Hindawi
Journal of Mathematics
Volume 2022, Article ID 9193446, 7 pages
https://doi.org/10.1155/2022/9193446

Research Article

On Classification of Semigroup $\mathcal{N}$ by Green's Theorem

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Received 27 February 2022; Accepted 22 April 2022; Published 30 June 2022

Academic Editor: Kenan Yildirim

In previous papers, it has been recently defined a new class of semigroups based on both Rees matrix and completely 0-simple semigroups. For this new structure, it has been introduced with certain basic properties and finiteness conditions. The main goal of the paper is to prove Green's Theorem for $\mathcal{N}$ by indicating the existence of Green's lemma. We also study generalized Green's relation over $\mathcal{N}$ and present some new findings. In particular, we classify our semigroup $\mathcal{N}$ associated with generalized Green's relation by constructing good homomorphism.

1. Introduction and Preliminaries

It is good to keep in mind the definition of the semigroup $\mathcal{N}$ given in [1], in order to grasp the importance of this paper. Since this paper is continuation of our work in [1], let us recall the basic facts regarding the semigroup $\mathcal{N}$.

The notation M_R represents the Rees matrix semigroup $M^0[S^0; I, J; P]$, and the notation M_C represents completely 0-simple semigroup $M^0[G^0; I, J; P']$.

Let us consider the mapping γ: $(M_R \times M_C) * (M_R \times M_C) \longrightarrow (M_R \times M_C)$

$$[(x, y, z), (a, b, c)] * [(k, l, m), (r, t, q)] = \begin{cases} ((x, y p_{zk} l, m), 0), & \text{if } p_{zk} \neq 0 \text{ and } p'_{cr} = 0, \\ (0, (a, b p_{cr}' t, q)), & \text{if } p_{zk} = 0 \text{ and } p'_{cr} \neq 0, \\ ((x, y p_{zk} l, m), (a, b p_{cr}' t, q)), & \text{if } p_{zk} \neq 0 \text{ and } p'_{cr} \neq 0, \\ (0_R, 0_C), & \text{if } p_{zk} = 0 \text{ and } p'_{cr} = 0, \end{cases} \tag{1}$$

for the elements $(x, y, z), (k, l, m) \in M_R$, and $(a, b, c), (r, t, q) \in M_C$. By the operation given in (14), the set $M_R \times M_C$ defines a semigroup $M^0[S^0, G^0; M_R, M_C; P, P']$.

We denote this new semigroup shortly by $\mathcal{N}$. The details of the element of $\mathcal{N}$ are as follows.

$$\mathcal{N} = \{(X_i, X_k)\} = \begin{cases} X_i = 0, & X_k \neq 0\,(1 \leq k \leq m) \text{ or} \\ X_i \neq 0, & X_k = 0\,(1 \leq i \leq n) \text{ or} \\ X_i \neq 0, & X_k \neq 0\,(1 \leq k \leq m) \text{ and } (1 \leq j \leq m) \text{ or} \\ X_i = 0, & X_k = 0. \end{cases} \tag{2}$$

Indeed, every X_i is an element of M_R, and X_k is an element of M_C. Moreover, the semigroup $\mathcal{N}$ is composed of the semigroup S and the group G. Furthermore, we can see the diagram in Figure 1 for $\mathcal{N}$.

Then, we obtained $\mathcal{N}$ which satisfies important homological properties and proved that the spined product of the (C)-inversive semigroup and the idempotent semigroup of $\mathcal{N}$ is isomorphic to the strictly inverse semigroup $\mathcal{N}$ in [1]. Furthermore, we gave some consequences of the results to make a detailed classification over $\mathcal{N}$ in the same reference. Despite the progress, several important issues remain to be addressed, which will be focused on them.

We now discuss a very helpful tool for the study of monoids/semigroups called Green's relations. Fundamental equivalence relations, $\mathscr{R}, \mathscr{L}, \mathscr{H}, \mathscr{D}$-relations, were first introduced and studied by Green in 1951. Then, using these equivalences, Green's lemma and theorem were proved. These relations have played a main role in the development of the semigroup theory. Green's relation is a subject in its own right, with a substantial body of literature, see for instance [2]. Especially, Green's relations may be used to depict the structure of regular semigroups (for example, see Theorem 2.1 in [3]). On the other hand, in [4, 5], the authors studied some Green's relations for semigroup and monoid structures. The Green's lemma and Green's theorem are the natural next steps in these relations. In [6], the author studied an application for Γ-semigroups techniques via the Green's theorem. In the present paper, our aim is to prove Green's lemma and Green's theorem for $\mathcal{N}$.

On the other hand, in [7], the authors introduced generalized Green's relation over a semigroup, namely, the generalized Green's relation. Because of this reason, replacing Green's relations upon a given semigroup is the most efficient method to study the generalized regular semigroups.

Generalized Green's relation, which is another equivalence class, is crucial for the classification of the semigroup class such as the abundant and adequate semigroup. One can regard the abundant semigroup as another kind of generalized regular semigroups. Obviously, an abundant semigroup is not necessarily a regular semigroup. In abundant semigroups, a homomorphic image of an abundant semigroup must not be abundant, and the notions of good homomorphism for abundant semigroups are introduced in [8, 9]. Another of our aim is to focus on the generalizations of Green's relations on a semigroup $\mathcal{N}$ and think of their applications such as classification of semigroups and good homomorphism in this paper.

The paper is structured as follows. We reveal important results related to Green's relations which are Green's lemma and Green's theorem for $\mathcal{N}$ (Proposition 1 and Theorem 1) in Section 2. In Section 3, by considering generalizations of Green's relations on a semigroup $\mathcal{N}$ as mentioned by Green's *relations (Lemma 5 and Theorem 2). Furthermore, we make new classifications for $\mathcal{N}$ associated with the abundant semigroup (Corollaries 1 and 2). Finally, we present other results which build a structure of abundant and adequate semigroups in terms of good homomorphism (Corollary 3) in the same section.

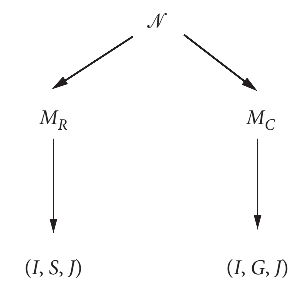

Figure 1: The diagram for the semigroup $\mathcal{N}$.

2. Green's Lemma and Green's Theorem for $\mathcal{N}$

A natural question in the theory of semigroups can be stated as follows: what kind of new material can be obtained for a semigroup S by considering the results on Green's relation of the same semigroup S? It is well known that certain equivalence relations $\mathscr{L}$ and $\mathscr{R}$ for an arbitrary semigroup S were first introduced by Green [10], in which those are defined in a very influential method to make a classification for a new semigroup. As a result of this fact, in [1], we studied $\mathscr{L}$ and $\mathscr{R}$-Green's relations for $\mathcal{N}$. On the other hand, we will prove Green's lemma and Green's theorem by developing the results which are obtained in [1].

Some basic facts about the behavior of these relations are summarized in the following lemma.

Lemma 1 (see [1]). *For* $[(\alpha,\beta,\gamma), (x,y,z)], [(\alpha',\beta',\gamma'), (x',y',z')] \in \mathcal{N}$

(i) $\gamma = \gamma', z = z', \beta\mathscr{L}\beta'$ *and* $y\mathscr{L}y' \Leftrightarrow [(\alpha,\beta,\gamma), (x,y,z)]\mathscr{L}\ [(\alpha',\beta',\gamma'), (x',y',z')]$.

(ii) $\alpha = \alpha', x = x', \beta\mathscr{R}\beta'$ *and* $y\mathscr{R}y' \Leftrightarrow [(\alpha,\beta,\gamma), (x,y,z)]\ \mathscr{R}[(\alpha',\beta',\gamma'), (x',y',z')]$.

For $N_1, N_2 \in \mathcal{N}$, we denote by $(N_1)_{\mathscr{L}}$ and $(N_2)_{\mathscr{L}}$ the $\mathscr{L}$-classes of $\mathcal{N}$ containing the elements N_1 and N_2, respectively. Furthermore, for $s_1, s_2 \in \mathcal{N}$, let us assume that $N_1 * s_1 = N_2$ and $N_2 * s_2 = N_1$.

Lemma 2. *Let* $x_1 \in (N_1)_{\mathscr{L}}$. *Then,* $(x_1 * s_1, N_1 * s_1) \in \mathscr{L}$ *and* $x_1 * s_1 \in (N_2)_{\mathscr{L}}$.

Proof. Let us choose an element $N_1 = [(a,b,c), (d,e,f)]$ such that $x_1 \in (N_1)_{\mathscr{L}}$. This clearly implies that $(x_1, N_1) \in \mathscr{L} \Rightarrow (x_1 * s_1, N_1 * s_1) \in \mathscr{L} \Rightarrow (x_1 * s_1, N_2) \in \mathscr{L} \Rightarrow x_1 * s_1 \in (N_2)_{\mathscr{L}}$.

Nevertheless, we can present this short proof in the following alternative way:

For $x_1 = [(a,b^*,c), (d,e^*,f)]$, we have $b^*\mathscr{L}b$ and $e^*\mathscr{L}e$ by Lemma 1 via by the definition of Green's relation [2], and we certainly have $\alpha b^* = b, \beta b = b^*$ and $\alpha' e^* = b, \beta' e = e^*$ such that $\alpha, \alpha', \beta, \beta' \in \mathcal{N}$. Thus,

$$x_1 * s_1 = [(a,b^*,c),(d,e^*,f)] * [(a',b',c),(d',e',f)] = \left[(a,b^* p_{ca'}b',c),\left(d,e^* p_{fd'}e',f\right)\right]. \quad (3)$$

Since $b^* p_{ca'}b' = \beta b p_{ca'}b' = \beta y \mathscr{L} y$ and $e^* p_{fd'}e' = \beta' e p_{fd'}e' = \beta l \mathscr{L} l$, we finally have $x_1 * s_1 \in (N_2)_{\mathscr{L}}$. □

Similarly as in Lemma 2, by choosing an element $N_2 = [(a,y,c),(d,l,f)]$ such that $x_2 \in (N_2)_{\mathscr{L}}$, we also obtain the following preresult.

Lemma 3. *Let $x_2 \in (N_2)_{\mathscr{L}}$. Then, $(x_2 * s_2, N_2 * s_2) \in \mathscr{L}$ and $x_2 * s_2 \in (N_1)_{\mathscr{L}}$.*

Proposition 1. *(Green lemma for $\mathscr{N}$) Let $N_1 = [(a,b,c),(d,e,f)]$ and $N_2 = [(a,y,c),(d,l,f)]$ be $\mathscr{R}$-equivalent elements of the semigroup $\mathscr{N}$, and let $s_1, s_2 \in \mathscr{N}$ such that $N_1 * s_1 = N_2, N_2 * s_2 = N_1$. Then, the mappings $x_1 \mapsto x_1 s_1 (x_1 \in (N_1)_{\mathscr{L}})$ and $x_2 \mapsto x_2 s_2 (x_2 \in (N_2)_{\mathscr{L}})$ are mutually inverse and one-to-one mappings of $(N_1)_{\mathscr{L}}$ upon $(N_2)_{\mathscr{L}}$ and $(N_2)_{\mathscr{L}}$ upon $(N_1)_{\mathscr{L}}$, respectively.*

Proof. By Lemmas 2 and 3, since $x_1 * s_1 \in (N_2)_{\mathscr{L}}$ ve $x_2 * s_2 \in (N_1)_{\mathscr{L}}$, one can obtain the following well-defined mappings:

$$\Gamma\colon (N_1)_{\mathscr{L}} \longrightarrow (N_2)_{\mathscr{L}}, \Gamma'\colon (N_2)_{\mathscr{L}} \longrightarrow (N_1)_{\mathscr{L}}, \quad x_1 \mapsto x_1 * s_1, \quad x_2 \mapsto x_2 * s_2. \quad (4)$$

Now, let us show these two mappings are mutually inverse. To do that let us take the elements $D \in (N_1)_{\mathscr{L}}$ and $D' \in (N_2)_L$. In fact $D \in (N_1)_{\mathscr{L}}$ implies $D = N_1$, or there exists $\alpha_3 \in \mathscr{N}$ such that $\alpha_3 * N_1 = D$, and similarly, $D' \in (N_2)_L$ implies $D' = N_2$, or there exists $t^* \in \mathscr{N}$ such that $t^* * N_2 = D'$. By considering the compositions

$$\Gamma' \circ \Gamma\colon (N_1)_{\mathscr{L}} \longrightarrow (N_2)_{\mathscr{L}}, \Gamma \circ \Gamma'\colon (N_2)_{\mathscr{L}} \longrightarrow (N_1)_{\mathscr{L}}, \quad x_1 \mapsto x_1 * s_1 * s_2, \quad x_2 \mapsto x_2 * s_2 * s_1, \quad (5)$$

of mappings Γ and Γ',

(i) for the case $D = N_1$, we have $(\Gamma' \circ \Gamma)(D) = (\Gamma' \circ \Gamma)(N_1) = \Gamma'(\Gamma(N_1)) = \Gamma'\ (N_1 \star s_1) = \Gamma'(N_2) = N_2 * s_2 = N_1 = D$, and

(ii) for the other case, we obtain $(\Gamma' \circ \Gamma)(D) = (\Gamma' \circ \Gamma)(\alpha_3 * N_1) = \Gamma'(\Gamma(\alpha_3 * N_1)) = \Gamma'(\alpha_3 * N_1 \star s_1) = \Gamma'\ (\alpha_3 * N_2) = \alpha_3 * N_2 * s_2 = \alpha_3 * N_1 = D$.

Therefore $\Gamma' \circ \Gamma$ is the identity mapping on $(N_1)_L$. In fact, a quite similar process can be applied on $\Gamma \circ \Gamma'$ to get the identity mapping. Hence, Γ and Γ' are mutually inverse mappings (in other words, $\mathscr{R}$-class preserving). Hence, Γ is one-to-one mapping of $(N_1)_L$ onto $(N_2)_L$, and Γ' is $(1-1)$ mapping of $(N_2)_L$ onto $(N_1)_L$. □

Theorem 1. *(Green theorem for $\mathscr{N}$) Let N_1 and N_3 be $\mathscr{R} \circ \mathscr{L}$ equivalent elements of a semigroup $\mathscr{N}$. Then, there exists $N_2 \in \mathscr{N}$ such that $N_1 \mathscr{R} N_2$ and $N_2 \mathscr{L} N_3$, and hence, $N_1 s_1 = N_2$, $N_2 s_2 = N_1$, $t N_2 = N_3$, $t' N_3 = N_2$ for some $s, s', \alpha_1, \alpha_2 \in \mathscr{N}$. The mappings*

$$\delta\colon (N_1)_{\mathscr{R}} \cap (N_1)_{\mathscr{L}} \longrightarrow (N_3)_{\mathscr{R}} \cap (N_3)_{\mathscr{L}} \delta'\colon (N_3)_{\mathscr{R}} \cap (N_3)_{\mathscr{L}} \longrightarrow (N_1)_{\mathscr{R}} \cap (N_1)_{\mathscr{L}}, \quad x_1 \mapsto \alpha_1 * x_1 * s_1 \quad x_1 \mapsto \alpha_2 * x_1 * s_2, \quad (6)$$

are mutually inverse, one-to-one mappings of $(N_1)_{\mathscr{R}} \cap (N_1)_{\mathscr{L}}$ and $(N_3)_{\mathscr{R}} \cap (N_3)_{\mathscr{L}}$ upon each other.

Proof. For an element $x_1 \in (N_1)_{\mathscr{R}} \cap (N_1)_{\mathscr{L}}$, clearly, we have $x_1 \in (N_1)_{\mathscr{R}}$ and $x_1 \in (N_1)_{\mathscr{L}}$. Therefore,

$$\begin{aligned} &x_1 \in (N_1)_{\mathscr{R}} \Rightarrow (x_1, N_1) \in \mathscr{R} \Rightarrow N_1 * z = x_1 \text{ and } x_1 * z' = N_1, \quad \text{for some } z, z' \in \mathscr{N}, \\ &x_1 \in (N_1)_{\mathscr{L}} \Rightarrow (x_1, N_1) \in \mathscr{L} \Rightarrow w * N_1 = x_1 \text{ and } w' * x_1 = N_1, \quad \text{for some } v, v' \in \mathscr{N}. \end{aligned} \quad (7)$$

After that we can indicate the element N_3 as

$$\begin{aligned} N_3 &= \alpha_1 * N_2 = \alpha_1 * (N_1 * s_1) = \alpha_1 * (x_1 * z') * s_1 = \alpha_1 * (v * N_1) * z' * s_1 = \\ &\alpha_1 * v * (N_2 * s_2) * z' * s_1 = \alpha_1 * v * (N_1 * s_1) * s_2 * z' * s_1 = \\ &\alpha_1 * (v * N_1) * s_1 * s_2 * z' * s_1 = \alpha_1 * x_1 * s_1 * (s_2 * z' * s_1), \end{aligned} \quad (8)$$

such that

$$\alpha_1 * x_1 * s_1 = \alpha_1 * (N_1 * z) * s_1 = \alpha_1 * (N_2 * s_2) * z * s_1 \\ = (\alpha_1 * N_2) * (s_2 * z * s_1) = N_3 * (s_2 * z * s_1). \quad (9)$$

Since $s_2 * z' * s_1$ and $s_2 * z * s_1$ in elements $\mathcal{N}$, it is satisfied that $N_3 = \alpha_1 * x_1 * s_1 * (s_2 * z' * s_1)$ and $\alpha_1 * x_1 * s_1 = N_3 * (s_2 * z * s_1)$, and we have $(\alpha_1 * x_1 * s_1, N_3) \in \mathcal{R}$, so $\alpha_1 * x_1 * s_1 \in (N_3)_{\mathcal{R}}$.

We can also indicate the element N_3 as

$$N_3 = \alpha_1 * N_2 = \alpha_1 * (N_1 * s_1) = \alpha_1 * (v' * x_1) * s_1 = \alpha_1 * v' * (N_1 * z) * s_1 \\ = \alpha_1 * v' * (N_2 * s_2) * z * s_1 = \alpha_1 * v' * (\alpha_2 * N_3) * s_2 * z * s_1 \\ = \alpha_1 * v' * \alpha_2 * (\alpha_1 * N_2) * s_2 * z * s_1 = \alpha_1 * v' * \alpha_2 * \alpha_1 * (N_2 * s_2) * z * s_1 \\ = \alpha_1 * v' * \alpha_2 * \alpha_1 * N_1 * z * s_1 = \alpha_1 * v' * \alpha_2 * \alpha_1 * (N_1 * z) * s_1 = (\alpha_1 * v' * \alpha_2) * (\alpha_1 * x_1 * s_1), \quad (10)$$

such that

$$\alpha_1 * x_1 * s_1 = \alpha_1 * (v * N_1) * s_1 = \alpha_1 * v * (N_1 * s_1) = \\ \alpha_1 * v * N_2 = \alpha_1 * v * (\alpha_2 * N_3) = (\alpha_1 * v * \alpha_2) * N_3. \quad (11)$$

Since $\alpha_1 * v' * \alpha_2$ and $\alpha_1 * v * \alpha_2$ in elements $\mathcal{N}$, it is satisfied that $(\alpha_1 * v' * \alpha_2) * (\alpha_1 * x_1 * s_1) = N_3$ and $(\alpha_1 * v * \alpha_2) * N_3 = \alpha_1 * x_1 * s_1$, and we have $(\alpha_1 * x_1 * s_1, N_3) \in \mathcal{L}$, so $\alpha_1 * x_1 * s_1 \in (N_3)_{\mathcal{L}}$. Hence, we obtain $\alpha_1 * x_1 * s_1 \in (N_3)_{\mathcal{R}} \cup (N_3)_{\mathcal{L}}$, and the whole process mentioned above implies that the mapping δ is well defined.

By considering an arbitrary element $x_1 \in (N_3)_{\mathcal{R}} \cap (N_3)_{\mathcal{L}}$ and applying a similar approach as above, one can also obtain the mapping δ' which is well defined.

Now, suppose that D is an element of $(N_1)_{\mathcal{R}} \cap (N_1)_{\mathcal{L}}$ which gives $D \in (N_1)_{\mathcal{R}}$ and $D \in (N_1)_{\mathcal{L}}$.

$$D \in (N_1)_{\mathcal{R}} \Rightarrow (D, N_1) \in \mathcal{R} \Rightarrow N_1 * u = D \text{ and } D * u' = N_1, \quad \text{for some } u, u' \in \mathcal{N}, \\ D \in (N_1)_{\mathcal{L}} \Rightarrow (D, N_1) \in \mathcal{L} \Rightarrow v * N_1 = D \text{ and } v' * D = N_1, \quad \text{for some } v, v' \in \mathcal{N}. \quad (12)$$

Then, we have

$$(\delta' \circ \delta)(D) = \delta'(\delta(D)) = \delta'(\alpha_1 * D * s_1) = \alpha_2 * (\alpha_1 * D * s_1) * s_2 \\ = \alpha_2 * \alpha_1 * (v * N_1) * s_1 * s_2 = \alpha_2 * \alpha_1 * v * (N_1 * s_1) * s_2 \\ = \alpha_2 * \alpha_1 * v * (N_2 * s_2) = \alpha_2 * \alpha_1 * (v * N_1) = \alpha_2 * \alpha_1 * D \\ = \alpha_2 * \alpha_1 * (N_1 * u) = \alpha_2 * \alpha_1 * (N_2 * s_2) * u \\ = \alpha_2 * (\alpha_1 * N_2) * s_2 * u = \alpha_2 * N_3 * s_2 * u \\ = (\alpha_2 * N_3) * s_2 * u = N_2 * s_2 * u = (N_2 * s_2) * u = N_1 * u = D. \quad (13)$$

With a similar approach as above, one can further get $(\delta \circ \delta')(D) = D$ by considering an element $D \in (N_3)_{\mathcal{R}} \cup (N_3)_{\mathcal{L}}$.

The mapping δ is one-to-one mapping of $(N_1)_{\mathcal{R}} \cap (N_1)_{\mathcal{L}}$ onto $(N_3)_R \cap (N_3)_L$ and the mapping δ' is a one-to-one mapping of $(N_3)_R \cap (N_3)_L$ onto $(N_1)_{\mathcal{R}} \cap (N_1)_{\mathcal{L}}$ so δ' and δ are mutually inverse.

The proof is now complete. □

3. Generalized Green's Relation for $\mathcal{N}$

Let us consider the semigroup S. $E(S)$ is the set of all idempotents. The element $e \in S$ is called a regular element if there exists an element $x \in S$ such that $exe = e$ and $xex = x$. S is said to be the regular semigroup if each element of S is regular. Green's relation may be used to describe the structure of regular semigroups [3]. In fact, the result related to regularity over $\mathcal{N}$ is given in [1]. However, in here, we will focus on the generalizations of Green's relations on a semigroup $\mathcal{N}$ and study their applications such as adequate and abundant semigroups. Obviously, these types of semigroups are natural generalizations of regular semigroups, and so, it would be appropriate to study $\mathcal{N}$ being abundant and adequate semigroups. We note that the index sets (used to expose the semigroup $\mathcal{N}$) are composed of one element in this section.

In [11, 12], Fountain observed that Green's $*$ relation (i.e., generalized Green's relation) may be studied to abundant semigroups, namely, superabundant semigroups. Many papers have been dedicated to generalizations and improvements of Green's $*$ relation in various contexts (see, for instance, [7, 13]).

In the following context, we will remind some fundamentals which are related to the terminology generalized Green's relations over a semigroup.

We consider the semigroup S to be a semigroup, and $\alpha, \beta \in S$. By [12], it is known that

$$\alpha\mathcal{L}^*\beta \Leftrightarrow \left[\left(u, v \in S^1\right)\alpha u = \alpha v \Leftrightarrow \beta u = \beta v\right], \tag{14}$$

$$\alpha\mathcal{R}^*\beta \Leftrightarrow \left[\left(u, v \in S^1\right)u\alpha = v\alpha \Leftrightarrow u\beta = v\beta\right]. \tag{15}$$

S is called the abundant semigroup if each $\mathcal{L}^*$-class and $\mathcal{R}^*$-class include an idempotent element. Moreover, if each $\mathcal{H}^*$-class ($\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$) of S includes an idempotent element, then S is called superabundant. If S is a superabundant semigroup and $E(S)$ (idempotent semigroup of S) forms a subsemigroup of S, then S is called a cyber-group.

On the other hand, in [14], Fountain introduced that the most important subclasses of abundant semigroups are perhaps the classes of adequate semigroups. An abundant semigroup whose idempotents are commutative is considered as adequate semigroup. Therefore, an adequate semigroup in the class of abundant semigroups can be regarded as a generalization of the inverse semigroup in the class of regular semigroups (cf. [15]).

If for any $x, y \in S$, $x\mathcal{L}^*y$ implies $\theta(x)\mathcal{L}^*\theta(y)$ and $x\mathcal{R}^*y$ implies $\theta(x)\mathcal{R}^*\theta(y)$, then a semigroup homomorphism $\theta\colon S \longrightarrow T$ is called good [9].

Definition 1 (see [2]). Let S be a semigroup. An element x in S is called left cancellative if $xy = xz$ implies $y = z$ for all y and z in S and right cancellative if $yx = zx$ implies $y = z$ for all y and z in S. If every element in S is left cancellative, then S is called a left cancellative semigroup. Similarly, S is called a right cancellative semigroup if every element in S is right cancellative.

Now, we will consider $\mathcal{L}^*$ and $\mathcal{R}^*$ equivalences for the semigroup $\mathcal{N}$ and so with the operation $*$ defined in (14). Before, let us recall the following lemma, which was proved in ([1], Lemma 3).

Unless stated otherwise, in this whole section, S will denote the semigroup placed in the Rees matrix semigroup M_R which was constructed for $\mathcal{N}$ as presented in the introduction of this paper.

Lemma 4 (see [1]). *The element* $[(x, y, z), (k, l, m)] \in \mathcal{N}$ *is an idempotent* $\Leftrightarrow S \cup \{0\}$, *which is a rectangular band, and* $p'_{mk} = l^{-1}$.

Lemma 5. *For the semigroup S,*

(i) if it is a right cancellative semigroup, then every element of $E(\mathcal{N})$ *is* $\mathcal{R}^*$ *equivalence, and*

(ii) if it is a left cancellative semigroup, then every element of $E(\mathcal{N})$ *is* $\mathcal{L}^*$ *equivalence.*

Proof. The proof will be given for just (i) since the $\mathcal{L}^*$ equivalence case can be obtained quite similarly as $\mathcal{R}^*$ equivalence.

Since index sets are composed of one element, the structure of two arbitrary elements in $E(\mathcal{N})$ are the form of $[(a, b, c), (a, d, c)]$ and $[(a, s, c), (a, g, c)]$ (see [1], Lemma 3). So, we have

$$\begin{aligned}[(a, b', c), (a, d', c)] \star [(a, b, c), (a, d, c)] &= [(a, b'', c), (a, d'', c)] * [(a, b, c), (a, d, c)] \\ &\Rightarrow [(a, b' p_{ca} b, c), (a, d' p'_{ca} d, c)] = [(a, b'' p_{ca} b, c), (a, d'' p'_{ca} d, c)].\end{aligned} \tag{16}$$

Since $[(a, b, c), (a, d, c)]$ is in $E(\mathcal{N})$, by Lemma 4, we will have

$$\begin{aligned} p'_{ca} &= d^{-1}, \\ d' p'_{ca} d = d'' p'_{ca} d &\Rightarrow d' = d''. \end{aligned} \tag{17}$$

In addition, by the assumption, since S is a right cancellative semigroup, we have $b' p_{ca} s = b'' p_{ca} s$, which implies $b' = b''$. So, by taking into account (15), we can write

$$\begin{aligned}[(a, b', c), (a, d', c)] * [(a, s, c), (a, g, c)] &= [(a, b'', c), (a, d'', c)] * [(a, s, c), (a, g, c)] \\ &\Rightarrow [(a, b' p_{ca} s, c), (a, d' p''_{ca} g, c)] = [(a, b'' p_{ca} s, c), (a, d'' p''_{ca} g, c)].\end{aligned} \tag{18}$$

All of the preceding progress results in $E(\mathcal{N})$ as $\mathcal{R}^*$ equivalence, as required.

One of the main theorems of this paper is the following. □

Theorem 2. *Let S be a cancellative semigroup. Then, the set $E(\mathcal{N})$ is not only an abundant semigroup but also a superabundant semigroup.*

Proof. Consider any two idempotent elements $\overline{x} = [(a, b, c), (a, d, c)]$ and $\overline{y} = [(a, s, c), (a, g, c)]$ in $E(\mathcal{N})$. By Lemma 4, we have $p_{ca}' = d^{-1}$ and $p_{ca}'' = g^{-1}$. In addition, since $\overline{x}$ and $\overline{y}$ are idempotents, $\mathcal{L}^*$, $\mathcal{R}^*$ and $\mathcal{H}^*$-classes consist of idempotent elements. According to Lemma 5 and the definition of an abundant semigroup, $E(\mathcal{N})$ is the abundant semigroup. Furthermore, according to the definition of the superabundant semigroup, $E(\mathcal{N})$ is the superabundant semigroup since $\mathcal{H}^*$-classes consist of idempotent elements.

The following consequences are immediate. □

Corollary 1. *If $\mathcal{N}$ is commutative, then $E(\mathcal{N})$ is an abundant semigroup as well as an adequate semigroup.*

Proof. Assume that $\mathcal{N}$ is commutative. Thus, the elements of $E(\mathcal{N})$ are commutative which gives the meaning of an adequate semigroup since an abundant semigroup whose idempotents are commutative is actually an adequate semigroup. □

Corollary 2. *For any $\mathcal{N}$, every $E(\mathcal{N})$ is a superabundant semigroup as well as a cyber-group.*

Proof. By Theorem 2, $E(\mathcal{N})$ is a superabundant semigroup of $\mathcal{N}$. Furthermore, since $E(\mathcal{N})$ is a subsemigroup of $\mathcal{N}$, we reach that it is also a cyber-group.

As we mentioned in Corollary 1, it is well known that every commutative abundant semigroup $\mathcal{N}$ is an adequate semigroup for $\mathcal{N}$. By utilizing this result, we show that the homomorphism of abundant semigroups for $\mathcal{N}$ is a good kind of homomorphism. Now, we let $\mathcal{N}^a$ and $\mathcal{N}^A$ denote the abundant semigroup and commutative abundant semigroup (adequate) for $\mathcal{N}$, respectively.

Our final result in this paper is as follows. □

Corollary 3. *If $f\colon \mathcal{N}^A \longrightarrow \mathcal{N}^a$ is a homomorphism, then f is a good homomorphism.*

Proof. Let $f\colon \mathcal{N}^A \longrightarrow \mathcal{N}^a$ be a homomorphism, then we have $f(A), f(B) \in \mathcal{N}^a$ for $A, B \in \mathcal{N}^A$. Since it is known that $A, B \in \mathcal{N}^A$ by using the definition of adequate semigroup, we have $A\mathcal{L}^*B$ and$A\mathcal{R}^*B$. Furthermore, since $f(A), f(B) \in \mathcal{N}^a$, we have $f(A)\mathcal{L}^*f(B)$ and $f(A)\mathcal{R}^*f(B)$, and so, f is a kind of good homomorphism. □

4. Summary and Conclusions

Green's relations are one of the interested areas in the semigroup theory. There are different types of it also, which are used for Green's theorem. In this paper, we prove Green's theorem over $\mathcal{N}$ by using Green's relations of the semigroup $\mathcal{N}$. Furthermore, we present the classification of the semigroup $\mathcal{N}$ considering generalized Green's relation. Finally, we obtain a new result about good homomorphism.

As future work, we are going to study some other properties and classification of the special semigroup $\mathcal{N}$ over semigroups.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Ministry of Education and Deanship of Scientific Research (DSR), King Abdulaziz University (KAU) (grant number IFPFP-272-22).

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