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Research Article

Some Oscillation Criteria for a Class of Third-Order Delay Differential Equations

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In this paper, we study oscillatory properties of solutions to a class of third-order differential equation $(a(t)((r(t)x'(t))')^{\alpha})' + (p(t)((r(t)x'(t)))')^{\alpha} + q(t)f(x(\sigma(t))) = 0$, where $f(x)/x^{\beta} \geq k > 0$ and α and β are quotients of odd positive integers. By using the generalized Riccati technique, we obtain some oscillation and asymptotic criteria when $\alpha \geq \beta$ and $\alpha < \beta$. Finally, some examples are given to show the effectiveness of the criteria obtained.

1. Introduction

Oscillation phenomena take part in different models from real-world applications (see, e.g., the papers [1, 2]), where oscillation and delay situations take part in models from mathematical biology when their formulation includes cross-diffusion terms. The oscillation theory of differential equations is an active area of research. Many scholars have studied the oscillation of solutions to various classes of differential equations and made some progress, especially for the first-order and second-order equations (see, e.g., the monographs [3, 4], the papers [5, 6], and the references cited therein). Third-order differential equations originate in many different fields of applied mathematics and physics, for example, the deflection of a buckling beam with a fixed or varying cross section, three-layer beam, electromagnetic waves, and the rising tide caused by gravitational blowing. The theory of third-order differential equations can be applied to many fields, such as to biology, population growth, engineering, generic repression, control theory, and climate model. Recently, a great deal of interest in asymptotic properties and oscillatory behavior of solutions to different classes of third-order delay differential equations has been manifested (see, e.g., the papers [7–19] and the references cited therein).

In particular, by using the Riccati transformation technique, the oscillation properties of the equation

$$\left(r|u'|^{\gamma-1}u'\right)'(t) + \left(q|y(\sigma)|^{\beta-1}y(\sigma)\right)(t) = 0, \tag{1}$$

have been considered by Liu et al. [20] under the conditions $\gamma \geq \beta$, $r\prime(t) > 0$, and $\sigma'(t) > 0$. Based on the properties of comformable fractional differential and integral, Feng and Meng have considered the following fractional-order dynamic equation on time scales [21]:

$$\begin{aligned}&\left(a(t)\left(\left[r(t)x^{(\alpha)}(t)\right]^{(\alpha)}\right)^{\gamma}\right)^{(\alpha)}\\&\quad + p(t)\left(\left[r(t)x^{(\alpha)}(t)\right]^{(\alpha)}\right)^{\gamma} + q(t)f(x(t)) = 0,\end{aligned} \tag{2}$$

and obtained some oscillation criteria where $f(x)/x^{\gamma} \geq L > 0$ for $x \neq 0$, and $\gamma \geq 1$ is a quotient of two odd positive integers. By using the integral averaging technique, the equation

$$\begin{aligned}&\left(r_2(\tau)\left(r_1(\tau)(y'(\tau))^{\alpha}\right)'\right)' + \phi(\tau, y'(\delta(\tau)))\\&\quad + q(\tau)f(y(\sigma(\tau))) = 0,\end{aligned} \tag{3}$$

has been studied by Moaaz et al. [22] under the conditions that $f(x)/x^{\beta} \geq k_2$, $\alpha \geq \beta$, and $\sigma(t)$ is nondecreasing. However, most of the oscillation criteria have been obtained under the condition $\alpha \geq \beta$ or $\alpha < \beta$. Motivated by the above works, we further study the oscillation criteria of third-order equation when $\alpha \geq \beta$ and $\alpha < \beta$.

In this paper, we mainly study the oscillation solution of the following third-order delay differential equation:

$$\left(a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\right)' + p(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha} + q(t)f(x(\sigma(t))) = 0, \tag{4}$$

where $t \geq t_0$ and α is a quotient of odd positive integers. Throughout this paper, we assume that

(A1) $a(t) \in C^1([t_0,\infty)t, n(q0,\infty))$,
$r(t) \in C^2([t_0,\infty)t, n(q0,\infty))$,
$p(t) \in C([t_0,\infty)t, n(q0,\infty))$,
$q(t) \in C([t_0,\infty)t, n(q0,\infty))$.

(A2) $\sigma(t) \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) \leq t$, $\sigma\prime(t) \geq 0$, and $\lim_{t\longrightarrow\infty}\sigma(t) = \infty$.

(A3) $f \in C(\mathbb{R},\mathbb{R})$ and $f(x)/x^{\beta} \geq k > 0$, and β is a quotient of odd positive integers.

A function $x(t)$ is called a solution of (4) if $x(t)$ satisfies (4) on $[Tx,\infty)$ and if $x(t)$, $r(t)x'(t)$ and $((r(t)x\prime(t))')^{\alpha} \in C^1([T_x,\infty],\mathbb{R})$ where $T_x \geq t_0$. We consider solutions of (4) which satisfy $\sup|x(t)|$: $T \leq t < \infty$, $T_x \geq t_x$. A solution of (4) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (4) is said to be oscillatory if all its solutions are oscillatory.

2. Preliminaries

In this section, we give the following lemmas which are used as tools in establishing new oscillation criteria for (4).

We introduce the following notation:

$$\begin{aligned}
v(t) &= \exp\left(\int_{t_0}^{t}\frac{p(s)}{a(s)}\mathrm{d}s\right),\\
\eta(t,t_0) &= \int_{t_0}^{t}\frac{1}{r(s)}\mathrm{d}s,\\
v(t,t_0) &= \int_{t_0}^{t}\frac{1}{(v(s)a(s))^{1/\alpha}}\mathrm{d}s,\\
R(t,t_0) &= \int_{t_0}^{t}\frac{v(s,t_0)}{r(s)}\mathrm{d}s,\\
Q(t) &= \begin{cases}\lambda_1, & \text{if}\beta > \alpha;\\ 1, & \text{if}\beta = \alpha;\\ \lambda_2 R^{\beta-\alpha/\alpha}(t,t_1), & \text{if}\beta < \alpha,\end{cases}
\end{aligned} \tag{5}$$

where λ_1 and λ_2 are positive real constants.

Lemma 1. *Assume that $x(t)$ is the positive solution of (4) and*

$$\int_{t_0}^{\infty}\frac{1}{(v(s)a(s))^{1/\alpha}}\mathrm{d}s = \infty, \tag{6}$$

$$\int_{t_0}^{\infty}\frac{1}{r(s)}\mathrm{d}s = \infty. \tag{7}$$

Then, one of the following two cases holds.

(I) $x(t) > 0$, $x\prime(t) > 0$, $(r(t)x\prime(t))' > 0$.

(II) $x(t) > 0$, $x\prime(t) < 0$, $(r(t)x\prime(t))' > 0$ for $t \geq t_1 \geq t_0$, where t_1 is sufficiently large.

Proof. Let $x(t)$ be the positive solution of (4). Without loss of generality, we may assume that there exists a sufficiently large t_1 such that $x(t) > 0$ and $x(\sigma(t)) > 0$.

From (4), we obtain

$$\begin{aligned}
\left(v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\right)' &= v(t)\left(a(t)\left(\left(r(t)x\prime(t)\right)'\right)^{\alpha}\right)' + v(t)p(t)\left(\left(r(t)x\prime(t)\right)'\right)^{\alpha}\\
&= -v(t)q(t)f(x(\sigma(t)))\\
&< 0.
\end{aligned} \tag{8}$$

Hence, $v(t)a(t)((r(t)x\prime(t))')^{\alpha}$ is strictly decreasing on $[t_1,\infty)$. Because $a(t) > 0$, $v(t) > 0$, we can deduce that $(r(t)x\prime(t))'$ is eventually of one sign.

We can claim that $(r(t)x\prime(t))' > 0$ for $t \geq t_2 \geq t_1$. In fact, if $(r(t)x\prime(t))' \leq 0$, according to the monotonicity of $v(t)a(t)((r(t)x\prime(t))')^{\alpha}$, we have

$$v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha} \leq v(t_2)a(t_2)\left(\left(r(t_2)x'(t_2)\right)'\right)^{\alpha} = -M, \tag{9}$$

for $M > 0$. Integrating (9) from t_2 to t, we obtain

$$r(t)x\prime(t) \leq r(t_2)x\prime(t_2) - \int_{t_2}^{t} M^{1/\alpha}(v(s)a(s))^{-1/\alpha}\mathrm{d}s. \tag{10}$$

Letting $t \longrightarrow \infty$, from (6), we have $r(t)x\prime(t) \longrightarrow -\infty$. Thus, there exists a sufficiently large t_3 such that $r(t)x\prime(t) < 0$ for $t \geq t_3 \geq t_2$. Furthermore, we have $r(t)x\prime(t) \leq r(t_3)x\prime(t_3)$. Dividing by $r(t)$ and integrating on $[t_3,t]$, we have

$$x(t) \leq x(t_3) + r(t_3)x\prime(t_3)\int_{t_3}^{t}\frac{1}{r(s)}\mathrm{d}s. \tag{11}$$

From (7), we have $\lim_{t\longrightarrow\infty}x(t) = -\infty$. This contradicts $x(t) > 0$, which implies $(r(t)x\prime(t))' > 0$. We complete the proof. □

Lemma 2. *Assume that $x(t)$ satisfies case (II). Suppose*

$$\int_{t_0}^{\infty}\frac{1}{r(v)}\int_{u}^{\infty}\left[\frac{1}{r(u)a(u)}\int_{u}^{\infty}v(s)q(s)\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u\mathrm{d}v=\infty, \tag{12}$$

and then $\lim_{t\longrightarrow\infty}x(t)=0$.

Proof. Let $x(t)$ be an eventually positive solution of equation (4). Because $x(t)$ has property (II), we obtain $\lim_{t\longrightarrow\infty}x(t)=l\geq 0$.

We claim $l=0$. Otherwise, if we assume $l>0$, then it follows that $x(t)\geq l$ for $t\geq t_1\geq t_0$.

From (A3), we have

$$\begin{aligned}\left(v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\right)'&=-v(t)q(t)f(x(\sigma(t)))\\&\leq -kv(t)q(t)x^{\beta}(\sigma(t))\\&\leq 0.\end{aligned} \tag{13}$$

Integrating the above inequality on $[t,\infty)$, we have

$$\begin{aligned}-v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}&\leq -\lim_{t\longrightarrow\infty}v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\\&\quad -k\int_{t}^{\infty}v(s)q(s)x^{\beta}(\sigma(s))\mathrm{d}s\\&\leq -k\int_{t}^{\infty}v(s)q(s)x^{\beta}(\sigma(s))\mathrm{d}s\\&\leq -kl^{\beta}\int_{t}^{\infty}v(s)q(s)\mathrm{d}s,\end{aligned} \tag{14}$$

which means

$$-\left(r(t)x'(t)\right)'\leq -k^{1/\alpha}l^{\beta/\alpha}\left[\frac{1}{v(t)a(t)}\int_{t}^{\infty}v(s)q(s)\mathrm{d}s\right]^{1/\alpha}. \tag{15}$$

Integrating inequality (15) on $[t,\infty)$, we obtain

$$\begin{aligned}r(t)x'(t)&\leq \lim_{t\longrightarrow\infty}r(t)x'(t)-k^{1/\alpha}l^{\beta/\alpha}\int_{t}^{\infty}\left[\frac{1}{v(u)a(u)}\int_{u}^{\infty}v(s)q(s)\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u\\&\leq -k^{1/\alpha}l^{\beta/\alpha}\int_{t}^{\infty}\left[\frac{1}{v(u)a(u)}\int_{u}^{\infty}v(s)q(s)\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u.\end{aligned} \tag{16}$$

Further integrate the above inequality on $[t_1,t]$ to get

$$x(t)\leq x(t_1)-k^{1/\alpha}l^{\beta/\alpha}\int_{t_1}^{t}\frac{1}{r(v)}\int_{v}^{\infty}\left[\frac{1}{v(u)a(u)}\int_{u}^{\infty}v(s)q(s)\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u\mathrm{d}v. \tag{17}$$

Letting $t\longrightarrow\infty$, by (12), we can deduce that $\lim_{t\longrightarrow\infty}x(t)=-\infty$, which leads to a contradiction. So, we have $l=0$, and then $\lim_{t\longrightarrow\infty}x(t)=0$. We complete the proof. □

Lemma 3. *Assume (6) and (7) hold. If $x(t)$ is a positive solution of equation (4) with case (I) for $t\geq t_1\geq t_0$, where t_1 is sufficiently large, then for $t\in[t_1,\infty)$, the following inequality holds:*

$$\begin{aligned}x'(t)&\geq\frac{v(t,t_1)}{r(t)}\left(v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\right)^{1/\alpha},\\x^{\beta-\alpha}(t)&\geq\begin{cases}m, & \text{if}\beta>\alpha,\\1, & \text{if}\beta=\alpha,\\MR^{\beta-\alpha}(t,t_1), & \text{if}\beta<\alpha,\end{cases}\end{aligned} \tag{18}$$

where m, M are positive real constants.

Proof. Let $x(t)$ be a positive solution of (4). We assume that there exists a $t_1\geq t_0$ such that $x(t)>0$ and $x(\sigma(t))>0$ for $t\geq t_1$. By Lemma 1, we know

$$\begin{aligned}r(t)x'(t)&\geq r(t)x'(t)-r(t_1)x'(t_1)=\int_{t_1}^{t}\left(r(s)x'(s)\right)'\mathrm{d}s\\&\geq\int_{t_1}^{t}\frac{\left(v(s)a(s)\left(\left(r(s)x'(s)\right)'\right)^{\alpha}\right)^{1/\alpha}}{\left(v(s)a(s)\right)^{1/\alpha}}\mathrm{d}s.\end{aligned} \tag{19}$$

From (13), we can see that $v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}$ is decreasing on $[t_1,\infty)$. Hence,

$$\begin{aligned}r(t)x'(t)&\geq\left(v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\right)^{1/\alpha}\int_{t_1}^{t}\frac{1}{\left(v(s)a(s)\right)^{1/\alpha}}\mathrm{d}s\\&=\left(v(t)a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\right)^{1/\alpha}v(t,t_1).\end{aligned} \tag{20}$$

Thus,

$$x'(t) \geq \frac{v(t,t_1)}{r(t)}\left(v(t)a(t)\left(\left(r(t)x\prime(t)\right)'\right)^{\alpha}\right)^{1/\alpha}. \quad (21)$$

Since $x(t)$ is positive and increasing, we have $x(t) \geq x(t_1) = m_1$ for $t \geq t_1 \geq t_0$. Moreover, due to the monotonicity of $v(t)a(t)((r(t)x'(t))')^{\alpha}$, we have

$$v(t)a(t)\left(\left(r(t)x\prime(t)\right)'\right)^{\alpha} \leq v(t_1)a(t_1)\left(\left(r(t_1)x'(t_1)\right)'\right)^{\alpha} = M_1, \quad (22)$$

where $M \geq 0$ for $t \geq t_2 \geq t_1 \geq t_0$. Hence,

$$\left(r(t)x\prime(t)\right)' \leq M_1^{1/\alpha}\frac{1}{\left(v(t)a(t)\right)^{1/\alpha}}. \quad (23)$$

Integrating the above inequality from t_1 to t, we have

$$\begin{aligned} r(t)x\prime(t) &\leq r(t_1)x\prime(t_1) + M_1^{1/\alpha}\int_{t_1}^{t}\frac{1}{\left(v(s)a(s)\right)^{1/\alpha}}ds \\ &= r(t_1)x\prime(t_1) + M_1^{1/\alpha}v(t,t_1) \\ &\leq \left(\frac{r(t_1)x\prime(t_1)}{v(t,t_1)} + M_1^{1/\alpha}\right)v(t,t_1) \\ &\leq \left(\frac{r(t_1)x\prime(t_1)}{v(t_2,t_1)} + M_1^{1/\alpha}\right)v(t,t_1) = M_1^{*}v(t,t_1), \end{aligned} \quad (24)$$

where $M_1^{*} = r(t_1)x\prime(t_1)/v(t_2,t_1) + M_1^{1/\alpha}$. Thus,

$$x\prime(t) \leq \frac{1}{r(t)}M_1^{*}v(t,t_1). \quad (25)$$

Furthermore,

$$\begin{aligned} x(t) &\leq x(t_1) + M_1^{*}\int_{t_1}^{t}\frac{1}{r(s)}v(s,t_1)ds \\ &\leq \left(\frac{x(t_1)}{R(t,t_1)} + M_1^{*}\right)R(t,t_1) \\ &\leq \left(\frac{x(t_1)}{R(t_2,t_1)} + M_1^{*}\right)R(t,t_1) \\ &= M_2R(t,t_1), \end{aligned} \quad (26)$$

where $M_2 = x(t_1)/R(t_2,t_1) + M_1^{*}$. Then, we have

$$x^{\beta-\alpha}(t) \geq \begin{cases} m, & \text{if}\beta > \alpha, \\ 1, & \text{if}\beta = \alpha, \\ MR^{\beta-\alpha}(t,t_1), & \text{if}\beta < \alpha, \end{cases} \quad (27)$$

for $t \geq t_2 \geq t_1$, where $m = m_1^{\beta-\alpha}$, $M = M_2^{\beta-\alpha}$. The proof is complete. □

Lemma 4 (see [23, Theorem 41]). *Assuming that A and B are nonnegative real numbers, then*

$$A^{\gamma} + (\gamma-1)B^{\gamma} - \gamma AB^{\gamma-1} \geq 0\ (\gamma > 1), \quad (28)$$

3. Oscillation Results

In this section, we establish some new oscillation criteria for (4). For the following theorem, we introduce a class of function $\mathscr{R}$. Let

$$\begin{aligned} D &= (t,s)\colon t_0 \leq s \leq t, \\ D_0 &= (t,s)\colon t_0 \leq s < t. \end{aligned} \quad (29)$$

The function $H \in C(D,\mathbb{R})$ is said to belong to the class $\mathscr{R}$, if

(i) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $(t,s) \in D_0$.

(ii) H has a continuous and nonpositive partial derivative $\partial H(t,s)/\partial s$ on D_0 with respect to s.

Theorem 1. *Assume that (6), (7), and (12) hold. If there exists a function $\rho(t) \in C^1([t_0.\infty),R_+)$ such that for all sufficiently large T, the following condition holds:*

$$\limsup_{t\longrightarrow\infty}\int_{T}^{t}\left[k\rho(s)v(s)q(s) - \frac{\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\alpha r(\sigma(s))}{\beta\sigma\prime(s)v(\sigma(s),t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{\rho_{+'}(s)}{\rho(s)}\right)^{\alpha+1}\right]ds = \infty, \quad (30)$$

where $\rho_{+'}(t) = \max(0,\rho\prime(t))$, then every solution of equation (4) is oscillatory or tends to zero.

Proof. Let $x(t)$ be a nonoscillatory solution of (4). Without loss of generality, we may assume that there exists a sufficiently large t_1 such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t > t_1$. According to Lemma 1, $x(t)$ is either the case of (I) or the case of (II).

Now we assume $x(t)$ satisfies the (I); define

$$w(t) = \rho(t)\frac{v(t)a(t)\left((r(t)x\prime(t))'\right)^{\alpha}}{x^{\beta}(\sigma(t))}\, t \geq t_1. \tag{31}$$

Then, $w(t) > 0$, and

$$w\prime(t) = \frac{\rho\prime(t)}{\rho(t)}w(t) + \rho(t)\frac{\left(v(t)a(t)\left((r(t)x'(t))'\right)^{\alpha}\right)'}{x^{\beta}(\sigma(t))} - \rho(t)\frac{\beta v(t)a(t)\left((r(t)x'(t))'\right)^{\alpha}x\prime(\sigma(t))\sigma\prime(t)}{x^{\beta+1}(\sigma(t))}. \tag{32}$$

From (13), we know

$$\begin{aligned} w\prime(t) &\leq \frac{\rho\prime(t)}{\rho(t)}w(t) - k\rho(t)v(t)q(t) - \rho(t)\frac{\beta v(t)a(t)\left((r(t)x'(t))'\right)^{\alpha}x\prime(\sigma(t))\sigma\prime(t)}{x^{\beta+1}(\sigma(t))} \\ &\leq \frac{\rho\prime(t)}{\rho(t)}w(t) - k\rho(t)v(t)q(t) - \beta\frac{x\prime(\sigma(t))\sigma\prime(t)}{x(\sigma(t))}w(t). \end{aligned} \tag{33}$$

By Lemma 3 and (A2), it is obvious that

$$\begin{aligned} x\prime(\sigma(t)) &\geq \frac{v(\sigma(t),t_1)}{r(\sigma(t))}\left(v(\sigma(t))a(\sigma(t))\left((r(\sigma(t))x\prime(\sigma(t)))'\right)^{\alpha}\right)^{1/\alpha} \\ &\geq \left(v(t)a(t)\left((r(t)x\prime(t))'\right)^{\alpha}\right)^{1/\alpha}\frac{v(\sigma(t),t_1)}{r(\sigma(t))}. \end{aligned} \tag{34}$$

Thus,

$$w\prime(t) \leq \frac{\rho\prime(t)}{\rho(t)}w(t) - k\rho(t)v(t)q(t) - \beta\frac{\sigma\prime(t)}{x(\sigma(t))}\frac{\left(v(t)a(t)\left((r(t)x\prime(t))'\right)^{\alpha}\right)^{1/\alpha}v(\sigma(t),t_1)}{r(\sigma(t))}w(t). \tag{35}$$

So, we can obtain

$$w\prime(t) \leq \frac{\rho\prime(t)}{\rho(t)}w(t) - k\rho(t)v(t)q(t) - \beta\frac{\sigma\prime(t)v(\sigma(t),t_1)}{r(\sigma(t))\rho^{1/\alpha}(t)}x^{\beta-\alpha/\alpha}(\sigma(t))w^{1+1/\alpha}(t). \tag{36}$$

By Lemma 3, we have

$$w\prime(t) \leq \frac{\rho_+'(t)}{\rho(t)}w(t) - k\rho(t)v(t)q(t) - \beta\frac{\sigma\prime(t)v(\sigma(t),t_1)}{r(\sigma(t))\rho^{1/\alpha}(t)}Q(\sigma(t))w^{1+1/\alpha}(t). \tag{37}$$

By Lemma 4, let

$$\gamma = 1 + \frac{1}{\alpha},$$

$$A = \left(\frac{\beta\sigma\prime(t)v(\sigma(t), t_1)Q(\sigma(t))}{r(\sigma(t))\rho^{1/\alpha}(t)}\right)\left(\frac{1}{\gamma}\right)w(t), \tag{38}$$

$$B = \left(\frac{r(\sigma(t))\rho^{1/\alpha}(t)}{\beta\sigma\prime(t)v(\sigma(t), t_1)Q(\sigma(t))}\right)^{\alpha/\gamma}\left(\frac{\rho_+'(t)}{\rho(t)}\right)^{\alpha}\frac{1}{\gamma^{\alpha}}.$$

Then,

$$\frac{\rho_+'(t)}{\rho(t)}w(t) - \beta\frac{\sigma\prime(t)v(\sigma(t), t_1)}{r(\sigma(t))\rho^{1/\alpha}(t)}Q(\sigma(t))w^{1+1/\alpha}(t) \le \frac{\rho(t)}{(\alpha+1)^{\alpha+1}}\left(\frac{\alpha r(\sigma(t))}{\beta\sigma\prime(t)v(\sigma(t), t_1)Q(\sigma(t))}\right)^{\alpha}\left(\frac{\rho_+'(t)}{\rho(t)}\right)^{\alpha+1}. \tag{39}$$

Combining (30) and (37), we obtain

$$w'(t) \le -k\rho(t)v(t)q(t) + \frac{\rho(t)}{(\alpha+1)^{\alpha+1}}\left(\frac{\alpha r(\sigma(t))}{\beta\sigma\prime(t)v(\sigma(t), t_1)Q(\sigma(t))}\right)^{\alpha}\left(\frac{\rho_+'(t)}{\rho(t)}\right)^{\alpha+1}. \tag{40}$$

Integrating (39) from t_1 to t,

$$\int_{t_1}^{t}\left[k\rho(s)v(s)q(s) - \frac{\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\alpha r(\sigma(s))}{\beta\sigma\prime(s)v(\sigma(s), t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{\rho_+'(s)}{\rho(s)}\right)^{\alpha+1}\right]ds \le w(t_1) - w(t) < w(t_1). \tag{41}$$

Letting $t \longrightarrow \infty$, we have

$$\limsup_{t\longrightarrow\infty}\int_{t_1}^{t}\left[k\rho(s)v(s)q(s) - \frac{\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\alpha r(\sigma(s))}{\beta\sigma\prime(s)v(\sigma(s), t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{\rho_+'(s)}{\rho(s)}\right)^{\alpha+1}\right]ds \le w(t_1) < \infty, \tag{42}$$

which contradicts condition (30).

If $y(t)$ satisfies (II), then from Lemma 2, it is obvious that $\lim_{t\longrightarrow\infty}x(t) = 0$. The proof is complete. □

Theorem 2. *Assume that (6), (7), and (12) hold. If there exist functions $\rho(t) \in C^1([t_0.\infty), R_+)$ and $H \in \mathscr{R}$ such that*

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[kH(t,s)\rho(s)v(s)q(s) - \left(\frac{\alpha r(\sigma(s))}{\beta H(t,s)\sigma\prime(s)v(\sigma(s), t)Q(\sigma(s))}\right)^{\alpha}\left(\frac{h_+(t,s)}{\alpha+1}\right)^{\alpha+1}\rho(s)\right]ds = \infty, \tag{43}$$

for $t \ge t_1 \ge t_0$, where $h(t,s) = H(t,s)\rho_+'(s)/\rho(s) + H_s'(t,s)$, $h_+(t,s) = \max(0, h(t,s))$, then every solution of equation (4) is oscillatory or tends to zero.

Proof. Let $x(t)$ be a nonoscillatory solution of (4). Without loss of generality, we may assume that there exists a sufficiently large t_1 such that $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t > t_1$.

According to Lemma 1, $x(t)$ is either the case of (I) or the case of (II). Now we assume $x(t)$ satisfies (I). Let $w(t)$ be defined as in Theorem 1; by (37), we have

$$k\rho(t)v(t)q(t) \le -w'(t) + \frac{\rho_+'(t)}{\rho(t)}w(t) - \beta\frac{\sigma'(t)v(\sigma(t),t_1)}{r(\sigma(t))\rho^{1/\alpha}(t)}Q(\sigma(t))w^{1+1/\alpha}(t). \quad (44)$$

Substituting t with s in the above inequality, multiplying each side by $H(t,s)$, and integrating from t_1 to t, we obtain

$$\begin{aligned} k\int_{t_1}^{t} H(t,s)\rho(s)v(s)q(s)\mathrm{d}s &\le -\int_{t_1}^{t} H(t,s)w'(s)\mathrm{d}s + \int_{t_1}^{t} H(t,s)\frac{\rho_+'(s)}{\rho(s)}w(s)\mathrm{d}s - \int_{t_1}^{t} H(t,s)\beta\frac{\sigma'(s)v(\sigma(s),t_1)}{r(\sigma(s))\rho^{1/\alpha}(s)}Q(\sigma(s))w^{1+1/\alpha}(s)\mathrm{d}s \\ &\le H(t,t_1)w(t_1) + \int_{t_1}^{t}\left[H(t,s)\frac{\rho_+'(s)}{\rho(s)} + H_s'(t,s)\right]w(s)\mathrm{d}s - \int_{t_1}^{t} H(t,s)\beta\frac{\sigma'(s)v(\sigma(s),t_1)}{r(\sigma(s))\rho^{1/\alpha}(s)}Q(\sigma(s))w^{1+1/\alpha}(s)\mathrm{d}s \\ &\le H(t,t_1)w(t_1) + \int_{t_1}^{t}\left[h_+(t,s)w(s) - H(t,s)\beta\frac{\sigma'(s)v(\sigma(s),t_1)}{r(\sigma(s))\rho^{1/\alpha}(s)}Q(\sigma(s))w^{1+1/\alpha}(s)\right]\mathrm{d}s. \end{aligned} \quad (45)$$

By Lemma 4, let

$$\gamma = 1 + \frac{1}{\alpha},$$
$$A = \left[H(t,s)\frac{\beta\sigma'(s)v(\sigma(s),t_1)}{r(\sigma(s))\rho^{1/\alpha}(s)}Q(\sigma(s))\right]^{1/\gamma} w(s),$$
$$B = \left[\frac{r(\sigma(s))\rho^{1/\alpha}(s)}{H(t,s)\beta\sigma'(s)v(\sigma(s),t_1)Q(\sigma(s))}\right]^{\alpha/\gamma}\left(\frac{h_+(t,s)}{\gamma}\right)^{\alpha}. \quad (46)$$

Then,

$$h_+(t,s)w(s) - H(t,s)\beta\frac{\sigma'(s)v(\sigma(s),t_1)}{r(\sigma(s))\rho^{1/\alpha}(s)}Q(\sigma(s))w^{1+1/\alpha}(s) \le \left(\frac{\alpha r(\sigma(s))}{\beta H(t,s)\sigma'(s)v(\sigma(s),t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{h_+(t,s)}{\alpha+1}\right)^{\alpha+1}\rho(s). \quad (47)$$

Combining (45) and (47), we have

$$k\int_{t_1}^{t} H(t,s)\rho(s)v(s)q(s)\mathrm{d}s \le H(t,t_1)w(t_1) + \int_{t_1}^{t}\left(\frac{\alpha r(\sigma(s))}{\beta H(t,s)\sigma'(s)v(\sigma(s),t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{h_+(t,s)}{\alpha+1}\right)^{\alpha+1}\rho(s)\mathrm{d}s. \quad (48)$$

Because $H_s'(t,s) \le 0$ for $t \ge t_1 \ge t_0$, we can get $H(t,t_0) \ge H(t,t_1) > 0$. Hence,

$$\int_{t_1}^{t}\left[kH(t,s)\rho(s)v(s)q(s)-\left(\frac{\alpha r(\sigma(s))}{\beta H(t,s)\sigma\prime(s)v(\sigma(s),t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{h_+(t,s)}{\alpha+1}\right)^{\alpha+1}\rho(s)\right]ds \\ \leq H(t,t_1)w(t_1)\leq H(t,t_0)w(t_1). \tag{49}$$

Dividing $H(t,t_0)$, we have

$$\frac{1}{H(t,t_0)}\int_{t_1}^{t}\left[kH(t,s)\rho(s)v(s)q(s)-\left(\frac{\alpha r(\sigma(s))}{\beta H(t,s)\sigma\prime(s)v(\sigma(s),t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{h_+(t,s)}{\alpha+1}\right)^{\alpha+1}\rho(s)\right]ds\leq w(t_1). \tag{50}$$

Letting $t\longrightarrow\infty$, we have

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_0)}\int_{t_1}^{t}\left[kH(t,s)\rho(s)v(s)q(s)-\left(\frac{\alpha r(\sigma(s))}{\beta H(t,s)\sigma\prime(s)v(\sigma(s),t_1)Q(\sigma(s))}\right)^{\alpha}\left(\frac{h_+(t,s)}{\alpha+1}\right)^{\alpha+1}\rho(s)\right]ds\leq w(t_1)<\infty, \tag{51}$$

which contradicts (43).

If $y(t)$ satisfies (II), then from Lemma 2, it is obvious that $\lim_{t\longrightarrow\infty}x(t)=0$. The proof is complete. □

4. Examples

Example 1. Consider the equation

$$\left(t^{\alpha-1/2}(x\prime\prime(t))^{\alpha}\right)'+\frac{1}{t}(x\prime\prime(t))^{\alpha}+\frac{1}{t^2}x^{\beta}\left(\frac{t}{\beta}\right)\left(e^{x(t/\beta)}+1\right)=0, \tag{52}$$

where $\beta,\alpha>1$, $a(t)=t^{\alpha-1/2}$, $r(t)=1$, $p(t)=1/t$, $q(t)=1/t^2$, $f(x)=x^{\beta}(e^{x}+1)$, $\sigma(t)=t/\beta$. Then, we have

$$1\leq v(t)=e^{\int_{t_0}^{t}p(s)/a(s)ds}=e^{\int_{t_0}^{t}1/\left(s^{\alpha+1/2}\right)ds}=e^{2/1-2\alpha\left(1/t^{\alpha-1/2}-1/t_0^{\alpha-1/2}\right)}\leq e^{2}. \tag{53}$$

Now we check conditions (6), (7), and (12).

$$\begin{aligned}
&\int_{t_0}^{\infty}\frac{1}{(v(s)a(s))^{1/\alpha}}ds\geq e^{-2/\alpha}\int_{t_0}^{\infty}\frac{1}{s^{1-1/2\alpha}}ds=\infty,\\
&\int_{t_0}^{\infty}\frac{1}{r(s)}ds=\int_{t_0}^{\infty}ds=\infty,\\
&\int_{t_0}^{\infty}\frac{1}{r(v)}\int_{u}^{\infty}\left[\frac{1}{r(u)a(u)}\int_{u}^{\infty}v(s)q(s)ds\right]^{1/\alpha}dudv\\
&\geq\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{u^{\alpha-1/2}}\int_{u}^{\infty}\frac{1}{s^2}ds\right]^{1/\alpha}dudv\\
&=\int_{t_0}^{\infty}\int_{v}^{\infty}\frac{1}{u^{1+1/2\alpha}}dudv\\
&=2\alpha\int_{t_0}^{\infty}\frac{1}{v^{1/2\alpha}}dv=\infty.
\end{aligned} \tag{54}$$

So, conditions (6), (7), and (12) hold. Because $v(\infty, t_0) = \infty$, there exists a sufficiently large $T \geq t_1$ such that $v(t, t_1) \geq v(t/\beta, t_1) \geq 1$ for $t \in [T, \infty)$. Furthermore, $f(x)/x^\beta \geq k = 1$, and letting $\rho(t) = t$, we have the following three situations.

For $\beta > \alpha$,

$$\begin{aligned}
&\limsup_{t\longrightarrow\infty} \int_{t_1}^{t} \left[k\rho(s)v(s)q(s) - \frac{\rho(s)}{(\alpha+1)^{\alpha+1}} \left(\frac{\alpha r(\sigma(s))}{\beta\sigma\prime(s)v(\sigma(s),t_1)Q(\sigma(s))} \right)^{\alpha} \left(\frac{\rho_+'(s)}{\rho(s)} \right)^{\alpha+1} \right] \mathrm{d}s \\
&\geq \limsup_{t\longrightarrow\infty} \int_{t_1}^{T} \left[\frac{1}{s} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\alpha}{\lambda_1 v(s/\beta, t_1)} \right)^{\alpha} \cdot \frac{1}{s^\alpha} \right] \mathrm{d}s + \limsup_{t\longrightarrow\infty} \int_{T}^{t} \left[\frac{1}{s} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\alpha}{\lambda_1} \right)^{\alpha} \cdot \frac{1}{s^\alpha} \right] \mathrm{d}s \\
&= \infty.
\end{aligned} \tag{55}$$

For $\beta = \alpha$, from the above, it is obvious that (30) holds.

For $\beta < \alpha$,

$$\begin{aligned}
&v(t,t_1) \leq \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)} \mathrm{d}s \leq \int_{t_1}^{t} \frac{1}{s^{1-1/2\alpha}} \mathrm{d}s \leq 2\alpha t^{1/2\alpha}, \\
&R(t,t_1) \leq \int_{t_1}^{t} 2\alpha \cdot s^{1/2\alpha} \mathrm{d}s \leq 2\alpha \cdot t^{1/2\alpha+1}, \\
&\limsup_{t\longrightarrow\infty} \int_{t_1}^{t} \left[k\rho(s)v(s)q(s) - \frac{\rho(s)}{(\alpha+1)^{\alpha+1}} \left(\frac{\alpha r(\sigma(s))}{\beta\sigma\prime(s)v(\sigma(s),t_1)Q(\sigma(s))} \right)^{\alpha} \left(\frac{\rho_+'(s)}{\rho(s)} \right)^{\alpha+1} \right] \mathrm{d}s \\
&\geq \limsup_{t\longrightarrow\infty} \int_{t_1}^{\infty} \left[\frac{1}{s} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(\frac{\alpha\lambda_2}{v(s/\beta,t_1)} \right)^{\alpha} R^{\alpha-\beta}\left(\frac{s}{\beta}, t_1 \right) \cdot \frac{1}{s^\alpha} \right] \mathrm{d}s \\
&\geq \limsup_{t\longrightarrow\infty} \int_{t_1}^{T} \left[\frac{1}{s} - \frac{\left(2\alpha^2\lambda_2\right)^{\alpha}}{(\alpha+1)^{\alpha+1}(2\alpha)^{\beta}} \left(\frac{1}{v(s/\beta,t_1)} \right)^{\alpha} \cdot \frac{1}{s^{\beta/2\alpha+\beta-1/2}} \right] \mathrm{d}s \\
&\quad + \limsup_{t\longrightarrow\infty} \int_{T}^{t} \left[\frac{1}{s} - \frac{\left(2\alpha^2\lambda_2\right)^{\alpha}}{(\alpha+1)^{\alpha+1}(2\alpha)^{\beta}} \cdot \frac{1}{s^{\beta/2\alpha+\beta-1/2}} \right] \mathrm{d}s \\
&= \infty.
\end{aligned} \tag{56}$$

From above, we know

$$\limsup_{t\longrightarrow\infty} \int_{t_1}^{t} \left[k\rho(s)v(s)q(s) - \frac{\rho(s)}{(\alpha+1)^{\alpha+1}} \left(\frac{\alpha r(\sigma(s))}{\beta\sigma\prime(s)v(\sigma(s),t_1)Q(\sigma(s))} \right)^{\alpha} \left(\frac{\rho_+'(s)}{\rho(s)} \right)^{\alpha+1} \right] \mathrm{d}s = \infty. \tag{57}$$

Therefore, by Theorem 1, every solution of (52) is oscillatory or tends to zero.

Example 2. Consider the equation

$$\left(t^{\alpha-1/2} \left(\left(\frac{1}{t} x\prime(t) \right)' \right)^{\alpha} \right)' + \frac{1}{t} \left(\left(\frac{1}{t} x\prime(t) \right)' \right)^{\alpha} + \frac{1}{t^2} x^{\beta}\left(\frac{t}{\beta} \right) e^{x^2(t/\beta)} = 0, \tag{58}$$

where $a(t) = t^{\alpha-1/2}$, $r(t) = 1/t$, $p(t) = 1/t$, $q(t) = 1/t^2$, $f(x) = x^\beta e^{x^2}$, $\sigma(t) = t/\beta$, which satisfy conditions

(A1)–(A3). Similar to the Example 1, we can easily verify that (6), (7), and (12) hold. Furthermore, we choose $\rho(t) = t$, $H(t,s) = t - s$. By Theorem 2, we have $h(t,s) = t/s - 2$ and

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[kH(t,s)\rho(s)v(s)q(s)-\left(\frac{\alpha r(\sigma(s))}{\beta H(t,s)\sigma\prime(s)v(\sigma(s),t)Q(\sigma(s))}\right)^{\alpha}\left(\frac{h_+(t,s)}{\alpha+1}\right)^{\alpha+1}\rho(s)\right]\mathrm{d}s=\infty. \tag{59}$$

Then, by Theorem 2, we know that any solution of (58) is oscillatory or tends to zero.

5. Conclusions

In this work, by using the generalized Riccati technique, we establish some new oscillation criteria of (4) and give some examples to verify the oscillation criteria. These new oscillation criteria complement some known results for third-order delay differential equations. Furthermore, in future work we will study the forced oscillatory behavior of this equation with forcing term.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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