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Research Article

Spectrum of Superhypergraphs via Flows

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For any $n \in \mathbb{N}$ and given nonempty subset V, the concept of n-superhypergraphs is introduced by Florentin Smarandache based on $P^n(V)$ (n-th power set of V). In this paper, we present the novel concepts *supervertices*, *superedges*, and *superhypergraph* via the concept of *flow*. This study computes the number of superedges of any given superhypergraphs, and based on the numbers of superedges and partitions of an underlying set of superhypergraph, we obtain the number of all superhypergraphs on any nonempty set. As a main result of the research, this paper is introducing the *incidence matrix* of superhypergraph and computing the *characteristic polynomial* for the incidence matrix of superhypergraph, so we obtain the *spectrum* of superhypergraphs. The flow of superedges plays the main role in computing of spectrum of superhypergraphs, so we compute the spectrum of superhypergraphs in some types such as *regular flow*, *regular reversed flow*, and *regular two-sided flow*. The new conception of *superhypergraph* and computation of the spectrum of superhypergraphs are introduced firstly in this paper.

1. Introduction and Preliminaries

The theory of graph is a main and important theory for modeling the real problem in the world, and this theory extends in past years in this regard. The disadvantage of a graph is that it cannot connect more than two elements, so this problem causes weakness in this theory. Berge generalized the theory of graphs to the mathematical concept of theory of hypergraphs with the motivation that hypergraphs solve the conflicts, defects, and shortcomings of graph theory around 1960 [1]. Hypergraphs have some applications in other sciences and the real-world, one of the applications of hypergraphs is a simulation for complex hypernetworks. Today, hypergraphs have a vital role and important performances, so are used in complex hypernetworks such as computer science, wireless sensor hypernetwork, and social hypernetworks. In this regards there has been a lot of research about using hypergraphs to problems in real-world such as hypergraph matching via game-theoretic hypergraph clustering [2], hypergraph matching via game-theoretic hypergraph clustering [3], hypergraph-based centrality metrics for maritime container service networks, a worldwide application [4], clustering ensemble via structured hypergraph learning [5], and hypergraph neural network for skeleton-based action recognition [6]. There is some main connection between graphs and hypergraphs via the mathematical computational tools and basic theorems in which these connections facilitate the modeling of other sciences with mathematics. Further materials regarding graphs and hypergraphs are available such as extending factorizations of complete uniform hypergraphs [7], finding perfect matchings in bipartite hypergraphs [8], graphs and hypergraphs [1], resilient hypergraphs with fixed matching number [9], on the spectrum of hypergraphs [10], on the distance spectrum of minimal cages and associated distance biregular graphs [11], on the spectrum of the perfect matching derangement graph [12], and probabilistic refinement of the asymptotic spectrum of graphs [13]. Recently, Hamidi and Saeid computed eigenvalues of discrete complete hypergraphs and partitioned hypergraphs. They defined positive equivalence relation on hypergraphs that establishes a connection between hypergraphs and graphs, and it makes a connection between a spectrum of graphs and a spectrum of the quotient of any hypergraphs. They studied the construct spectrum of path trees via the quotient of partitioned hypergraphs [14]. A hypergraph on any given set considers a relationship between elements and the set (as objects or hyper vertices) and describes this relationship if it is a weighted hypergraph. It is an ideal condition if proper weights are known, but in most situations, the weights may not be known, and the relationships are hesitant in a natural sense. With the advent of the fuzzy graph, the importance of this theory increased and fuzzy graph as a generalization of a graph provides more information in real-life problems. Based on Zadeh's fuzzy relations [15], the notion of hypergraph has been extended in the fuzzy theory and the concept of fuzzy hypergraph was provided by Kaufmann [16]. Recently, some researchers investigated the concept of fuzzy hypergraphs and applications such as fuzzy hypergraphs and related extensions [17], an algorithm to compute the strength of competing interactions in the Bering sea based on Pythagorean fuzzy hypergraphs [18] and bipolar fuzzy soft information applied to hypergraphs [19]. Recently, Smarandache introduced a new the concept as a generalization of hypergraphs to n-superhypergraph, plithogenic n-superhypergraph {with supervertices (that are groups of vertices) and hyperedges {defined on power set of power set...} that is the most general form of a graph as today}, which have several properties and are connected with the real-world [20]. Indeed, n-superhypergraphs are a generalization of hypergraphs, with the advantage that they can communicate between the hyperedges.

Regarding these points, we consider a nonempty set and make a partition of the given set into some subsets, and relate this subset together with some maps. Indeed, subsets will call supervertices, the mapping between them will call superedges or flows and the system with supervertices and superedges will call a quasi superhypergraph. The main motivation of this the concept is a generalization of graphs to hypernetworks such that all elements be related together. In hypergraph theory, any hypergraph can relate a set of elements, while without any details that it makes some conflicts, defects, and shortcomings in the hypergraph theory. Thus, by introducing superhypergraph, we try to eliminate defects of graph (sometimes graph structures give very limited information about complex networks) structures and hypergraph structures (although the hypergraph structures are for covering graph defects in the applications but in hypergraphs, the relation between of vertices cannot be described in full details). As a main result of the research, this paper is introducing the incidence matrix of superhypergraph and computing the characteristic polynomial for the incidence matrix of superhypergraph, so we obtain the spectrum of superhypergraphs. Indeed, we computed the number of superedges of any given superhypergraphs and based on superedges and partitions of an underlying set of superhypergraph, we obtained the number of all superhypergraphs on any nonempty set. The flow of superedges plays the main role in computing of spectrum of superhypergraphs, so we computed the spectrum of superhypergraphs in some types regular flow, regular reversed flow, and regular two-sided flow.

Definition 1. [1] Let X be a finite set. A hypergraph on X is a pair $H = (X, \{E_i\}_{i=1}^m)$ such that for all $1 \leq i \leq m, \varnothing \neq E_i \subseteq X$ and $\cup_{i=1}^m E_i = X$. The elements $x_1, x_2, \ldots, x_n$ of X are called vertices, and the sets $E_1, E_2, \ldots, E_m$ are called the hyperedges of the hypergraph H. In hypergraphs, hyperedges can contain an element (loop) two elements (edge) or more than three elements. A hypergraph $H = (X, \{E_i\}_{i=1}^m)$ is called a complete hypergraph, if for any $x, y \in X$ there is $1 \leq i \leq m$ such that $\{x, y\} \subseteq E_i$. A hypergraph $H = (X, \{E_i\}_{i=1}^n)$ is called as a joint complete hypergraph, if $|X| = n$ for all $1 \leq i \leq n, |E_i| = i$ and $E_i \subseteq E_{i+1}$ element (loop). If for all $1 \leq k \leq m$, $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph and n rows representing the vertices $x_1, x_2, \ldots, x_n$, where for all $1 \leq i \leq n$ and for all $1 \leq j \leq m$, we have $m_{ij} = 1$ if $x_i \in E_j$ and $m_{ij} = 0$ if $x_i \notin E_j$.

Definition 2. [20] Let $m \in \mathbb{N}$ and $V = \{v_1, v_2, \ldots, v_m\}$ be a set of vertices, that contains single vertices (the classic alones), indeterminate vertices (unclear, vague, unknown) and null vertices (unknown, empty). Consider $P(V)$ as the power set of V, $P^2(V) = P(P(V))\ldots$, and $P^n(V) = P(P^{n-1}(V))$ be the n-power set of the set V. Then, the n-superhypergraph ($n-$SHG) is an ordered pair n-SHG $= (G_n, E_n)$, where for any $n \in \mathbb{N}, G_n \subseteq P^n(V)$ is the set of vertices and $E_n \subseteq P^n(V)$ is the set of edges. The set G_n contains some type of vertices, such as single vertices (the classical ones), indeterminate vertices (unclear, vague, partially unknown), null vertices (totally unknown, empty), and supervertices (or subset vertex), i.e., two or more (single, indeterminate, or null) vertices together as a group (organization). An n-supervertex is a collection of many vertices such that at least one is a $(n-1)$-supervertex and all other supervertices in to the collection if any have the order $r \leq n-1$. The set of edges E_n contains some type of edges such as single edges (the classic alones), indeterminate (unclear, vague, partially unknown), null-edge (empty, totally unknown), hyperedge (containig three or more single vertices), superedge (containing two vertices atleast one of them being a super vertex), n-superedge (containing two vertices, atleast one being an $n-$ super vertex and the other of order $r-$ super vertex with $r \leq n$), superhyperedge (containing three or more vertices, at least one being a supervertex, n-superhyperedge (containing three or more vertices, atleast one being an $n-$ super vertex and the other $r-$ super vertices with $r \leq n$), multiedges (two or more edges connecting the same two vertices), and loop (an edge that connects an element).

2. On (Quasi) Superhypergraph

In this section, we introduce the concepts of supervertex, superedge, superhypergraph, and investigate their properties. For any given superhypergraph, the lower and upper bound of the set of their superedges is computed and proved in a theorem. Also, we computed and proved the number of all superhypergraphs constructed on any given nonempty set. In the following, for any nonempty set X, will denote $P^*(X) = \{Y | \varnothing \neq Y \subseteq X\}$.

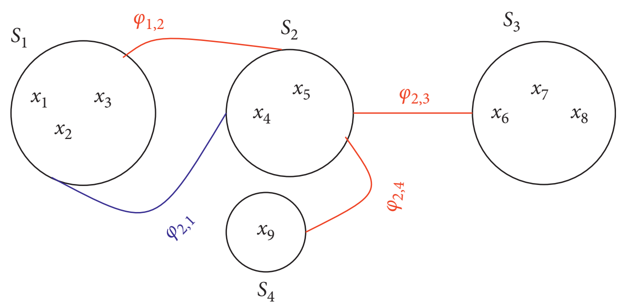

FIGURE 1: Superhypergraph $H = (X, \{S_i\}_{i=1}^{4}, \{\varphi_{1,2}, \varphi_{2,1}, \varphi_{2,3}, \varphi_{2,4}\})$.

In what follows, based on the concept of n-superhypergraph [20], recall, define, and investigate a special case in n-superhypergraphs as the notation of quasi superhypergraph.

Definition 3. Let X be a nonempty set. Then,

(i) $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ is called a quasi superhypergraph, if $\{\varphi_{i,j}\}_{i,j} \neq \varnothing$ and $X = \cup_{i=1}^{n} S_i$, where $k \geq 2$,

(ii) for all $1 \leq i \leq k, S_i \in P^*(X)$, is called a supervertex and for any $i \neq j$, the map $\varphi_{i,j}\colon S_i \longrightarrow S_j$ (say S_i links to S_j) is called a superedge,

(iii) the quasi superhypergraph $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ is called a superhypergraph, if for any $S_i \in P^*(X)$, there exists at least one $S_j \in P^*(X)$ such that S_i links to S_j (it is not necessary all super vertices be linked).

(iv) The superhypergraph $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ is called a trivial superhypergraph, if $k = 1$ (S_1 can't link to itself).

Example 1. Let $X = \{x_i\}_{i=1}^{9}$. Then, $H = (X, \{S_i\}_{i=1}^{4}, \{\varphi_{1,2}, \varphi_{2,1}, \varphi_{2,3}, \varphi_{2,4}\})$ is a quasi superhypergraph (there is no any link between of S_4 and S_3) in Figure 1, where

$$\begin{aligned}
\varphi_{1,2} &= \{(x_1, x_4), (x_2, x_5), (x_3, x_4)\},\\
\varphi_{2,1} &= \{(x_4, x_3), (x_5, x_2)\},\\
\varphi_{2,3} &= \{(x_4, x_6), (x_5, x_7)\},\\
\varphi_{2,4} &= \{(x_5, x_9), (x_4, x_9)\}.
\end{aligned} \tag{1}$$

Let $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ be a superhypergraph. We will denote $\Phi(H) = \{\varphi_{i,j} \mid i, j \geq 1\}$ by the set of all superedges of superhypergraph H. In what follows, compute and prove the lower bound and upper bound of $\Phi(H)$, as set of all superedges of superhypergraph.

Theorem 1. *Let $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ be a superhypergraph. Then,*

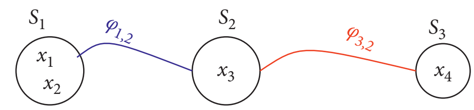

FIGURE 2: Superhypergraph $H = (X, \{S_i\}_{i=1}^{3}, \{\varphi_{1,2}, \varphi_{3,2}\})$ with $|\Phi(H)| = 2$.

$$(k-1) \leq |\Phi(H)| \leq \sum_{1 \leq i \neq j \leq n} |S_i|^{|S_j|}. \tag{2}$$

Proof. Let $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ be a superhypergraph. Since, by definition, for any $S_i \in P^*(X)$, there exists at least one $S_j \in P^*(X)$ such that S_i links to S_j, we have $(k-1)$ superedges. In addition, let $\Phi(S_i, S_j) = \{\varphi_{i,j}\colon S_i \longrightarrow S_j \mid i, j\}$. For all $1 \leq i \neq j \leq n$, in one case, if $S_i \cap S_j = \varnothing$, then, $|\Phi(S_i, S_j)| = |S_i|^{|S_j|} + |S_j|^{|S_i|}$. Hence, if $|\Phi(S_1, S_2, \ldots, S_{k-1})| = \sum_{1 \leq i \neq j \leq k-1} |S_i|^{|S_j|}$, then, for all $1 \leq i \neq j \leq n, S_i \cap S_j = \varnothing$ implies that

$$\begin{aligned}
|\Phi(S_1, S_2, \ldots, S_{k-1}, S_k)| &= |\Phi((S_1, S_2, \ldots, S_{k-1}), S_k)|\\
&= |\Phi(S_1, S_2, \ldots, S_{k-1})|\\
&\quad + \sum_{1 \leq i \leq k-1} \left(|S_i|^{|S_k|} + |S_k|^{|S_i|}\right)\\
&= \sum_{1 \leq i \neq j \leq n} |S_i|^{|S_j|}.
\end{aligned} \tag{3}$$

In another case, if there exists some $1 \leq i \neq j \leq n$ such that $S_i \cap S_j \neq \varnothing$, since for all $1 \leq i \neq j \leq n$, $S_i \cap S_j \subseteq S_i$, for all $1 \leq t \leq n$, we get that $\Phi(S_i \cap S_j, S_t) \subseteq \Phi(S_i, S_t)$ and $\Phi(S_i \cap S_j, S_t) \subseteq \Phi(S_j, S_t)$. So, in any cases, $|\Phi| \leq \sum_{1 \leq i \neq j \leq n} |S_i|^{|S_j|}$. □

Example 2. Let $|X| = \{x_i\}_{i=1}^{4}$ and $H = (X, \{S_i\}_{i=1}^{3}, \{\varphi_{i,j}\}_{i,j})$ be a superhypergraph, where $|S_1| = 2, |S_2| = 1, |S_3| = 1$. Then, by Theorem 1, $2 \leq |\Phi(H)| \leq 8 = (2^1 + 2^1 + 1^2 + 1^1 + 1^2 + 1^1)$, where

$$\begin{aligned}
\varphi_{1,2} &= \{(x_1, x_3), (x_2, x_3)\},\\
\varphi_{1,3} &= \{(x_1, x_4), (x_2, x_4)\},\\
\varphi_{2,1} &= \{(x_3, x_1)\},\\
\varphi'_{2,1} &= \{(x_3, x_2)\},\\
\varphi_{3,1} &= \{(x_4, x_1)\},\\
\varphi'_{3,1} &= \{(x_4, x_2)\},\\
\varphi_{2,3} &= \{(x_3, x_4)\},\\
\varphi_{3,2} &= \{(x_4, x_3)\}.
\end{aligned} \tag{4}$$

The superhypergraph $H = (X, \{S_i\}_{i=1}^{3}, \{\varphi_{1,2}\}_{3,2})$ with minimum superedges is shown in Figure 2.

Let X be a nonempty set, $\mathcal{SH} = \{H | H \text{ is a superhypergraph on } X\}$ and $\mathcal{SH}(n_1, n_2, \ldots, n_k) = \left\{(X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j}) \in \mathcal{SH} \,\|\, |S_i| = n_i \text{ and for all } i \neq j, \; S_i \cap S_j = \varnothing\right\}$. In what follows, compute and prove the number of $\mathcal{SH}(n_1, n_2, \ldots, n_k)$, as the set of all superhypergraphs based on any nonempty set X, where $|X| = n$.

Theorem 2. *Let X be a nonempty set, $r, n \in \mathbb{N}$ and $|X| = n$.*

(i) $|\mathcal{SH}(n)| = 1$.

(ii) If $\sum_{i=1}^r n_i = n$ and $m = |\{i | n_i = n_j\}|$, then, $|\mathcal{SH}(n_1, n_2, \ldots, n_r)| = (1/m!) \prod_{i=1}^r \binom{n - \sum_{j=1}^{i-1} n_j}{n_i} (\sum_{1 \leq i \neq j \leq r} n_i^{n_j})$.

Proof

(i) By definitions is clear.

(ii) Let $|X| = n$. Since $\mathcal{SH}(n_1, n_2, \ldots, n_k) = \left\{(X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j}) \in \mathcal{SH} \,\|\, |S_i| = n_i \text{ and for all } i \neq j, \; S_i \cap S_j = \varnothing\right\}$, we get that $\{S_1, S_2, \ldots, S_k\} \in \mathcal{P}(X)$ is a partition of set X, where $|S_i| = n_i$. Consider S_1, then, the numbers of selected vertices in S_1 is equal to $\binom{n}{n_1}$. Because for any $1 \leq i \leq k$, $S_i \cap S_j = \varnothing$, the number of selected vertices in S_2 is equal to $\binom{n - n_1}{n_2}$. It follows that the numbers of ways to chosen the selected vertices between S_1, S_2 is equal to $\binom{n}{n_1}\binom{n - n_1}{n_2}$ for the case $n_1 \neq n_2$ and is equal to $(1/2!)\binom{n}{n_1}\binom{n - n_1}{n_2}$, for the case $n_1 = n_2$. Thus, in the process of doing so and by induction, for all $1 \leq i \leq k$, $|\mathcal{P}(X)| = (1/m!) \prod_{i=1}^k \binom{n - \sum_{j=1}^{i-1} n_j}{n_i}$, where $m = |\{i | n_i = n_j\}|$. In addition, by Theorem 1, we have $|\Phi(H)| \leq \sum_{1 \leq i \neq j \leq n} |S_i|^{|S_j|}$, so $|\mathcal{SH}(n_1, n_2, \ldots, n_r)| = (1/m!) \prod_{i=1}^r \binom{n - \sum_{j=1}^{i-1} n_j}{n_i} (\sum_{1 \leq i \neq j \leq r} n_i^{n_j})$. □

Theorem 3. *Let X be a nonempty set and $|X| = n$. If $\alpha = \{(n_1, n_2, \ldots, n_r) | \sum_{i=1}^r n_i = n, n_i, r \in \mathbb{N}\}$ and $m = |\{i | n_i = n_j\}|$, then,*

$$|\mathcal{SH}(X)| = \sum_{\alpha} \left(\frac{1}{m!} \prod_{i=1}^{r} \binom{n - \sum_{j=1}^{i-1} n_j}{n_i} \left(\sum_{1 \leq i \neq j \leq r} n_i^{n_j} \right) \right). \tag{5}$$

Proof. It is clear by Theorem 2. □

Example 3. Let X be an arbitrary set and $|X| = 4$. Then,

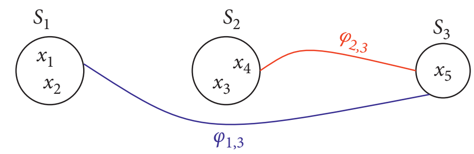

Figure 3: Superhypergraph $H = (X, \{S_i\}_{i=1}^3, \{\varphi_{1,3}, \varphi_{2,3}\})$.

$$\begin{aligned}
|\mathcal{SH}(X)| &= |\mathcal{SH}(4)| + |\mathcal{SH}(3,1)| + |\mathcal{SH}(2,2)| \\
&\quad + |\mathcal{SH}(2,1,1)| + |\mathcal{SH}(1,1,1,1)| \\
&= 1 + \binom{4}{3}\binom{1}{1}\left(3^1 + 1^3\right) \\
&\quad + \frac{1}{2!}\binom{4}{2}\binom{2}{2}\left(2^2 + 2^2\right) \\
&\quad + \frac{1}{2!}\binom{4}{2}\binom{2}{1}\binom{1}{1}\left(2^1 + 2^1 + 1^2 + 1^1 + 1^2 + 1^1\right) \\
&\quad + \frac{1}{4!}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}\left(1^1 + 1^1 + 1^1\right)4 = 101.
\end{aligned} \tag{6}$$

3. Incidence Matrix of Superhypergraphs

In this section, we introduce a square matrix as incidence matrix associate with any given superhypergraph with sign function. Indeed, in the incidence matrix associate to any given superhypergraph the domain and range of any map determine the sign function.

Let $H = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ be a superhypergraph and $\Psi(H) \subseteq \Phi(H)$. Then, we have the following concepts.

Definition 4. Let $H = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ be a superhypergraph and $|X| = n$. Define $A_{(|S_1|, |S_2|, \ldots, |S_k|)} = (a_{ij})_{n \times (k + |\Psi(H)|)}$ as incidence matrix of H with $k + |\Psi(H)|$ columns representing the supervertices $S_1, S_2, \ldots, S_k$, superedges $\varphi_{i,j}$ and n rows representing the vertices $x_1, x_2, \ldots, x_n$, where $n = (k + |\Psi(H)|)$ and

$$a_{ij} = \begin{cases} 1, & \text{if } x_i \in S_i \text{ or } x_i \in \text{Domain}(\varphi_{i,j}), \\ -1, & \text{if } x_i \in \text{Range}(\varphi_{i,j}), \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Example 4. Let $X = \{x_i\}_{i=1}^5$. Then, $H = (X, \{S_i\}_{i=1}^3, \{\varphi_{1,3}, \varphi_{2,3}\})$ is a superhypergraph in Figure 3, where $\varphi_{1,3} = \{(x_1, x_5), (x_2, x_5)\}$ and $\varphi_{2,3} = \{(x_3, x_5), (x_4, x_5)\}$.

Then, $A_{(2,2,1)}$ is the incidence matrix of H.

$$A_{(2,2,1)} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & \phi_{1,3} & \phi_{2,3} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \end{array}. \tag{8}$$

3.1. Characteristic Polynomial for the Incidence Matrix of Superhypergraph. In this subsection, we compute the characteristic polynomial of the incidence matrix of any given superhypergraph. Let $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ be a superhypergraph and so investigate the spectrum of the superhypergraph.

From now on, let $P_{(|S_1|,|S_2|,\ldots,|S_k|)}(x)$ be the *characteristic polynomial* of the incidence matrix $A_{(|S_1|,|S_2|,\ldots,|S_k|)}$ corresponding to superhypergraph H and $E(A_{(|S_1|,|S_2|,\ldots,|S_k|)}) = \{x | \text{is an eigenvalue of } A_{(|S_1|,|S_2|,\ldots,|S_k|)}\}$. In addition, for any S_i, S_j, if $\Phi(S_i, S_j) = \{\varphi_{i,j} | \varphi_{i,j}\colon S_i \longrightarrow S_j, i, j \geq 1\}$, will say S_i flows to S_j and will denote by $S_i \rightsquigarrow S_j$. In this case, will denote $A_{(|S_1|,|S_2|,\ldots,|S_k|)}$ by $A^{\rightsquigarrow}_{(|S_1|,|S_2|,\ldots,|S_k|)}$ and $P_{(|S_1|,|S_2|,\ldots,|S_k|)}(x)$ by $P^{\rightsquigarrow}_{(|S_1|,|S_2|,\ldots,|S_k|)}(x)$.

Theorem 4. *Let $|S_1| = 1, |S_2| = n$, $n \geq 2$ and $S_1 \rightsquigarrow S_2$. Then,*

(i) $P^{\rightsquigarrow}_{1,n}(x) = (-x)^{n-1}(x^2 + (n-3)x - (n-2))$.

(ii) Spec $(A^{\rightsquigarrow}_{1,n}) = \begin{pmatrix} 0 & 1 & 2-n \\ n-1 & 1 & 1 \end{pmatrix}$.

Proof

(i) Let $n = 2$. It is easy to see that $P^{\rightsquigarrow}_{1,2}(x) = (-x)(x^2 - x)$. Suppose that $k \geq 3$ and $P^{\rightsquigarrow}_{1,k-1}(x) = (-x)^{k-2}(x^2 + (k-4)x - (k-3))$. Then, $A^{\rightsquigarrow}_{1,k} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where $c_3 = \cdots = c_{k+1} = [1, \underbrace{-1, \ldots, -1}_{k\text{-times}}]^t$. It follows that

$$P^{\rightsquigarrow}_{1,k}(x) = \det\left(A^{\rightsquigarrow}_{1,k} - I_{(k+1)\times(k+1)}X\right) = \det\left(B^{\rightsquigarrow}_{1,k}\right), \tag{9}$$

such that $B^{\rightsquigarrow}_{1,k} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B^{\rightsquigarrow}_{1,k} = [c_1', c_2', \ldots, c_k', c_{k+1}']$ that for all $i \in \{3, \ldots, k+1\}, c_i' = [1, \underbrace{-1, \ldots,}\ -1_{(i-2)\text{-times}}, -1 - x, \underbrace{-1, \ldots, -1}_{(k-i+1)\text{-times}}]^t$. Now, consider $D^{\rightsquigarrow}_{1,k} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j \neq k+1, \\ -c_k' + c_{k+1}', & j = k+1. \end{cases} \tag{10}$$

Thus, $D^{\rightsquigarrow}_{1,k} = [c_1', c_2', \ldots, c_{k-1}', c_k', c_{k+1}'']$, where $c_{k+1}'' = [\underbrace{0, \ldots, 0}_{(k-1)\text{-times}}, x, -x]^t = (x)[\underbrace{0, \ldots, 0}_{(k-1)\text{-times}}, 1, -1]^t$. Based on induction assumption and computations of determinant based on column c_{k+1}'' in matrix $D^{\rightsquigarrow}_{1,k}$, have

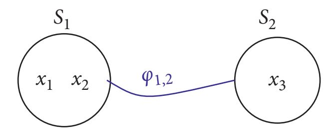

Figure 4: Superhypergraph $H = (X, \{S_i\}_{i=1}^{2}, \{\varphi_{1,2}\})$.

$$\begin{aligned} P^{\rightsquigarrow}_{k,1}(x) &= \det\left(D^{\rightsquigarrow}_{1,k} - I_{(k+1)\times(k+1)}X\right) \\ &= x\left(-P^{\rightsquigarrow}_{1,k-1}(x) + (-x)^{k-1} + (-x)^{k-2}\right) \\ &= (-x)^{k-1}\left(x^2 + (k-3)x - (k-2)\right). \end{aligned} \tag{11}$$

(ii) It is clear by item (i). □

Theorem 5. *Let $|S_1| = n, |S_2| = 1$, $n \geq 2$ and $S_1 \rightsquigarrow S_2$. Then, $P^{\rightsquigarrow}_{n,1}(x) = (-x)^{n-1}(x^2 - (n-2)x - 2)$.*

Proof. Let $n = 2$. It is easy to see that $P^{\rightsquigarrow}_{2,1}(x) = (-x)(x^2 - 2)$. Suppose that $k \geq 3$ and $P^{\rightsquigarrow}_{k-1,1}(x) = (-x)^{k-2}(x^2 - (k-3)x - 2)$. Then, $A^{\rightsquigarrow}_{k,1} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where $c_3 = \cdots = c_{k+1} = [\underbrace{1, 1, \ldots, 1}_{k\text{-times}}, -1]^t$. It follows that

$$P^{\rightsquigarrow}_{k,1}(x) = \det\left(A^{\rightsquigarrow}_{k,1} - I_{(k+1)\times(k+1)}X\right) = \det\left(B^{\rightsquigarrow}_{k,1}\right), \tag{12}$$

such that $B^{\rightsquigarrow}_{k,1} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B^{\rightsquigarrow}_{k,1} = [c_1', c_2', \ldots, c_k', c_{k+1}']$ that for all $i \in \{3, 4, \ldots, k\}, c_i' = [\underbrace{1, 1, \ldots, 1}_{\{i-1\}\text{-times}}, 1 - x, \underbrace{1, 1, \ldots, 1}_{\{k-i\}\text{-times}}, -1]^t$ and $c_{k+1}' = [\underbrace{1, 1, \ldots, 1}_{k\text{-times}}, -1 - x]^t$. Consider $D^{\rightsquigarrow}_{k,1} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j \neq k, \\ c_k' - c_{k+1}', & j = k. \end{cases} \tag{13}$$

Thus, $D^{\rightsquigarrow}_{k,1} = [c_1', c_2', \ldots, c_{k-1}', c_k'', c_{k+1}']$, where $c_k'' = [\underbrace{0, \ldots, 0}_{\{k-1\}\text{-times}}, -x, x]^t = (-x)[\underbrace{0, \ldots, 0}_{\{k-1\}\text{-times}}, 1, -1]^t$. Based on induction assumption and computations of determinant based on column c_k'' in matrix $D^{\rightsquigarrow}_{k,1}$, have

$$P^{\rightsquigarrow}_{k,1}(x) = \det\left(D^{\rightsquigarrow}_{k,1} - I_{(k+1)\times(k+1)}X\right) = -xP^{\rightsquigarrow}_{k-1,1}(x) + (-x)^k. \tag{14}$$

□

Theorem 6. *Let $|S_1| = n, |S_2| = 2$, $n \geq 2$ and $S_1 \rightsquigarrow S_2$. Then, $P^{\rightsquigarrow}_{n,2}(x) = (-x)^n(x^2 - (n-3)x - 4)$.*

Proof. Let $n = 2$. It is easy to see that $P^{\rightsquigarrow}_{2,1}(x) = x^2(x^2 + x - 4)$. Suppose that $k \geq 3$ and $P^{\rightsquigarrow}_{k-1,1}(x) = (-x)^{k-1}(x^2 - (k-4)x - 4)$.Then, $A^{\rightsquigarrow}_{k,2} = [c_1, c_2, \ldots, c_k, c_{k+1}, c_{k+2}]$, where $c_3 = \cdots = c_{k+2} = [\underbrace{1, 1, \ldots, 1}_{k\text{-times}}, -1, -1]^t$. It follows that

$$P^{\rightsquigarrow}_{k,2}(x) = \det\left(A^{\rightsquigarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = \det\left(B^{\rightsquigarrow}_{k,2}\right), \tag{15}$$

such that $B^{\rightsquigarrow}_{k,2} = [b_{ij}]_{(k+2)\times(k+2)}$, where $B^{\rightsquigarrow}_{k,2} = [c_1', c_2', \ldots, c_k', c_{k+1}', c_{k+2}']$ that for all $i \in \{3, 4, \ldots, k\} c_i' =$

$[\underbrace{1,1,\ldots,1}_{\{i-1\}-\text{times}}, 1-x, \underbrace{1,1,\ldots,1}_{\{k-i\}-\text{times}}, -1,-1]^t, c_{k+1}{}' = [\underbrace{1,1,\ldots,1}_{k-\text{times}}, -1-x,, -1]^t$ and $c'_{k+2} = [\underbrace{1,1,\ldots,1}_{k-\text{times}}, -1, -1-x]^t$. In addition, $D^{\rightsquigarrow}_{k,2} = (d_{ij})_{(k+2)\times(k+2)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j \neq k, \\ c'_k - c'_{k+2}, & j = k. \end{cases} \tag{16}$$

Thus, $D^{\rightsquigarrow}_{k,2} = [c'_1, c'_2, \ldots, c''_k, c_{k+1}{}', c_{k+2}{}']$, where $c''_k = [\underbrace{0,\ldots,0}_{\{k-1\}-\text{times}}, -x, 0, x]^t = (-x)[\underbrace{0,\ldots,0}_{\{k-1\}-\text{times}}, 1, 0, -1]^t$. Based on induction assumption and computations of determinant based on column c''_k in matrix $D^{\rightsquigarrow}_{k,2}$ show that

$$P^{\rightsquigarrow}_{k,2}(x) = \det\left(D^{\rightsquigarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = -xP^{\rightsquigarrow}_{k-1,2}(x) + (-x)^{k+1}. \tag{17}$$

□

Theorem 7. *Let $|S_1| = n, |S_2| = 3$, $n \geq 2$ and $S_1 \rightsquigarrow S_2$. Then, $P^{\rightsquigarrow}_{n,3}(x) = (-x)^{n+1}(x^2 - (n-4)x - 6)$.*

Proof. It is similar to Theorem 6. □

Corollary 1. *Let $|S_1| = n, |S_2| = m$, $n \geq 2, m \geq 1$ and $S_1 \rightsquigarrow S_2$. Then,*

(i) $P^{\rightsquigarrow}_{n,m}(x) = (-x)^{n+m-2}(x^2 - (n-m-1)x - 2m)$.

(ii) Spec $(A^{\rightsquigarrow}_{n,m}) = (0\ \ (\alpha + \sqrt{\alpha^2 + 8m})/2\ \ (\alpha - \sqrt{\alpha^2 + 8m})/2 n + m - 21)$, where $\alpha = n - m - 1$.

(iii) $\sum_{x \in E(A^{\rightsquigarrow}_{n,m})} x = 0$ if and only if $n = m + 1$.

Example 5. Let $X = \{x_1, x_2, x_3\}$. Then, $H = (X, \{S_i\}_{i=1}^2, \{\varphi_{1,2}\})$ is a superhypergraph as shown in Figure 4 and incidence matrix of $A^{\rightsquigarrow}_{2,1}$ as follows.

$$\begin{array}{cc} & \begin{array}{ccc} S_1 & S_2 & \phi_{1,2} \end{array} \\ M_H = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \end{array}. \tag{18}$$

Thus, by Theorem 5, we have $P^{\rightsquigarrow}_{2,1}(x) = (-x)(x^2 - 2)$ and so $\text{Spec}(A^{\rightsquigarrow}_{2,1}) = \begin{pmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$.

3.2. Characteristic Polynomial for the Incidence Matrix of Superhypergraph with Reverse Flows. In this subsection, we compute the characteristic polynomial and spectrum of the superhypergraph in the reverse flows to the previous section.

For any S_i, S_j, if $\Phi(S_j, S_i) = \{\varphi_{i,j} | \varphi_{i,j}: S_j \longrightarrow S_i, i, j \geq 1\}$, will say S_j flows to S_i and will denote by $S_i \leftarrow S_j$ (reverse flows to $S_i \rightsquigarrow S_j$) and so $A_{|S_1|,|S_2|,\ldots,|S_k|}$ by $A^{\leftarrow}_{|S_1|,|S_2|,\ldots,|S_k|}$ and $P_{|S_1|,|S_2|,\ldots,|S_k|}(x)$ by $P^{\leftarrow}_{|S_1|,|S_2|,\ldots,|S_k|}(x)$.

Theorem 8. *Let $|S_1| = 1, |S_2| = n$, $n \geq 2$ and $S_1 \leftarrow S_2$. Then,*

(i) $P^{\leftarrow}_{1,n}(x) = (-x)^{n-1}(x^2 - (n+1)x + n)$.

(ii) Spec $(A^{\leftarrow}_{1,n}) = \begin{pmatrix} 0 & 1 & n \\ n-1 & 1 & 1 \end{pmatrix}$.

Proof

(i) Let $n = 2$. It is easy to see that $P^{\leftarrow}_{1,2}(x) = (-x)(x^2 - 3x + 2)$. Suppose that $k \geq 3$ and $P^{\leftarrow}_{1,k-1}(x) = (-x)^{k-2}(x^2 - kx + (k-1))$. Then, $A^{\leftarrow}_{1,k} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where $c_3 = \cdots = c_{k+1} = [-1, \underbrace{1,\ldots,1}_{k-\text{times}}]^t$. It follows that

$$P^{\leftarrow}_{1,k}(x) = \det\left(A^{\leftarrow}_{1,k} - I_{(k+1)\times(k+1)}X\right) = \det\left(B^{\leftarrow}_{1,k}\right), \tag{19}$$

such that $B^{\leftarrow}_{1,k} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B^{\leftarrow}_{1,k} = [c'_1, c'_2, \ldots, c'_k, c'_{k+1}]$ that for all $i \in \{3, 4, \ldots, k+1\}, c'_i = [-1, \underbrace{1,\ldots,1}_{(i-2)-\text{times}}, \underbrace{1-x}_{i-\text{th}}, \underbrace{1,\ldots,1}_{(k-i+1)-\text{times}}]^t$. Now, consider $D^{\leftarrow}_{1,k} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j \neq k+1, \\ -c'_k + c'_{k+1}, & j = k+1. \end{cases} \tag{20}$$

Thus, $D^{\leftarrow}_{1,k} = [c'_1, c'_2, \ldots, c'_{k-1}, c'_k, c_{k+1}{}'']$, where $c_{k+1}{}'' = [\underbrace{0,\ldots,0}_{(k-1)-\text{times}}, x, -x]^t = (x)[\underbrace{0,\ldots,0}_{(k-1)-\text{times}}, 1, -1]^t$. Based on induction assumption and computations of determinant based on column c''_{k+1} in matrix $D^{\leftarrow}_{1,k}$, have

$$\begin{aligned} P^{\leftarrow}_{1,k}(x) &= \det\left(D^{\leftarrow}_{1,k} - I_{(k+1)\times(k+1)}X\right) \\ &= x\left(-P^{\leftarrow}_{1,k-1}(x) - (-x)^{k-1} - (-x)^{k-2}\right) \\ &= (-x)^{k-1}\left(x^2 - (k+1)x + k\right). \end{aligned} \tag{21}$$

(ii) It is clear by item (i). □

Theorem 9. *Let $|S_1| = n, |S_2| = 1$, $n \geq 2$ and $S_1 \leftarrow S_2$. Then, $P^{\leftarrow}_{n,1}(x) = (-x)^{n-1}(x^2 + (n-4)x + 2)$.*

Proof. Let $n = 2$. It easy is to see that $P^{\leftarrow}_{2,1}(x) = (-x)(x^2 - 2x + 2)$. Suppose that $k \geq 3$ and $P^{\leftarrow}_{k-1,1}(x) = (-x)^{k-2}(x^2 + (k-5)x + 2)$.Then, $A^{\leftarrow}_{k,1} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where $c_3 = \cdots = c_{k+1} = [\underbrace{-1,-1,\ldots,-1}_{k\text{ times}}, 1]^t$. It follows that

$$P^{\leftarrow}_{k,1}(x) = \det\left(A^{\leftarrow}_{k,1} - I_{(k+1)\times(k+1)}X\right) = \det\left(B^{\leftarrow}_{k,1}\right), \tag{22}$$

such that $B^{\leftarrow}_{k,1} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B^{\leftarrow}_{k,1} = [c'_1, c'_2, \ldots, c'_k, c'_{k+1}]$ that for all $i \in \{3, \ldots, k\}, c'_i = [\underbrace{-1,\ldots,-1}_{(i-1)-\text{times}}, \underbrace{-1-x}_{i^{\text{th}}}, \underbrace{-1,\ldots,-1}_{(k-i)-\text{times}}, 1]^t$ and $c'_{k+1} = [\underbrace{-1,-1,\ldots,-1}_{k\text{ times}}, 1-x]^t$. Moreover, consider $D^{\leftarrow}_{k,1} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j \neq k, \\ c'_k - c'_{k+1}, & j = k. \end{cases} \tag{23}$$

Thus, $D^{\leftarrow}_{k,1} = [c_1', c_2', \ldots, c_{k-1}', c_k'', c_{k+1}']$, where $c_k'' = [\underbrace{0, \ldots, 0}_{k-1 \text{ times}}, -x, x]^t = (-x)[\underbrace{0, \ldots, 0}_{k-1 \text{ times}}, 1, -1]^t$. Based on induction assumption and computations of determinant based on column c_k'' in matrix $D^{\leftarrow}_{k,1}$, have

$$P^{\leftarrow}_{k,1}(x) = \det\left(D^{\leftarrow}_{k,1} - I_{(k+1)\times(k+1)}X\right) = -xP^{\leftarrow}_{k-1,1}(x) + (-1)^{k+1}x^k. \tag{24}$$

□

Theorem 10. *Let* $|S_1| = n, |S_2| = 2, n \geq 2$ *and* $S_1 \leftarrow S_2$. *Then,* $P^{\leftarrow}_{n,2}(x) = (-x)^n (x^2 + (n-5)x + 4)$.

Proof. Let $n = 2$. It is easy to see that $P^{\leftarrow}_{2,1}(x) = x^2(x^2 - 3x + 4)$. Suppose that $k \geq 3$ and $P^{\leftarrow}_{k-1,1}(x) = (-x)^{k-1}(x^2 + (k-6)x + 4)$.Then, $A^{\leftarrow}_{k,2} = [c_1, c_2, \ldots, c_k, c_{k+1}, c_{k+2}]$, where $c_3 = \cdots = c_{k+2} = [\underbrace{-1, -1, \ldots, -1}_{k \text{ times}}, 1, 1]^t$. It follows that

$$P^{\leftarrow}_{k,2}(x) = \det\left(A^{\leftarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = \det\left(B^{\leftarrow}_{k,2}\right), \tag{25}$$

such that $B^{\leftarrow}_{k,2} = [b_{ij}]_{(k+2)\times(k+2)}$, where $B^{\leftarrow}_{k,2} = [c_1', c_2', \ldots, c_k', c_{k+1}', c_{k+2}']$ that for all $i \in \{3, \ldots, k\}, c_i' = [\underbrace{-1, \ldots, -1}_{(i-1)-\text{times}}, \underbrace{-1-x}_{i^{\text{th}}}, \underbrace{-1, \ldots, -1}_{(k-i)-\text{times}}, 1, 1]^t, c_{k+1}' = [\underbrace{-1, \ldots, -1}_{k-\text{times}}, 1-x, 1]^t$, and $c_{k+2}' = [\underbrace{-1, \ldots, -1}_{k-\text{times}}, 1, 1-x]^t$. Consider $D^{\leftarrow}_{k,2} = (d_{ij})_{(k+2)\times(k+2)}$ where

$$d_{ij} = \begin{cases} b_{ij}, & j \neq k, \\ c_k' - c_{k+1}', & j = k. \end{cases} \tag{26}$$

Thus, $D^{\leftarrow}_{k,2} = [c_1', c_2', \ldots, c_k'', c_{k+1}', c_{k+2}']$, where $c_k'' = [\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, -x, x, 0]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, 1, -1, 0]^t$. Based on induction assumption and computations of determinant based on column c_k'' in matrix $D^{\leftarrow}_{k,2}$ show that

$$P^{\leftarrow}_{k,2}(x) = \det\left(D^{\leftarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = (-x)P^{\leftarrow}_{k-1,2}(x) + (-1)^k x^{k+1}. \tag{27}$$

□

Theorem 11. *Let* $|S_1| = n, |S_2| = 3,\ n \geq 2$ *and* $S_1 \leftarrow S_2$. *Then,* $P_{n,3}(x) = (-x)^{n+1}(x^2 + (n-6)x + 6)$.

Proof. It is similar to Theorem 6. □

Corollary 2. *Let* $|S_1| = n, |S_2| = m,\ n \geq 2, m \geq 1$ *and* $S_1 \leftarrow S_2$. *Then,*

(i) $P^{\leftarrow}_{n,m}(x) = (-x)^{n+m-2}(x^2 + (n-m-3)x + 2m)$.

(ii) Spec $(A^{\leftarrow}_{n,m}) = \begin{pmatrix} 0 & (-\alpha + \sqrt{\alpha^2 - 8m})/2 & (-\alpha - \sqrt{\alpha^2 - 8m})/2 \\ n+m-2 & 1 & 1 \end{pmatrix}$, *where* $\alpha = n - m - 3$.

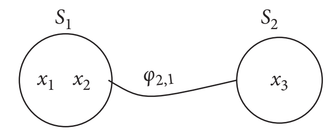

Figure 5: Superhypergraph $H = (X, \{S_i\}_{i=1}^2, \{\varphi_{2,1}\})$.

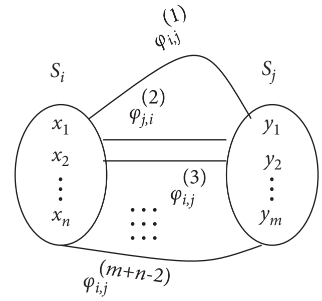

Figure 6: $H = (X, \{S_i, S_j\}, \{\varphi^{(1)}_{i,j}, \varphi^{(2)}_{j,i}, \ldots, \varphi^{(m+n-2)}_{j,i}\})$.

(iii) $\sum_{x \in E(A_{n,m})} x = 0$ *if and only if* $n = m + 3$.

Example 6. Let $X = \{x_1, x_2, x_3\}$. Then, $H = (X, \{S_i\}_{i=1}^2, \{\varphi_{2,1}\})$ is a superhypergraph as shown in Figure 5 and incidence matrix of $A_{2,1}$ as follows.

$$M_H = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & \varphi_{2,1} \end{array} \\ \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}. \tag{28}$$

Thus, by Theorem 9, we have $P_{2,1}(x) = (-x)(x^2 - 2x + 2)$ and so Spec $(A^{\leftarrow}_{2,1}) = \begin{pmatrix} 0 & 1+i & 1-i \\ 1 & 1 & 1 \end{pmatrix}$.

3.3. *Characteristic Polynomial for the Incidence Matrix of Superhypergraph with Two-Sided Flows.*

In this subsection, we compute the characteristic polynomial and spectrum of the superhypergraph with two-sided flows.

For any S_i, S_j, if $\Phi = \{\varphi_{i,j} | \varphi_{i,j}: S_i \leftrightarrow S_j, i, j \geq 1\}$, will say S_i flows to S_j by two-sided and will denote by $S_i \leftrightsquigarrow S_j$ and so $A_{|S_1|,|S_2|,\ldots,|S_k|}$ by $A^{\leftrightsquigarrow}_{|S_1|,|S_2|,\ldots,|S_k|}$ and $P_{|S_1|,|S_2|,\ldots,|S_k|}(x)$ by $P^{\leftrightsquigarrow}_{|S_1|,|S_2|,\ldots,|S_k|}(x)$. When we will show that $S_i \leftrightsquigarrow S_j$ of type $[+, -, +, -, \ldots]$, it means that the first map flows from S_i to S_j, the second map flows from S_j to S_i, the third map flows from S_i to S_j, etc., respectively. We will show it in Figure 6, where, n is an odd.

Theorem 12. *Let* $|S_1| = 1, |S_2| = n,\ n \geq 3$ *and* $S_1 \leftrightsquigarrow S_2$ *of type* $[+, -, +, -, \ldots]$. *Then,*

(i) $P^{\leftrightsquigarrow}_{1,n}(x) = \begin{cases} -(x^{2k})(x-1) & n = 2k \\ x^{2k}(x-1)^2 & n = 2k+1 \end{cases}$.

(ii) if n is an even, then Spec $(A^{\leftrightsquigarrow}_{1,n}) = \begin{pmatrix} 0 & 1 \\ n & 1 \end{pmatrix}$.

(iii) if n is an odd, then Spec $(A_{1,n}^{\leftrightsquigarrow}) = \begin{pmatrix} 0 & 1 \\ n-1 & 2 \end{pmatrix}$.

Proof

(i) Let $n=3$ and $n=4$. It is easy to see that $P_{1,3}^{\leftrightsquigarrow}(x) = x^2(x-1)^2$ and $P_{1,4}^{\leftrightsquigarrow}(x) = -x^4(x-1)$. Suppose that $k \geq 3$ is an odd and $P_{1,k-1}^{\leftrightsquigarrow}(x) = x^{2k-2}(x-1)^2$. Then, $A_{1,k}^{\leftrightsquigarrow} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where for all $i \in \{3,5,7,\ldots,k\}$, $c_i = [1, \underbrace{-1,\ldots,-1}_{k\text{-times}}]^t$ and where for all $i \in \{4,8,\ldots,k+1\}$, $c_i = [-1, \underbrace{1,\ldots,1}_{k\text{-times}}]^t$. It follows that

$$P_{1,k}^{\leftrightsquigarrow}(x) = \det\left(A_{1,k}^{\leftrightsquigarrow} - I_{(k+1)\times(k+1)}X\right) = \det\left(B_{1,k}^{\leftrightsquigarrow}\right), \tag{29}$$

such that $B_{1,k}^{\leftrightsquigarrow} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B_{1,k}^{\leftrightsquigarrow} = [c_1', c_2', \ldots, c_k', c_{k+1}']$, where for all $i \in \{3,5,7,\ldots,k\}, c_i' = [1, \underbrace{-1,\ldots,-1}_{(i-2)\text{-times}}, \underbrace{-1-x}_{(i)^{\text{th}}}, \underbrace{-1,\ldots,-1}_{(k-i+1)\text{-times}}]^t$ and for all $i \in \{4,8,\ldots, k+1\}, c_i' = [-1, \underbrace{1,\ldots,1}_{(i-2)\text{-times}}, \underbrace{1-x}_{(i)^{\text{th}}}, \underbrace{1,\ldots,1}_{(k-i+1)\text{-times}}]^t$. Now, consider $D_{1,k}^{\leftrightsquigarrow} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & 1 \leq j \leq k-1, \\ c_k' + c_{k-1}', & j = k, \\ c_{k+1}' - c_{k-1}', & j = k+1. \end{cases} \tag{30}$$

Thus, $D_{1,k}^{\leftrightsquigarrow} = [c_1', c_2', \ldots, c_{k-1}', c_k'', c_{k+1}'']$, where $c_k'' = [\underbrace{0,\ldots,0}_{(k-2)\text{-times}}, -x, -x, 0]^t = (-x)[\underbrace{0,\ldots,0}_{(k-2)\text{-times}}, 1, 1, 0]^t$ and $c_{k+1}'' = [\underbrace{0,\ldots,0}_{(k-2)\text{-times}}, x, 0, -x]^t = (-x)[\underbrace{0,\ldots,0}_{(k-2)\text{-times}}, -1, 0, 1]^t$. Based on induction assumption and computations of determinant based on column c_{k+1}'' in matrix $D_{1,k}^{\leftrightsquigarrow}$, have

$$P_{1,k}^{\leftrightsquigarrow}(x) = \det\left(D_{1,k}^{\leftrightsquigarrow} - I_{(k+1)\times(k+1)}X\right) = \left(x^2\right)\left(P_{1,k-2}^{\leftrightsquigarrow}(x)\right) = x^{2k}(x-1)^2. \tag{31}$$

In a similar way, if n is an even, we get that $P_{1,n}^{\leftrightsquigarrow}(x) = -(x^n)(x-1)$.

(ii) and (iii) They are clear by item (i) and (ii). □

Theorem 13. *Let $|S_1| = n, |S_2| = 1, n \geq 3$ and $S_1 \leftrightsquigarrow S_2$ of type $[+,-,+,-,\ldots]$. Then,*

$$P_{n,1}^{\leftrightsquigarrow}(x) = \begin{cases} (-x)^{2k-1}\left(x^2-2\right), & n = 2k, \\ (x)^{2k}\left(x^2-3x+2\right), & n = 2k+1. \end{cases} \tag{32}$$

Proof. Let $n=3$. It is easy to see that $P_{3,1}^{\leftrightsquigarrow}(x) = x^2(x^2-3x+2)$. Suppose that $k \geq 3$ is an odd and $P_{k-1,1}^{\leftrightsquigarrow}(x) = x^{2k-2}(x^2-3x+2)$. Then, $A_{k,1}^{\leftrightsquigarrow} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where for all $i \in \{3,5,\ldots,k\}, c_i = [\underbrace{1,\ldots,1}_{k\text{-times}}, -1]^t$ and where for all $i \in \{4,6,\ldots,k+1\}, c_i = [\underbrace{-1,\ldots,-1}_{k\text{-times}}, 1]^t$. It follows that

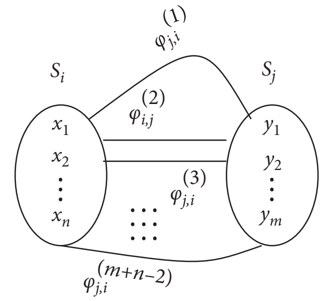

Figure 7: $H = \left(X, \{S_i, S_j\}, \{\varphi_{i,j}^{(1)}, \varphi_{j,i}^{(2)}, \ldots, \varphi_{j,i}^{(m+n-2)}\}\right)$.

$$P_{k,1}^{\leftrightsquigarrow}(x) = \det\left(A_{k,1}^{\leftrightsquigarrow} - I_{(k+1)\times(k+1)}X\right) = \det\left(B_{k,1}^{\leftrightsquigarrow}\right), \tag{33}$$

such that $B_{k,1}^{\leftrightsquigarrow} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B_{k,1}^{\leftrightsquigarrow} = [c_1', c_2', \ldots, c_k', c_{k+1}']$, where for all $i \in \{3,5,\ldots,k\}$, $c_i' = [\underbrace{1,\ldots,1}_{(i-1)\text{-times}}, \underbrace{1-x}_{(i)^{\text{th}}}, \underbrace{1,\ldots,1}_{(k-i)\text{-times}}, -1]^t$ and for all $i \in \{4,8,\ldots,k-1\}$, $c_i' = [\underbrace{-1,\ldots,-1}_{(i-1)\text{-times}}, \underbrace{-1-x}_{(i)^{\text{th}}}, \underbrace{-1,\ldots,-1}_{(k-i)\text{-times}}, 1]^t$ and for $i = k+1$, $c_i' = [\underbrace{-1,\ldots,-1}_{k\text{-times}}, 1-x]^t$. Now, consider $D_{k,1}^{\leftrightsquigarrow} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j = 1,\ldots,k-2,k+1, \\ c_k' + c_{k-1}', & j = k-1, \\ c_k' + c_{k+1}', & j = k. \end{cases} \tag{34}$$

Thus, $D_{k,1}^{\leftrightsquigarrow} = [c_1', c_2', \ldots, c_{k-2}', c_{k-1}'', c_k'', c_{k+1}']$, where $c_{k-1}'' = [\underbrace{0,\ldots,0}_{(k-2)\text{-times}}, -x, -x, 0]^t = (-x)[\underbrace{0,\ldots,0}_{(k-2)\text{-times}}, 1, 1, 0]^t$ and $c_k'' = [\underbrace{0,\ldots,0}_{(k-1)\text{-times}}, -x, -x]^t = (-x)[\underbrace{0,\ldots,0}_{(k-1)\text{-times}}, 1, 1]^t$. Based on induction assumption and computations of determinant based on column c_k'' in matrix $D_{k,1}^{\leftrightsquigarrow}$, we have

$$\begin{aligned} P_{k,1}^{\leftrightsquigarrow}(x) &= \det\left(D_{k,1}^{\leftrightsquigarrow} - I_{(k+1)\times(k+1)}X\right) = \left(x^2\right)\left(P_{k-2,1}^{\leftrightsquigarrow}(x)\right) \\ &= x^{2k}\left(x^2-3x+2\right). \end{aligned} \tag{35}$$

In a similar way, if k is an even, we get that $P_{k,1}^{\leftrightsquigarrow}(x) = (-x)^{2k-1}(x^2-2)$.

(ii) and (iii) They are clear by item (i) and (ii). □

Theorem 14. *Let $|S_1| = n, |S_2| = 2, n \geq 3$ and $S_1 \leftrightsquigarrow S_2$ of type $[+,-,+,-,\ldots]$. Then,*

$$P_{n,2}^{\leftrightsquigarrow}(x) = \begin{cases} x^{2k+1}(x-1), & n = 2k, \\ -x^{2k+2}(x-2), & n = 2k+1. \end{cases} \tag{36}$$

Proof. Let $n=3$. It is easy to see that $P_{3,1}^{\leftrightsquigarrow}(x) = (-x)^4(x-2)$. Suppose that $k \geq 3$ is an odd and $P_{k-1,2}^{\leftrightsquigarrow}(x) = (-x)^{2k}(x-2)$. Then, $A_{k,2}^{\leftrightsquigarrow} = [c_1, c_2, \ldots, c_k+1, c_{k+2}]$, where for all $i \in \{3,5,\ldots,k+2\}, c_i = [\underbrace{1,\ldots,1}_{k\text{-times}}, -1, -1]^t$ and where for all $i \in \{4,6,\ldots,k+1\}$, $c_i = [\underbrace{-1,\ldots,-1}_{k\text{-times}}, 1, 1]^t$. It follows that

$$P^{\leftrightsquigarrow}_{k,2}(x) = \det\left(A^{\leftrightsquigarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = \det\left(B^{\leftrightsquigarrow}_{k,2}\right), \tag{37}$$

such that $B^{\leftrightsquigarrow}_{k,2} = [b_{ij}]_{(k+2)\times(k+2)}$, where $B^{\leftrightsquigarrow}_{k,2} = [c'_1, c'_2, \ldots, c'_{k+1}, c'_{k+2}]$, where for all $i \in \{3, 5, \ldots, k\}, c'_i = [\underbrace{1, \ldots, 1}_{(i-1)-\text{times}}, \underbrace{1-x}_{(i)^{\text{th}}}, \underbrace{1, \ldots, 1}_{(k-i)-\text{times}}, -1, -1]^t$ and for $i = k+2, c'_i = [\underbrace{1, \ldots, 1}_{(k-2)-\text{times}}, -1, -1-x]^t$ and for all $i \in \{4, 8, \ldots, k+1\}, c'_i = [\underbrace{-1, \ldots, -1}_{(i-1)-\text{times}}, \underbrace{-1-x}_{(i)^{\text{th}}}, \underbrace{-1, \ldots, -1}_{(k-i)-\text{times}}, 1, 1]^t$. Now, consider $D^{\leftrightsquigarrow}_{k,2} = (d_{ij})_{(k+2)\times(k+2)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j = 1, \ldots, k-2, k+1, k+2, \\ c'_k + c'_{k-1}, & j = k-1, \\ c'_k + c'_{k+1}, & j = k. \end{cases} \tag{38}$$

Thus, $D^{\leftrightsquigarrow}_{k,2} = [c'_1, c'_2, \ldots, c'_{k-2}, c''_{k-1}, c''_k, c'_{k+1}, c'_{k+2}]$, where $c''_{k-1} = [\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, -x, -x, 0, 0]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, 1, 1, 0, 0]^t$ and $c''_k = [\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, -x, -x, 0]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, 1, 1, 0]^t$. Based on induction assumption and computations of determinant based on column c''_k in matrix $D^{\leftrightsquigarrow}_{k,2}$, we have

$$\begin{aligned} P^{\leftrightsquigarrow}_{k,2}(x) &= \det\left(D^{\leftrightsquigarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = \left(x^2\right)\left(P^{\leftrightsquigarrow}_{k-2,2}(x)\right) \\ &= (-x)^{2k+2}(x-2). \end{aligned} \tag{39}$$

In a similar way, if k is an even, we get that $P^{\leftrightsquigarrow}_{k,2}(x) = x^{2k+1}(x-1)$.

(ii) and (iii) They are clear by item (i) and (ii). □

Theorem 15. *Let $|S_1| = n, |S_2| = 3$, $n \geq 2$ and $S_1 \leftrightsquigarrow S_2$ of type $[+, -, +, -, \ldots]$. Then,*

$$P^{\leftrightsquigarrow}_{n,3}(x) = \begin{cases} (-x)^{2k+1}\left(x^2 - 2\right), & n = 2k\,(k \geq 1), \\ (-x)^{2k+2}\left(x^2 - 3x + 2\right), & n = 2k+1\,(k \geq 1). \end{cases} \tag{40}$$

Proof. It is similar to Theorem 14. □

Corollary 3. *Let $|S_1| = n, |S_2| = m$, $n \geq 2, m \geq 1$ and $S_1 \leftrightsquigarrow S_2$ of type $[+, -, +, -, \ldots]$. If m is an odd, then,*

(i) $P^{\leftrightsquigarrow}_{n,m}(x) = \begin{cases} (-x)^{2k+m-2}(x^2-2) & n = 2k\,(k\geq 1) \\ x^{2k+m-1}(x^2-3x+2) & n = 2k+1\,(k\geq 1) \end{cases}$,

(ii) If n is an odd, then, Spec $(A^{\leftrightsquigarrow}_{n,m}) = \begin{pmatrix} 0 & 1 & 2 \\ n+m-2 & 1 & 1 \end{pmatrix}$,

(iii) If n is an even, then Spec $(A^{\leftrightsquigarrow}_{n,m}) = \begin{pmatrix} 0 & -\sqrt{2} & \sqrt{2} \\ n+m-2 & 1 & 1 \end{pmatrix}$,

(iv) $\sum_{x \in E(A_{n,m})} x \neq 0$.

Corollary 4. *Let $|S_1| = n, |S_2| = m$, $n \geq 2, m \geq 1$ and $S_1 \leftrightsquigarrow S_2$ of type $[+, -, +, -, \ldots]$. If m is an even, then,*

(i) if n is an even, then, $P^{\leftrightsquigarrow}_{n,m}(x) = \begin{cases} (-x)^{2k+m-1}(x-1) & n = 2k\,(k \geq 1) \\ -(x^{2k+m})(x-2) & n = 2k+1\,(k \geq 1) \end{cases}$,

(ii) If n is an odd, then, Spec $(A^{\leftrightsquigarrow}_{n,m}) = \begin{pmatrix} 0 & 2 \\ n+m-1 & 1 \end{pmatrix}$,

(iii) If n is an even, then, Spec $(A^{\leftrightsquigarrow}_{n,m}) = \begin{pmatrix} 0 & 1 \\ n+m-1 & 1 \end{pmatrix}$,

(iv) $\sum_{x \in E(A_{n,m})} x \neq 0$.

When we will show that $S_i \leftrightsquigarrow S_j$ of type $[-, +, -, +, \ldots]$, it means that the first map flows from S_j to S_i, the second map flows from S_i to S_j, the third map flows from S_j to S_i, etc., respectively. We will show it in Figure 7, where, n is an odd.

Theorem 16. *Let $|S_1| = 1, |S_2| = n$, $n \geq 3$ and $S_1 \leftrightsquigarrow S_2$ of type $[-, +, -, +, \ldots]$. Then,*

(i) $P^{\leftrightsquigarrow}_{1,n}(x) = \begin{cases} -(x^{2k-1})(x^2 - 3x + 2) & n = 2k \\ x^{2k}(x-1)^2 & n = 2k+1 \end{cases}$,

(ii) if n is an even, then, Spec $(A^{\leftrightsquigarrow}_{1,n}) = \begin{pmatrix} 0 & 1 & 2 \\ n-1 & 1 & 1 \end{pmatrix}$,

(iii) if n is an odd, then, Spec $(A^{\leftrightsquigarrow}_{1,n}) = \begin{pmatrix} 0 & 1 \\ n-1 & 2 \end{pmatrix}$.

Proof

(i) Let $n = 3$ and $n = 4$. It is easy to see that $P^{\leftrightsquigarrow}_{1,3}(x) = x^2(x-1)^2$ and $P^{\leftrightsquigarrow}_{1,4}(x) = -x^3(x^2 - 3x + 2)$. Suppose that $k \geq 3$ is an even and $P^{\leftrightsquigarrow}_{1,k-1}(x) = -x^{2k-3}(x^2 - 3x + 2)$. Then, $A^{\leftrightsquigarrow}_{1,k} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where for all $i \in \{3, 5, 7, \ldots, k\}$, $c_i = [-1, \underbrace{1, \ldots, 1}_{k-\text{times}}]^t$ and where for all $i \in \{4, 6, 8, \ldots, k\}$, $c_i = [1, \underbrace{-1, \ldots, -1}_{k-\text{times}}]^t$. It follows that

$$P^{\leftrightsquigarrow}_{1,k}(x) = \det\left(A^{\leftrightsquigarrow}_{1,k} - I_{(k+1)\times(k+1)}X\right) = \det\left(B^{\leftrightsquigarrow}_{1,k}\right), \tag{41}$$

such that $B^{\leftrightsquigarrow}_{1,k} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B^{\leftrightsquigarrow}_{1,k} = [c'_1, c'_2, \ldots, c'_k, c'_{k+1}]$, where for all $i \in \{4, 6, 8, \ldots, k\}$, $c'_i = [1, \underbrace{-1, \ldots, -1}_{(i-2)-\text{times}}, \underbrace{-1-x}_{(i)^{\text{th}}}, \underbrace{-1, \ldots, -1}_{(k+1-i)-\text{times}}]^t$ and for all $i \in \{3, 5, 7, \ldots, k\}, c'_i = [-1, \underbrace{1, \ldots, 1}_{(i-2)-\text{times}}, \underbrace{1-x}_{(i)^{\text{th}}}, \underbrace{1, \ldots, 1}_{(k+1-i)-\text{times}}]^t$. Now, consider $D^{\leftrightsquigarrow}_{1,k} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & 1 \leq j \leq k-1, \\ c'_k + c'_{k-1}, & j = k, \\ c'_{k+1} - c'_{k-1}, & j = k+1. \end{cases} \tag{42}$$

Thus, $D^{\leftrightsquigarrow}_{1,k} = [c'_1, c'_2, \ldots, c'_{k-1}, c''_k, c''_{k+1}]$, where $c''_k = [\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, -x, -x, 0]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, 1, 1, 0]^t$ and $c''_{k+1} = [\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, x, 0, -x]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, -1, 0, 1]^t$. Based on induction assumption and computations of determinant based on column c''_{k+1} in matrix $D^{\leftrightsquigarrow}_{1,k}$, we have

$$\begin{aligned} P^{\leftrightsquigarrow}_{1,k}(x) &= \det\left(D^{\leftrightsquigarrow}_{1,k} - I_{(k+1)\times(k+1)}X\right) = \left(x^2\right)\left(P^{\leftrightsquigarrow}_{1,k-2}(x)\right) \\ &= -\left(x^{2k-1}\right)\left(x^2 - 3x + 2\right). \end{aligned} \tag{43}$$

In a similar way, if n is an odd, we get that $P^{\leftrightsquigarrow}_{1,n}(x) = x^{n-1}(x-1)^2$.

(ii) and (iii) They are clear by item (i). □

Theorem 17. *Let $|S_1| = n, |S_2| = 1$, $n \geq 3$ and $S_1 \leftrightsquigarrow S_2$ of type $[-,+,-,+,\ldots]$. Then,*

$$P^{\leftrightsquigarrow}_{n,1}(x) = \begin{cases} (-x)^{2k-1}\left(x^2 - 2x + 2\right), & n = 2k, \\ (-x)^{2k}\left(x^2 + x - 2\right), & n = 2k+1. \end{cases} \tag{44}$$

Proof. Let $n = 3, n = 4$. It is easy to see that $P^{\leftrightsquigarrow}_{3,1}(x) = x^2(x^2 + x - 2)$ and $P^{\leftrightsquigarrow}_{4,1}(x) = (-x)^3(x^2 - 2x + 2)$ Suppose that $k \geq 3$ is an even and $P^{\leftrightsquigarrow}_{k-1,1}(x) = (-x)^{2k-3}(x^2 - 2x + 2)$. Then, $A^{\leftrightsquigarrow}_{k,1} = [c_1, c_2, \ldots, c_k, c_{k+1}]$, where for all $i \in \{3, 5, \ldots, k+1\}, c_i = [\underbrace{-1, \ldots, -1}_{k-\text{times}}, 1]^t$ and where for all $i \in \{4, 6, \ldots, k\}, c_i = [\underbrace{1, \ldots, 1}_{k-\text{times}}, -1]^t$. It follows that

$$P^{\leftrightsquigarrow}_{k,1}(x) = \det\left(A^{\leftrightsquigarrow}_{k,1} - I_{(k+1)\times(k+1)}X\right) = \det\left(B^{\leftrightsquigarrow}_{k,1}\right), \tag{45}$$

such that $B^{\leftrightsquigarrow}_{k,1} = [b_{ij}]_{(k+1)\times(k+1)}$, where $B^{\leftrightsquigarrow}_{k,1} = [c'_1, c'_2, \ldots, c'_k, c_{k+1}']$, where for all $i \in \{4, 6, \ldots, k\}$, $c'_i = [\underbrace{1, \ldots, 1}_{(i-1)-\text{times}}, \underbrace{1-x}_{(i)^{\text{th}}}, \underbrace{1, \ldots, 1}_{(k-i)-\text{times}}, -1]^t$ and for all $i \in \{3, 5, \ldots, k-1\}, c'_i = [\underbrace{-1, \ldots, -1}_{(i-1)-\text{times}}, -1-x, \underbrace{-1, \ldots, -1}_{(k-i)-\text{times}}, 1]$ and for $i = k+1, c'_i = [\underbrace{-1, \ldots, -1}_{k-\text{times}}, 1-x]^t$. Now, consider $D^{\leftrightsquigarrow}_{k,1} = (d_{ij})_{(k+1)\times(k+1)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j = 1, \ldots, k-2, k+1, \\ c'_k + c'_{k-1}, & j = k-1, \\ c'_k + c'_{k+1}, & j = k. \end{cases} \tag{46}$$

Thus, $D^{\leftrightsquigarrow}_{k,1} = [c'_1, c'_2, \ldots, c_{k-2}', c_{k-1}'', c''_k, c_{k+1}']$, where $c_{k-1}'' = [\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, -x, -x, 0]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, 1, 1, 0]^t$ and $c''_k = [\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, -x, -x]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, 1, 1]^t$. Based on induction assumption and computations of determinant based on column c''_k in matrix $D^{\leftrightsquigarrow}_{k,1}$, we have

$$\begin{aligned} P^{\leftrightsquigarrow}_{k,1}(x) &= \det\left(D^{\leftrightsquigarrow}_{k,1} - I_{(k+1)\times(k+1)}X\right) = \left(x^2\right)\left(P^{\leftrightsquigarrow}_{k-2,1}(x)\right) \\ &= (-x)^{2k-1}\left(x^2 - 2x + 2\right). \end{aligned} \tag{47}$$

In a similar way, if k is an odd, we get that $P^{\leftrightsquigarrow}_{k,1}(x) = (-x)^{2k}(x^2 + x - 2)$.

(ii) and (iii) They are clear by item (i) and (ii). □

Theorem 18. *Let $|S_1| = n, |S_2| = 2$, $n \geq 3$ and $S_1 \leftrightsquigarrow S_2$ of type $[-,+,-,+,\ldots]$. Then,*

$$P^{\leftrightsquigarrow}_{n,2}(x) = \begin{cases} x^{2k+1}(x-1), & n = 2k, \\ (-x)^{2k+3}, & n = 2k+1. \end{cases} \tag{48}$$

Proof. Let $n = 3$. It is easy to see that $P^{\leftrightsquigarrow}_{3,2}(x) = (-x)^5$ and for $n = 4$, $P^{\leftrightsquigarrow}_{4,2}(x) = x^5(x-1)$. Suppose that $k \geq 3$ is an odd and $P^{\leftrightsquigarrow}_{k-1,2}(x) = (-x)^{2k+1}$. Then, $A^{\leftrightsquigarrow}_{k,2} = [c_1, c_2, \ldots, c_{k+1}, c_{k+2}]$, where for all $i \in \{3, 5, \ldots, k+2\}, c_i = [\underbrace{-1, \ldots, -1}_{k-\text{times}}, 1, 1]^t$ and where for all $i \in \{4, 6, \ldots, k+1\}, c_i = [\underbrace{1, \ldots, 1}_{k-\text{times}}, -1, -1]^t$. It follows that

$$P^{\leftrightsquigarrow}_{k,2}(x) = \det\left(A^{\leftrightsquigarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = \det\left(B^{\leftrightsquigarrow}_{k,2}\right), \tag{49}$$

such that $B^{\leftrightsquigarrow}_{k,2} = [b_{ij}]_{(k+2)\times(k+2)}$, where $B^{\leftrightsquigarrow}_{k,2} = [c'_1, c'_2, \ldots, c'_{k+1}, c'_{k+2}]$, where for all $i \in \{3, 5, \ldots, k\}$, $c'_i = [\underbrace{-1, \ldots, -1}_{(i-1)-\text{times}}, \underbrace{-1-x}_{(i)^{\text{th}}}, \underbrace{-1, \ldots, -1}_{(k-i)-\text{times}}, 1, 1]^t$ and for $i = k+2$, $c'_i = [\underbrace{-1, \ldots, -1}_{k-\text{times}}, 1, 1-x]^t$ and for all $i \in \{4, 8, \ldots, k-1\}$, $c'_i = [\underbrace{1, \ldots, 1}_{(k-2)-\text{times}}, \underbrace{1-x}_{(i)^{\text{th}}}, \underbrace{1, \ldots, 1}_{(k-i)-\text{times}}, -1, -1]^t$ and for $i = k+1$, $c'_i = [\underbrace{1, \ldots, 1}_{k-\text{times}}, -1-x, -1]^t$. Now, consider $D^{\leftrightsquigarrow}_{k,2} = (d_{ij})_{(k+2)\times(k+2)}$, where

$$d_{ij} = \begin{cases} b_{ij}, & j = 1, \ldots, k-2, k+1, k+2, \\ c'_k + c'_{k-1}, & j = k-1, \\ c'_k + c'_{k+1}, & j = k. \end{cases} \tag{50}$$

Thus, $D^{\leftrightsquigarrow}_{k,2} = [c'_1, c'_2, \ldots, c'_{k-2}, c_{k-1}'', c''_k, c'_{k+1}, c'_{k+2}]$, where $c_{k-1}'' = [\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, -x, -x, 0, 0]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-2)-\text{times}}, 1, 1, 0, 0]^t$ and $c''_k = [\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, -x, -x, 0]^t = (-x)[\underbrace{0, \ldots, 0}_{(k-1)-\text{times}}, 1, 1, 0]^t$. Based on induction assumption and computations of determinant based on column c''_k in matrix $D^{\leftrightsquigarrow}_{k,2}$, we have

$$P^{\leftrightsquigarrow}_{k,2}(x) = \det\left(D^{\leftrightsquigarrow}_{k,2} - I_{(k+2)\times(k+2)}X\right) = \left(x^2\right)\left(P^{\leftrightsquigarrow}_{k-2,2}(x)\right) = (-x)^{2k+3}. \tag{51}$$

In a similar way, if k is an even, we get that $P^{\leftrightsquigarrow}_{k,2}(x) = x^{2k+1}(x-1)$.

(ii) and (iii) They are clear by item (i) and (ii). □

Theorem 19. *Let $|S_1| = n, |S_2| = 3$, $n \geq 2$ and $S_1 \leftrightsquigarrow S_2$ of type $[-,+,-,+,\ldots]$. Then,*

$$P^{\leftrightsquigarrow}_{n,3}(x) = \begin{cases} -x^{2k+1}\left(x^2 - 2x + 2\right), & n = 2k, \\ x^{2k+2}\left(x^2 + x - 2\right), & n = 2k+1. \end{cases} \tag{52}$$

Proof. It is similar to Theorem 18. □

Corollary 5. *Let $|S_1| = n, |S_2| = m$, $n \geq 2, m \geq 1$ and $S_1 \leftrightsquigarrow S_2$ of type $[-,+,-,+,\ldots]$. If m is an odd, then,*

(i) $P^{\leftrightsquigarrow}_{n,m}(x)=\begin{cases}-x^{2k+m-2}(x^2-2x+2) & n=2k\,(k\geq 1)\\ x^{2k+m-1}(x^2+x-2) & n=2k+1\,(k\geq 1)\end{cases}$,

(ii) If n is an odd, then, Spec $(A^{\leftrightsquigarrow}_{n,m})=\begin{pmatrix}0 & -2 & 1\\ n+m-2 & 1 & 1\end{pmatrix}$,

(iii) If n is an even, then, Spec $(A^{\leftrightsquigarrow}_{n,m})=\begin{pmatrix}0 & 1+i & 1-i\\ n+m-2 & 1 & 1\end{pmatrix}$,

(iv) $\sum_{x\in E(A_{n,m})}x\neq 0$.

Corollary 6. *Let* $|S_1|=n, |S_2|=m$, $n\geq 2, m\geq 1$ *and* $S_1\leftrightsquigarrow S_2$ *of type* $[-,+,-,+,\ldots]$. *If* m *is an even, then,*

(i) $P^{\leftrightsquigarrow}_{n,m}(x)=\begin{cases}x^{2k+m-1}(x-1) & n=2k\,(k\geq 1)\\ -(x^{2k+m+1}) & n=2k+1\,(k\geq 1)\end{cases}$,

(ii) If n is an odd, then, Spec $(A^{\leftrightsquigarrow}_{n,m})=\begin{pmatrix}0\\ n+m\end{pmatrix}$,

(iii) If n is an even, then, Spec $(A^{\leftrightsquigarrow}_{n,m})=\begin{pmatrix}0 & 1\\ n+m-1 & 1\end{pmatrix}$,

(iv) $\sum_{x\in E(A_{n,m})}x\neq 0$.

4. Conclusions and Future Works

The current paper has introduced a novel concept of superhypergraphs as a generalization of graphs. The advantage of the notation of superhypergraphs is that it considers the relationship between a set of elements separately and as a whole, and this helps to eliminate the defects of graphs and superhypergraphs. The notation of superhypergraphs can be useful tools in modeling the real issues in engineering sciences and other sciences, especially network-related issues. For any given superhypergraph, the lower and upper bound of the number of the set of their superedges is computed and so it is computed and proved the number of all superhypergraphs constructed on any given nonempty set. Polynomial characteristics and eigenvalues of a matrix that represents a superhypergraph can provide useful information about the superhypergraph. The concept of the incidence matrix of superhypergraphs is presented and the characteristic polynomial of the incidence matrix of superhypergraphs and spectrum of superhypergraphs is analyzed and computed. It is shown that the spectrum of superhypergraphs depended on to flows of their maps between supervertices and the spectrum of superhypergraphs varies with the change of direction of flows. We presented and computed the spectrum of superhypergraphs with some types of flows such as one-sided flows, left to the right flows, right to left flows, and two-sided reverse flows. We hope that these results are helpful for further studies in the theory of graphs, hypergraphs, and superhypergraphs. In our future studies, we hope to obtain more results regarding domination sets and domination numbers of superhypergraphs, fuzzy superhypergraphs, and obtain some results in this regard and their applications in the real-world.

Data Availability

The data used to support the findings of this study are included within this article and can be obtained from the corresponding author upon request for more details on the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] C. Berge, *Graphs and Hypergraphs*, Springer, North Holland, Netherlands, 1979.

[2] J. Hou, M. Pelillo, and H. Yuan, "Hypergraph matching via game-theoretic hypergraph clustering," *Pattern Recognition*, vol. 125, Article ID 108526, 2022.

[3] Y. Qian, H. Meng, W. Lu, Z. Liao, Y. Ding, and H. Wu, "Identification of DNA-binding proteins via hypergraph based laplacian support vector machine," *Current Bioinformatics*, vol. 17, no. 1, pp. 108–117, 2022.

[4] D. Tocchi, C. Sys, A. Papola, F. Tinessa, F. Simonelli, and V. Marzano, "Hypergraph-based centrality metrics for maritime container service networks: a worldwide application," *Journal of Transport Geography*, vol. 98, Article ID 103225, 2022.

[5] P. Zhou, X. Wang, L. Du, and X. Li, "Clustering ensemble via structured hypergraph learning," *Information Fusion*, vol. 78, pp. 171–179, 2022.

[6] X. Hao, Ji. Li, Y. Guo, Ta. Jiang, and M. Yu, "Hypergraph neural network for skeleton-based action recognition," *IEEE Transactions on Image Processing*, vol. 30, pp. 2263–2275, 2021.

[7] M. Amin Bahmanian and M. Newman, "Extending factorizations of complete uniform hypergraphs," *Combinatorica*, vol. 38, no. 6, pp. 1309–1335, 2018.

[8] C. Annamalai, "Finding perfect matchings in bipartite hypergraphs," *Combinatorica*, vol. 38, no. 6, pp. 1285–1307, 2018.

[9] P. Frankl, "Resilient hypergraphs with fixed matching number," *Combinatorica*, vol. 38, no. 5, pp. 1079–1094, 2018.

[10] A. Banerjee, "On the spectrum of hypergraphs," *Linear Algebra and Its Applications*, vol. 614, no. 1, pp. 82–110, 2021.

[11] A. Howlader and P. Panigrahi, "On the distance spectrum of minimal cages and associated distance biregular graphs," *Linear Algebra and Its Applications*, vol. 636, no. 1, pp. 115–133, 2022.

[12] P. Renteln, "On the spectrum of the perfect matching derangement graph," *Journal of Algebraic Combinatorics*, 2022.

[13] P. Vrana, "Probabilistic refinement of the asymptotic spectrum of graphs," *Combinatorica*, vol. 41, no. 6, pp. 873–904, 2021.

[14] M. Hamidi and A. Saeid, "Accessible spectrum of graphs," *Applicable Analysis and Discrete Mathematics*, vol. 15, no. 1, pp. 001–026, 2021.

[15] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[16] A. Kaufmann, *Introduction Ci la Thiorie des Sous-Ensemble Flous*, Masson, Paris, France, 1977.

[17] M. Akram and A. Luqman, "Fuzzy hypergraphs and related extensions," *Studies in Fuzziness and Soft Computing*, vol. 390, 2020.

[18] H. S. Nawaz, M. Akram, and J. C. R. Alcantud, "An algorithm to compute the strength of competing interactions in the Bering Sea based on Pythagorean fuzzy hypergraphs," *Neural Computing & Applications*, vol. 34, no. 2, pp. 1099–1121, 2022.
[19] M. Sarwar, M. Akram, and S. Shahzadi, "Bipolar fuzzy soft information applied to hypergraphs," *Soft Computing*, vol. 25, no. 5, pp. 3417–3439, 2021.
[20] F. Smarandache, "Extension of HyperGraph to n-SuperHyperGraph and to plithogenic n-SuperHyperGraph, and extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra," *Neutrosophic Sets and Systems*, vol. 33, pp. 290–296, 2020.