.6x_1^2+x_2^2+0.5$ and $P(x_1,x_2)=0.1x_1^2-0.2x_1x_2+x_1+0.3$ be given over Δ. The computed Bernstein coefficients of $F\cdot P$ of degree $L_F+L_P=4$ are as follows:

$$C_\gamma(F\cdot P,4,\Delta)=\begin{pmatrix}0.15 & 0.15 & 0.2 & 0.3 & 0.45\\ 0.275 & 0.3167 & 0.4917 & 0.75 & \\ 0.3783 & 0.6283 & 1.045 & & \\ 0.31 & 1.015 & & & \\ -0.14 & & & & \end{pmatrix}. \tag{36}$$

The number of computed coefficients of degree 4 is $D=\binom{4+2}{2}=15$.

In the following theorem, we generalize the method of fast determination of Bernstein enclosure boundary to the case of monotonicity.

Theorem 2. *The multi-indices $\widehat{\alpha}^*$ and $\widehat{\alpha}^+$ of the minimum and maximum coefficients of a Bernstein polynomial $F(x)$ over a simplex with power form coefficients $c_{\widehat{\beta}^{(r)}}>0$ ($c_{\widehat{\beta}^{(r)}}<0$ analogous) for all $r=1,\ldots,t$ satisfy*

$$\widehat{\alpha}^*=\min_{r=1,\ldots,t}\widehat{\alpha}^{*^{(r)}}, \tag{37}$$

$$\widehat{\alpha}^+=\max_{r=1,\ldots,t}\widehat{\alpha}^{+^{(r)}}. \tag{38}$$

Proof. We give the proof of (37), and the proof of (38) is entirely analogous. Assume that $c_{\widehat{\beta}^{(r)}}>0$, $r=1,\ldots,t$. We proceed by contradiction and assume there is some $\widehat{\alpha}^*$, $0\le|\widehat{\alpha}^*|\le l$, from which

$$\widehat{\alpha}^*<\widehat{\alpha}^{*^{(r)}},\quad r\in\{1,\ldots,t\}. \tag{39}$$

It follows that

$$\widehat{\alpha}^*=e_0. \tag{40}$$

Therefore, we deduce that $|\widehat{\alpha}^{*^{(r)}}|>0$, for all $r\in\{1,\ldots,t\}$. Then, the coefficients of Bernstein $C^{(r)}_{\widehat{\alpha}^{(r)}}(p,l,\Delta)$ are decreasing with respect to $\widehat{\alpha}^{(r)}$ and $|\widehat{\alpha}^{*^{(r)}}|=l$ for all $r\in\{1,\ldots,t\}$. Additionally, the coefficients $C_{\widehat{\alpha}}(F,l,\Delta)=\sum_{r=1}^t C^{(r)}_{\widehat{\alpha}^{(r)}}(F,l,\Delta)$ are decreasing with respect to $\widehat{\alpha}$ and so $|\widehat{\alpha}^*|>0$, which is a contradiction of (40). □

Example 4. Let $F=3x^7+x^5+6x^8$ be a polynomial power form given over $I=[0,1]$. By application of Theorem 2 to F, we directly find that $\widehat{\alpha}^+=8$ and $\widehat{\alpha}^*=0$. Subsequently, the respective maximum and minimum Bernstein values appear at the corresponding values of $\widehat{\alpha}^+$ and $\widehat{\alpha}^*$, which can be computed using (17).

Corollary 1. *Let $F(x)$ be a (multivariate) power form polynomial of t terms with coefficients $c_{\widehat{\beta}^{(r)}}>0$, $\forall r=1,\ldots,t$. Then, for all $r\in\{1,\ldots,t\}$, the multi-indices of minimum and maximum Bernstein coefficients are appeared as $\widehat{\alpha}^{*^{(r)}}=e_0$ and $\widehat{\alpha}^{+^{(r)}}=\widehat{\alpha}^{(r)}$, for some $\widehat{\alpha}^{(r)}\ge\widehat{\beta}^{(r)}$ with $|\widehat{\alpha}^{(r)}|=l$. If the coefficients $a_{\widehat{\beta}^{(r)}}<0$, $\forall r=1,\ldots,t$, then we have $\widehat{\alpha}^{+^{(r)}}=e_0$, and $\widehat{\alpha}^{*^{(r)}}=\widehat{\alpha}^{(r)}$, for some $\widehat{\alpha}^{(r)}\ge\widehat{\beta}^{(r)}$ with $|\widehat{\alpha}^{(r)}|=l$.*

Corollary 2. *By Lemma 1 and Theorem 2, the number of coefficients that are needed to determine the enclosure boundary of $n-$ dimensional polynomials over a simplex does not exceed $\binom{1+n}{1}$.*

5. Conclusions

In this work, we considered computing the minimum and maximum Bernstein coefficients (enclosure boundary) that optimize the range of polynomials over a simplex. We reduced the high complexity of computing all n-dimensional Bernstein coefficients of degree L to $\binom{1+n}{1}$. Therefore, we covered the cases of monotonicity, dominance, and multivariate monotonicity of Bernstein coefficients. Finally, the index of enclosure boundary was directly determined and only the enclosure value computed. In the future work, we consider generalizing this method to new classes of functions such as rational polynomial functions over triangles.

Data Availability

The authors confirm that the data supporting the findings of this study are included within the thesis https://kops.uni-konstanz.de/handle/123456789/20898.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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