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Research Article

Computing the Normalized Laplacian Spectrum and Spanning Tree of the Strong Prism of Octagonal Network

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Spectrum analysis and computing have expanded in popularity in recent years as a critical tool for studying and describing the structural properties of molecular graphs. Let O_n^2 be the strong prism of an octagonal network O_n. In this study, using the normalized Laplacian decomposition theorem, we determine the normalized Laplacian spectrum of O_n^2 which consists of the eigenvalues of matrices $\mathscr{L}_A$ and $\mathscr{L}_S$ of order $3n+1$. As applications of the obtained results, the explicit formulae of the degree-Kirchhoff index and the number of spanning trees for O_n^2 are on the basis of the relationship between the roots and coefficients.

1. Introduction

Graphs are a convenient way to depict chemical structures, where atoms are associated with vertices, while chemical bonds are associated with edges. This manifestation carries a wealth of knowledge about the molecule's chemical characteristics. In quantitative structure-activity/property relationship (QSAR/QSPR) studies, one may see that many chemical and physical properties of molecules are closely correlated with graph-theoretical parameters known as topological indices. One such graph-theoretical parameter is the multiplicative degree-Kirchhoff index (see [1]). In statistical physics (see [2]), the enumeration of spanning trees in a graph is a crucial problem. It is interesting to note that the multiplicative degree-Kirchhoff index is closely related to the number of spanning trees in a graph. The normalized Laplacian acts as a link between them.

Let G be an n-vertex simple, undirected, and connected graph with the vertex set of $V(G)$ and an edge set of $E(G)$. For standard notation and terminology, one may refer to the recent papers (see [3, 4]). The *(combinatorial) Laplacian matrix* of graph G is specified as $L_G = D_G - A_G$, where D_G is the vertex degree diagonal matrix of order n and $A(G)$ is an *adjacency matrix* of order n.

The normalized Laplacian is defined by

$$(\mathscr{L}_G)_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -\dfrac{1}{\sqrt{d_{v_i} d_{v_j}}}, & \text{if } i \neq j, v_i \sim v_j, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Evidently, $L(G) = D(G) - A(G)$ and $\mathscr{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2}$. As we all know, the normalized Laplacian technique is useful for analyzing the structural features of nonregular graphs. In reality, the interaction between a graph's structural features and its eigenvalues is the focus of spectral graph theory. For more information, see recent articles [5–8] or the book [9].

Many parameters were used to characterize and describe the structural features of graphs in chemical graph theory. The Wiener index [10, 11] was a well-known distance-based index, as it is known as $W(G) = \sum_{i<j} d_{ij}$. Eventually, Gutman [12] defined the Gutman index as follows:

$$\mathrm{Gut}(G) = \sum_{i<j} d_i d_j d_{ij}. \tag{2}$$

In accordance with electrical network theory, Klein and Randić [13] presented a new distance function called resistance distance that is denoted as r_{ij}. The resistance distance in electrical networks is between two arbitrary vertices i and j when every edge is replaced by a unit resistor. Klein and Ivanciuc [14] called it the Kirchhoff index, the total sum of resistance distances between each pair of vertices of G, which is $Kf(G) = \sum_{i<j} r_{ij}$. Later, the degree-Kirchhoff index was established by Chen and Zhang [1] and denoted by $Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}$.

Because of their practical uses in physics, chemistry, and other sciences, the Kirchhoff index and the degree-Kirchhoff index have gained a lot of attention. Klein and Lovász [15, 16] separately established that

$$Kf(G) = n\sum_{k=2}^{n} \frac{1}{\nu_k}, \tag{3}$$

where $0 = \nu_1 < \nu_2 \leq \cdots \leq \nu_n$ are the eigenvalues of $L(G)$. According to Chen [17], the degree-Kirchhoff index is,

$$Kf^*(G) = 2m\sum_{k=1}^{n} \frac{1}{\nu_k}, \tag{4}$$

where $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$ are the eigenvalues of $\mathscr{L}(G)$.

Since the Kirchhoff index and multiplicative degree-Kirchhoff index have been widely used in the domains of physics, chemistry, and network science. During the previous few decades, many scientists have been working on explicit formulae for the Kirchhoff and degree-Kirchhoff indices of graphs with particular structures, such as cycles [18], complete multipartite graphs [19], generalized phenylene [20], crossed octagonal [21], hexagonal chains [22], pentagonal-quadrilateral network [23], and so on. Other research on the Kirchhoff index and the multiplicative degree-Kirchhoff index of a graph has been published (see [24–31]). In organic chemistry, polyomino systems have received a lot of attention, especially in polycyclic aromatic compounds. Tree-like octagonal networks are condensed into octagonal networks that belong to the polycyclic conjugated hydrocarbons' family. The octagonal system without any branches is known as a linear octagonal network [32]. As shown in Figure 1, a linear octagonal network could also be created from a linear polyomino network by adding additional points to the line according to specified rules.

The *strong product* between the graphs G and H is denoted by $G\boxtimes H$, where the vertex set $V(G\boxtimes H)$ is $V_G \times V_H$ and $(a, x)(b, y)$ is an edge of $G\boxtimes H$ if $a = b$ and x is adjacent

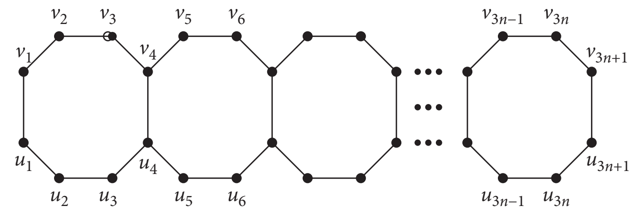

Figure 1: Graph O_n with labeled vertices.

to y in H or $x = y$ and a is adjacent to b in G or $xy \in E(H)$ and $ab \in E_G$. In particular, the strong product of K_2 and G is known as the strong prism of G. Recently, Li [33] and Ali [34] calculated the resistance distance-based parameters of the strong prism of unique graphs, such as strong prism of S_n and $L_n\boxtimes K_2$, respectively. Let O_n^2 be the strong prism of K_2 and O_n, denoted by $O_n^2 = K_2\boxtimes O_n$, as shown in Figure 2. Obviously, $|E(O_n^2)| = 34n + 6$ and $|V(O_n^2)| = 12n + 4$.

In this paper, motivated by [34–36], we derive an explicit analytical expression for the multiplicative degree-Kirchhoff index and also spanning trees of O_n^2.

2. Preliminaries

In this section, we start by going over some basic notation and then introduce a suitable technique. Given the square matrix R having order n, we refer to $R[i_1, i_2, \ldots, i_k]$ as the submatrix of R that results from deleting the i_1th, i_2th, …, i_kth columns and rows. Let $\Phi(R) = \det(xI_n - R)$ be the *characteristic polynomial* of the square matrix R. The labeled vertices of O_n^2 are as depicted in Figure 2 and $V_1 = \{u_1, \ldots, u_{3n+1}, v_1, \ldots, v_{3n+1}\}$ and $V_2 = \{u_1', \ldots, u_{3n+1}', v_1', \ldots, v_{3n+1}'\}$. The normalized Laplacian matrix $\mathscr{L}(O_n^2)$ could be represented as a block matrix below:

$$\mathscr{L}\left(O_n^2\right) = \begin{pmatrix} \mathscr{L}_{V_{11}}\left(O_n^2\right) & \mathscr{L}_{V_{12}}\left(O_n^2\right) \\ \mathscr{L}_{V_{21}}\left(O_n^2\right) & \mathscr{L}_{V_{22}}\left(O_n^2\right) \end{pmatrix}. \tag{5}$$

It is simple to verify that $\mathscr{L}_{V_{12}}(O_n^2) = \mathscr{L}_{V_{21}}(O_n^2)$ and $\mathscr{L}_{V_{11}}(O_n^2) = \mathscr{L}_{V_{22}}(O_n^2)$.

Let

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} I_{6n+2} & \frac{1}{\sqrt{2}} I_{6n+2} \\ \frac{1}{\sqrt{2}} I_{6n+2} & -\frac{1}{\sqrt{2}} I_{6n+2} \end{pmatrix}. \tag{6}$$

Then,

$$T\mathscr{L}\left(O_n^2\right)T' = \begin{pmatrix} \mathscr{L}_A\left(O_n^2\right) & 0 \\ 0 & \mathscr{L}_S\left(O_n^2\right) \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} \mathscr{L}_A\left(O_n^2\right) &= \mathscr{L}_{V_{11}} + \mathscr{L}_{V_{12}}, \\ \mathscr{L}_S\left(O_n^2\right) &= \mathscr{L}_{V_{11}} - \mathscr{L}_{V_{12}}. \end{aligned} \tag{8}$$

Huang et al. obtained the following lemma.

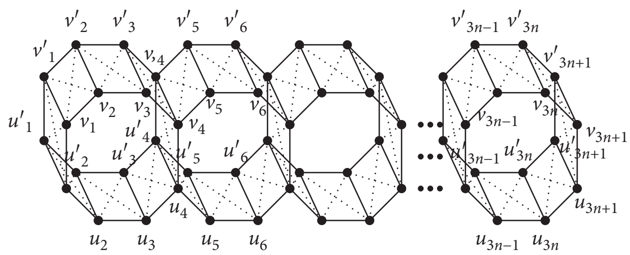

FIGURE 2: Graph O_n^2 with labeled vertices.

Lemma 1 (see [8]). *Let G be a graph and let $\mathscr{L}_A(O_n^2)$ and $\mathscr{L}_S(O_n^2)$ be as described above. Then, we have $\Phi(\mathscr{L}(O_n^2)) = \Phi(\mathscr{L}_A)\cdot\Phi(\mathscr{L}_S)$.*

Lemma 2 (see [1]). *Let $\rho_1 \le \rho_2 \le \cdots \le \rho_n$ be the eigenvalues of $\mathscr{L}(G)$; then, the degree-Kirchhoff index can also be written as $Kf^*(G) = 2m\sum_{i=2}^{n} 1/\rho_i$.*

Lemma 3 (see [17]). *Let G be n-vertex connected graph of size m; then, the spanning trees is $\tau(G) = 1/2m\prod_{i=1}^{n} d_i \prod_{k=2}^{n}\rho_k$.*

3. Main Results

In this section, we are committed to the explicit analytical solution for the multiplicative degree-Kirchhoff index, as well as the spanning tree of O_n^2. In terms of the role of normalized Laplacian $\mathscr{L}$, the following block matrices of $\mathscr{L}_{V_{11}}(O_n^2)$ and $\mathscr{L}_{V_{12}}(O_n^2)$ are obtained according to equation (8).

$$L_{V_{11}}(O_n^2) = \begin{pmatrix}
1 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{5} & 1 & -\frac{1}{5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{5} & 1 & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\
-\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 1 & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{5} & 1 & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{35}} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & 1
\end{pmatrix}_{(6n+2)\times(6n+2)},$$

.

$$L_{V_{12}}\left(O_n^2\right)=\begin{pmatrix} -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & -\frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\ -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{35}} & -\frac{1}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & -\frac{1}{5} \end{pmatrix}_{(6n+2)\times(6n+2)}. \tag{9}$$

By equation (8), we have a matrix of order $6n+2$:

$$L_A\left(O_n^2\right) = 2\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & \cdots & 0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \\ -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{pmatrix}, \tag{10}$$

and $\mathscr{L}_S(O_n^2) =$ diag$(6/5, 6/5, 6/5, 8/7, \ldots, 6/5, \quad 6/5, 8/7, 6/5, 6/5, 8/7, \ldots, 8/7, 6/5, 6/5, 6/5)Q$, a diagonal matrix with order $6n + 2$.

The normalized Laplacian spectrum of O_n^2 is constructed by the eigenvalues of $\mathscr{L}_A(O_n^2)$ and $\mathscr{L}_S(O_n^2)$, according to

Lemma 1. Given the fact that $\mathscr{L}_S(O_n^2)$ is just a diagonal matrix of order $6n + 2$, it is obvious that $6/5$ with multiplicity $4n + 4$ and $8/7$ with multiplicity $2n - 2$ are the eigenvalues of $\mathscr{L}_S(O_n^2)$.

Let

$$A=\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{pmatrix}_{(3n+1)\times(3n+1)},$$

$$C=\begin{pmatrix} -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} \end{pmatrix}_{(3n+1)\times(3n+1)}. \tag{11}$$

Thus, $(1/2)\mathscr{L}_A$ could be represented by the block matrix below:

$$\frac{1}{2}\mathscr{L}_A=\begin{pmatrix} A & C \\ C & A \end{pmatrix}. \tag{12}$$

Let

$$T=\begin{pmatrix} \frac{1}{\sqrt{2}}I_{3n+1} & \frac{1}{\sqrt{2}}I_{3n+1} \\ \frac{1}{\sqrt{2}}I_{3n+1} & -\frac{1}{\sqrt{2}}I_{3n+1} \end{pmatrix}. \tag{13}$$

Then,

$$T\left(\frac{1}{2}\mathscr{L}_A\right)T'=\begin{pmatrix} A+C & 0 \\ 0 & A-C \end{pmatrix}, \tag{14}$$

where T' indicates the transposition of T. Let $P=A+C$ and $Q=A-C$. Then,

$$P=\begin{pmatrix} \frac{1}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & \frac{1}{5} \end{pmatrix}_{(3n+1)\times(3n+1)},$$

$$Q=\begin{pmatrix} \frac{3}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{4}{7} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{5} & \frac{3}{5} \end{pmatrix}_{(3n+1)\times(3n+1)}. \tag{15}$$

By Lemma 1, it is simple to verify that the eigenvalues of $(1/2)\mathscr{L}_A$ consist of those of P and Q. Suppose that the eigenvalues of P and Q are denoted by γ_i and $\xi_j\,(i,j=1,2,\ldots,3n+1)$ with $\gamma_1\le\gamma_2\le\cdots\le\gamma_{3n+1}$ and $\xi_1\le\xi_2\le\cdots\le\xi_{3n+1}$, respectively. Then, the eigenvalues of $\mathscr{L}_A$ are $2\gamma_1,2\gamma_2,\ldots,2\gamma_{3n+1}$ and $2\xi_1,2\xi_2,\ldots,2\xi_{3n+1}$.where $0=\gamma_1<\gamma_2\le\cdots\le\gamma_{3n+1}$ and $0<\xi_1\le\xi_2\le\cdots\le\xi_{3n+1}$ are eigenvalues of P and Q, respectively.

Lemma 4. *Suppose that O_n^2 is the strong product of octagonal network. Then,*

$$Kf^*\left(O_n^2\right) = 2(34n+6)\left[(4n+4)\frac{5}{6}+(2n-2)\frac{7}{8}+\frac{1}{2}\sum_{i=2}^{3n+1}\frac{1}{\gamma_i}+\frac{1}{2}\sum_{j=1}^{3n+1}\frac{1}{\xi_j}\right], \tag{16}$$

On the basis of the relation between the coefficients and roots of $\Phi(P)$ (resp. $\Phi(Q)$), the formulae of $\sum_{i=2}^{3n+1} 1/\gamma_i$ (resp. $\sum_{j=1}^{3n+1} 1/\xi_j$) are obtained in the next lemmas.

Lemma 5. *Suppose that $0=\gamma_1<\gamma_2\le\cdots\le\gamma_{3n+1}$ are described as above. Then,*

$$\sum_{i=2}^{3n+1}\frac{1}{\gamma_i} = \frac{1359n^3+1115n^2+434n}{14(17n+3)}. \tag{17}$$

Suppose that $\Phi(P) = x^{3n+1}+a_1x^{3n}+\cdots+a_{3n-1}x^2 + a_{3n}x = x(x^{3n}+a_1x^{3n-1}+\cdots+a_{3n-1}x+a_{3n})$. Then, $\gamma_2,\gamma_3,\ldots,\gamma_{3n+1}$ satisfy the equation below:

$$x^{3n}+a_1x^{3n-1}+\cdots+a_{3n-1}x+a_{3n}=0, \tag{18}$$

so $1/\gamma_2, 1/\gamma_3,\ldots,1/\gamma_{3n+1}$ satisfy the equation below:

$$a_{3n}x^{3n}+a_{3n-1}x^{3n-1}+\cdots+a_1x+1=0. \tag{19}$$

Hence, by Vieta's theorem, we obtain

$$\sum_{i=2}^{3n+1}\frac{1}{\gamma_i} = \frac{(-1)^{3n-1}a_{3n-1}}{(-1)^{3n}a_{3n}}. \tag{20}$$

For the sake of convenience, consider W_i of P, which is the ith order principal submatrix generated by the first i columns and rows, $i=1,2,\ldots,3n$. Let $w_i=\det W_i$. Then,

$$\begin{aligned}
w_1 &= \frac{1}{5},\\
w_2 &= \frac{1}{25},\\
w_3 &= \frac{1}{125},\\
w_4 &= \frac{1}{875},\\
w_5 &= \frac{1}{4375},\\
w_6 &= \frac{1}{21875},
\end{aligned}
\qquad
\begin{cases}
w_{3i} = \dfrac{2}{5}w_{3i-1}-\dfrac{1}{25}w_{3i-2}, & \text{for } 1\le i\le n,\\[2ex]
w_{3i+1} = \dfrac{2}{7}w_{3i}-\dfrac{1}{35}w_{3i-1}, & \text{for } 1\le i\le n-1,\\[2ex]
w_{3i+2} = \dfrac{2}{5}w_{3i+1}-\dfrac{1}{35}w_{3i}, & \text{for } 1\le i\le n-1.
\end{cases} \tag{21}$$

By explicit calculation, these general formulae can be obtained as follows:

$$\begin{cases}
w_{3i} = \dfrac{7}{5}\left(\dfrac{1}{175}\right)^i, & \text{for } 1\le i\le n,\\[2ex]
w_{3i+1} = \dfrac{1}{5}\left(\dfrac{1}{175}\right)^i, & \text{for } 0\le i\le n-1,\\[2ex]
w_{3i+2} = \dfrac{1}{25}\left(\dfrac{1}{175}\right)^i, & \text{for } 0\le i\le n-1.
\end{cases} \tag{22}$$

The structure and determinant of matrix $\mathscr{L}_A$ are preserved by a permutation similarity transformation of a square matrix, and one gets $\det U_{3n+1-i} = \det W_{3n+1-i}$. We have

$$\begin{aligned}
(-1)^{3n}a_{3n} &= \sum_{i=1}^{3n+1}\det P[i] = \sum_{i=2}^{3n}\det P[i] + 2w_{3n}\\
&= \sum_{k=1}^{n}\det P[3k] + \sum_{k=1}^{n-1}\det P[3k+1] + \sum_{k=0}^{n-1}\det P[3k+2] + 2w_{3n}\\
&= \sum_{k=1}^{n} w_{3(k-1)+2}\cdot w_{3(n-k)+1} + \sum_{k=1}^{n-1} w_{3k}\cdot w_{3(n-k)} + \sum_{k=0}^{n-1} w_{3k+1}\cdot w_{3(n-k-1)+2} + 2w_{3n}\\
&= \frac{17n+3}{625}\left(\frac{1}{175}\right)^{n-1},
\end{aligned} \tag{23}$$

as desired.

Claim 1. $(-1)^{3n-1}a_{3n-1} = 1359n^3 + 1115n^2 + 434n/8750\,(1/175)^{n-1}$.

Proof of Claim 1. Noticing that $(-1)^{3n-1}a_{3n-1}$ is equal to the sum of all principal minors of P with $3n-1$ columns and rows, we have

$$(-1)^{3n-1}a_{3n-1} = \sum_{1\le i<j}^{3n+1} \begin{vmatrix} W_{i-1} & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & U \end{vmatrix}, \quad 1 \le i < j \le 3n+1, \tag{24}$$

where

$$Z = \begin{pmatrix} k_{i+1,i+1} & -\frac{1}{\sqrt{35}} & \cdots & 0 \\ -\frac{1}{\sqrt{35}} & k_{i+2,i+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{j-1,j-1} \end{pmatrix},$$

$$U = \begin{pmatrix} k_{j+1,j+1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & k_{3n,3n} & -\frac{1}{5} \\ 0 & \cdots & -\frac{1}{5} & k_{3n+1,3n+1} \end{pmatrix}. \tag{25}$$

Note that

$$(-1)^{3n-1}a_{3n-1} = \sum_{1\le i<j}^{3n+1} \det P[i,j] = \sum_{1\le i<j}^{3n+1} w_{i-1}\cdot w_{3n+1-j}\cdot \det Z. \tag{26}$$

□

Remark 1. If $1 \le i < i+1 = j \le 3n+1$, then Z is an empty matrix and let $\det Z = 1$. By equation (26), there are different possibilities which can be selected for i and j. Therefore, all these cases are classified as follows.

Case 1. Let $i = 3p$ and $j = 3q$, for $1 \le i < j \le 3n+1$. So, $1 \le p < q \le n$:

$$\det Z = \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{vmatrix}_{(3q-3p-1)} = (3q-3p+1)\left(\frac{1}{175}\right)^{q-p}. \tag{27}$$

Case 2. Let $i = 3p$ and $j = 3q+1$, for $1 \le i < i+1 < j \le 3n+1$. So, $1 \le p \le q \le n-1$:

$$\det Z = \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{vmatrix}_{(3q-3p)} = \frac{(3q-3p+2)}{7}\left(\frac{1}{175}\right)^{q-p}. \tag{28}$$

Case 3. Let $i = 3p$ and $j = 3q + 2$, for $1 \le i < j \le 3n + 1$. So, $1 \le p \le q \le n - 1$:

$$\det Z = \begin{vmatrix} \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(3q-3p+1)} = \frac{(3q-3p+3)}{35}\left(\frac{1}{175}\right)^{q-p}. \tag{29}$$

Case 4. Let $i = 3p + 1$ and $j = 3q$, for $1 \le i < j \le 3n + 1$. So, $0 \le p < q \le n$:

$$\det Z = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{vmatrix}_{(3q-3p-2)} = 35(3q-3p-1)\left(\frac{1}{175}\right)^{q-p}. \tag{30}$$

Case 5. Let $i = 3p+1$ and $j = 3q+1$, for $1 \le i < j \le 3n+1$. So, $0 \le p < q \le n-1$:

$$\det Z = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{vmatrix}_{(3q-3p-1)} = 7(3q-3p)\left(\frac{1}{175}\right)^{q-p}. \tag{31}$$

Case 6. Let $i = 3p+1$ and $j = 3q+2$, for $1 \le i < i+1 < j \le 3n+1$. So, $0 \le p \le q \le n-1$:

$$\det Z=\begin{vmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(3q-3p)} = (3q-3p+1)\left(\frac{1}{175}\right)^{q-p}. \quad (32)$$

Case 7. Let $i=3p+2$ and $j=3q$, for $1\le i<i+1<j\le 3n+1$. So, $0\le p<p+1<q\le n$:

$$\det Z=\begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{5} \end{vmatrix}_{(3q-3p-3)} = (3q-3p+1)\left(\frac{1}{175}\right)^{q-p}. \quad (33)$$

Case 8. Let $i=3p+2$ and $j=3q+1$, for $1\le i<j\le 3n+1$. So, $0\le p<q\le n-1$:

$$\det Z = \begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{5} & \frac{2}{5} \end{vmatrix}_{(3q-3p-2)} = 35(3q-3p-1)\left(\frac{1}{175}\right)^{q-p}. \tag{34}$$

Case 9. Let $i = 3p + 2$ and $j = 3q + 2$, for $1 \leq i < j \leq 3n + 1$. So, $0 \leq p < q \leq n - 1$:

$$\det Z = \begin{vmatrix} \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{35}} & \frac{2}{7} \end{vmatrix}_{(3q-3p-1)} = 5(3q-3p)\left(\frac{1}{175}\right)^{q-p}. \tag{35}$$

Combining these results with equation (26) and Cases 1–9 yields

$$(-1)^{3n-1}a_{3n-1} = \sum_{1\le i<j\le 3n+1} w_{i-1}\cdot w_{3n+1-j}\cdot \det Z_{j-1-i} = E_1+E_2+E_3, \quad (36)$$

where

$$\begin{aligned}
E_1 &= \sum_{1\le p<q\le n}\det P[3p,3q] + \sum_{1\le p\le q\le n-1}\det P[3p,3q+1] + \sum_{1\le p\le q\le n-1}\det P[3p,3q+2] \\
&\quad + \sum_{1\le p\le n}\det P[3p,3n+1] \\
&= \frac{n(n-1)(n+2)}{250}\left(\frac{1}{175}\right)^{n-1} + \frac{n^2(n-1)}{250}\left(\frac{1}{175}\right)^{n-1} \\
&\quad + \frac{n\left(n^2-1\right)}{250}\left(\frac{1}{175}\right)^{n-1} + \frac{3n^2+n}{350}\left(\frac{1}{175}\right)^{n-1} \\
&= \frac{21n^3+15n^2-16n}{1750}\left(\frac{1}{175}\right)^{n-1}, \\
E_2 &= \sum_{1\le p<q\le n}\det P[3p+1,3q] + \sum_{1\le p<q\le n-1}\det P[3p+1,3q+1] + \sum_{1\le p\le q\le n-1}\det P[3p+1,3q+2] \\
&\quad + \sum_{1\le p\le n-1}\det P[3p+1,3n+1] + \sum_{1\le q\le n}\det P[1,3q] + \sum_{1\le q\le n-1}\det P[1,3q+1] \\
&\quad + \sum_{0\le q\le n-1}\det P[1,3q+2] + Z_{3n-1} \\
&= \frac{7n^2(n-1)}{250}\left(\frac{1}{175}\right)^{n-1} + \frac{49n(n-1)(n-2)}{1250}\left(\frac{1}{175}\right)^{n-1} + \frac{7n(n-1)^2}{250}\left(\frac{1}{175}\right)^{n-1} \\
&\quad + \frac{21n(n-1)}{250}\left(\frac{1}{175}\right)^{n-1} + \frac{n(3n+1)}{50}\left(\frac{1}{175}\right)^{n-1} + \frac{21n(n-1)}{250}\left(\frac{1}{175}\right)^{n-1} \\
&\quad + \frac{n(3n-1)}{50}\left(\frac{1}{175}\right)^{n-1} + \frac{3n}{25}\left(\frac{1}{175}\right)^{n-1} \\
&= \frac{119n^3+108n^2+73n}{1250}\left(\frac{1}{245}\right)^{n-1}, \\
E_3 &= \sum_{0\le p<q\le n}\det P[3p+2,3q] + \sum_{0\le p<q\le n-1}\det P[3p+2,3q+1] + \sum_{0\le p<q\le n-1}\det P[3p+2,3q+2] \\
&\quad + \sum_{0\le p\le n-1}\det P[3p+2,3n+1] \\
&= \frac{n(n+1)(n+3)}{8750}\left(\frac{1}{175}\right)^{n-1} + \frac{7n^2(n-1)}{250}\left(\frac{1}{175}\right)^{n-1} + \frac{n\left(n^2-1\right)}{50}\left(\frac{1}{175}\right)^{n-1} \\
&\quad + \frac{n(3n+1)}{50}\left(\frac{1}{175}\right)^{n-1} \\
&= \frac{421n^3+284n^2+3n}{8750}\left(\frac{1}{175}\right)^{n-1}.
\end{aligned} \quad (37)$$

Substituting E_1, E_2, and E_3 in Equation (36), we get Claim 2.

Also, we can get Lemma 5 by combining Claims 1 and 2.

Lemma 6. *Let* $0 < \xi_1 < \xi_2 \leq \cdots \leq \xi_{3n+1}$ *be the eigenvalues of* Q *as above. Then,*

$$\sum_{j=1}^{3n+1} \frac{1}{\xi_j} = \frac{5(\eta_1 + \eta_2)}{(45 + 13\sqrt{15})(4 + \sqrt{15})^{n-1} + (45 - 13\sqrt{15})(4 - \sqrt{15})^{n-1}}, \tag{38}$$

where
$\eta_1 = (1500 + 401\sqrt{15} + n(1605 + 397\sqrt{15}))(4 + \sqrt{15})^{n-1}$
and
$\eta_2 = (1500 - 401\sqrt{15} + n(1605 - 397\sqrt{15}))(4 - \sqrt{15})^{n-1}$.

Proof . Suppose that $\Phi(Q) = x^{3n+1} + b_1 x^{3n} + \cdots + b_{3n} x + b_{3n+1}$.

So, $1/\xi_1, 1/\xi_2, \ldots, 1/\xi_{3n+1}$ satisfy the equation below:

$$b_{3n+1} x^{3n+1} + b_{3n} x^{3n} + \cdots + b_1 x + 1 = 0. \tag{39}$$

By Vieta's theorem, we obtain

$$\begin{aligned}\sum_{j=1}^{3n+1} \frac{1}{\xi_j} &= \frac{\sum_{j=1}^{3n+1} \xi_1 \cdots \xi_{j-1} \xi_{j+1} \cdots \xi_{3n+1}}{\prod_{j=1}^{3n+1} \xi_j}\\ &= \frac{(-1)^{3n} b_{3n}}{(-1)^{3n+1} b_{3n+1}} = \frac{(-1)^{3n} b_{3n}}{\det Q}.\end{aligned} \tag{40}$$

In order to find $(-1)^{3n} b_{3n}$ and detQ in (40), consider R_i of Q, which is the ith order principal submatrix generated by the first i columns and rows, $1 \leq i \leq 3n$. Let $r_i = \det R_i$. Then, $r_1 = 3/5$, $r_2 = 1/5$, $r_3 = 7/125$, $r_4 = 23/875$, $r_5 = 39/4375$, $r_6 = 11/4375$, and

$$\begin{cases} r_{3i} = \dfrac{2}{5} r_{3i-1} - \dfrac{1}{25} r_{3i-2}, & \text{for } 1 \leq i \leq n,\\[2ex] r_{3i+1} = \dfrac{4}{7} r_{3i} - \dfrac{1}{35} r_{3i-1}, & \text{for } 1 \leq i \leq n-1,\\[2ex] r_{3i+2} = \dfrac{2}{5} r_{3i+1} - \dfrac{1}{35} r_{3i}, & \text{for } 1 \leq i \leq n-1.\end{cases} \tag{41}$$

Similar to the method used as described above, we have

$$\begin{cases} r_{3i} = \dfrac{35 + 7\sqrt{15}}{50} \left(\dfrac{4 + \sqrt{15}}{175}\right)^i + \dfrac{35 - 7\sqrt{15}}{50} \left(\dfrac{4 - \sqrt{15}}{175}\right)^i, & \text{for } 1 \leq i \leq n,\\[2ex] r_{3i+1} = \dfrac{75 + 19\sqrt{15}}{750} \left(\dfrac{4 + \sqrt{15}}{175}\right)^i + \dfrac{75 - 19\sqrt{15}}{750} \left(\dfrac{4 - \sqrt{15}}{175}\right)^i, & \text{for } 0 \leq i \leq n-1,\\[2ex] r_{3i+2} = \dfrac{45 + 11\sqrt{15}}{150} \left(\dfrac{4 + \sqrt{15}}{175}\right)^i + \dfrac{45 - 11\sqrt{15}}{150} \left(\dfrac{4 - \sqrt{15}}{175}\right)^i, & \text{for } 0 \leq i \leq n-1.\end{cases} \tag{42}$$

□

Fact 1. $\det Q = 45 + 13\sqrt{15}/9375 \quad (4 + \sqrt{15}/175)^{n-1} + 45 - 13\sqrt{15}/9375(4 - \sqrt{15}/175)^{n-1}$.

Proof . Fact 1. Expanding detQ along the last row, we have

$$\begin{aligned}\det Q &= \frac{3}{5}\det r_{3n} - \frac{1}{25}\det r_{3n-1} = \frac{3}{5}\left[\frac{35 + 7\sqrt{15}}{50}\left(\frac{4 + \sqrt{15}}{175}\right)^n + \frac{35 - 7\sqrt{15}}{50}\left(\frac{4 - \sqrt{15}}{175}\right)^n\right]\\ &\quad - \frac{1}{25}\left[\frac{45 + 11\sqrt{15}}{150}\left(\frac{4 + \sqrt{15}}{175}\right)^{n-1} + \frac{45 - 11\sqrt{15}}{150}\left(\frac{4 - \sqrt{15}}{175}\right)^{n-1}\right]\\ &= \frac{45 + 13\sqrt{15}}{9375}\left(\frac{4 + \sqrt{15}}{175}\right)^{n-1} + \frac{45 - 13\sqrt{15}}{9375}\left(\frac{4 - \sqrt{15}}{175}\right)^{n-1}.\end{aligned} \tag{43}$$

□

Fact 2.

$$(-1)^{3n}b_{3n}=\frac{1500+401\sqrt{15}+n(1605+397\sqrt{15})}{1875}\left(\frac{4+\sqrt{15}}{175}\right)^{n-1}+\frac{1500-401\sqrt{15}+n(1605-397\sqrt{15})}{1875}\left(\frac{4-\sqrt{15}}{175}\right)^{n-1}. \quad (44)$$

Proof of Fact 2. Noting that $(-1)^{3n}b_{3n}$ is the summation of all principal minors of Q with $3n$ columns and rows, we have

$$\begin{aligned}(-1)^{3n}b_{3n}&=\sum_{i=1}^{3n+1}\det Q[i]=\sum_{i=1}^{3n+1}\det\begin{pmatrix}R_{i-1}&0\\0&S_{3n+1-i}\end{pmatrix}\\&=\sum_{i=1}^{3n+1}\det r_{i-1}\cdot\det s_{3n+1-i},\end{aligned} \quad (45)$$

where

$$S_{3n+1-i}=\begin{pmatrix}l_{i+1,i+1}&\cdots&0&0\\ \vdots&\ddots&\vdots&\vdots\\ 0&\cdots&l_{3n,3n}&-\frac{1}{\sqrt{35}}\\ 0&\cdots&-\frac{1}{\sqrt{35}}&l_{3n+1,3n+1}\end{pmatrix}. \quad (46)$$

The structure and determinant of matrix $\mathscr{L}_A$ are preserved by a permutation similarity transformation of a square matrix, and one gets $\det S_{3n+1-i}=\det R_{3n+1-i}$. In line with Equation (45), we have

$$\begin{aligned}(-1)^{3n}b_{3n}&=\sum_{i=1}^{3n+1}\det Q[i]=\sum_{p=1}^{n}\det Q[3p]+\sum_{p=0}^{n-1}\det Q[3p+1]+\sum_{p=0}^{n-1}\det Q[3p+2]+r_{3n}\\&=\sum_{p=1}^{n}r_{3(p-1)+2}\cdot r_{3(n-p)+1}+\sum_{p=1}^{n-1}r_{3p}\cdot r_{3(n-p)}+\sum_{p=0}^{n-1}r_{3p+1}\cdot r_{3(n-p-1)+2}+2r_{3n}.\end{aligned} \quad (47)$$

The following forms can also be generated by using the above equations:

$$\begin{aligned}\sum_{p=1}^{n}r_{3(p-1)+2}\cdot r_{3(n-p)+1}&=n\left[\frac{150+37\sqrt{15}}{375}\left(\frac{4+\sqrt{15}}{175}\right)^{n-1}+\frac{150-37\sqrt{15}}{375}\left(\frac{4-\sqrt{15}}{175}\right)^{n-1}\right]\\&\quad+\frac{14\sqrt{15}}{3}\left(\frac{4+\sqrt{15}}{175}\right)^{n}-\frac{14\sqrt{15}}{3}\left(\frac{4-\sqrt{15}}{175}\right)^{n},\end{aligned} \quad (48)$$

$$\begin{aligned}\sum_{p=1}^{n-1}r_{3p}\cdot r_{3(n-p)}&=(n-1)\left[\frac{35+7\sqrt{15}}{25}\left(\frac{4+\sqrt{15}}{175}\right)^{n}+\frac{35-7\sqrt{15}}{25}\left(\frac{4-\sqrt{15}}{175}\right)^{n}\right]\\&\quad+\frac{\sqrt{15}}{1875}\left(\frac{4+\sqrt{15}}{175}\right)^{n-1}-\frac{\sqrt{15}}{1875}\left(\frac{4-\sqrt{15}}{175}\right)^{n-1},\end{aligned} \quad (49)$$

$$\begin{aligned}\sum_{p=0}^{n-1}r_{3p+1}\cdot r_{3(n-p-1)+2}&=n\left[\frac{150+37\sqrt{15}}{375}\left(\frac{4+\sqrt{15}}{175}\right)^{n-1}+\frac{150-37\sqrt{15}}{375}\left(\frac{4-\sqrt{15}}{175}\right)^{n-1}\right]\\&\quad+\frac{14\sqrt{15}}{3}\left(\frac{4+\sqrt{15}}{175}\right)^{n}-\frac{14\sqrt{15}}{3}\left(\frac{4-\sqrt{15}}{175}\right)^{n},\end{aligned} \quad (50)$$

$$2r_{3n}=\frac{35+7\sqrt{15}}{25}\left(\frac{4+\sqrt{15}}{175}\right)^{n}+\frac{35-7\sqrt{15}}{25}\left(\frac{4-\sqrt{15}}{175}\right)^{n}. \quad (51)$$

We can obtain the desired result of Fact 2 by substituting equations (48)–(51) into (47).

In view of (40), Facts 1 and 2 and Lemma 6 hold immediately. □

The following theorem is derived from Lemmas 4–6.

Theorem 1. *Let* $O_n^2 = K_2 \boxtimes O_n$. *Then,*

$$Kf^*\left(O_n^2\right) = \frac{4077n^3 + 10604n^2 + 4844n + 399}{21} + (34n+6)\left[\frac{(-1)^{3n} b_{3n}}{\det Q}\right], \tag{52}$$

where

$$(-1)^{3n} b_{3n} = \frac{1500 + 401\sqrt{15} + n(1605 + 397\sqrt{15})}{1875}\left(\frac{4+\sqrt{15}}{175}\right)^{n-1} + \frac{1500 - 401\sqrt{15} + n(1605 - 397\sqrt{15})}{1875}\left(\frac{4-\sqrt{15}}{175}\right)^{n-1}$$
$$\det Q = \frac{45 + 13\sqrt{15}}{9375}\left(\frac{4+\sqrt{15}}{175}\right)^{n-1} + \frac{45 - 13\sqrt{15}}{9375}\left(\frac{4-\sqrt{15}}{175}\right)^{n-1}. \tag{53}$$

The explicit formulae of the spanning trees of O_n^2 are given below.

Theorem 2. *Let* $O_n^2 = K_2 \boxtimes O_n$. *Then,*

$$\tau\left(O_n^2\right) = \frac{2^{(16n-3)} \cdot 3^{(4n+3)}}{35}\left[(45 + 13\sqrt{15})(4+\sqrt{15})^{n-1} + (45 - 13\sqrt{15})(4-\sqrt{15})^{n-1}\right]. \tag{54}$$

Proof. By Lemma 2, we have $(6/5)^{(4n+4)} \cdot (8/7)^{(2n-2)} \prod_{i=2}^{3n+1} 2\gamma_i \prod_{i=1}^{3n+1} 2\xi_i \prod_{v \in V_{O_n^2}} d_{O_n^2} = 2|E_{O_n^2}|\tau(O_n^2)$. Note that

$$\prod_{v \in V_{O_n^2}} d_{O_n^2} = 5^{8n+8} \cdot 7^{4n-4},$$
$$\left|E_{O_n^2}\right| = 34n + 6,$$
$$\prod_{i=2}^{3n+1} \gamma_i = \frac{17n+3}{625}\left(\frac{1}{175}\right)^{n-1},$$
$$\prod_{i=1}^{3n+1} \xi_i = \det Q = \frac{45 + 13\sqrt{15}}{9375}\left(\frac{4+\sqrt{15}}{175}\right)^{n-1} + \frac{45 - 13\sqrt{15}}{9375}\left(\frac{4-\sqrt{15}}{175}\right)^{n-1}. \tag{55}$$

Hence, Theorem 2 immediately follows, along with Lemma 2. □

4. Conclusion

In this study, we consider O_n^2, which is the strong prism of the octagonal network. Using the normalized Laplacian theorems, we have determined the multiplicative degree-Kirchhoff index and the spanning tree of O_n^2. New discoveries, developments, and advancements in research are still required. In the near future, we will be exploring a more complex chemistry network.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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