Hindawi
Journal of Mathematics
Volume 2022, Article ID 9288157, 10 pages
https://doi.org/10.1155/2022/9288157

Research Article

Impact of Multiplicative Noise on the Exact Solutions of the Fractional-Stochastic Boussinesq-Burger System

Wael W. Mohammed,[1,2] **Farah M. Al-Askar**,[3] **and M. El-Morshedy**[4,5]

[1]*Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[3]*Department of Mathematical Science, Collage of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[4]*Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*
[5]*Department of Statistics and Computer Science, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

Correspondence should be addressed to Wael W. Mohammed; wael.mohammed@mans.edu.eg

Received 24 May 2022; Revised 26 August 2022; Accepted 7 September 2022; Published 22 September 2022

Academic Editor: Arzu Akbulut

In this paper, we consider the fractional-stochastic Boussinesq-Burger system (FSBBS) generated by the multiplicative Brownian motion. The Jacobi elliptic function techniques are used to create creative elliptic, hyperbolic, and rational fractional-stochastic solutions for FSBBS. Furthermore, we draw 2D and 3D graphs by using the MATLAB Package for some obtained solutions of FSBBS to discuss the influence of the Brownian motion on these solutions. Finally, we indicate that the Brownian motion stabilizes the solutions of FSBBS around zero.

1. Introduction

Nonlinear partial differential equations (NLPDEs) have grown in popularity in the area of nonlinear science, owing to their large variety of uses in economics [1], engineering [2], civil engineering [3], soil mechanics [4], physics [5], quantum mechanics [6], statistical mechanics [7], solid-state physics [8], population ecology [9], etc. Solitons are among the most common in the setting of NLPDE solutions, and they are essential for understanding nonlinear physical phenomena. Solitons are utilized to understand the properties of nonlinear media in various areas including quantum electronics, plasma physics, nonlinear optics, and fluid dynamics [10–13]. Recently, the searching of exact soliton solutions to NLPDEs has become an enthralling research topic in engineering and applied sciences. Many techniques have been used to determine exact solutions for NLPDE including tanh-sech [14, 15], Darboux transformation [16], sine-cosine [17, 18], $\exp(-\phi(\varsigma))$-expansion [19], (G'/G)-expansion [20–22], Lie symmetry analysis method [23], improved F-expansion method [24, 25], Hirota's function [26], the Jacobi elliptic function [27, 28], and perturbation [29, 30].

The fractional differential equation is extensively used in fluid mechanics, solid state physics, optical fibers, neural physics, quantum field theory, mathematical biology, plasma physics, and other areas [31–34]. Researchers recommend fractional-order derivative over ordinary order derivative because integer-order derivative is essentially a local operator, but fractional-order derivative is so much more. Also, they explain physical phenomena such as quantum mechanics, diffusion, gravity, heat, elasticity, fluid dynamics, electrodynamics, electrostatics, and sound. Recently, the exact solutions with conformable derivative have been obtained in many papers for instance [35–40].

On the other hand, a wide variety of complex nonlinear physical phenomena can be represented using stochastic partial differential equations (SPDEs). These kind of equations can be found in many fields, such as physics and finance.

On the other side, stochastic partial differential equations (SPDEs) can be used to represent a wide range of complicated nonlinear physical processes. These kind of equations appear in a variety of areas including engineering, geophysical, biology, climate dynamics, finance, and physics [41–43].

To realize a higher level of qualitative accord, we take the following fractional-stochastic Boussinesq-Burger system (FSBBS) perturbed in the itô sense by multiplicative noise:

$$d\Phi + \left[2\Phi\mathbb{D}_x^{\alpha}\Phi - \frac{1}{2}\mathbb{D}_x^{\alpha}\Psi\right]dt = \sigma\Phi d\mathbb{B}, \tag{1}$$

$$d\Psi + \left[2\mathbb{D}_x^{\alpha}(\Phi\Psi) - \frac{1}{2}\mathbb{D}_x^{3\alpha}\Phi\right]dt = \sigma\Psi d\mathbb{B}, \tag{2}$$

where $\Phi(x, t)$ denotes the horizontal velocity field. $\mathbb{D}^{\alpha}$ is the conformable derivative (CD) [44]. $\Psi(x, t)$ is the height of the water surface above the bottom horizontal level. $\mathbb{B}(t)$ is a Brownian motion (BM) and σ is the noise strength.

The Boussinesq–Burgers system (BBS), with $\alpha = 1$ and $\sigma = 0$, appears in fluid flow research and explains how shallow water waves spread. Due to the importance of BBS, many researchers have created its exact solutions by using various methods such as Hirota method [45], Lie symmetry method [46], sine-Gordon expansion method [47], Jacobi elliptic function method [48], extended homogeneous balance [49], Darboux transformation [50], The modified exp $(-\phi(\zeta))$-expansion function method [51], and Exp-function method [52]. On the other side, many techniques have been documented for fractional BBS, including a domain decomposition method [53] and generalized Kudryashov method [54]. The exact solutions of the FSBBS (1-2) have not yet been studied.

Our novelty of this paper is to find the exact fractional stochastic solutions of FSBBS (1-2). In the presence of a stochastic term and the fractional space, this study is the first to obtain analytical solutions to the FSBBS (1-2). Numerous solutions, including those involving elliptic, trigonometric, rational, and hyperbolic functions, can be obtained using the Jacobi elliptic function technique. Moreover, we utilize MATLAB to build 2D and 3D figures for some of the obtained solutions in this study to display how the BM influences on the solutions of FSBBS (1-2).

The layout of the document is as follows: in Sec. 2, we define and give some properties of the CD and BM. In Sec. 3, we use an effective wave transformation to establish the FSBBS (1-2) wave equation. In Sec.4, we use the Jacobi elliptic function method to generate the analytic of FSBBS (1-2). While, in Sec.5, the effect of the BM on the solutions obtained is studied. In Sec. 6, the document's conclusion is shown.

2. Preliminaries

In this section, we define and clarify some characteristics of the BM and CD. In the following, we define BM $\mathbb{B}(t)$ as:

Definition 1 (see [55]). Stochastic process $\{\mathbb{B}(t)\}_{t\geq 0}$ is said a BM if the following conditions satisfy: $B(t)$ is continuous function of $t \geq 0$; $B(0) = 0$; for $\tau_1 < \tau_2$,$B(\tau_2) - B(\tau_1)$ is independent; and $B(\tau_2) - B(\tau_1)$ has a Gaussian distribution $\aleph(0, \tau_2 - \tau_1)$.

Lemma 2 (see [55]). $\mathbb{E}(e^{\sigma\mathbb{B}(t)}) = e^{((1/2)\sigma^2 t)}$ *for* $\sigma \geq 0$.

Definition 3 (see [44]). Let $\phi : (0,\infty) \longrightarrow \mathbb{R}$, then the CD of ϕ of order $\alpha \in (0, 1]$ is defined as

$$\mathbb{D}_x^{\alpha}\phi(x) = \lim_{\kappa\longrightarrow 0} \frac{\phi\left(x + \kappa x^{1-\alpha}\right) - \phi(x)}{\kappa}. \tag{3}$$

Let us go through some of the CD's features. If a, b are constant, then

(1) $\mathbb{D}_x^{\alpha}[a] = 0$,

(2) $\mathbb{D}_x^{\alpha}[x^b] = bx^{b-\alpha}$,

(3) $\mathbb{D}_x^{\alpha}[a\Theta_1(x) + b\Theta_2(x)] = a\mathbb{D}_x^{\alpha}\Theta_1(x) + b\mathbb{D}_x^{\alpha}\Theta_2(x)$,

(4) $\mathbb{D}_x^{\alpha}\Theta(x) = x^{1-\alpha}(d\Theta/dx)$,

(5) $\mathbb{D}_x^{\alpha}(\Theta_1 \circ \Theta_2)(x) = x^{1-\alpha}\Theta_2'(x)\Theta_1'(\Theta_2(x))$.

3. Wave Equation of FSBBS

The next wave transformation is used

$$\begin{aligned}\Phi(x, t) &= \varphi(\xi)e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)}, \Psi(x, t)\\ &= \psi(\xi)e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)}, \xi = \frac{1}{\alpha}x^{\alpha} + \omega t,\end{aligned} \tag{4}$$

in order to attain the wave equation of FSBBS (1-2). Where ω is a constant, φ and ψ are deterministic functions. Plugging Equation (4) into Equations (1) and (2) and utilizing

$$\begin{aligned}d\Phi &= \left[\omega\varphi' dt + \sigma\varphi d\mathbb{B}\right]e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)},\\ d\Psi &= \left[\omega\psi' dt + \sigma\psi d\mathbb{B}\right]e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)},\\ \mathbb{D}_x^{\alpha}\Phi &= \varphi' e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)}, \mathbb{D}_x^{\alpha}\Psi = \psi' e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)},\\ \mathbb{D}_x^{3\alpha}\Phi &= \varphi''' e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)}, \mathbb{D}_x^{\alpha}(\Phi\Psi) = (\varphi\psi)' e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)},\end{aligned} \tag{5}$$

we get

$$\omega\varphi' + 2\varphi\varphi' e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)} - \frac{1}{2}\psi' = 0, \tag{6}$$

$$\omega\psi' + 2(\varphi\psi)' e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)} - \frac{1}{2}\varphi''' = 0. \tag{7}$$

Taking expectation $\mathbb{E}(\cdot)$ for Equations (6) and (7), we attain

$$\omega\varphi' + 2\varphi\varphi' e^{-((1/2)\sigma^2 t)}\mathbb{E}\left(e^{\sigma\mathbb{B}(t)}\right) - \frac{1}{2}\psi' = 0, \quad (8)$$

$$\omega\psi' + 2(\varphi\psi)' e^{-((1/2)\sigma^2 t)}\mathbb{E}\left(e^{\sigma\mathbb{B}(t)}\right) - \frac{1}{2}\varphi''' = 0. \quad (9)$$

Since $\mathbb{B}(t)$ is a Gaussian distribution, then $\mathbb{E}(e^{\sigma\mathbb{B}(t)}) = e^{((\sigma^2/2)\,t)}$. Now Equations (8) and (9) become

$$\omega\varphi' + 2\varphi\varphi' - \frac{1}{2}\psi' = 0, \quad (10)$$

$$\omega\psi' + (\varphi\psi)' - \frac{1}{2}\varphi''' = 0. \quad (11)$$

Integrating Equations (10) and (11) and putting the constants of integration equal zero, we have

$$\psi = 2\omega\varphi + 2\varphi^2, \quad (12)$$

$$\omega\psi + 2(\varphi\psi) - \frac{1}{2}\varphi'' = 0. \quad (13)$$

Plugging Equation (12) into (13), we attain

$$\varphi'' - 8\varphi^3 - 12\omega\varphi^2 - 4\omega^2\varphi = 0. \quad (14)$$

4. Exact Solutions of FSBBS

We use the Jacobi elliptic functions approach described by Peng [56] to find the solutions of Equation (14). Consequently, we can therefore derive the exact solutions of FSBBS (1-2).

4.1. Jacobi Elliptic Functions Method. First, we suppose the solutions of Equation (14) are

$$\varphi(\xi) = \sum_{i=1}^{N} a_i\chi^i, \quad (15)$$

where χ is the solution of

$$\chi' = \sqrt{\frac{1}{2}p\chi^4 + q\chi^2 + r}, \quad (16)$$

where r, q and p are real parameters.

We note from the next Table 1 that Equation (16) has different types of solutions relying on r, q and p:

Where $dn(\xi, m) = dn(\xi, m)$, $cn(\xi) = cn(\xi, m)$, $sn(\xi) = sn(\xi, m)$, for $0 < \mathrm{m} < 1$ are the Jacobi elliptic functions (JEFs). If $\mathrm{m} \longrightarrow 1$, then JEFs are converted into the hyperbolic functions as follows:

$$dn(\xi) \longrightarrow \text{sech}\,(\xi) sn(\xi) \longrightarrow \tanh\,(\xi), cn(\xi) \longrightarrow \text{sech}\,(\xi),$$
$$cs(\xi) \longrightarrow \text{csch}\,(\xi), ds \longrightarrow \text{csch}\,(\xi). \quad (17)$$

Table 1: All possible solutions for Equation (16).

Case	p	q	r	$\chi(\xi)$
1	$2\mathrm{m}^2$	$-(1+\mathrm{m}^2)$	1	$sn(\xi)$
2	2	$2\mathrm{m}^2-1$	$-\mathrm{m}^2(1-\mathrm{m}^2)$	$ds(\xi)$
3	2	$2-\mathrm{m}^2$	$(1-\mathrm{m}^2)$	$cs(\xi)$
4	$-2\mathrm{m}^2$	$2\mathrm{m}^2-1$	$(1-\mathrm{m}^2)$	$cn(\xi)$
5	-2	$2-\mathrm{m}^2$	(m^2-1)	$dn(\xi)$
6	$\frac{\mathrm{m}^2}{2}$	$\frac{(\mathrm{m}^2-2)}{2}$	$\frac{1}{4}$	$\frac{sn(\xi)}{1 \pm dn(\xi)}$
7	$\frac{\mathrm{m}^2}{2}$	$\frac{(\mathrm{m}^2-2)}{2}$	$\frac{\mathrm{m}^2}{4}$	$\frac{sn(\xi)}{1 \pm dn(\xi)}$
8	$\frac{-1}{2}$	$\frac{(\mathrm{m}^2+1)}{2}$	$\frac{-(1-\mathrm{m}^2)^2}{4}$	$\mathrm{m}cn(\xi) \pm dn(\xi)$
9	$\frac{\mathrm{m}^2-1}{2}$	$\frac{(\mathrm{m}^2+1)}{2}$	$\frac{(\mathrm{m}^2-1)}{4}$	$\frac{dn(\xi)}{1 \pm sn(\xi)}$
10	$\frac{1-\mathrm{m}^2}{2}$	$\frac{(1-\mathrm{m}^2)}{2}$	$\frac{(1-m^2)}{4}$	$\frac{cn(\xi)}{1 \pm sn(\xi)}$
11	$\frac{(1-\mathrm{m}^2)^2}{2}$	$\frac{(1-\mathrm{m}^2)^2}{2}$	$\frac{1}{4}$	$\frac{sn(\xi)}{dn \pm cn(\xi)}$
12	2	0	0	$\frac{c}{\xi}$
13	0	1	0	ce^{ξ}

4.2. Solutions of FSBBS. By balancing φ'' with φ^3 in Equation (14), we can calculate the parameter N as

$$N + 2 = 3N \Rightarrow N = 1. \quad (18)$$

Hence, Equation (15) with $N = 1$ becomes

$$\varphi = a_0 + a_1\chi. \quad (19)$$

Differentiating Equation (19) twice, we have, by using (16),

$$\varphi'' = a_1 q\chi + a_1 p\chi^3. \quad (20)$$

Substituting Equation (19) and Equation (20) into Equation (14) we obtain

$$\begin{aligned}&\left(a_1 p - 8a_1^3\right)\chi^3 - \left(24a_0a_1^2 + 12\omega a_1^2\right)\chi^2\\ &\quad + \left(a_1 q - 24a_0^2 a_1 - 24\omega a_0 a_1 - 4\omega^2 a_1\right)\chi\\ &\quad - \left(8a_0^3 + 12\omega a_0^2 + 4\omega^2 a_0\right) = 0.\end{aligned} \quad (21)$$

Equating each coefficient of χ^k, for $k = 0, 1, 2, 3$, to zero,

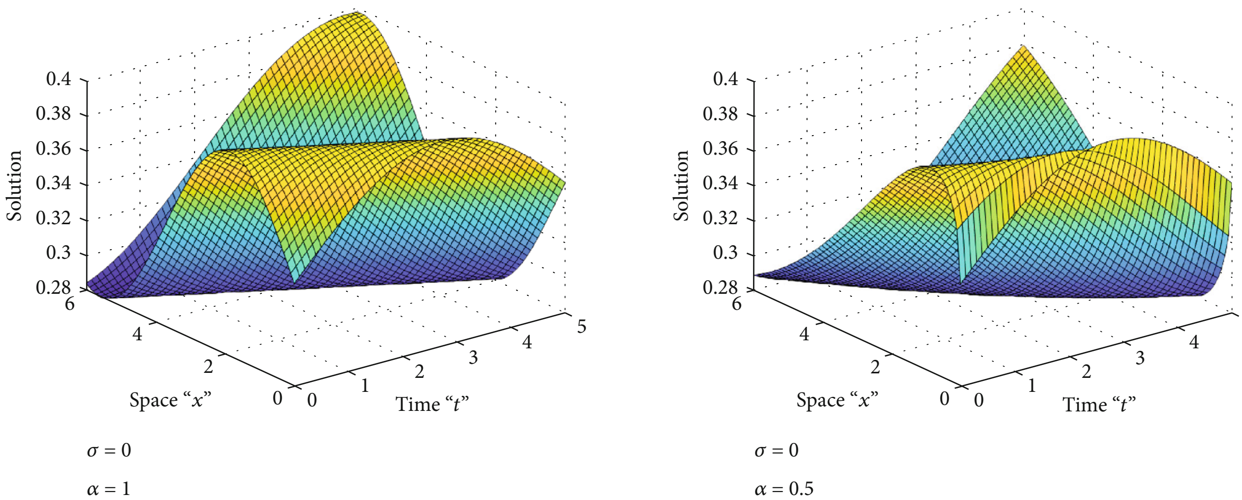

FIGURE 1: 3D-graph of Equation (33) with $\sigma = 0$ and various values of $\alpha = 1, 0.5$.

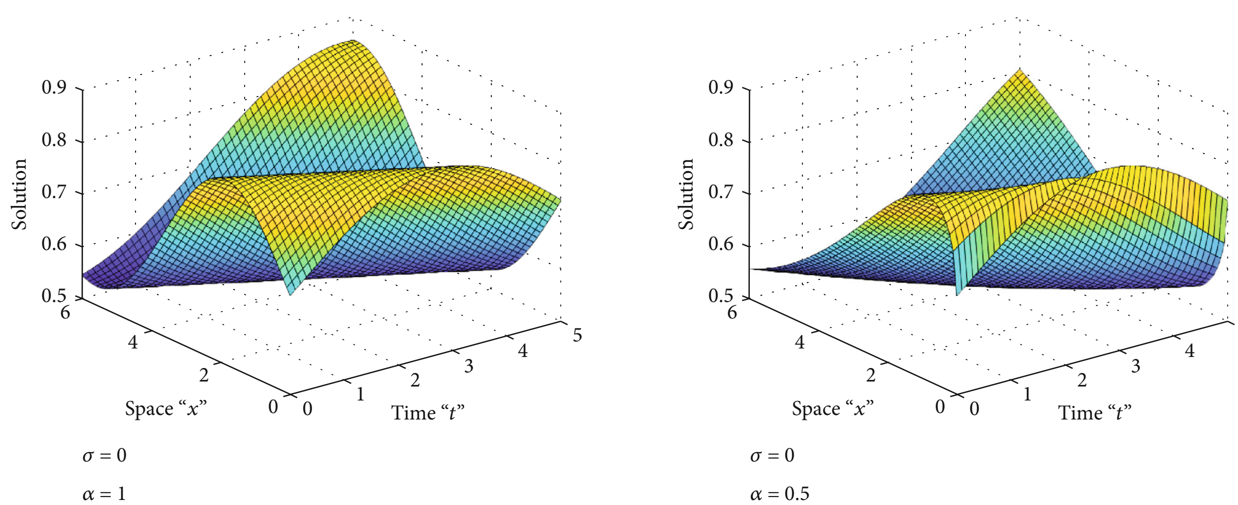

FIGURE 2: 3D-graph of Equation (34) with $\sigma = 0$ and various values of $\alpha = 1, 0.5$.

we have

$$\begin{aligned} a_1 p - 8a_1^3 &= 0, \\ 24a_0a_1^2 + 12\omega a_1^2 &= 0, \\ a_1 q - 24a_0^2 a_1 - 24\omega a_0 a_1 - 4\omega^2 a_1 &= 0, \end{aligned} \tag{22}$$

and

$$8a_0^3 + 12\omega a_0^2 + 4\omega^2 a_0 = 0. \tag{23}$$

We get by solving these equations:

$$a_0 = \frac{1}{2}\sqrt{-\frac{1}{2}q}, a_1 = \pm\sqrt{\frac{p}{8}}, \omega = \pm\sqrt{-\frac{1}{2}q}, \tag{24}$$

for $p > 0$ and $q \leq 0$. Then, the Equation (14) has the solutions:

$$\varphi(\xi) = \sqrt{-\frac{1}{8}q} \pm \sqrt{\frac{p}{8}}\chi(\xi). \tag{25}$$

Therefore, by utilizing (4) and (12), the solution of the FSBBS (2-1) are

$$\begin{aligned} \Phi(x,t) &= \left[\sqrt{-\frac{1}{8}q} + \sqrt{\frac{p}{8}}\chi(\xi)\right] e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}, \\ \Psi(x,t) &= \left[-\frac{3q}{4} + \sqrt{-pq}\chi(\xi) + \frac{p}{4}\chi^2\right] e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}. \end{aligned} \tag{26}$$

There are numerous cases, by using the previous Table 1, for $q \leq 0$, $p > 0$ and r as follows:

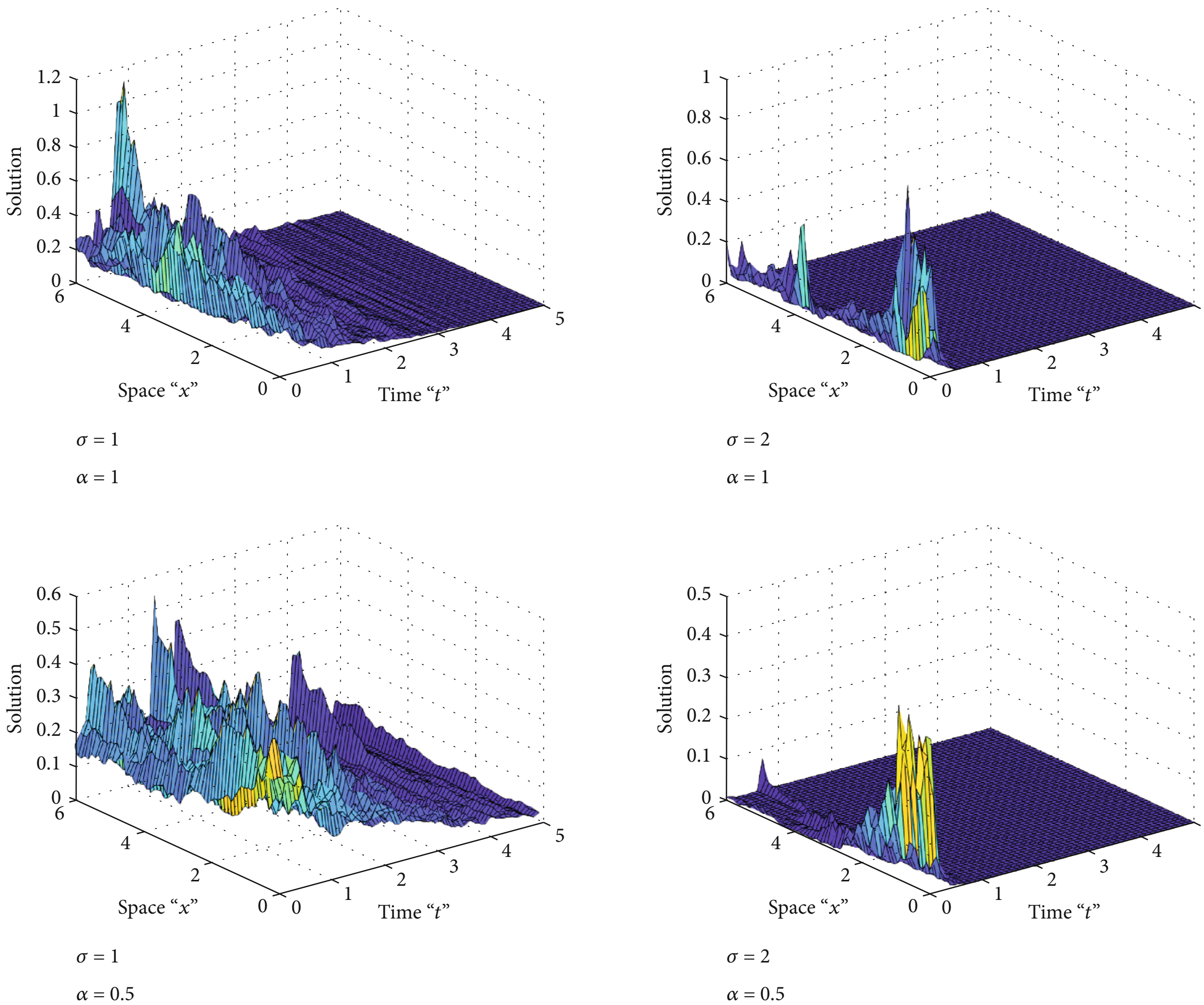

FIGURE 3: 3D-graph of Equation (33) with $\sigma = 1, 2$ and $\alpha = 1, 0.5$.

Case 1. If $q = -(1+m^2), p = 2m^2$ and $r = 1$, then $\chi(\xi) = sn(\xi)$. Therefore, the FSBBS (1-2) has the solution

$$\Phi(x,t) = \left[\sqrt{-\frac{1}{8}q} + \sqrt{\frac{p}{8}}sn\left(\frac{1}{\alpha}x^\alpha + \sqrt{\frac{-q}{2}}t\right)\right]e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)}, \tag{27}$$

$$\Psi(x,t) = \left[-\frac{3q}{4} + \sqrt{-pq}sn\left(\frac{x^\alpha}{\alpha} + \sqrt{\frac{-q}{2}}t\right) + \frac{p}{4}sn^2\left(\frac{x^\alpha}{\alpha} + \sqrt{\frac{-q}{2}}t\right)\right]e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)}. \tag{28}$$

If m⟶1, then Equations (27) and (28) degenerates to

$$\begin{aligned}\Phi(x,t) &= \left[\sqrt{-\frac{1}{8}q} + \sqrt{\frac{p}{8}}\tanh\left(\frac{1}{\alpha}x^\alpha + \sqrt{\frac{-q}{2}}t\right)\right]e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)},\\ \Psi(x,t) &= \left[-\frac{3q}{4} + \sqrt{-pq}\tanh\left(\frac{1}{\alpha}x^\alpha + \sqrt{\frac{-q}{2}}t\right) + \frac{p}{4}\tanh^2\left(\frac{1}{\alpha}x^\alpha + \sqrt{\frac{-q}{2}}t\right)\right]e^{\left(\sigma\mathbb{B}(t)-\left((1/2)\sigma^2 t\right)\right)}.\end{aligned} \tag{29}$$

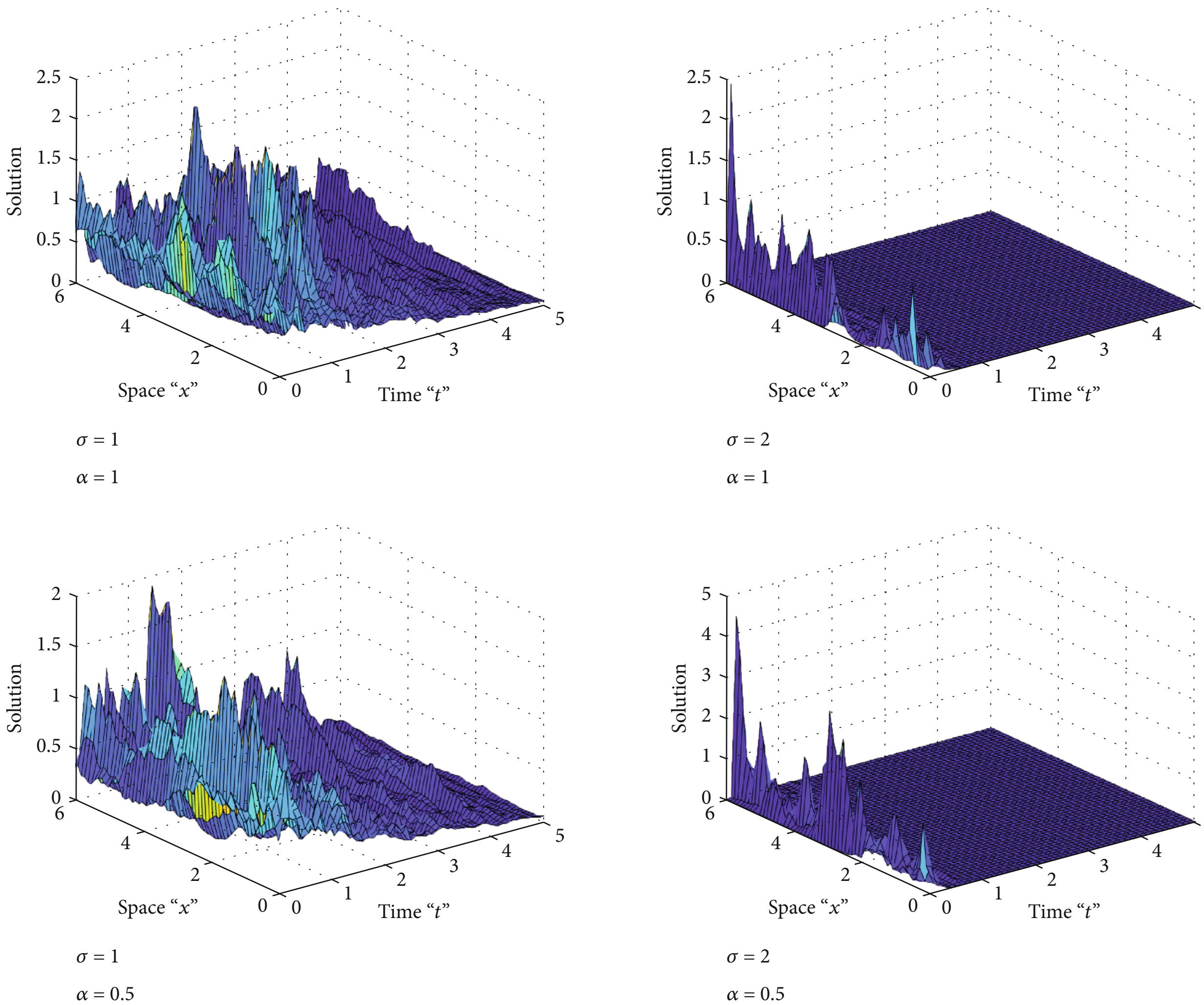

FIGURE 4: 3D-graph of Equation (34) with $\sigma = 1, 2$ and $\alpha = 1, 0.5$.

Case 2. If $q = 2\mathrm{m}^2 - 1$ for $m \le (1/\sqrt{2}), p = 2$ and $r = -\mathrm{m}^2(1 - m^2)$, then $\chi(\xi) = ds(\xi)$. So, the FSBBS (1-2) has the solution:

$$\Phi(x,t) = \left[\sqrt{-\frac{1}{8}q} + \sqrt{\frac{p}{8}} ds\left(\frac{1}{\alpha}x^{\alpha} + \sqrt{\frac{-q}{2}}t\right)\right] e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}, \tag{30}$$

$$\Psi(x,t) = \left[-\frac{3q}{4} + \sqrt{-pq} ds\left(\frac{x^{\alpha}}{\alpha} + \sqrt{\frac{-q}{2}}t\right) + \frac{p}{4} ds^2 \cdot \left(\frac{x^{\alpha}}{\alpha} + \sqrt{\frac{-q}{2}}t\right)\right] e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}. \tag{31}$$

If m⟶1, then Equations (30) and (31) degenerates to

$$\begin{aligned}
\Phi(x,t) &= \left[\sqrt{-\frac{1}{8}q} + \sqrt{\frac{p}{8}} \operatorname{csch}\left(\frac{x^{\alpha}}{\alpha} + \sqrt{\frac{-q}{2}}t\right)\right] e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}, \\
\Psi(x,t) &= \left[-\frac{3q}{4} + \sqrt{-pq} \operatorname{csch}\left(\frac{x^{\alpha}}{\alpha} + \sqrt{\frac{-q}{2}}t\right) + \frac{p}{4} \operatorname{csc h}^2\left(\frac{x^{\alpha}}{\alpha} + \sqrt{\frac{-q}{2}}t\right)\right] e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}.
\end{aligned} \tag{32}$$

Case 3. If $q = ((m^2 - 2)/2)$,$p = (m^2/2)$ and $r = (1/4)$ (or $r = (m^2/4)$), then $\chi(\xi) = ((sn(\xi))/(1 \pm dn(\xi)))$. Therefore, the FSBBS (1-2) has the solution:

$$\Phi(x,t) = \left[\sqrt{-\frac{1}{8}q} + \sqrt{\frac{p}{8}}\frac{\left(\mathrm{sn}\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)}{\left(1 \pm \mathrm{dn}\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)}\right]e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}, \tag{33}$$

$$\Psi(x,t) = \left[-\frac{3q}{4} + \sqrt{-pq}\frac{\left(\mathrm{sn}\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)}{\left(1 \pm \mathrm{dn}\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)} + + \frac{p}{4}\frac{\left(\mathrm{sn}^2\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)}{\left(1 \pm \mathrm{dn}\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)^2}\right]e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}. \tag{34}$$

If m $\longrightarrow$ 1, then Equations (33) and (34) degenerates to

$$\Phi(x,t) = \left[\sqrt{-\frac{1}{8}q} + \sqrt{\frac{p}{8}}\frac{\left(\tanh\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)}{\left(1 \pm \mathrm{sech}\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)}\right]e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}, \tag{35}$$

$$\Psi(x,t) = \left[-\frac{3q}{4} + \sqrt{-pq}\frac{\left(\tanh\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)}{\left(1 \pm \mathrm{sech}\left(((1/\alpha)x^\alpha) + \sqrt{(-q/2)}t\right)\right)} + + \frac{p}{4}\frac{\left(\tanh^2\left((x^\alpha/\alpha) + \sqrt{(-q/2)}t\right)\right)}{\left(1 \pm \mathrm{sech}\left((x^\alpha/\alpha) + \sqrt{(-q/2)}t\right)\right)^2}\right]e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}. \tag{36}$$

Case 4. If $q = 0$,$p = 2$ and $r = 0$, then $\chi(\xi) = (C/\xi)$. Hence, the FSBBS (1-2) has the solution:

$$\begin{aligned}\Phi(x,t) &= \left[\frac{\alpha C}{2}x^{-\alpha}\right]e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)},\\ \Psi(x,t) &= \left[\frac{\alpha C}{2}x^{-2\alpha}\right]e^{\left(\sigma\mathbb{B}(t) - \left((1/2)\sigma^2 t\right)\right)}.\end{aligned} \tag{37}$$

Remark 4. If we set $\sigma = 0$ and $\alpha = 1$ in Equations (27) and (36), then we get the same results as reported in [48].

5. The Influence of Fractional Derivative and Noise

Here, the influence of noise on the achieved solutions of FSBBS (1-2) is explained. For various values of α (the fractional derivative order) and σ (noise strength), some graphs are provided using the MATLAB tools.

Firstly the Fractional Derivative Influence. In Figures 1 and 2, if $\sigma = 0$ and m = 0.4, we can observe that the surface shrinks when α is decreasing:

Secondly the Noise Influence. In Figures 3 and 4, when noise is introduced, the surface flattens significantly if its strength is increased $\sigma = 1, 2$

In Figure 5, we introduce 2D-graph of the $\Phi(x,t)$ in (33) with $\alpha = 1$ and with $\sigma = 0, 0.5, 1, 2$, which highlights the previous outcomes:

We may deduct from Figures 1–5 that:

(1) When the fractional-order α increases, the surface expands,

(2) The multiplicative noise stabilizes the solutions of FSBBS at zero.

This results show that it is important to add the stochastic term into the Boussinesq-Burger equation in order to obtain accurate solutions.

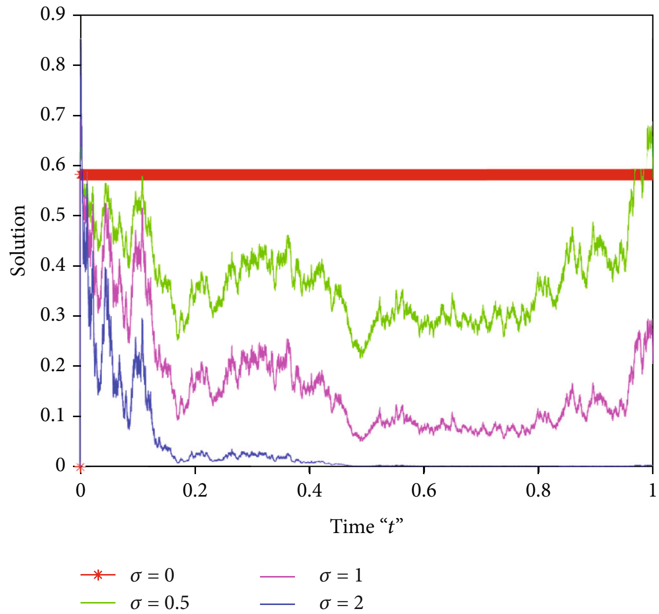

Figure 5: 2D graph of Equation (33)

6. Conclusions

In this article, the exact fractional-stochastic solutions of the fractional-stochastic Boussinesq-Burger system (1-2) driven by multiplicative noise were successfully obtained by using the Jacobi elliptic function method. Numerous analytical solutions for FSBBS (1-2) including elliptic, trigonometric, rational, and hyperbolic functions can be determined using the Jacobi elliptic function method. Because of the importance of FSBBS in fluid flow research and in explaining the propagation of shallow water waves, the acquired solutions are far more beneficial and efficient in understanding several critical complicated physical phenomena. In addition, we utilized the MATLAB package to demonstrate how multiplicative noise and fractional derivative influenced the solutions of FSBBS. As a result, we concluded that the stabilization of the solutions of the FSBBS (1-2) is affected by the multiplicative noise.

Data Availability

All data are available in this paper

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by the Princess Nourah Bint Abdulrahman University Researchers Supporting Project (No. PNURSP2022R273) of Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia.

References

[1] E. Scalas, R. Gorenflo, and F. Mainardi, "Fractional calculus and continuous-time finance," *Physica A*, vol. 284, no. 1-4, pp. 376–384, 2000.

[2] J. Sabatier, O. P. Agrawal, and J. A. T. Machado, *Advances in Fractional Calculus*, Springer, Dordrecht, 2007.

[3] L. Baudouin, A. Rondepierre, and S. Neild, "Robust control of a cable from a hyperbolic partial differential equation model," *IEEE Transactions on Control Systems Technology*, vol. 27, no. 3, pp. 1343–1351, 2019.

[4] L. Shao, X. Guo, S. Liu, and G. Zheng, *Effective Stress and Equilibrium Equation for Soil Mechanics*, CRC Press, Boca Raton, FL, USA, 2017.

[5] F. M. Al-Askar, W. W. Mohammed, and M. Alshammari, "Impact of Brownian motion on the analytical solutions of the space-fractional stochastic approximate long water wave equation," *Symmetry*, vol. 14, no. 4, p. 740, 2022.

[6] N. Laskin, "Nonlocal quantum mechanics: fractional calculus approach," in *Volume 5 Applications in Physics, Part B*, pp. 207–236, Walter de Gruyter, 2019.

[7] E. Barkai, R. Metzler, and J. Klafter, "From continuous time random walks to the fractional Fokker-Planck equation," *Physical Review E*, vol. 61, no. 1, pp. 132–138, 2000.

[8] Z. He, Z. Han, J. Yuan et al., "Quantum plasmonic control of trions in a picocavity with monolayer WS2," *Science Advances*, vol. 5, no. 10, article eaau8763, 2019.

[9] Y. Lin and J. Gao, "Research on diffusion effect of ecological population model based on delay differential equation," *Caribbean Journal of Science*, vol. 52, pp. 333–335, 2019.

[10] A. Biswas and D. Milovic, "Bright and dark solitons of the generalized nonlinear Schrödinger's equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 6, pp. 1473–1484, 2010.

[11] A. M. Wazwaz, “Bright and dark optical solitons for (2+1)-dimensional Schrodinger (NLS) equations in the anomalous dispersion regimes and the normal dispersive regimes,” *Optik*, vol. 192, article 162948, 2019.

[12] H. Rezazadeh, A. Korkmaz, A. E. Achab, W. Adel, and A. Bekir, “New travelling wave solution-based new Riccati equation for solving KdV and modified KdV equations,” *Applied Mathematics and Nonlinear Sciences*, vol. 6, no. 1, pp. 447–458, 2021.

[13] H. U. Rehman, M. Bibi, M. S. Saleem, H. Rezazadeh, and W. Adel, “New optical soliton solutions of the Chen Lee Liu equation,” *International Journal of Modern Physics B*, vol. 35, no. 18, p. 2150184, 2021.

[14] A. M. Wazwaz, “The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations,” *Applied Mathematics and Computation*, vol. 167, no. 2, pp. 1196–1210, 2005.

[15] W. Malfliet and W. Hereman, “The tanh method. I. Exact solutions of nonlinear evolution and wave equations,” *Physica Scripta*, vol. 54, no. 6, pp. 563–568, 1996.

[16] M. Wen-Xiu and B. Sumayah, “A binary darboux transformation for multicomponent NLS equations and their reductions,” *Analysis and Mathematical Physics*, vol. 11, no. 2, p. 44, 2021.

[17] A. M. Wazwaz, “A sine-cosine method for handlingnonlinear wave equations,” *Mathematical and Computer Modelling*, vol. 40, no. 5-6, pp. 499–508, 2004.

[18] C. Yan, “A simple transformation for nonlinear waves,” *Physics Letters A*, vol. 224, no. 1-2, pp. 77–84, 1996.

[19] K. Khan and M. A. Akbar, “The $\exp(-\Phi(\varsigma))$-expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation,” *International Journal of Dynamical Systems and Differential Equations*, vol. 5, no. 1, pp. 72–83, 2014.

[20] M. L. Wang, X. Z. Li, and J. L. Zhang, “The (G’G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics,” *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.

[21] H. Zhang, “New application of the G’G-expansion method,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 8, pp. 3220–3225, 2009.

[22] W. W. Mohammed, M. Alesemi, S. Albosaily, N. Iqbal, and M. El-Morshedy, “The exact solutions of stochastic fractional-space Kuramoto-Sivashinsky equation by using (G’G)-Expansion Method,” *Mathematics*, vol. 9, no. 21, p. 2712, 2021.

[23] M. S. Hashemi, A. Haji-Badali, and P. Vafadar, “Group invariant solutions and conservation laws of the Fornberg– Whitham equation,” *Zeitschrift für Naturforschung A*, vol. 69, no. 8-9, pp. 489–496, 2014.

[24] A. Akbulut, M. S. Hashemi, and H. Rezazadeh, “New conservation laws and exact solutions of coupled burgers’ equation,” *Waves in Random and Complex Media*, pp. 1–20, 2021.

[25] A. Akbulut and S. M. R. Islam, “Study on the Biswas-Arshed equation with the Beta time derivative,” *International Journal of Applied and Computational Mathematics*, vol. 8, no. 4, p. 167, 2022.

[26] R. Hirota, “Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons,” *Physical Review Letters*, vol. 27, no. 18, pp. 1192–1194, 1971.

[27] Z. L. Yan, “Abundant families of Jacobi elliptic function solutions of the (2+1)-dimensional integrable Davey-Stewartson-type equation via a new method,” *Chaos, Solitons and Fractals*, vol. 18, no. 2, pp. 299–309, 2003.

[28] F. M. Al-Askar and W. W. Mohammed, “The analytical solutions of the stochastic fractional RKL equation via Jacobi elliptic function method,” *Advances in Mathematical Physics*, vol. 2022, Article ID 1534067, 8 pages, 2022.

[29] W. W. Mohammed, “Stochastic amplitude equation for the stochastic generalized Swift-Hohenberg equation,” *Journal of the Egyptian Mathematical Society*, vol. 23, no. 3, pp. 482–489, 2015.

[30] W. W. Mohammed, “Amplitude equation with quintic nonlinearities for the generalized Swift-Hohenberg equation with additive degenerate noise,” *Advances in Difference Equations*, vol. 2016, no. 1, Article ID 84, 2016.

[31] S. B. Yuste, L. Acedo, and K. Lindenberg, “Reaction front in anA+B→Creaction-subdiffusion process,” *Physical Review E*, vol. 69, no. 3, article 036126, 2004.

[32] W. W. Mohammed, O. Bazighifan, M. M. Al-Sawalha, A. O. Almatroud, and E. S. Aly, “The influence of noise on the exact solutions of the stochastic fractional-space chiral nonlinear Schrödinger equation,” *Fractal and Fractional*, vol. 5, no. 4, p. 262, 2021.

[33] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert, “The fractional-order governing equation of Lévy motion,” *Water Resources Research*, vol. 36, no. 6, pp. 1413–1423, 2000.

[34] N. Iqbal, R. Wu, and W. W. Mohammed, “Pattern formation induced by fractional cross-diffusion in a 3-species food chain model with harvesting,” *Mathematics and Computers in Simulation*, vol. 188, pp. 102–119, 2021.

[35] H. Tajadodi, Z. A. Khan, A. R. Irshad, J. F. Gómez-Aguilare, A. Khan, and H. Khan, “Exact solutions of conformable fractional differential equations,” *Results in Physics.*, vol. 22, article 103916, 2021.

[36] H. Bulut, T. A. Sulaiman, and H. M. Baskonus, “Dark, bright optical and other solitons with conformable space-time fractional second-order spatiotemporal dispersion,” *Optik*, vol. 163, pp. 1–7, 2018.

[37] H. Rezazadeh, H. Tariq, M. Mirzazadeh, and Q. Zhou, “New exact solutions of nonlinear conformable time-fractional Phi-4 equation,” *Chinese Journal of Physics*, vol. 56, no. 6, pp. 2805–2816, 2018.

[38] S. W. Yao, T. Rasool, R. Hussain, H. Rezazadeh, and M. Inc, “Exact soliton solutions of conformable fractional coupled Burger's equation using hyperbolic funtion approach,” *Results in Physics*, vol. 30, article 104776, 2021.

[39] M. A. Shallal, K. K. Ali, K. R. Raslan, H. Rezazadeh, and A. Bekir, “Exact solutions of the conformable fractional EW and MEW equations by a new generalized expansion method,” *Journal of Ocean Engineering and Science*, vol. 5, no. 3, pp. 223–229, 2020.

[40] A. Zafar, M. Raheel, A. Bekir, and W. Razzaq, “The conformable space-time fractional Fokas-Lenells equation and its optical soliton solutions based on three analytical schemes,” *International Journal of Modern Physics B*, vol. 35, no. 1, p. 2150004, 2021.

[41] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, 1998.

[42] P. Imkeller and A. H. Monahan, “Conceptual stochastic climate models,” *Stochastics and Dynamics*, vol. 2, no. 3, pp. 311–326, 2002.

[43] W. W. Mohammed and D. Blömker, “Fast-diffusion limit with large noise for systems of stochastic reaction-diffusion equations,” *Stochastic Analysis and Applications*, vol. 34, no. 6, pp. 961–978, 2016.

[44] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, “A new definition of fractional derivative,” *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[45] P. Wang, B. Tian, W. Liu, X. Lü, and Y. Jiang, “Lax pair, Bäcklund transformation and multi-soliton solutions for the Boussinesq–Burgers equations from shallow water waves,” *Applied Mathematics and Computation*, vol. 218, no. 5, pp. 1726–1734, 2011.

[46] I. E. Mhlanga and C. M. Khalique, “Exact solutions of generalized Boussinesq-Burgers equations and (2+1)-dimensional Davey-Stewartson equations,” *Journal of Applied Mathematics*, vol. 2012, Article ID 389017, 8 pages, 2012.

[47] K. K. Ali, R. Yilmazer, and H. Bulut, “Analytical Solutions to the Coupled Boussinesq–Burgers Equations via Sine-Gordon Expansion Method,” in *4th International Conference on Computational Mathematics and Engineering Sciences (CMES-2019)*, pp. 233–240, Springer, 2020.

[48] A. A. Rady and M. Khalfallah, “On soliton solutions for Boussinesq-Burgers equations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 4, pp. 886–894, 2010.

[49] K. Mohammed, “Exact traveling wave solutions of the Boussinesq-Burgers equation,” *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 666–671, 2009.

[50] X. Li and A. Chen, “Darboux transformation and multi-soliton solutions of Boussinesq-Burgers equation,” *Physics Letters A*, vol. 342, no. 5-6, pp. 413–420, 2005.

[51] T. A. Sulaiman, H. Bulut, A. Yokus, and H. M. Baskonus, “On the exact and numerical solutions to the coupled Boussinesq equation arising in ocean engineering,” *Indian Journal of Physics*, vol. 93, no. 5, pp. 647–656, 2019.

[52] L. K. Ravi, S. S. Ray, and S. Sahoo, “New exact solutions of coupled Boussinesq–Burgers equations by Exp-function method,” *Journal of Ocean Engineering and Science.*, vol. 2, no. 1, pp. 34–46, 2017.

[53] L. Meenatchi and M. Kaliyappan, “Solving time-fractional nonlinear coupled Boussinesq-Burgers equations arise in propagation of shallow water waves using adomian decomposition method,” *AIP Conference Proceedings*, vol. 2095, article 030015, 2019.

[54] M. M. A. Khater and D. Kumar, “New exact solutions for the time fractional coupled Boussinesq–Burger equation and approximate long water wave equation in shallow water,” *Journal of Ocean Engineering and Science*, vol. 2, no. 3, pp. 223–228, 2017.

[55] O. Calin, *An informal introduction to stochastic calculus with applications*, World Scientific Publishing Co. Pte. Ltd, 2015.

[56] Y. Z. Peng, “Exact solutions for some nonlinear partial differential equations,” *Physics Letters A*, vol. 314, no. 5-6, pp. 401–408, 2003.