Computing the Normalized Laplacian Spectrum and Spanning Tree of the Strong Prism of Octagonal Network

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1.Introduction

Graphs are a convenient way to depict chemical structures, where atoms are associated with vertices, while chemical bonds are associated with edges. This manifestation carries a wealth of knowledge about the molecule's chemical characteristics. In quantitative structure-activity/property relationship (QSAR/QSPR) studies, one may see that many chemical and physical properties of molecules are closely correlated with graph-theoretical parameters known as topological indices. One such graph-theoretical parameter is the multiplicative degree-Kirchhoff index (see [1]). In statistical physics (see [2]), the enumeration of spanning trees in a graph is a crucial problem. It is interesting to note that the multiplicative degree-Kirchhoff index is closely related to the number of spanning trees in a graph. The normalized Laplacian acts as a link between them.

Let G be an n-vertex simple, undirected, and connected graph with the vertex set of V(G) and an edge set of E(G). For standard notation and terminology, one may refer to the recent papers (see [3, 4]). The (combinatorial) Laplacian matrix of graph G is specified as \( L_G = D_G - A_G \), where \( D_G \) is the vertex degree diagonal matrix of order n and \( A(G) \) is an adjacency matrix of order n.

The normalized Laplacian is defined by

\[
(L^*_G)_{ij} = \begin{cases} 
1, & \text{if } i = j, \\
-\frac{1}{\sqrt{d_i d_j}}, & \text{if } i \neq j, v_i \sim v_j, \\
0, & \text{otherwise}.
\end{cases}
\]

(1)

Evidently, \( L(G) = D(G) - A(G) \) and \( L^*_G = (D(G))^{-1/2} L(G) (D(G))^{-1/2} \). As we all know, the normalized Laplacian technique is useful for analyzing the structural features of nonregular graphs. In reality, the interaction between a
graph’s structural features and its eigenvalues is the focus of spectral graph theory. For more information, see recent articles [5–8] or the book [9].

Many parameters were used to characterize and describe the structural features of graphs in chemical graph theory. The Wiener index [10, 11] was a well-known distance-based index, as it is known as \( W(G) = \sum_{i<j} d_{ij} \). Eventually, Gutman [12] defined the Gutman index as follows:

\[
Gut(G) = \sum_{i<j} d_i d_j \tag{2}
\]

In accordance with electrical network theory, Klein and Randić [13] presented a new distance function called resistance distance that is denoted as \( r_{ij} \). The resistance distance in electrical networks is between two arbitrary vertices \( i \) and \( j \) when every edge is replaced by a unit resistor. Klein and Ivanciuc [14] called it the Kirchhoff index, the total sum of resistance distances between each pair of vertices of \( G \), which is \( K_f(G) = \sum_{i<j} r_{ij} \). Later, the degree-Kirchhoff index was established by Chen and Zhang [1] and denoted by \( K_f \). Since the Kirchhoff index and multiplicative degree-Kirchhoff index and have gained a lot of attention. Klein and Lovász [15, 16] separately established that

\[
K_f(G) = n \sum_{k=2}^{n} \frac{1}{\sqrt{\lambda_k}} \tag{3}
\]

where \( 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \) are the eigenvalues of \( L(G) \). According to Chen [17], the degree-Kirchhoff index is,

\[
K_f^+(G) = 2m \sum_{k=1}^{n} \frac{1}{\sqrt{\lambda_k}} \tag{4}
\]

where \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) are the eigenvalues of \( \mathcal{L}(G) \).

Since the Kirchhoff index and multiplicative degree-Kirchhoff index have been widely used in the domains of physics, chemistry, and network science. During the previous few decades, many scientists have been working on explicit formulae for the Kirchhoff and degree-Kirchhoff indices of graphs with particular structures, such as cycles [18], complete multipartite graphs [19], generalized phenylene [20], crossed octagonal [21], hexagonal chains [22], pentagonal-quadrilateral network [23], and so on. Other research on the Kirchhoff index and the multiplicative degree-Kirchhoff index of a graph has been published (see [24–31]). In organic chemistry, polyomino systems have received a lot of attention, especially in polycyclic aromatic compounds. Tree-like octagonal networks are condensed into octagonal networks that belong to the polycyclic conjugated hydrocarbons’ family. The octagonal system without any branches is known as a linear octagonal network [32]. As shown in Figure 1, a linear octagonal network could also be created from a linear polyomino network by adding additional points to the line according to specified rules.

The strong product between the graphs \( G \) and \( H \) is denoted by \( G \bowtie H \), where the vertex set \( V(G \bowtie H) \) is \( V_G \times V_H \) and \( (a, x)(b, y) \) is an edge of \( G \bowtie H \) if \( a = b \) and \( x \) is adjacent to \( y \) in \( H \) or \( x = y \) and \( a \) is adjacent to \( b \) in \( G \) or \( xy \in E(H) \) and \( ab \in E_G \). In particular, the strong product of \( K_n \) and \( G \) is known as the strong prism of \( G \). Recently, Li [33] and Ali [34] calculated the resistance distance-based parameters of the strong prism of unique graphs, such as strong prism of \( S_n \) and \( L_{p\bowtie K_2} \), respectively. Let \( O_n \) be the strong prism of \( K_2 \) and \( O_n \), denoted by \( O_n = K_2 \bowtie O_n \), as shown in Figure 2. Obviously, \( |E(O_n^2)| = 34n + 6 \) and \( |V(O_n^2)| = 12n + 4 \).

In this paper, motivated by [34–36], we derive an explicit analytical expression for the multiplicative degree-Kirchhoff index and also spanning trees of \( O_n \).

**2. Preliminaries**

In this section, we start by going over some basic notation and then introduce a suitable technique. Given the square matrix \( R \) having order \( n \), we refer to \( R[i_1, i_2, \ldots, i_k] \) as the submatrix of \( R \) that results from deleting the \( i_1 \text{th}, i_2 \text{th}, \ldots, i_k \text{th} \) columns and rows. Let \( \Phi(R) = \det(xI_n - R) \) be the characteristic polynomial of the square matrix \( R \). The labeled vertices of \( O_n^2 \) are as depicted in Figure 2 and \( V_1 = \{ u_1, \ldots, u_{3n+1}, v_1, \ldots, v_{3n+1} \} \) and \( V_2 = \{ u'_1, \ldots, u'_{3n+1}, v'_1, \ldots, v'_{3n+1} \} \). The normalized Laplacian matrix \( \mathcal{L}(O_n^2) \) could be represented as a block matrix below:

\[
\mathcal{L}(O_n^2) = \begin{pmatrix}
\mathcal{L}_{V_11}(O_n^2) & \mathcal{L}_{V_12}(O_n^2) \\
\mathcal{L}_{V_21}(O_n^2) & \mathcal{L}_{V_22}(O_n^2) 
\end{pmatrix}.
\]

It is simple to verify that \( \mathcal{L}_{V_12}(O_n^2) = \mathcal{L}_{V_21}(O_n^2) \) and \( \mathcal{L}_{V_11}(O_n^2) = \mathcal{L}_{V_22}(O_n^2) \).

Let \( T = \begin{pmatrix} \frac{1}{\sqrt{2}}I_{6n+2} & \frac{1}{\sqrt{2}}I_{6n+2} \\
\frac{1}{\sqrt{2}}I_{6n+2} & -\frac{1}{\sqrt{2}}I_{6n+2} \end{pmatrix} \).

Then,

\[
T\mathcal{L}(O_n^2)T' = \begin{pmatrix}
\mathcal{L}_A(O_n^2) & 0 \\
0 & \mathcal{L}_S(O_n^2)
\end{pmatrix},
\]

where

\[
\mathcal{L}_A(O_n^2) = \mathcal{L}_{V_11} + \mathcal{L}_{V_12},
\]

\[
\mathcal{L}_S(O_n^2) = \mathcal{L}_{V_11} - \mathcal{L}_{V_12}.
\]

Huang et al. obtained the following lemma.
Lemma 1 (see [8]). Let $G$ be a graph and let $L_A(O_2^n)$ and $L_S(O_2^n)$ be as described above. Then, we have $\Phi(L(O_2^n)) \cdot \Phi(L_A) \cdot \Phi(L_S)$.

Lemma 2 (see [1]). Let $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n$ be the eigenvalues of $L(G)$; then, the degree-Kirchhoff index can also be written as $Kf^*(G) = 2m \sum_{i=1}^{n} 1/\rho_i$.

Lemma 3 (see [17]). Let $G$ be an $n$-vertex connected graph of size $m$; then, the spanning trees is $\tau(G) = 1/2m \prod_{i=1}^{n} d_i \prod_{k=2}^{n} \rho_k$.

3. Main Results

In this section, we are committed to the explicit analytical solution for the multiplicative degree-Kirchhoff index, as well as the spanning tree of $O_2^n$. In terms of the role of normalized Laplacian $L$, the following block matrices of $L_{V_1}(O_2^n)$ and $L_{V_2}(O_2^n)$ are obtained according to equation (8).

$$
L_{V_1}(O_2^n) =
\begin{pmatrix}
1 & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{5} & 1 & \frac{1}{5} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{5} & 1 & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & 1 & \cdots & 0 & 0 & 0 & 0 & \frac{1}{7} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & 1 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 1 & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & 0 & \frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{35}} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 1 & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 1 & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{7} & \cdots & 0 & 0 & 0 & \cdots & 1 & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 & \cdots & 1 & \frac{1}{5} \cdot (6n+2) \cdot (6n+2),
\end{pmatrix}
$$

\[ \text{Figure 2: Graph } O_2^n \text{ with labeled vertices.} \]
By equation (8), we have a matrix of order $6n + 2$:
and \( \mathcal{L}_S(O_n^2) = \text{diag}(6/5, 6/5, 6/5, 8/7, \ldots, 6/5, 6/5, 6/5, 8/7, 8/7, \ldots, 8/7, 6/5, 6/5, 6/5)Q \), a diagonal matrix with order \( 6n + 2 \).

The normalized Laplacian spectrum of \( O_n^2 \) is constructed by the eigenvalues of \( L_A(O_n^2) \) and \( \mathcal{L}_S(O_n^2) \), according to Lemma 1. Given the fact that \( \mathcal{L}_S(O_n^2) \) is just a diagonal matrix of order \( 6n + 2 \), it is obvious that \( 6/5 \) with multiplicity \( 4n + 4 \) and \( 8/7 \) with multiplicity \( 2n - 2 \) are the eigenvalues of \( \mathcal{L}_S(O_n^2) \).

Let

\[
L_A(O_n^2) = 2 \begin{pmatrix}
\frac{2}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{3}{7} & \cdots & 0 & 0 & 0 & 0 & \frac{1}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & \frac{1}{5} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 2 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 2 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 2 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 2 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 2 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{5} & 2 & \frac{1}{5} & 0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]
Let

\[
A = \begin{pmatrix}
\frac{2}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & \frac{3}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{2}{5}
\end{pmatrix}_{(3n+1) \times (3n+1)},
\]

\[
C = \begin{pmatrix}
-\frac{1}{5} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \frac{1}{7} & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{1}{5}
\end{pmatrix}_{(3n+1) \times (3n+1)}.
\]

Thus, \((1/2) \mathbf{A}\) could be represented by the block matrix below:

\[
\frac{1}{2} \mathbf{A} = \begin{pmatrix}
A & C \\
C & A
\end{pmatrix}.
\]

Let

\[
T = \frac{1}{\sqrt{2}} \mathbf{I}_{3n+1} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}_{3n+1}.
\]

Then,

\[
T \left( \frac{1}{2} \mathbf{A} \right) T' = \begin{pmatrix}
A + C & 0 \\
0 & A - C
\end{pmatrix},
\]

where \(T'\) indicates the transposition of \(T\). Let \(P = A + C\) and \(Q = A - C\). Then,

\[
P = \begin{pmatrix}
\frac{1}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{1}{5}
\end{pmatrix}_{(3n+1) \times (3n+1)},
\]

\[
Q = \begin{pmatrix}
\frac{3}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & \frac{4}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{1}{5}
\end{pmatrix}_{(3n+1) \times (3n+1)}.
\]

By Lemma 1, it is simple to verify that the eigenvalues of \((1/2) \mathbf{A}\) consist of those of \(P\) and \(Q\). Suppose that the eigenvalues of \(P\) and \(Q\) are denoted by \(\gamma_i\) and \(\xi_j\) \((i, j = 1, 2, \ldots, 3n + 1)\) with \(\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{3n+1}\) and \(\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{3n+1}\), respectively. Then, the eigenvalues of \(\mathbf{A}\) are \(2\gamma_1, 2\gamma_2, \ldots, 2\gamma_{3n+1}\) and \(2\xi_1, 2\xi_2, \ldots, 2\xi_{3n+1}\), where \(0 = \gamma_1 < \gamma_2 \leq \cdots \leq \gamma_{3n+1}\) and \(0 < \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{3n+1}\) are eigenvalues of \(P\) and \(Q\), respectively.
Lemma 4. Suppose that $O_n^2$ is the strong product of octagonal network. Then,

$$K_f^*(O_n^2) = 2(34n + 6) \left( (4n + 4) \frac{5}{6} + (2n - 2) \frac{7}{8} + \frac{1}{2} \sum_{j=2}^{3n+1} \frac{1}{j} + \frac{1}{2} \sum_{j=2}^{3n+1} \frac{1}{j} \right)$$

(16)

On the basis of the relation between the coefficients and roots of $\Phi(P)$ (resp. $\Phi(Q)$), the formulae of $\sum_{i=2}^{3n+1} 1/y_i$ (resp. $\sum_{i=2}^{3n+1} 1/\xi_i$) are obtained in the next lemmas.

Lemma 5. Suppose that $0 = y_1 < y_2 \leq \cdots \leq y_{3n+1}$ are described as above. Then,

$$\sum_{i=2}^{3n+1} \frac{1}{y_i} = \frac{1359n^3 + 1115n^2 + 434n}{14(17n + 3)}$$

(17)

Suppose that $\Phi(P) = x^{3n+1} + a_{3n} x^{3n} + \cdots + a_{3n-1} x + a_{3n}$. Then,

$$y_2, y_3, \ldots, y_{3n+1}$$

satisfy the equation below:

$$(3n + 1) x^{3n} + a_{3n} x^{3n-1} + \cdots + a_{1} x + 1 = 0.$$

(18)

By explicit calculation, these general formulae can be obtained as follows:

$$w_3 = \frac{1}{125},$$

$$w_4 = \frac{1}{875},$$

$$w_5 = \frac{1}{4375},$$

$$w_6 = \frac{1}{21875},$$

(21)

$$w_{3i} = \frac{2}{5} w_{3i-1} - \frac{1}{25} w_{3i-2}, \quad \text{for } 1 \leq i \leq n,$$

$$w_{3i+1} = \frac{2}{7} w_{3i} - \frac{1}{35} w_{3i-1}, \quad \text{for } 1 \leq i \leq n - 1,$$

$$w_{3i+2} = \frac{2}{5} w_{3i+1} - \frac{1}{35} w_{3i}, \quad \text{for } 1 \leq i \leq n - 1.$$

(22)

The structure and determinant of matrix $Z_\Delta$ are preserved by a permutation similarity transformation of a square matrix, and one gets $\det U_{3n+1-i} = \det W_{3n+1-i}$. We have

$$(-1)^{3n} a_{3n} = \sum_{i=1}^{3n+1} \det P[i] = \sum_{i=2}^{3n} \det P[i] + 2w_{3n}$$

$$= \sum_{k=1}^{n} \det P[3k] + \sum_{k=1}^{n-1} \det P[3k + 1] + \sum_{k=0}^{n-1} \det P[3k + 2] + 2w_{3n}$$

(23)

$$= \sum_{k=1}^{n} w_{3(k-1)+2} \cdot w_{3(m-k)+1} + \sum_{k=1}^{n} w_{3k} \cdot w_{3(n-k)} + \sum_{k=0}^{n-1} w_{3k+1} \cdot w_{3(n-k)+2} + 2w_{3n}$$

$$= \frac{17n + 3}{625} \left( \frac{1}{175} \right)^{n-1},$$

as desired.
Claim 1. \((-1)^{3n-1} a_{3n-1} = 1359n^3 + 1115n^2 + 434n/8750\) \((1/175)^n-1\).

Proof of Claim 1. Noticing that \((-1)^{3n-1} a_{3n-1}\) is equal to the sum of all principal minors of \(P\) with \(3n-1\) columns and rows, we have

\[
(-1)^{3n-1} a_{3n-1} = \sum_{1 \leq i < j} W_{i,j-1} 0 0 Z 0, \quad 1 \leq i < j \leq 3n + 1, \quad (24)
\]

where

\[
Z = \begin{pmatrix}
\frac{1}{\sqrt{35}} & \frac{1}{\sqrt{35}} & \cdots & 0 \\
\frac{1}{\sqrt{35}} & \frac{1}{\sqrt{35}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\sqrt{35}}
\end{pmatrix},
\]

and

\[
U = \begin{pmatrix}
0 & \cdots & 0 & 0 \\
0 & \cdots & \frac{1}{5} & k_{3n,1} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \frac{1}{5} & k_{3n+1,1}
\end{pmatrix}.
\]

(25)

Note that

\[
(-1)^{3n-1} a_{3n-1} = \sum_{1 \leq i < j} \det P[i,j] = \sum_{1 \leq i < j} w_{i-1} \cdot w_{j+1} \cdots \det Z.
\]

(26)

Remark 1. If \(1 \leq i < i + 1 = j < 3n + 1\), then \(Z\) is an empty matrix and let \(\det Z = 1\). By equation (26), there are different possibilities which can be selected for \(i\) and \(j\). Therefore, all these cases are classified as follows.

Case 1. Let \(i = 3p\) and \(j = 3q\), for \(1 \leq i < j \leq 3n + 1\). So, \(1 \leq p < q \leq n\):

\[
\begin{pmatrix}
\frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & \frac{2}{5} & \frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{35}} & \frac{2}{5}
\end{pmatrix}
= (3q - 3p + 1) \left(\frac{1}{175}\right)^{q-p}.
\]

(27)

Case 2. Let \(i = 3p\) and \(j = 3q + 1\), for \(1 \leq i < i + 1 < j \leq 3n + 1\). So, \(1 \leq p < q \leq n - 1\):
Case 3. Let $i = 3p$ and $j = 3q + 2$, for $1 \leq i < j \leq 3n + 1$. So, $1 \leq p \leq q \leq n - 1$:

\[
\begin{vmatrix}
\frac{2}{7} & \frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{\sqrt{35}} & \frac{2}{5} & \frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{35}} & \frac{2}{7}
\end{vmatrix} = \frac{(3q - 3p + 2)}{7} \left( \frac{1}{175} \right)^{q-p}. \tag{28}
\]

Case 4. Let $i = 3p + 1$ and $j = 3q$, for $1 \leq i < j \leq 3n + 1$. So, $0 \leq p < q \leq n$:

\[
\begin{vmatrix}
\frac{2}{7} & \frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{\sqrt{35}} & \frac{2}{5} & \frac{1}{5} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & \frac{2}{7} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & \frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{35}} & \frac{2}{7}
\end{vmatrix} = \frac{(3q - 3p + 3)}{35} \left( \frac{1}{175} \right)^{q-p}. \tag{29}
\]
Case 5. Let \( i = 3p + 1 \) and \( j = 3q + 1 \), for \( 1 \leq i < j \leq 3n + 1 \). So, \( 0 \leq p < q \leq n - 1 \):

\[
\begin{vmatrix}
\frac{2}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{35}} & \frac{2}{5}
\end{vmatrix}
= 35(3q - 3p - 1) \left( \frac{1}{175} \right)^{q-p} .
\]

(30)

Case 6. Let \( i = 3p + 1 \) and \( j = 3q + 2 \), for \( 1 \leq i < i + 1 < j \leq 3n + 1 \). So, \( 0 \leq p \leq q \leq n - 1 \):

\[
\begin{vmatrix}
\frac{2}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{2}{5}
\end{vmatrix}
= 7(3q - 3p) \left( \frac{1}{175} \right)^{q-p} .
\]

(31)
Case 7. Let $i = 3p + 2$ and $j = 3q$, for $1 \leq i < i + 1 < j \leq 3n + 1$. So, $0 \leq p < p + 1 < q \leq n$:

$$
\begin{vmatrix}
\frac{2}{5} & \frac{1}{5} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{5} & \frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{35}} & \frac{2}{7} \\
\end{vmatrix}
$$

$$
\text{det}Z = (3q - 3p + 1) \left( \frac{1}{175} \right)^{q-p}. \tag{32}
$$

Case 8. Let $i = 3p + 2$ and $j = 3q + 1$, for $1 \leq i < j \leq 3n + 1$. So, $0 \leq p < q \leq n - 1$:

$$
\begin{vmatrix}
\frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & \frac{1}{5} & \cdots & 0 & 0 \\
0 & 0 & -\frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{7} & -\frac{1}{\sqrt{35}} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{35}} & \frac{2}{5} \\
\end{vmatrix}
$$

$$
\text{det}Z = (3q - 3p + 1) \left( \frac{1}{175} \right)^{q-p}. \tag{33}
$$
Case 9. Let $i = 3p + 2$ and $j = 3q + 2$, for $1 \leq i < j \leq 3n + 1$. So, $0 \leq p < q - 1$:

\[
\begin{vmatrix}
\frac{2}{5} & -\frac{1}{\sqrt{35}} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{35}} & \frac{2}{7} & -\frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{35}} & \frac{2}{5} & -\frac{1}{5} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{5} & \frac{2}{5} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \frac{2}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{5} & \frac{2}{5}
\end{vmatrix} = 35 (3q - 3p - 1) \left( \frac{1}{175} \right)^{q-p}.
\]  
(34)

Combining these results with equation (26) and Cases 1–9 yields

Combining these results with equation (26) and Cases 1–9 yields
\[
(-1)^{3n-1} a_{3n-1} = \sum_{1 \leq i < j \leq 3n+1} w_{i-1} \cdot w_{3n+1-j} \cdot \det Z_{j-1-i} \quad \text{where}
\]
\[
E_1 = E_2 + E_3,
\]

\[
E_1 = \sum_{1 \leq p < q \leq n} \det P[3p, 3q] + \sum_{1 \leq p < q \leq n-1} \det P[3p, 3q + 1] + \sum_{1 \leq p < q \leq n-1} \det P[3p, 3q + 2] + \sum_{1 \leq p \leq n} \det P[3p, 3n + 1]
\]
\[
= \frac{n(n-1)(n+2)}{250} \left( \frac{1}{175} \right)^{n-1} + \frac{n^2(n-1)}{250} \left( \frac{1}{175} \right)^{n-1}
\]
\[
+ \frac{n^2(n-1) + 3n^n}{250} + \frac{n(n-1)}{175} \left( \frac{1}{175} \right)^{n-1}
\]
\[
= \frac{21n^3 + 15n^2 - 16n}{1750} \left( \frac{1}{175} \right)^{n-1},
\]

\[
E_2 = \sum_{1 \leq p < q \leq n} \det P[3p + 1, 3q] + \sum_{1 \leq p < q \leq n-1} \det P[3p + 1, 3q + 1] + \sum_{1 \leq p < q \leq n-1} \det P[3p + 1, 3q + 2] + \sum_{1 \leq p < q \leq n-1} \det P[1, 3q] + \sum_{1 \leq q \leq n-1} \det P[1, 3q + 1]
\]
\[
+ \sum_{0 \leq q \leq n-1} \det P[1, 3q + 2] + Z_{3n-1}
\]
\[
= \frac{7n^2(n-1)}{250} \left( \frac{1}{175} \right)^{n-1} + \frac{49n(n-1)(n-2)}{1250} \left( \frac{1}{175} \right)^{n-1} + \frac{7n(n-1)^2}{250} \left( \frac{1}{175} \right)^{n-1}
\]
\[
+ \frac{21n(n-1)}{250} \left( \frac{1}{175} \right)^{n-1} + \frac{n(3n+1)}{50} \left( \frac{1}{175} \right)^{n-1} + \frac{21n(n-1)}{250} \left( \frac{1}{175} \right)^{n-1}
\]
\[
+ \frac{n(3n-1)}{50} \left( \frac{1}{175} \right)^{n-1} + \frac{3n}{25} \left( \frac{1}{175} \right)^{n-1}
\]
\[
= \frac{119n^3 + 108n^2 + 73n}{1250} \left( \frac{1}{175} \right)^{n-1},
\]

\[
E_3 = \sum_{0 \leq p < q \leq n} \det P[3p + 2, 3q] + \sum_{0 \leq p < q \leq n-1} \det P[3p + 2, 3q + 1] + \sum_{0 \leq p < q \leq n-1} \det P[3p + 2, 3q + 2] + \sum_{0 \leq p < q \leq n-1} \det P[3p + 2, 3n + 1]
\]
\[
+ \sum_{0 \leq p \leq n-1} \det P[3p + 2, 3n + 1]
\]
\[
= \frac{n(n+1)(n+3)}{8750} \left( \frac{1}{175} \right)^{n-1} + \frac{7n^2(n-1)}{250} \left( \frac{1}{175} \right)^{n-1} + \frac{n(n^2-1)}{50} \left( \frac{1}{175} \right)^{n-1}
\]
\[
+ \frac{n(3n+1)}{50} \left( \frac{1}{175} \right)^{n-1}
\]
\[
= \frac{421n^3 + 284n^2 + 3n}{8750} \left( \frac{1}{175} \right)^{n-1}.
\]
Substituting $E_1$, $E_2$, and $E_3$ in Equation (36), we get Claim 2.

Also, we can get Lemma 5 by combining Claims 1 and 2.

**Lemma 6.** Let $0 < \xi_1 < \xi_2 \leq \cdots \leq \xi_{3n+1}$ be the eigenvalues of $Q$ as above. Then,

$$5(\eta_1 + \eta_2) = (45 - 13\sqrt{15})(4 - \sqrt{15})^{n-1}.$$  

**Proof.** Suppose that $\Phi(Q) = x^{3n+1} + b_1 x^{3n} + \cdots + b_{3n} x + b_{3n+1}$. So, $1/\xi_1, 1/\xi_2, \ldots, 1/\xi_{3n+1}$ satisfy the equation below:

$$b_{3n+1}x^{3n+1} + b_{3n}x^{3n} + \cdots + b_1 x + 1 = 0. \quad (39)$$

By Vieta’s theorem, we obtain

$$\sum_{j=1}^{3n+1} \frac{1}{\xi_j} = \frac{\sum_{j=1}^{3n+1} \xi_{j-1} \xi_{j+1} \cdots \xi_{3n+1}}{\prod_{j=1}^{3n+1} \xi_j}, \quad (40)$$

where

$$\eta_1 = (1500 + 401\sqrt{15} + n(1605 + 397\sqrt{15})) (4 + \sqrt{15})^{n-1}$$

and

$$\eta_2 = (1500 - 401\sqrt{15} + n(1605 - 397\sqrt{15})) (4 - \sqrt{15})^{n-1}.$$

In order to find $(-1)^{3n}b_{3n}$ and $\det Q$ in (40), consider $R_i$ of $Q$, which is the $i$th order principal submatrix generated by the first $i$ columns and rows, $1 \leq i \leq 3n$. Let $r_i = \det R_i$. Then, $r_1 = 3/5$, $r_2 = 1/5$, $r_3 = 7/125$, $r_4 = 23/875$, $r_5 = 39/4375$, $r_6 = 11/4375$, and

$$
\begin{align*}
r_{3i} &= \frac{2}{5} r_{3i-1} - \frac{1}{25} r_{3i-2}, \quad \text{for } 1 \leq i \leq n, \\
r_{3i+1} &= \frac{4}{7} r_{3i} - \frac{1}{35} r_{3i-1}, \quad \text{for } 1 \leq i \leq n-1, \\
r_{3i+2} &= \frac{2}{5} r_{3i+1} - \frac{1}{35} r_{3i}, \quad \text{for } 1 \leq i \leq n-1.
\end{align*}
$$

Similar to the method used as described above, we have

$$
\begin{align*}
r_{3i} &= \frac{35 + 7\sqrt{15}}{50} \left(\frac{4 + \sqrt{15}}{175}\right)^i + \frac{35 - 7\sqrt{15}}{50} \left(\frac{4 - \sqrt{15}}{175}\right)^i, \quad \text{for } 1 \leq i \leq n, \\
r_{3i+1} &= \frac{75 + 19\sqrt{15}}{750} \left(\frac{4 + \sqrt{15}}{175}\right)^i + \frac{75 - 19\sqrt{15}}{750} \left(\frac{4 - \sqrt{15}}{175}\right)^i, \quad \text{for } 0 \leq i \leq n-1, \\
r_{3i+2} &= \frac{45 + 11\sqrt{15}}{150} \left(\frac{4 + \sqrt{15}}{175}\right)^i + \frac{45 - 11\sqrt{15}}{150} \left(\frac{4 - \sqrt{15}}{175}\right)^i, \quad \text{for } 0 \leq i \leq n-1.
\end{align*}
$$

**Fact 1.** $\det Q = 45 + 13\sqrt{15}/9375 \left(4 + \sqrt{15}/175\right)^{n-1} + 45 - 13\sqrt{15}/9375 \left(4 - \sqrt{15}/175\right)^{n-1}$.

**Proof.** Fact 1. Expanding $\det Q$ along the last row, we have

$$
\begin{align*}
\det Q &= \frac{3}{5} \det r_{3n} - \frac{1}{25} \det r_{3n-1} = \frac{3}{5} \left[ \det r_{3n} - \frac{1}{25} \det r_{3n-1} \right] \\
&= \frac{3}{5} \left[ \frac{35 + 7\sqrt{15}}{50} \left(\frac{4 + \sqrt{15}}{175}\right)^n + \frac{35 - 7\sqrt{15}}{50} \left(\frac{4 - \sqrt{15}}{175}\right)^n \right] \\
&\quad - \frac{1}{25} \left[ \frac{45 + 11\sqrt{15}}{150} \left(\frac{4 + \sqrt{15}}{175}\right)^{n-1} + \frac{45 - 11\sqrt{15}}{150} \left(\frac{4 - \sqrt{15}}{175}\right)^{n-1} \right] \\
&= \frac{45 + 13\sqrt{15}}{9375} \left(\frac{4 + \sqrt{15}}{175}\right)^{n-1} + \frac{45 - 13\sqrt{15}}{9375} \left(\frac{4 - \sqrt{15}}{175}\right)^{n-1}.
\end{align*}
$$

□
Fact 2.

\[-1^3b_{3n} = \frac{1500 + 401\sqrt{15} + n(1605 + 397\sqrt{15})}{1875} \left(4 + \sqrt{15}\right)^{n-1} + \frac{1500 - 401\sqrt{15} + n(1605 - 397\sqrt{15})}{1875} \left(4 - \sqrt{15}\right)^{n-1}.\]

(44)

Proof of Fact 2. Noting that \(-1^3b_{3n}\) is the summation of all principal minors of \(Q\) with 3\(n\) columns and rows, we have

\[-1^3b_{3n} = \sum_{i=1}^{3n+1} \det Q[i] = \sum_{i=1}^{3n+1} \det \left( \begin{array}{cc} R_{i-1} & 0 \\ 0 & S_{3n+1-i} \end{array} \right).\]

(45)

The structure and determinant of matrix \(S_{3n+1-i}\) are preserved by a permutation similarity transformation of a square matrix, and one gets \(\det S_{3n+1-i} = \det R_{3n+1-i}\). In line with Equation (45), we have

\[-1^3b_{3n} = \sum_{i=1}^{3n+1} \det Q[i] = \sum_{p=1}^{n} \det Q[3p] + \sum_{p=0}^{n-1} \det Q[3p + 1] + \sum_{p=0}^{n-1} \det Q[3p + 2] + r_{3n}.\]

(47)

The following forms can also be generated by using the above equations:

\[\sum_{p=1}^{n} r_{3(p-1)+2} \cdot r_{3(n-p)+1} = n \left[ \frac{150 + 37\sqrt{15}}{375} \left(4 + \sqrt{15}\right)^{n-1} + \frac{150 - 37\sqrt{15}}{375} \left(4 - \sqrt{15}\right)^{n-1} \right] + \frac{14\sqrt{15}}{3} \left(4 + \sqrt{15}\right)^n - \frac{14\sqrt{15}}{3} \left(4 - \sqrt{15}\right)^n,\]

(48)

\[\sum_{p=1}^{n-1} r_{3p} \cdot r_{3(n-p)} = (n-1) \left[ \frac{35 + 7\sqrt{15}}{25} \left(4 + \sqrt{15}\right)^n + \frac{35 - 7\sqrt{15}}{25} \left(4 - \sqrt{15}\right)^n \right] + \sqrt{15} \left(4 + \sqrt{15}\right)^{n-1} - \sqrt{15} \left(4 - \sqrt{15}\right)^{n-1},\]

(49)

\[\sum_{p=0}^{n-1} r_{3p+1} \cdot r_{3(n-p)+2} = n \left[ \frac{150 + 37\sqrt{15}}{375} \left(4 + \sqrt{15}\right)^{n-1} + \frac{150 - 37\sqrt{15}}{375} \left(4 - \sqrt{15}\right)^{n-1} \right] + \frac{14\sqrt{15}}{3} \left(4 + \sqrt{15}\right)^n - \frac{14\sqrt{15}}{3} \left(4 - \sqrt{15}\right)^n,\]

(50)

\[2r_{3n} = \frac{35 + 7\sqrt{15}}{25} \left(4 + \sqrt{15}\right)^n + \frac{35 - 7\sqrt{15}}{25} \left(4 - \sqrt{15}\right)^n.\]

(51)

We can obtain the desired result of Fact 2 by substituting equations (48)--(51) into (47). In view of (40), Facts 1 and 2 and Lemma 6 hold immediately. \(\square\)
The following theorem is derived from Lemmas 4–6.

**Theorem 1.** Let \( O^2_n = K_2 \otimes O_n \). Then,

\[
Kf^*(O^2_n) = \frac{4077n^3 + 10604n^2 + 4844n + 399}{21} + (34n + 6) \left[ \frac{(-1)^{3n} b_{3n}}{\det Q} \right],
\]

where

\[
(-1)^{3n} b_{3n} = \frac{1500 + 401\sqrt{15} + n(1605 + 397\sqrt{15})}{1875} \left( \frac{4 + \sqrt{15}}{175} \right)^{n-1} + \frac{1500 - 401\sqrt{15} + n(1605 - 397\sqrt{15})}{1875} \left( \frac{4 - \sqrt{15}}{175} \right)^{n-1}
\]

\[
\det Q = \frac{45 + 13\sqrt{15}}{9375} \left( \frac{4 + \sqrt{15}}{175} \right)^{n-1} + \frac{45 - 13\sqrt{15}}{9375} \left( \frac{4 - \sqrt{15}}{175} \right)^{n-1}.
\]

The explicit formulae of the spanning trees of \( O^2_n \) are given below.

**Theorem 2.** Let \( O^2_n = K_2 \otimes O_n \). Then,

\[
\tau(O^2_n) = 2^{\frac{16n+3}{9}} \cdot 3^{\frac{4n+3}{2}} \left[ (45 + 13\sqrt{15})(4 + \sqrt{15})^{n-1} + (45 - 13\sqrt{15})(4 - \sqrt{15})^{n-1} \right].
\]

**Proof.** By Lemma 2, we have \((6/5)^{4n+4} \cdot (8/7)^{(2n-2)} \prod_{i=2}^{3n+1} y_i \prod_{i=1}^{3n+1} 2\xi_i \prod_{v \in V(G)} d_{O^2_n} = 2|E_{O^2_n}| \tau(O^2_n)\). Note that

\[
\prod_{v \in V(G)} d_{O^2_n} = 5^{8n+8} \cdot 7^{4n-4},
\]

\[
|E_{O^2_n}| = 34n + 6,
\]

\[
\prod_{i=2}^{3n+1} y_i = \frac{17n + 3}{625} \left( \frac{1}{175} \right)^{n-1},
\]

\[
\prod_{i=1}^{3n+1} \xi_i = \det Q = \frac{45 + 13\sqrt{15}}{9375} \left( \frac{4 + \sqrt{15}}{175} \right)^{n-1} + \frac{45 - 13\sqrt{15}}{9375} \left( \frac{4 - \sqrt{15}}{175} \right)^{n-1}.
\]

Hence, Theorem 2 immediately follows, along with Lemma 2. \( \square \)
4. Conclusion

In this study, we consider $O_2^3$, which is the strong prism of the octagonal network. Using the normalized Laplacian theorems, we have determined the multiplicative degree-Kirchhoff index and the spanning tree of $O_2^3$. New discoveries, developments, and advancements in research are still required. In the near future, we will be exploring a more complex chemistry network.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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