Hindawi
Journal of Mathematics
Volume 2022, Article ID 9244325, 6 pages
https://doi.org/10.1155/2022/9244325

Research Article

Equal-Square Graphs Associated with Finite Groups

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Received 26 October 2021; Accepted 20 January 2022; Published 24 February 2022

Academic Editor: Gohar Ali

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The graphical representation of finite groups is studied in this paper. For each finite group, a simple graph is associated for which the vertex set contains elements of group such that two distinct vertices x and y are adjacent iff $x^2 = y^2$. We call this graph an equal-square graph of the finite group G, symbolized by $ES(G)$. Some interesting properties of $ES(G)$ are studied. Moreover, examples of equal-square graphs of finite cyclic groups, groups of plane symmetries of regular polygons, group of units $U(n)$, and the finite abelian groups are constructed.

1. Introduction

Graphs are studied very extensively by taking into account some characteristics and properties, which are used to prepare set of vertices and constructing edges. Almost all fields of sciences and social sciences are studied in the prospect of theory of graphs. Even the abstract areas of mathematics are linked with graphs. Discrete mathematics, combinatorics, and number theory play a very significant role in this study; also, concepts of abstract algebraic structures are studied with the help of graphs.

Visualizing groups using graphs is a rapidly growing trend in algebraic graph theory [1–3]. The automorphism group of a graph and the Cayley graph of a group motivates to study the interplay between graphs and groups, see [4–8]. There exists numerous ways to associate a graph with certain groups by taking subgroups or elements as vertices, and two vertices are adjacent iff they satisfy a certain relation. Such graphs are characterized through the set of vertices and the adjacency relation between the vertices. In recent years, graph theory and its applications associated to algebraic structures are studied extensively. To contribute in such a commendable area, we introduce here a new type of combination. We construct a graph whose vertex set is a finite group G and has two distinct vertices a and b adjacent iff $a^2 = b^2$. We call this graph an equal-square graph of G and will be represented by $ES(G)$.

A large amount of literature is devoted to study the graphs associated to finite groups, for instance, commuting graphs [9–15], noncommuting graphs [16, 17], intersection graphs [18], prime graphs [19–21], conjugacy class graphs [22], power graphs [23–25], inverse graphs [26], quadratic residues graphs [27], order divisor graphs [28], and square graphs [29].

For reader's facilitation, some basic terminologies are recalled here from [30–32]. The dihedral group, symbolized by D_n, is the group of plane symmetries of an n-sided regular polygon $(n \geq 3)$. This group has order $2n$ and is denoted by $D_n = \langle u, v | u^n = v^2 = (uv)^2 = e \rangle$. The group of units of the ring $\mathbb{Z}_n$ is symbolized by $U(n)$, i.e., $U(n) = \{\overline{y} \in \mathbb{Z}_n | (y, n) = 1\}$. For some fixed prime p, a group G is called a p-group if, for each $x \in G$, we have $|x| = p^k$ for some $k \in \mathbb{Z}^+$. For some fixed prime p, an abelian p-group of exponent p is called elementary abelian p-group. A finite elementary abelian p-group is isomorphic to $\mathbb{Z}_p^k = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$ (direct sum of k-copies), for some $k \in \mathbb{Z}^+$. Elementary abelian 2-groups are sometimes called Boolean groups, cf. [33].

A graph G that contains only vertices without any edge is known as an empty graph. Two vertices having an edge or edges between them are termed as adjacent vertices. An undirected graph G that does not contain parallel edges, loops, or multiple edges is called a simple graph. A simple graph of order n in which every two vertices are adjacent to each other is called the complete graph, symbolized by K_n. A graph in which no edges cross each other is called a planar graph. If we can partition the vertex set of a graph G into two nonempty disjoint subsets such that each edge has one vertex in each partition, then the graph G is called a bipartite graph. The disjoint union of graphs G_1 and G_2 is the graph $G_1 + G_2$ such that $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. The graph $kG = G + G + \cdots + G$ denotes the disjoint union of k copies of graph G.

Our goal in this work is to associate a finite group with new graph representation. Corresponding to each finite group G, we will denote by $ES(G)$, the equal-square graph of G, whose vertex set consists of all the elements of G such that two distinct vertices a and b are adjacent iff $a^2 = b^2$ (Definition 1). Some examples in this regard are $ES(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = K_4$ (Example 1) and $ES(D_6) = K_8 + 2K_2$ (Example 3). In Section 2, we describe some basic properties of $ES(G)$ and investigate that when $ES(G)$ is connected or complete. We obtain the following results in this section: $|G|$ is odd iff $ES(G)$ is empty (Theorem 1). $ES(G)$ is connected iff $ES(G)$ is complete iff G is elementary abelian 2-group (Theorem 2). We conclude that, for every finite group G, $ES(G)$ is either a complete graph or it is a disjoint union of complete graphs (Remark 2). In Section 3, we completely describe the equal-square graph of cyclic groups. We obtain the following results in this section: $ES(G) = nK_2$, for every cyclic group G of order $2n$ (Theorem 3). However, in general, the converse is not true (Remark 3). For every prime $p \neq 2$, $ES(\mathbb{F}_{p^n}^*) = (p^n - 1)/2K_2$ (Corollary 1). In Section 4 we describe the equal-square graph of the dihedral group. We prove that $ES(D_n) = K_{n+1} + (n-1)K_1$ if n is odd and $ES(D_n) = K_{n+2} + ((n-2)/2)K_2$ if n is even (Theorem 4). In Section 5, we characterize the equal-square graph of the group of units $U(n)$ (Theorems 5–7).

In Section 6, we completely describe the equal-square graph of the finite abelian groups. Mainly, we obtain the following results in this section. If $G_i = \langle a_i \rangle$ are cyclic groups of order n_i, for each $1 \leq i \leq k$, then $ES(\oplus_{i=1}^{k} G_i) = (n_1 n_2 \cdots n_k / 2^r) K_{2^r}$, where r denotes the number of even integers in the set $\{n_1, n_2, \ldots, n_k\}$ (Theorem 9). If $G \cong \mathbb{Z}_{2^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{\beta_t}} \oplus \mathbb{Z}_{p_1}\beta_{11} \oplus \cdots \oplus \mathbb{Z}_{p_1}\beta_{1t_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}\beta_{k1} \oplus \cdots \oplus \mathbb{Z}_{p_k}\beta_{kt_k}$, where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_t \geq 1$ with $\beta_1 + \beta_2 + \cdots + \beta_t = \alpha$ and $\beta_{i1} \geq \beta_{i2} \geq \cdots \geq \beta_{it_i} \geq 1$ with $\beta_{i1} + \beta_{i2} + \cdots + \beta_{it_i} = \alpha_i$, for each i; $1 \leq i \leq k$, then $ES(G) = (|G|/2^t) K_{2^t}$ (Theorem 10). If G is an abelian group of finite order $|G| = 2^\alpha \lambda$, provided λ is odd and $\alpha \geq 1$, then we will have $ES(G) = (|G|/2^t) K_{2^t}$ for each partition $\alpha = d_1 + d_2 + \cdots + d_t$ of the integer α of length t (Corollary 2). If G is an abelian group of finite order $|G| = 2^\alpha \lambda$, provided λ is odd and $\alpha \geq 1$, then G can have α distinct (nonisomorphic) equal-square graphs (Corollary 3). All possible equal-square graphs of an abelian group of order 36000 are described (Example 6 and Table 1). The equal-square graphs of small order groups are described at the end (Example 7 and Table 2).

Throughout this paper, all the groups will be finite. Moreover, all the graphs will be finite and simple. Any unexplained material will be standard as in [30–32].

2. Properties of $ES(G)$

This section presents some basic properties of $ES(G)$. We determine that the groups with odd order have empty equal-square graphs. Furthermore, we investigate that when $ES(G)$ is connected or complete. We start by introducing the equal-square graph $ES(G)$ and some of its examples.

Definition 1. Let G be a finite group with group operation written multiplicatively. A graph will be called an equal-square graph if its vertex set consists of entire elements of G and has two distinct The equal-square graph $ES(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ of the Klein group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-square graph of the group G.

Example 1. The equal-square graph $ES(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ of the Klein – 4 group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is presented in Figure 1.

Example 2. The dihedral group $D_6 = \langle a, b | a^6 = b^2 = (ab)^2 = e \rangle$ of order 12 is obtained by the symmetries of a regular hexagon in the plane. The equal-square graph $ES(D_6)$ of D_6 is as in Figure 2.

Example 3. The dihedral group $D_4 = \langle a, b | a^4 = b^2 = (ab)^2 = e \rangle$ (group of symmetries of a square in plane) and the group of quaternions Q_8 have the same equal-square graph, as shown in Figure 3.

Theorem 1. *A group G has odd order iff $ES(G)$ is empty.*

Proof. Suppose G has odd order, and let $x^2 = y^2$, for some $x, y \in G$. Then, $|x| = m_1$ and $|y| = m_2$, for some odd positive integers m_1 and m_2. We can assume that $2 \leq m_1 \leq m_2$. Now, $x^2 = y^2$ implies that $x^{2m_2} = y^{2m_2} = e$, and hence, $m_1 | 2m_2$. Since m_2 is odd, so $m_1 | m_2$. Similarly, $m_2 | m_1$, and hence, $m_1 = m_2$. We can write $m_1 = m_2 = 2k + 1$ for some positive integer k. However, then, $x = xe = xx^{2k+1} = (x^2)^{k+1} = (y^2)^{k+1} = y^{2k+1} y = ey = y$. Hence, $ES(G)$ is empty.

Conversely, suppose $ES(G)$ is empty. If $|G|$ is even, Cauchy's theorem ensures that G has an element of order 2 which is adjacent to e, a contradiction. □

Remark 1. We have seen in the above result that the groups having odd order have an empty equal-square graph. Therefore, we are mostly interested in the equal-square graph of the groups having even order.

Theorem 2. *The following assertions are equivalent for a group G with $|G| \geq 2$.*

(a) $ES(G)$ is connected

(b) $ES(G)$ is complete

(c) G is elementary abelian 2-group

Table 1: Equal-square graph of abelian group of order 36000.

Partitions of $\alpha = 5$	Length of partitions	Equal-square graph
$1+1+1+1+1$	5	$(36000/2^5)K_{2^5} = 1125K_{32}$
$1+1+1+2$	4	$(36000/2^4)K_{2^4} = 2250K_{16}$
$1+1+3, 2+2+1$	3	$(36000/2^3)K_{2^3} = 4500K_8$
$1+4,\ 2+3$	2	$(36000/2^2)K_{2^2} = 9000K_4$
5	1	$(36000/2)K_2 = 18000K_2$

Table 2: Equal-square graphs of small-order groups.

Order	Groups	Equal-square graphs
1	$\{e\}$	K_1
2	$\mathbb{Z}_2$	K_2
3	$\mathbb{Z}_3$	$3K_1$
4	$\mathbb{Z}_4, \mathbb{Z}_2\oplus\mathbb{Z}_2$	$2K_2, K_4$
5	$\mathbb{Z}_5$	$5K_1$
6	$\mathbb{Z}_6, S_3$	$3K_2, K_4 + 2K_1$
7	$\mathbb{Z}_7$	$7K_1$
8	$\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2, \mathbb{Z}_2\oplus\mathbb{Z}_4, \mathbb{Z}_8, D_4, \mathbb{Q}_8$	$K_8, 2K_4, 4K_2, K_6 + K_2$
9	$\mathbb{Z}_9, \mathbb{Z}_3\oplus\mathbb{Z}_3$	$9K_1$
10	$\mathbb{Z}_{10}, D_5$	$5K_2, K_6 + 4K_1$

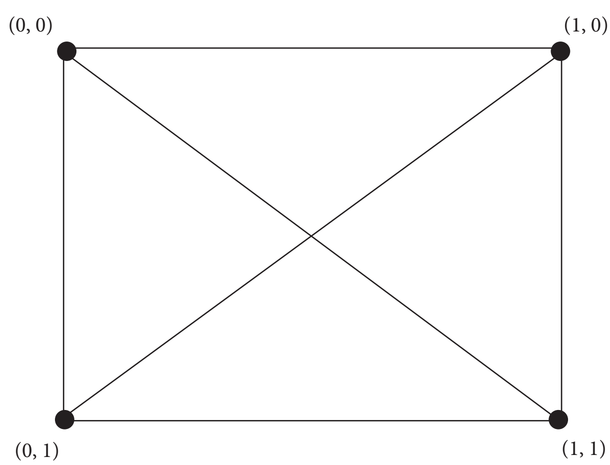

Figure 1: $ES(\mathbb{Z}_2\oplus\mathbb{Z}_2)$.

Proof

(a) ⇒ (b): suppose $ES(G)$ is connected. Then, for each $a \in G$, there exist $a_1, a_2, , , , a_n \in G$ such that $a - a_1 - a_2 - \cdots - a_n - e$. This implies that $a^2 = a_1^2 = a_2^2 = \cdots = e^2 = e$. We conclude that $a^2 = e$, for each $a \in G$, and hence, $x^2 = y^2$, for all $x, y \in G$. It follows completeness of $ES(G)$.

(b) ⇒ (c): if $ES(G)$ is complete, then $x^2 = y^2$, for all $x, y \in G$. In particular, $x^2 = e$, for all $x \in G$. This implies that each nonidentity element in G has order 2. This implies that G is abelian and hence an elementary abelian 2-group.

(c) ⇒ (a): G is elementary abelian 2-group. Then, each nonidentity element in G has order 2. This implies that $x^2 = e$, for all $x \in G$. Hence, each vertex in G is adjacent to e and so $ES(G)$ is connected. □

Example 4. The equal-square graph of an elementary abelian 2-group is given by

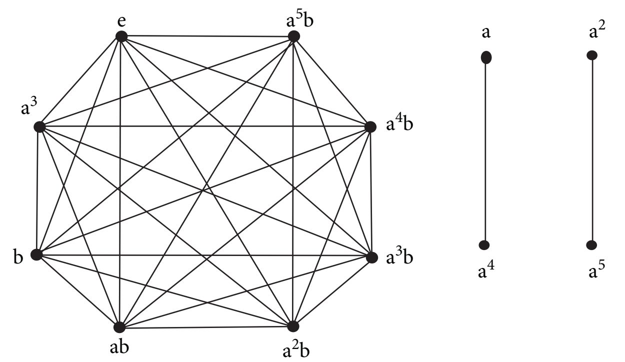

Figure 2: $ES(D_6)$.

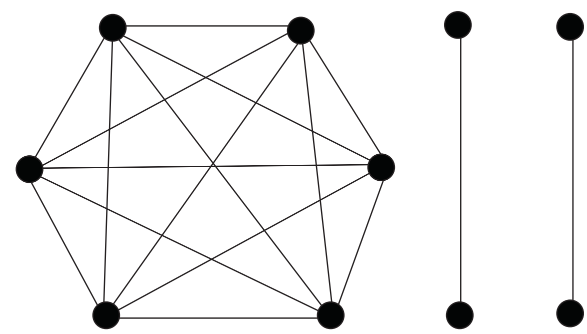

Figure 3: $ES(D_4) \cong ES(Q_8)$.

$$ES\Big(\underbrace{\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\cdots\oplus\mathbb{Z}_2}_{m-\text{times}}\Big) = K_{2^m}. \tag{1}$$

Remark 2. Let G be a finite group. If we define $G^2 = \{x^2 | x \in G\}$, then clearly $1 \le |G^2| \le |G|$. Moreover, if $|G^2| = 1$, i.e., $G^2 = \{e\}$, then $ES(G)$ is a complete graph. If $|G^2| = 2$, then $ES(G)$ is a disjoint union of two complete graphs. If $|G^2| = 3$, then $ES(G)$ is a disjoint union of three complete graphs and so on. If $|G^2| = |G|$, then $ES(G)$ is empty, i.e., $ES(G) = |G|K_1$. We conclude that, for a finite group G, $ES(G)$ is either a complete graph or it is a disjoint union of complete graphs.

3. Equal-Square Graph of Cyclic Groups

In this section, we completely describe the equal-square graph of cyclic groups. Recall that $G_1 + G_2$ denotes the disjoint union of graphs G_1 and G_2. Moreover, $kG = G + G + \cdots + G$ denotes the disjoint union of k copies of graph G.

Theorem 3. *For a cyclic group G of order $2n$, $ES(G) = nK_2$.*

Proof. Let $G = \langle a \rangle = \{e, a, a^2, a^3, \ldots, a^n, \ldots, a^{2n-1}\}$. Then, $(a^i)^2 = (a^{n+i})^2$, where $0 \le i \le n-1$. Now, let a^i be adjacent to a^j such that $0 \le i \le j \le 2n-1$, then $(a^i)^2 = (a^j)^2 \Rightarrow a^{2i} = a^{2j} \Rightarrow a^{2(j-i)} = e \Rightarrow |a|$ divides $2(j-i) \Rightarrow 2n | 2(j-i) \Rightarrow n | (j-i), 0 \le j-i \le 2n-1 \Rightarrow j-i = n$ or 0, then $j = i+n$ or $j = i$. Therefore, each a^i is adjacent to only one vertex, i.e., a^{n+i}, where $0 \le i \le n-1$. Hence, $ES(G) = nK_2$. □

Remark 3. The group $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ is not cyclic, but $ES(\mathbb{Z}_3 \oplus \mathbb{Z}_6) = 9K_2$. This implies that the converse of the above statement in general does not hold.

Corollary 1. *If $p \neq 2$ is prime and $\mathbb{F}_{p^n}$ denotes the finite field of order p^n, then*

$$ES\left(\mathbb{F}_{p^n}^*\right) = \frac{p^n - 1}{2} K_2. \quad (2)$$

Proof. Apply Theorem 3 and the fact that the multiplicative group $\mathbb{F}_{p^n}^*$ of every finite field $\mathbb{F}_{p^n}$ is a cyclic group, cf. Theorem 22.2 of [31]. □

4. Equal-Square Graph of Dihedral Groups

In this section, we completely describe the equal-square graph of the group of plane symmetries of n-gon $(n \geq 3)$.

Theorem 4. *The equal-square graph of the dihedral group $D_n = \langle \alpha, \beta | \alpha^n = \beta^2 = (\alpha\beta)^2 = e \rangle$ is given by*

$$ES(D_n) = \begin{cases} K_{n+2} + \left(\dfrac{n-2}{2}\right) K_2, & \text{if } n \text{ is even}, \\ K_{n+1} + (n-1) K_1, & \text{if } n \text{ is odd}. \end{cases} \quad (3)$$

Proof. Suppose n is even. Let $|a^k| = 2$, for some k, where $1 \leq k \leq n-1$. Then, $n/(n,k) = 2$ implies that $k = n/2$. Also, $|a^i b| = 2$, for all i, $0 \leq i \leq n-1$. Thus, we have total $n+1$ elements of order 2, namely, $a^{n/2}, b, ab, a^2b, , a^3b, \ldots, a^{n-1}b$. Hence, by joining all these to identity, we get a complete subgraph K_{n+2} in $ES(D_n)$. Now, we are left with $n-2$ elements, namely, $a^1, a^2, \ldots, a^{(n/2)-1}, a^{(n/2)+1}, \ldots, a^{n-1}$ in D_n. Now, by taking squares of all these remaining elements, we see that a is adjacent to $a^{(n/2)-1}$, a^2 is adjacent to $a^{(n/2)+1}$, a^3 is adjacent to $a^{(n/2)+3}$, and so on. In this way, we get disjoint union of $(n-2)/2$ copies of K_2 in $ES(D_n)$. Hence, $ES(D_n) = K_{n+2} + ((n-2)/2) K_2$.

Now, suppose that n is odd. Then, no power of a has order two. Thus, the only elements of order two will be $b, ab, a^2b, , a^3b, \ldots, a^{n-1}b$, which by joining to identity, form a complete subgraph K_{n+1} in $ES(D_n)$. Then, we are left with $n-1$ elements, namely, $a^1, a^2, \ldots, a^{n-1}$ in D_n. Suppose $(a^i)^2 = (a^j)^2$, for some $i \neq j$, with $1 \leq i, j \leq n-1$. Then, $n | 2(i-j)$. Since n is odd, so $n | i-j$. This implies that $a^{i-j} = e$, and thus, $a^i = a^j$. Hence, $ES(D_n) = K_{n+1} + (n-1) K_1$. □

5. Equal-Square Graph of the Group of Units $U(n)$

This section completely describes the equal-square graph of the group of unit elements of the ring $\mathbb{Z}_n$. By using the results proved in [27], we can completely describe the equal-square graph of the group of unit elements of the ring $\mathbb{Z}_n$ as follows.

Theorem 5. *Let $U(n)$ be the group of units of the ring $\mathbb{Z}_n$.*

(a) *$ES(U(2))$ is the empty graph*
(b) *$ES(U(2^2)) = K_2$*
(c) *$ES(U(2^s)) = 2^{s-3} K_4$, for each integer $s \geq 3$*
(d) *$ES(U(p)) = ((p-1)/2) K_2$, for each prime $p \neq 2$*
(e) *$ES(U(p^s)) = ((p^{s-1}(p-1))/2) K_2$, for each prime $p \neq 2$ and for each positive integer s*

Proof. Apply Theorem 2.1 of [27]. □

Theorem 6. *Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$; $p_i s$ are distinct odd primes and $\alpha_i s \in \mathbb{Z}^+$. Then,*

$$ES(U(n)) = \frac{\phi(n)}{2^m} K_{2^m}. \quad (4)$$

Proof. Apply Theorem 2.2 of [27]. □

Theorem 7. *Let $n = 2^s \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$; $p_i s \neq 2$ are distinct primes and $\alpha_i s \geq 1$, $s \geq 0$ are integers. Then,*

$$ES(U(n)) = \begin{cases} \dfrac{\phi(n)}{2^m} K_{2^m}, & \text{if } s = 0 \text{ or } 1, \\ \dfrac{\phi(n)}{2^{m+1}} K_{2^{m+1}}, & \text{if } s = 2, \\ \dfrac{\phi(n)}{2^{m+2}} K_{2^{m+2}}, & \text{if } s \geq 3. \end{cases} \quad (5)$$

Proof. Apply Theorem 2.3 of [27]. □

6. Equal-Square Graph of Finite Abelian Groups

In this section, we completely describe the equal-square graph of the finite abelian groups. Let Γ be the collection of all finite simple graphs (including the empty graph). First, we recall some definitions and results from [34].

Some basic properties of the strong product are recalled below.

Proposition 1 (see Sections 4.1, 4.2, and 5.2 of [34]). *Let "+" and "⊠" be the operations of disjoint union and strong product defined on Γ, respectively. The following assertions hold.*

(a) *$G_1 \boxtimes G_2 \cong G_2 \boxtimes G_1$, for all $G_1, G_2 \in \Gamma$, i.e., ⊠ is commutative*
(b) *$G_1 \boxtimes (G_2 \boxtimes G_3) \cong (G_1 \boxtimes G_2) \boxtimes G_3$, for all $G_1, G_2, G_3 \in \Gamma$, i.e., ⊠ is associative*
(c) *$G_1 \boxtimes (G_2 + G_3) = (G_1 \boxtimes G_2) + (G_1 \boxtimes G_3)$, for all $G_1, G_2, G_3 \in \Gamma$, i.e., ⊠ is distributive over +*
(d) *$K_1 \boxtimes G \cong G$, for all $G \in \Gamma$, i.e., the trivial graph K_1 is a unity in Γ*

Definition 2 (see Sections 4.2 and 5.2 of [34]). For graphs $G_1, G_2, G_3, \ldots, G_k$ in Γ, the strong product

$\boxtimes_{i=1}^{k} G_i = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k \in \Gamma$ is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2) \times \cdots \times V(G_k)$ such that two distinct vertices $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are adjacent iff $a_i b_i \in E(G_i)$ or $a_i = b_i$, for each i; $1 \leq i \leq k$. The kth power of a graph $G \in \Gamma$ with respect to $\boxtimes$ is symbolized by $G^{\boxtimes,k}$ and is defined as $G^{\boxtimes,k} = \underbrace{G \boxtimes G \boxtimes \cdots \boxtimes G}_{k-\text{times}}$.

Lemma 1. *$rK_m \boxtimes sK_n = rsK_{mn}$, for all $m, n, r, s \in \mathbb{Z}^+$.*

Proof. We have $K_m \boxtimes K_n = K_{mn}$, see Exercise 4.7 of [34]. Therefore,

$$\begin{aligned} rK_m \boxtimes sK_n &= \underbrace{(K_m + K_m + \cdots + K_m)}_{r-\text{times}} \boxtimes \underbrace{(K_n + K_n + \cdots + K_n)}_{s-\text{times}} \\ &= \underbrace{(K_m \boxtimes K_n) + (K_m \boxtimes K_n) + \cdots + (K_m \boxtimes K_n)}_{rs-\text{times}} \\ &= \underbrace{K_{mn} + K_{mn} + \cdots + K_{mn}}_{rs-\text{times}} \\ &= rsK_{mn}. \end{aligned} \tag{6}$$

□

Theorem 8. *For any two groups G and H in Γ, $ES(G \oplus H) = ES(G) \boxtimes ES(H)$.*

Proof. Clearly, we have that $V(ES(G \oplus H)) = V(ES(G) \boxtimes ES(H)) = G \oplus H$. Let $(a, b), (a', b') \in G \oplus H$ such that $(a, b), (a', b') \in E(ES(G \oplus H))$. Then, $(a, b)^2 = (a', b')^2$, and hence, $a^2 = a'^2$ and $b^2 = b'^2$. From $a^2 = a'^2$, we get that either $a = a'$ or $aa' \in E(G)$. Also, from $b^2 = b'^2$, we get that either $b = b'$ or $bb' \in E(H)$. Combining, we obtain that $a = a'$ and $bb' \in E(H)$, or $aa' \in E(G)$ and $b = b'$, or $aa' \in E(G)$ and $bb' \in E(H)$. Hence, $(a, b)(a', b') \in E(ES(G) \boxtimes ES(H))$.

Conversely, we suppose that $(a, b), (a', b') \in G \oplus H$ such that $(a, b)(a', b') \in E(ES(G) \boxtimes ES(H))$. This implies that $a = a'$ and $bb' \in E(H)$, or $aa' \in E(G)$ and $b = b'$, or $aa' \in E(G)$ and $bb' \in E(H)$. From this, we conclude that either $a = a'$ or $a^2 = a'^2$ and also either $b = b'$ or $b^2 = b'^2$. Therefore, $a^2 = a'^2$ and $b^2 = b'^2$, and hence, $(a, b)(a', b') \in E(ES(G \oplus H))$. □

Theorem 9. *Let $G_i = \langle a_i \rangle$ be cyclic groups of order n_i for each $1 \leq i \leq k$. Then,*

$$ES\left(\bigoplus_{i=1}^{k} G_i\right) = \frac{n_1 n_2 \cdots n_k}{2^r} K_{2^r}, \tag{7}$$

where r denotes the number of even integers in the set $\{n_1, n_2, \ldots, n_k\}$.

Proof. Applying Theorem 8, we obtain

$$ES\left(\bigoplus_{i=1}^{k} G_i\right) = \mathop{\boxtimes}_{i=1}^{k} ES(G_i). \tag{8}$$

Also, by Theorems 1 and 3, we have, for each $1 \leq i \leq n$,

$$ES(G_i) = \begin{cases} n_i K_1, & \text{if } n_i \text{ is odd}, \\ \dfrac{n_i}{2} K_2, & \text{if } n_i \text{ is even}. \end{cases} \tag{9}$$

Combining the above two equations and then by using Lemma 1, we obtain

$$ES\left(\bigoplus_{i=1}^{k} G_i\right) = \frac{n_1 n_2 \cdots n_k}{2^r} K_{2^r}. \tag{10}$$

□

Examples 5. We can immediately determine the equal-square graph of the direct product of cyclic groups as follows:

$$\begin{aligned} ES(\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_6) &= \frac{3 \cdot 4 \cdot 5 \cdot 6}{2^2} K_{2^2} = 90K_4, \\ ES(\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{12}) &= \frac{2 \cdot 6 \cdot 8 \cdot 9 \cdot 12}{2^4} K_{2^4} = 648K_{16}, \end{aligned} \tag{11}$$

Let G be a finite abelian group with $|G| = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct odd primes and $\alpha, \alpha_i s$ are positive integers. Then, G can be uniquely expressed as $G \cong \mathbb{Z}_{2^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{\beta_t}} \oplus \mathbb{Z}_{p_1} \beta_{11} \oplus \cdots \oplus \mathbb{Z}_{p_1} \beta_{1t_1} \oplus \cdots \oplus \mathbb{Z}_{p_k} \beta_{k1} \oplus \cdots \oplus \mathbb{Z}_{p_k} \beta_{kt_k}$, where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_t \geq 1$ with $\beta_1 + \beta_2 + \cdots + \beta_t = \alpha$ and $\beta_{i1} \geq \beta_{i2} \geq \cdots \geq \beta_{it_i} \geq 1$ with $\beta_{i1} + \beta_{i2} + \cdots + \beta_{it_i} = \alpha_i$, for each i; $1 \leq i \leq k$, cf. Section 5.2, Theorem 5 [30].

Theorem 10. *With notations above, $ES(G) = (|G|/2^t) K_{2^t}$.*

Proof. Applying Theorem 8, we obtain that $ES(G) = ES(\mathbb{Z}_{2^{\beta_1}}) \boxtimes \cdots \boxtimes ES(\mathbb{Z}_{2^{\beta_t}}) \boxtimes ES(\mathbb{Z}_{p_1} \beta_{11}) \boxtimes \cdots \boxtimes ES(\mathbb{Z}_{p_1} \beta_{1t_1}) \boxtimes \cdots \boxtimes ES(\mathbb{Z}_{p_k} \beta_{k1}) \boxtimes \cdots \boxtimes ES(\mathbb{Z}_{p_k} \beta_{kt_k})$. Then, by using Theorems 1 and 3, we get that $ES(G) = 2^{\beta_1 - 1} K_2 \boxtimes \cdots \boxtimes 2^{\beta_t - 1} K_2 \boxtimes p_1 \beta_{11} K_1 \boxtimes \cdots \boxtimes p_1 \beta_{1t_1} K_1 \boxtimes \cdots \boxtimes p_k \beta_{k1} K_1 \boxtimes \cdots \boxtimes p_k \beta_{kt_k} K_1$. Furthermore, by using Lemma 1, we get that $ES(G) = (2^{\beta_1 + \cdots + \beta_t - t} p_1^{\alpha_1} \cdots p_k^{\alpha_k}) K_{2^t} = (|G|/2^t) K_{2^t}$. □

Corollary 2. *Let G be an abelian group with order $|G| = 2^{\alpha} \lambda$, where λ is odd and $\alpha \geq 1$. Then, we have the equation $ES(G) = (|G|/2^t) K_{2^t}$ for each partition $\alpha = d_1 + d_2 + \cdots + d_t$ of the integer α of length t.*

Proof. We can write $\lambda = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct odd primes and $\alpha_i s$ are nonnegative integers. In light of Theorem 10, for each partition $\alpha = d_1 + d_2 + \cdots + d_t$ of length t, we get that $ES(G) = (|G|/2^t) K_{2^t}$. □

Corollary 3. *Let G be an abelian group with order $|G| = 2^{\alpha} \lambda$, where λ is odd and $\alpha \geq 1$. Then, it can have exactly α distinct (nonisomorphic) equal-square graphs.*

Example 6. All possible equal-square graphs of an abelian group of order $36000 = 2^5 3^2 5^3$ are described in Table 1.

Example 7. The equal-square graphs of small order groups are described in Table 2.

7. Concluding Remarks

Graph theory is an interesting area of research. A lot of topics in different areas of research are beautifully presented in the form of graphs. This work relates the finite group structures with simple graphs in the form of equal-square graphs which is defined by a particular adjacency relation on vertices. We have presented several examples in the support of results proved in this paper. Also, we have constructed tables for different well-known groups of finite order which provide corresponding equal-square graphs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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