Research Article

Coefficient Inequalities for a Subclass of Symmetric $q$-Starlike Functions Involving Certain Conic Domains

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In this paper, we make use of a certain Ruscheweyh-type $q$-differential operator to introduce and study a new subclass of $q$-starlike symmetric functions, which are associated with conic domains and the well-known celebrated Janowski functions in $\mathbb{D}$.

We then investigate many properties for the newly defined functions class, including for example coefficients inequalities, the Fekete–Szegő Problems, and a sufficient condition. There are also relevant connections between the results provided in this study and those in a number of other published articles on this subject.

1. Introduction, and Preliminaries

Let $SU (\mathbb{D})$ be a class of analytic functions where $\mathbb{D}$ is the open unit disk and is given by the following equation:
\begin{equation}
\mathbb{D} = \left\{ z; \ z \in \mathbb{C}, \ |z| < 1 \right\},
\end{equation}

and let $f \in SU$ be those functions in the open unit disk $\mathbb{D}$ which are normalized by the following equation:
\begin{align}
& f (0) = 0, \\
& f' (0) = 1,
\end{align}

thus, we have the following series form for $f \in SU$
\begin{equation}
f (z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.
\end{equation}

Moreover, all normalized univalent functions in $\mathbb{D}$ are contained in the set $SU \subset SU$. For two given functions $g_1, g_2 \in SU$, we say that $g_1$ is subordinate to $g_2$, written symbolically as $g_1 \prec g_2$, if there exist a Schwarz function $w$, which is holomorphic in $\mathbb{D}$ with
\begin{equation}
w (0) = 0, \quad |w (z)| < 1,
\end{equation}

so that
\begin{equation}
g_1 (z) = g_2 (w (z)) \quad (z \in \mathbb{D}).
\end{equation}

Moreover, if the function $g_2$ is univalent in $\mathbb{D}$, then the following equivalence hold true:
\begin{equation}
g_1 \prec g_2 \Leftrightarrow g_1 (0) = g_2 (0), \quad g_1 (\mathbb{D}) \subset g_2 (\mathbb{D}).
\end{equation}
Let \( \mathcal{P} \) be the class of Carathéodory function, an analytic function \( \chi \in \mathcal{P} \) if
\[
\chi(z) = 1 + \sum_{n=1}^{\infty} \chi_n z^n,
\]
such that
\[
\chi(0) = 1, \quad \Re \{\chi(z)\} > 0 \quad (\forall \ z \in \mathbb{D}).
\]

**Definition 1.** A given function \( p \) is said to be in the class \( \mathcal{P}(\mathfrak{X}, \mathfrak{Y}) \) if
\[
p(z) = \frac{1 + \mathfrak{X}z}{1 + \mathfrak{Y}z} \quad (-1 \leq \mathfrak{Y} < \mathfrak{X} \leq 1).
\]

Janowski [1] investigated the class of functions \( \mathcal{P}(\mathfrak{X}, \mathfrak{Y}) \) and found that \( p(z) \in \mathcal{P}(\mathfrak{X}, \mathfrak{Y}) \) if and only if there exist a function \( \chi \in \mathcal{P} \) such that
\[
p(z) = \frac{(\mathfrak{X} + 1)\chi(z) - (\mathfrak{X} - 1)}{(\mathfrak{Y} + 1)\chi(z) - (\mathfrak{Y} - 1)} \quad (-1 \leq \mathfrak{Y} < \mathfrak{X} \leq 1).
\]

**Definition 2.** A function \( f \) of the form (3) be in the functions class \( \mathcal{S}^* (\mathfrak{X}, \mathfrak{Y}) \) if and only if
\[
\frac{zf'(z)}{f(z)} = \frac{(\mathfrak{X} + 1)\chi(z) - (\mathfrak{X} - 1)}{(\mathfrak{Y} + 1)\chi(z) - (\mathfrak{Y} - 1)} \quad (-1 \leq \mathfrak{Y} < \mathfrak{X} \leq 1).
\]

Historically speaking, Kanas and Wiśniewska were the first (see [2], [3]), (see also [4]) who introduced and defined the class of \( \lambda \)-uniformly convex functions \( (\lambda - \mathcal{U}C\mathcal{V}) \) and \( \lambda \)-starlike functions \( (\lambda - \mathcal{S}\mathcal{T}) \) subject to the conic domain \( \Omega_\lambda \), where
\[
\Omega_\lambda = \left\{ u + iv: u > \lambda \sqrt{(u - 1)^2 + v^2}, \ u > 0 \right\}, \lambda \geq 0.
\]

Moreover if \( \lambda \), is fixed and \( (\lambda = 0) \) then \( \Omega_\lambda \) denote the conic region bounded by the imaginary axis, if \( \lambda = 1 \), we have a parabola, if \( 0 < \lambda < 1 \) this domain represents the right branch of hyperbola and for \( \lambda > 1 \) an ellipse.

For these conic areas, the following functions serve as extremal functions

\[
p_\lambda(z) = \begin{cases} 
1 + z & (\lambda = 0), \\
1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right)^2 & (\lambda = 1), \\
1 + \frac{2}{1 - \lambda} \sinh^2 \left( \frac{\lambda}{2} \arccos \lambda \right) \arctan \frac{\sqrt{\lambda}}{1 - \sqrt{\lambda}} & (0 \leq \lambda < 1), \\
1 + \frac{1}{\lambda^2 - 1} \sin \left( \frac{\pi}{2K(k)} \int_0^{\pi(\sqrt{\lambda})} \frac{\text{d}t}{\sqrt{1 - t^2 \sqrt{1 - \kappa^2 t^2}}} \right) + \frac{1}{\lambda^2 - 1} & (\lambda > 1),
\end{cases}
\]

where
\[
u(z) = \frac{z - \sqrt{k}}{1 - \sqrt{k}z} \quad (\forall \ z \in \mathbb{D}),
\]

and \( \kappa \in (0, 1) \) is chosen such that \( \lambda = \cos h (\pi K'(\kappa)/(4K(\kappa))) \). Here, \( K(\kappa) \) is Legendre’s complete elliptic integral of first kind and \( K'(\kappa) = K(\sqrt{1 - \kappa^2}) \), that is \( K'(t) \) is the complementary integral of \( K(t) \).

The function \( p_\lambda(z) \) in [5] be given as follows:
\[
p_\lambda(z) = 1 + c_1 z + c_{21} z^2 + c_{31} z^3 + \cdots,
\]

where
\[
c_{11} = \begin{cases} 
\frac{8(\arccos k)^2}{\pi^2(1 - k^2)} & 0 \leq \lambda < 1, \\
\frac{4}{\pi^2} & \lambda = 1, \\
\frac{\pi^2}{4(\kappa^2 - 1)R^2(t)\sqrt{1 + t}} & \lambda > 1,
\end{cases}
\]

and
\[
c_{21} = c_1(c_1),
\]
\[
\begin{align*}
\zeta_1 &= \begin{cases} 
\left(\frac{2}{3}\right)[(2/\pi) \arccos \kappa]^2 + 2 & 0 \leq \lambda < 1, \\
\frac{2}{3} & \lambda = 1, \\
\frac{4R^2(t)(t^2 + 6t + 1) - \pi^2}{24R^2(t)(1 + t)^{\sqrt{t}}} & \lambda > 1,
\end{cases} \\
&= \begin{cases} 
\left(\frac{2}{3}\right)[(2/\pi) \arccos \kappa]^2 + 2 & 0 \leq \lambda < 1, \\
\frac{2}{3} & \lambda = 1, \\
\frac{4R^2(t)(t^2 + 6t + 1) - \pi^2}{24R^2(t)(1 + t)^{\sqrt{t}}} & \lambda > 1,
\end{cases} \\
&= \frac{4R^2(t)(t^2 + 6t + 1) - \pi^2}{24R^2(t)(1 + t)^{\sqrt{t}}} \lambda > 1,
\end{align*}
\]

and \( t \in (0, 1) \).

The following is defined by Noor et al. [6], who combine the ideas of Janowski functions and conic regions.

\[
\Omega_{1}[\mathcal{X}, \mathcal{Y}] = \left\{ w : \mathcal{R} \left( \frac{\mathcal{Y} - 1)w - (\mathcal{X} - 1)}{(\mathcal{Y} + 1))w - (\mathcal{X} + 1)) \right) > \lambda \left| \frac{(\mathcal{Y} - 1)w - (\mathcal{X} - 1)}{(\mathcal{Y} + 1))w - (\mathcal{X} + 1)) \right| \right\},
\]

or equivalently for \( w = u + iv \), we have the following equation:

\[
\left[(\mathcal{Y}^2 - 1)(u^2 + v^2) - 2(\mathcal{X}\mathcal{Y} - 1)u + (\mathcal{X}^2 - 1)^2 \right] > \lambda \left[(-2(\mathcal{Y} + 1))(u^2 + v^2) + 2(\mathcal{X} + \mathcal{Y} + 2)u - 2(\mathcal{X} + 1)^2 + 4(\mathcal{X} - \mathcal{Y})^2v^2 \right].
\]

Definition 3. A function \( p \) from the functions class \( \mathcal{P} \) be in the functions class \( \lambda - \mathcal{P}(\mathcal{X}, \mathcal{Y}) \), if

\[
p(z) = \frac{(\mathcal{X} + 1)p_1(z) - (\mathcal{X} - 1)}{(\mathcal{Y} + 1)p_1(z) - (\mathcal{Y} - 1)} \left( -1 \leq \mathcal{Y} < \mathcal{X} \leq 1, \right)
\]

where \( p_1(z) \) is defined by (13).

Geometrically for the domain \( \Omega_{1}[\mathcal{X}, \mathcal{Y}] \), we have the following

\[
f(e^{ij}z) = e^{ij}f(z).
\]

The set of all \( (j,i) \)-symmetric functions is denote by \( \delta^{(j,i)} \). First of all, if \( j = 0 \) and \( i = 2 \), then \( \delta^{(j,i)} \) is called even symmetric functions. Secondly, if \( j = 1 \) and \( i = 2 \), then \( \delta^{(j,i)} \) is called odd symmetric functions. Thirdly, if \( j = 1 \), then \( \delta^{(j,i)} \) is called \( i \)-symmetric functions.

Theorem 1 (see [7]). For mapping \( f : \mathbb{D} \longrightarrow \mathbb{C} \), there is just one series of \( (j,i) \)-symmetrical functions \( f_{j,i}(z) \) exists that given as follows:

\[
f(z) = \sum_{j=0}^{i-1} f_{j,i}(z).
\]

From equation (25), we can get the following equation:

\[
f_{j,i}(z) = \frac{1}{i} \sum_{v=0}^{i-1} e^{-vj} f(e^{v}z) = \frac{1}{i} \sum_{v=0}^{i-1} e^{-vj} \left( \sum_{n=1}^{\infty} a_n(e^{v}z)^n \right).
\]
\[ f_{j,i}(z) = \sum_{n=0}^{\infty} \psi_n a_n z^n, \quad a_1 = 1, \quad \psi_1 = 1, \quad (27) \]

where \( f \in \mathcal{A} \) and
\[ e = e^{2\pi il \alpha} \quad (\mathbb{N}_0 = \{1, 2, 3, \ldots\}), \quad (28) \]
\[ \psi_n = \frac{1}{i} \sum_{l=0}^{n-1} e^{\alpha (l-j)^2} = \begin{cases} 1, & n = li + j \\ 0, & n \neq li + j \end{cases}, \quad (29) \]

2. Some Basic Concepts of \( q \)-Calculus

The usage of the quantum (or \( q \) -) calculus in many diverse areas of mathematics and physics is quite significant. In the theory of Univalent functions, Srivastava [8] firstly apply the \( q \)-calculus in order to put the foundation of a new direction for other researchers. Motivated by [8] of Srivastava, many researchers have worked on this direction. For example, the convolution theory, enable us to investigate various properties of analytic functions. Due to the large range of applications of \( q \)-calculus and the importance of \( q \)-operators instead of regular operators, many researchers have explored \( q \)-calculus in depth, such as, Kanas and Reducanu [9], Muhammad and Sokol [10] and Noor et al. [11–15]. Also in \( q \)-calculus, many researchers have worked on this direction. For example, the convolution theory, enable us to investigate various properties of analytic functions. Due to the large range of applications of \( q \)-calculus and the importance of \( q \)-operators instead of regular operators, many researchers have explored \( q \)-calculus in depth, such as, Kanas and Reducanu [9], Muhammad and Sokol [10] and Noor et al. [11–15]. Also in [1–5, 9, 16–22], Ahmad et al. see also [21], have used the \( q \)-derivative operator to define a new subclass \( q \)-meromorphic starlike functions. They also developed some remarkable results for their defined classes of analytic functions. In addition, Srivastava [23] see also [8, 12–15, 23–26] recently published survey-cum-expository review paper this might be useful for researchers and scholars working on these subjects. For some recent and related study about \( q \)-series, we may refer the interested readers to see [27–29].

**Definition 5** (see [30]). Let \( q \in (0, 1) \) and \( q \)-integer \( n \), be defined as follows:
\[ [n, q] = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \quad [0, q] = 0. \quad (30) \]

**Definition 6.** We define the \( q \)-shifted factorial as follows:
\[ p_{\lambda,q}(x, \xi, z) = \frac{(x(1+q)+(-q+3)p_1(z)-(x(1+q)+(-q+3))}{(\xi(1+q)+(-q+3)p_1(z)-(\xi(1+q)+(-q+3))}, \quad \lambda \geq 0. \quad (39) \]

and \( p_1(z) \) is defined by (13).

Geometrically, we have the following equation:
\[ \Omega_{\lambda,q}(x, \xi) = \{w = u + iv: \Re (\xi) > \lambda |\xi - 1|\}. \quad (40) \]

where
\[ [0, q]! = 1, \quad [n, q]! = [1, q][2, q] \ldots [n, q]. \quad (31) \]

It could be seen that
\[ \lim_{q \to 1} [n, q] = n, \quad (32) \]
\[ \lim_{q \to 1} [n, q]! = n!. \]

In general \([t, q] = 1 - q^t/1-q\).

**Definition 7** (see [20]). For an analytic function \( f \), the \( q \)-deformation or \( q \)-generalization of derivative is defined by the following equation:
\[ \partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \in \mathbb{D}), \quad (33) \]
\[ \partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n, q] a_n z^{n-1}, \quad (34) \]
and
\[ \partial_q z^n = [n, q] z^{n-1}. \quad (35) \]

**Definition 8.** The generalized \( q \)-Pochhammer symbol is given by the following equation:
\[ [t, q]_n = [t, q][t+1, q][t+2, q] \ldots [t+n-1, q]. \quad (36) \]

and \( q \)-gamma function be given as follows:
\[ \Gamma_q(t + 1) = [t, q] \Gamma_q(t), \quad (37) \]
\[ \Gamma_q(1) = 1. \]

**Definition 9** (see [31]). A function \( p \in \lambda \cdot \mathcal{P}_q(x, \xi) \) if and only if
\[ p(z) < p_{\lambda,q}(x, \xi, z), \quad (38) \]
where
\[ \Psi = \frac{(\xi(1+q)+(-q+3)w(z)-(x(1+q)+(-q+3))}{(\xi(1+q)+(-q+3)w(z)-(x(1+q)+(-q+3))}, \quad (41) \]

For more detail (see [31]).
Definition 10 (see [9]). For \( f \in \mathcal{A} \), the Ruscevvyh-q differential operator be defined as follows:

\[
R_q^{1} f(z) = f(z) * \mathcal{H}_{q,\lambda_1+1}(z), \quad (z \in \mathbb{D}, \lambda_1 > -1), \tag{42}
\]

where

\[
\mathcal{H}_{q,\lambda_1+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda_1 + n)}{[n-1, q]!} \phi_q(1 + \lambda_1) z^n
\]

\[
= z + \sum_{n=2}^{\infty} \frac{[\lambda_1 + 1, q]_{n-1}}{[n-1, q]!} \phi_q(1 + \lambda_1) z^n
\]

\[
= z + \sum_{n=2}^{\infty} \phi_{q,n-1} z^n,
\]

with

\[
\phi_{q,n} = \frac{[\lambda_1 + 1, q]_{n-1}}{[n-1, q]!} \phi_q(1 + \lambda_1) \tag{44}
\]

By “*” we mean convolution (or Hadamard product). Moreover from (42), we have the following equation:

\[
R_q^{0} f(z) = f(z), \quad R_q^{1} f(z) = z\partial_q f(z), \tag{45}
\]

and

\[
R_q^{m} f(z) = z\partial_q^m f(z), \quad (m \in \mathbb{N}). \tag{46}
\]

Making use of (42) and (43), the power series of \( R_q^{1} f(z) \) is given by the following equation:

\[
R_q^{1} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda_1 + n)}{[n-1, q]!} \phi_q(1 + \lambda_1) a_{n+1} z^n
\]

\[
= z + \sum_{n=2}^{\infty} \frac{[\lambda_1 + 1, q]_{n-1}}{[n-1, q]!} \phi_q(1 + \lambda_1) a_{n+1} z^n
\]

\[
= z + \sum_{n=2}^{\infty} \phi_{q,n-1} a_{n+1} z^n.
\]

Similarly,

\[
R_q^{1} f(z) = f(z) * \mathcal{H}_{q,\lambda_1+1}(z), \quad (z \in \mathbb{D}, \lambda_1 > -1),
\]

where \( \phi_{q,n} \) is given by (44) and \( \psi_n \) is given by (49).

Note that,

\[
\lim_{q \to 1} \mathcal{H}_{q,\lambda_1+1}(z) = \frac{z}{(1 - z)^{\lambda_1+1}}. \tag{49}
\]

and

\[
\lim_{q \to 1} R_q^{1} f(z) = f(z) * \frac{z}{(1 - z)^{\lambda_1+1}}. \tag{50}
\]

When \( q \to 1^- \), Ruscveyh-q differential operator reduce to Ruscveyh differential operator [14]. Also, note that

\[
z\partial_q (R_q^{1} f(z)) = \left(1 + \frac{[\lambda_1, q]}{q^{\lambda_1}}\right) R_q^{1} f(z) - \frac{[\lambda_1, q]}{q^{\lambda_1}} R_q^{1} f(z), \tag{52}
\]

and

\[
z\partial_q (R_q^{1} f(z)) = \left(1 + \frac{[\lambda_1, q]}{q^{\lambda_1}}\right) R_q^{1} f(z) - \frac{[\lambda_1, q]}{q^{\lambda_1}} R_q^{1} f(z). \tag{53}
\]

Definition 11. A given function \( f \) is said to be in the functions class \( \lambda \cdot \mathcal{Y}^{(\lambda_1, \lambda)}(\lambda_1, \lambda, \mathcal{Y}) \), \( \lambda \geq 0 \), \( -1 \leq \mathcal{Y} < \lambda \leq 1 \), if

\[
\mathcal{R}(\phi(\lambda_1, \mathcal{Y}, q, \lambda_1)) > \lambda |\phi(\lambda_1, \mathcal{Y}, q, \lambda_1) - 1|, \tag{54}
\]

or equivalently

\[
\frac{z\partial_q R_q^{1} f(z)}{R_q^{1} f(z)} \in \lambda - \mathcal{R}_q(\lambda_1, \mathcal{Y}), \tag{55}
\]

where

\[
\varphi(\lambda_1, \mathcal{Y}, q) = \frac{\mathcal{Y}(1 + q) - (-q + 3)}{\mathcal{Y}(1 + q) + (-q + 3)} \left( z\partial_q R_q^{1} f(z)/R_q^{1} f(z) \right) - \left( \mathcal{Y}(1 + q) - (-q + 3) \right) \left( z\partial_q R_q^{1} f(z)/R_q^{1} f(z) \right).
\]
Each of the following special case of the above-defined functions class $\lambda-\mathcal{V}^{(j)}_q(\lambda_1, \mathcal{X}, \mathcal{Y})$ is worthy of note.

(i) One can easily seen that
\[
\lim_{q \to 1-} \lambda - \mathcal{V}^{(j)}_q (0, \mathcal{X}, \mathcal{Y}) = \lambda - \mathcal{V}^{(j)} (\mathcal{X}, \mathcal{Y}),
\]
where $\lambda-\mathcal{V}^{(j)} (\mathcal{X}, \mathcal{Y})$ is the functions classes studied by Al-Sarari and Latha (see [18]).

(ii) If we set
\[
q \to 1-,
\lambda_1 = 0,
j = 1.
\]

in Definition 11, we have class $\lambda-\mathcal{V}^0 (\mathcal{X}, \mathcal{Y}, i)$ introduced in [17].

(iii) If we put
\[
q \to 1-,
\lambda_1 = 0,
j = 1 = i.
\]

we get the functions class $\lambda-\mathcal{V}^0 (\mathcal{X}, \mathcal{Y})$ (see [30]).

(iv) We see that
\[
\lim_{q \to 1-} \lambda 0\mathcal{V}^{(1,1)}_q (0, 1, -1) = \lambda 0\mathcal{V}^0,
\]
where Kanas and Wisniowska [3] have studied the class $\lambda-\mathcal{V}^0$. 

(v) If we set
\[
q \to 1-,
\mathcal{Y} = 1 = i = j = \lambda_1 + 1,
\mathcal{X} = 1 - 2\alpha.
\]

in Definition 11, we get the class $\mathcal{V}^0 (\mathcal{X}, \mathcal{Y}, \alpha)$ (see [24]).

(vi) It can be easily seen that
\[
\lim_{q \to 1-} 00\mathcal{V}^{(1,1)}_q (0, \mathcal{X}, \mathcal{Y}) = \mathcal{V}^0 (\mathcal{X}, \mathcal{Y}),
\]
where the functions class $\mathcal{V}^0 (\mathcal{X}, \mathcal{Y})$ is studied in [1].

3. A Set of Lemmas

Lemma 1 (see [13]). Let $\chi(z) = 1 + \sum_{n=1}^\infty \chi_n z^n \in \mathcal{F}(z) = 1 + \sum_{n=1}^\infty C_n z^n$. If $\mathcal{F}(z)$ is convex univalent in $\mathbb{D}$, then
\[
|\chi_n| \leq |C_n|, \quad n \geq 1.
\]

Lemma 2 (see [32]). If $\chi(z) = 1 + \chi_1 z + \chi_2 z^2 + \cdots$ is an analytic function with positive real part in $\mathbb{D}$, then
\[
|\chi_2 - \nu \chi_1^2| \leq \begin{cases} -4\nu + 2 & \text{if } \nu < 0, \\ 2 & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2 & \text{if } \nu > 1. \end{cases}
\]

The equality holds for
\[
\chi(z) = \frac{1 + z}{1 - z}, \quad \text{when } \nu < 0 \text{ or } \nu > 1,
\]
or one of its relations. If $\nu = 0$, the equality holds if and only if
\[
\chi(z) = \left(\frac{1 + \frac{1}{2}z}{1 - z}\right) \frac{1 + z}{1 - z} \left(\frac{1 - \frac{1}{2}z}{1 + z}\right) \frac{1 - z}{1 + z}, \quad (0 \leq \lambda \leq 1),
\]
or one of its relations. If $\nu = 0$, the equality holds if and only if
\[
\chi(z) = \frac{1 + z^2}{1 - z}, \quad \text{when } 0 < \nu < 1.
\]

Lemma 3 (see [32]). If the function $\chi(z)$ given by (8) belongs to the class $\mathcal{P}$ of analytic functions and have positive real part in $\mathbb{D}$, then
\[
|\chi_n| \leq 2, \quad n \in \mathbb{N}.
\]

The equality holds for
\[
f(z) = \frac{1 + z}{1 - z}.
\]

Lemma 4 (see [32]). If the function $\chi(z)$ given by (8) belongs to the class $\mathcal{P}$ of analytic functions and have positive real part in $\mathbb{D}$, then
\[
|c_2 - \mu c_1^2| \leq 2 \max \{1, |2\mu - 1|\}, \quad \forall \mu \in \mathbb{C}.
\]

Lemma 5. Let $\lambda \in [0, \infty)$ be fixed and
\[
\rho_{\lambda,q} (\mathcal{X}, \mathcal{Y}, z) = \frac{(\mathcal{X}(1 + q) + (-q + 3)) \rho_{\lambda} (z) - (\mathcal{X}(1 + q) - (-q + 3))}{(\mathcal{Y}(1 + q) + (-q + 3)) \rho_{\lambda} (z) - (\mathcal{Y}(1 + q) - (-q + 3))}.
\]
then

\[ p_{\lambda q}(X, Y, z) = 1 + Y_5(q)_{c_5}z + \left[Y_5(q)_{c_2}Y_6(q)_{c_3}\right]z^2 + \cdots, \]

(74) where \( Y_5(q), Y_6(q), c_5, \) and \( c_3 \) given by (83), (84), (16), and (17).

Proof. From (39), we have the following equation:

\[
p_{\lambda q}(X, Y, z) = \frac{(X + q + (-q + 3))p_1(z) - (X + q - (-q + 3))}{(Y + q + (-q + 3))p_1(z) - (Y + q - (-q + 3))},
\]

\[
= \left[\frac{X(1 + q) + (-q + 3)}{Y(1 + q) - (-q + 3)}\right]p_1(z) - (X(1 + q) - (-q + 3))]
\]

\[
\cdot \left[\frac{1}{Y(1 + q) - (-q + 3)}\right]^{-1}
\]

\[
= \frac{(X(1 + q) + (-q + 3))}{Y(1 + q) - (-q + 3)} \left[1 - \frac{(X(1 + q) + (-q + 3))}{Y(1 + q) - (-q + 3)}p_1(z)\right]
\]

\[
\cdot \left[1 + \sum_{n=1}^{\infty} \left(\frac{(X(1 + q) + (-q + 3))}{Y(1 + q) - (-q + 3)}p_1(z)\right)^n\right]
\]

\[
= \frac{(X(1 + q) + (-q + 3))}{Y(1 + q) - (-q + 3)} + Y_1(q)p_1(z) + Y_2(q)(p_1(z))^2 + Y_3(q)(p_1(z))^3 + \cdots,
\]

where

\[
Y_1(q) = \frac{(X(1 + q) + (-q + 3))(Y(1 + q) + (-q + 3))}{(Y(1 + q) - (-q + 3))^2} - \frac{(X(1 + q) + (-q + 3))}{(Y(1 + q) - (-q + 3)),}
\]

(76)

\[
Y_2(q) = \frac{(X(1 + q) + (-q + 3))(Y(1 + q) + (-q + 3))^2}{(Y(1 + q) - (-q + 3))^3} - \frac{(X(1 + q) + (-q + 3))(Y(1 + q) - (-q + 3))}{(Y(1 + q) - (-q + 3))^2},
\]

and

\[
Y_3(q) = \frac{(X(1 + q) + (-q + 3))(Y(1 + q) + (-q + 3))^3}{(Y(1 + q) - (-q + 3))^4} - \frac{(X(1 + q) + (-q + 3))(Y(1 + q) - (-q + 3))^2}{(Y(1 + q) - (-q + 3))^3},
\]

(77)

By using (15) and (75), we have the following equation:
\[ p_{\lambda,q}(X, Y, z) = \sum_{n=1}^{\infty} -2(\Psi(1+q) + (-q+3))^{n-1} \]

\[ \frac{2n((X(1+q) - \Psi(1+q))(\Psi(1+q) + (-q+3))^{n-1}}{(\Psi(1+q) - (-q+3))^{n+1}} \]

\[ + \sum_{n=1}^{\infty} \zeta_{1,z} + \sum_{n=1}^{\infty} 2n((X(1+q) - \Psi(1+q))(\Psi(1+q) + (-q+3))^{n-1} \]

\[ \times (\Psi(1+q) - (-q+3))^{n+1} \]

\[ \sum_{n=1}^{\infty} \frac{2n((X(1+q) - \Psi(1+q))(\Psi(1+q) + (-q+3))^{n-1}}{(\Psi(1+q) - (-q+3))^{n+1}} \]

\[ \sum_{n=1}^{\infty} \zeta_{21} + \sum_{n=1}^{\infty} Y_{4}(q)\zeta_{1}^{2} \] \[ z^{2} + \cdots, \]

where

\[ Y_{4}(q) = \frac{2n(\Psi(1+q) + (-q+3))(\Psi(1+q) - (3-q) + \Psi(1+q))^{n}}{(\Psi(1+q) - (3-q))^{n+2}} \]

The series

\[ \sum_{n=1}^{\infty} -2((3-q) + \Psi(1+q))^{n-1} \]

and

\[ \sum_{n=1}^{\infty} 2n((X(1+q) - \Psi(1+q))(\Psi(1+q) + (-q+3))^{n-1} \]

\[ \times (\Psi(1+q) - (-q+3))^{n+1} \]

are convergent and convergent to 1, \((1+q)(X - \Psi)/4\) and \((\Psi(1+q) + (-q+3))(X - \Psi)(1+q)/8\), respectively. Therefore (78) becomes

\[ p_{\lambda,q}(X, Y, z) = 1 + Y_{5}(q)\zeta_{1,z} + \{Y_{5}(q)\zeta_{21} - Y_{6}(q)\zeta_{1}^{2}\}z^{2} + \cdots, \]

(82)

where

\[ Y_{5}(q) = \frac{(X - \Psi)(1+q)}{4} \]

(83)

and

\[ Y_{6}(q) = \frac{(3-q) + \Psi(1+q))(X - \Psi)(1+q)}{8} \]

(84)

which is our required result.

Remark 1. If we set \(A = 1\), \(B = -1\) and let \(q \rightarrow 1\), in Lemma 5, we will arrive it that of a result given by Sim et al. [25].

Lemma 6. Let \(\chi(z) = 1 + \sum_{n=1}^{\infty} \chi_{n}z^{n} \in \lambda \cdot P_{\lambda,q}(X, Y)\), then

\[ |\chi_{n}| \leq Y_{5}(q)|\zeta_{1}| = \frac{(X - \Psi)(1+q)}{4}|\zeta_{1}|, \quad n \geq 1. \]

(85)

Proof. By Definition 9 and \(\chi(z) \in \lambda \cdot P_{\lambda,q}(X, Y)\) if and only if

\[ \chi(z) \prec p_{\lambda,q}(X, Y, z)(\lambda \geq 0) \]

(86)

\[ p_{\lambda,q}(X, Y, z) \]

is given by (39).

By using (82) and (86), we have the following equation:

\[ \chi(z) \times 1 + Y_{5}(q)|\zeta_{1}|z + \{Y_{5}(q)\zeta_{21} - Y_{6}(q)\zeta_{1}^{2}\}z^{2} + \cdots. \]

(87)

Now by using Lemma 1 on (87), we have the following equation:

\[ |\chi_{n}| \leq Y_{5}(q)|\zeta_{1}| = \frac{(X - \Psi)(1+q)}{4}|\zeta_{1}|, \]

(88)

which is our required result.
4. Main Results

**Theorem 2.** An analytic function $f$ of the form (3) belongs to the class $k-\mathcal{F}_q^{(j)}(\lambda_1, \mathfrak{X}, \mathfrak{Y})$, if the following condition is satisfied:

$$
\sum_{n=2}^{\infty} (\mathcal{F}_1 + \mathcal{F}_2)\varphi_{q,n-1}|a_n| \leq |\mathfrak{Y} - \mathfrak{X}|(1 + q),
$$

where

$$
\mathcal{F}_1 = 2(k + 1)(3 - q)|\psi_n - [n, q]|,
$$

and $\psi_n$ is given by (29) and

$$
(1 + \lambda)|\varphi(\mathfrak{X}, \mathfrak{Y}, q, \lambda_1) - 1| = \frac{2(1 + \lambda)(3 - q)(R_1^1 f_{j,i}(z) - z\partial_{\alpha} R_1^1 f(z))}{(3 - q + \mathfrak{Y}(1 + q)z\partial_{\alpha} R_1^1 f(z) - (\mathfrak{X}(1 + q) + (-q + 3))R_1^1 f_{j,i}(z)}
$$

$$
\leq \frac{2(1 + \lambda)(3 - q)\sum_{n=2}^{\infty} |\psi_n - [n, q]| |\varphi_{q,n-1}|a_n|}{(|\mathfrak{Y} - \mathfrak{X}|(1 + q) - \sum_{n=2}^{\infty} \Lambda_q \varphi_{q,n-1}|a_n|}
$$

where

$$
\Lambda_q = [((3 - q + \mathfrak{Y}(1 + q))[n, q]) - (\mathfrak{X}(1 + q) + (-q + 3))\psi_n].
$$

Expression in (93) is bounded above by 1 if

$$
\sum_{n=2}^{\infty} (\mathcal{F}_1 + \mathcal{F}_2)\varphi_{q,n-1}|a_n| \leq |\mathfrak{Y} - \mathfrak{X}|(1 + q),
$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by (90) and (91) respectively. Thus, we have completed the proof of our Theorem.

If in Theorem 1 we put

$$
q \longrightarrow 1 - \lambda_1 = 0,
$$

$$
i = 1
$$

then the result was reduced to the following known one.

**Corollary 1** (see [30]). An analytic function $f$ of the form (3) is in the class $\lambda-\mathcal{F}(\mathfrak{X}, \mathfrak{Y})$, if it hold the condition

$$
\sum_{n=2}^{\infty} (2(1 + \lambda)(n - 1) + n(\mathfrak{Y} + 1) - (\mathfrak{X} + 1))|a_n| \leq |\mathfrak{Y} - \mathfrak{X}|.
$$

If in Theorem 1, we set

$$
\lambda_1 + 1 = j = 1 = i = \mathfrak{X} = \mathfrak{Y} + 2.
$$

**Proof.** Suppose that (89) holds, then it is enough to show that

$$
\lambda|\varphi(\mathfrak{X}, \mathfrak{Y}, q, \lambda_1) - 1| - \Re\{\varphi(\mathfrak{X}, \mathfrak{Y}, q, \lambda_1) - 1\} < 1.
$$

where $\varphi(\mathfrak{X}, \mathfrak{Y}, q, \lambda_1)$ is given by (56).

Now, we have the following equation:

$$
\mathcal{F}_2 = |(\mathfrak{Y}(q + 1) - (3 - q))[n, q] - (\mathfrak{X}(q + 1) + (3 - q))\psi_n|.
$$

(91)

and let $q \longrightarrow 1 -$, then we get the following well-known consequence.

**Corollary 2** (see [2]). An analytic function $f$ of the form (3) is in the class $\lambda-\mathcal{D}$, if it satisfies the condition

$$
\sum_{n=2}^{\infty} |n + \lambda(n - 1)|a_n| \leq 1.
$$

(99)

If we put

$$
\lambda_1 + 1 = j = 1 = i = \mathfrak{Y} + 2,
$$

$$
\mathfrak{X} = 1 - 2\alpha(0 \leq \alpha < 1),
$$

(100)

and let $q \longrightarrow 1 -$, in Theorem 2, then we have the following known result.

**Corollary 3** (see [24]). An analytic function $f$ of the form (3) is in the class $\lambda-\mathcal{D}(\lambda, \alpha)$, if it satisfies the condition

$$
\sum_{n=2}^{\infty} |n(1 + \lambda) - (\lambda + \alpha)|a_n| \leq 1 - \alpha\left(\begin{array}{c}
0 \leq \alpha < 1, \\
\lambda \geq 0
\end{array}\right).
$$

(101)

**Theorem 3.** If $f(z) \in k-\mathcal{F}_q^{(j)}(\lambda_1, \mathfrak{X}, \mathfrak{Y})$, then
where \( k \geq 0, -1 \leq \mathbf{X} \leq 1 \) and \( \delta_k, \psi_n, \phi_{q,n-1} \) are defined by (16), (29) and (44).

**Proof.** Let \( f(z) = \lambda - \mathcal{Y}^{(jz)}_{q}(\lambda, \mathbf{X}, \mathbf{Y}) \) and

\[
\frac{z \partial_q^n f(z)}{R_q^n f_{jz}(z)} = \chi(z),
\]

which implies that

\[
z \partial_q^n f(z) = R_q^n f_{jz}(z) \chi(z).
\]

By using (47) and (48) on (104), we have the following equation:

\[
z + \sum_{m=2}^{\infty} ([m, q] - \psi_m) \phi_{q,m-1} a_n z^n
= \left( z + \sum_{m=2}^{\infty} [m, q] \psi_m \phi_{q,m-1} a_n z^n \right) \left( 1 + \sum_{n=1}^{\infty} c_n z^n \right).
\]

By making use of the well-known Cauchy Product formula on Right hand side of (105), we get the following equation:

\[
\sum_{m=2}^{\infty} ([m, q] - \psi_m) \phi_{q,m-1} a_n z^n
= \sum_{m=2}^{\infty} \left( \sum_{n=1}^{m-1} \phi_{q,n-1} \psi_n c_{n-p} a_p \right) z^n.
\]

On both sides of equation (106), comparing the coefficients of \( z^n \), we have the following equation:

\[
([m, q] - \psi_m) \phi_{q,m-1} a_n
= \sum_{p=1}^{m-1} \phi_{q,m-1} \psi_p c_{n-p} a_p, \quad a_1 = 1, \quad \psi_1 = 1, \quad \psi_0 = 1,
\]

which implies that

\[
|a_n| \leq \frac{1}{([m, q] - \psi_m) \phi_{q,m-1}} \sum_{p=1}^{m-1} |\psi_{n-1} \psi_p| |c_{n-p}|.
\]

By using Lemma 6 together with (108), we have the following equation:

\[
|a_n| \leq \frac{(\mathbf{X} - \mathbf{Y}) (1 + q)}{4([m, q] - \psi_m) \phi_{q,m-1}} \sum_{p=1}^{m-1} |\psi_{p-1} \psi_p| |a_p|.
\]

We claim that

\[
|a_n| \leq \frac{(\mathbf{X} - \mathbf{Y}) (1 + q)}{4([m, q] - \psi_m) \phi_{q,m-1}} \sum_{p=1}^{m-1} |\psi_{p-1} \psi_p| |a_p|.
\]
\[ |c_i (\mathbf{X - Y})(1 + q)| + 4 |m, q| - \psi_{m} |\Psi_{q,m-1} | \]
\[ \frac{1}{4} [m + 1, q] - \psi_{m+1} |\Psi_{q,m} | \]
\[ \prod_{p=1}^{m-1} \left( \frac{|c_i (\mathbf{X - Y})(1 + q)| + 4 |p, q| - \psi_{p} |\Psi_{q,p-1} |}{4 [p + 1, q] - \psi_{m+1} |\Psi_{q,p} |} \right) \]
\[ \geq \frac{1}{4} \left( [m, q] - \psi_{m+1} |\Psi_{q,m-1} | \right) \sum_{p=1}^{m-1} |\Psi_{q,p-1} | |a_p| . \]
\[ \text{Furthermore,} \]
\[ \prod_{p=1}^{m} \left( \frac{|c_i (\mathbf{X - Y})(1 + q)| + 4 |p, q| - \psi_{p} |\Psi_{q,p-1} |}{4 [p + 1, q] - \psi_{m+1} |\Psi_{q,p} |} \right) \]
\[ \geq \frac{1}{4} \left( [m, q] - \psi_{m+1} |\Psi_{q,m-1} | \right) \sum_{p=1}^{m-1} |\Psi_{q,p-1} | |a_p| . \]
\[ = \frac{1}{4} \left( [m, q] - \psi_{m+1} |\Psi_{q,m-1} | \right) \sum_{p=1}^{m} |\Psi_{q,p-1} | |a_p| . \]
\[ \text{(118)} \]

Corollary 4 (see [30]). An analytic function \( f \) of the form (3) belongs to the class \( \lambda, \delta \mathcal{T}(\mathbf{X, Y}) \), if it satisfies
\[ |a_n| \leq \prod_{j=0}^{n-1} \left( \frac{c_i (\mathbf{X - Y}) - 2 j \mathbf{Y}}{2 (j + 1)} \right) . \]
\[ \text{(122)} \]

If we set
\[ \lambda_1 = 0, \]
\[ j = 1 = i = \mathbf{X} = -\mathbf{Y}, \]
and let \( q \rightarrow 1^- \), in Theorem 2, we have the following result.

Corollary 5 (see [3]). An analytic function \( f \) of the form (3) belongs to the class \( \lambda, \delta \mathcal{T} \), if it satisfies
\[ |a_n| \leq \prod_{j=0}^{n-1} \left( \frac{c_i + j}{j + 1} \right) . \]
\[ \text{(124)} \]

If we set
\[ \lambda_1 = 0, \]
\[ j = 1 = i = \mathbf{X} = -\mathbf{Y}, \]
and let \( q \rightarrow 1^- \), also by taking \( \lambda = 0 \), then \( c_i = 2 \), in Theorem 2, we have the following result.

Corollary 6 (see [1]). An analytic function \( f \) of the form (3) is in the class \( \mathcal{S}^* (\mathbf{X, Y}) \), if it satisfies
\[ |a_n| \leq \prod_{j=0}^{n-1} \left( \frac{|\mathbf{X} - \mathbf{Y}| - j \mathbf{Y}}{2 (j + 1)} \right) . \]
\[ -1 \leq \mathbf{Y} \leq \mathbf{X} \leq 1. \]
\[ \text{(126)} \]

Theorem 4. If the function \( f(z) \) given by (3) belongs to the class \( k, \mathcal{F}^* (\mathbf{X, Y}) \), then
\[ |a_2| \leq \frac{(\mathbf{X} - \mathbf{Y})(1 + q) \delta_k}{2 (3 - q) (1 + q - \psi_{2}) \psi_{q,1}} , \]
\[ \text{(127)} \]

and

\[ |a_3| \leq \frac{(\mathbf{X} - \mathbf{Y})(1 + q)}{2 (3 - q) (1 + q - \psi_{2}) \psi_{q,2}} \]
\[ \left\{ \delta_k \frac{\delta_k}{2} - \frac{1}{2} \left( \frac{\mathbf{X} + \mathbf{Y} + 3 - q}{2 (3 - q)} - \psi_{2} \right) \right\} \]
\[ \text{(128)} \]

and let \( q \rightarrow 1^- \), we get the following known result. \( \square \)
Proof. Let \( \chi \in \mathcal{D} \) and defined by the following equation:
\[
\chi(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots ,
\]
(129)
where \( w(z) \) is a Schwarz function, such that \( w(0) = 0 \) and \(|w(z)| < 1\). It follows easily that
\[
w(z) = \frac{\chi(z) - 1}{\chi(z) + 1}
= \frac{c_1}{2} z + \left( \frac{1}{2} \frac{c_1^2}{4} \right) z^2 + \left( \frac{1}{2} \frac{c_1^3 c_2}{8} + \frac{c_1^3}{8} \right) z^3 \cdots .
\]
(130)

\[
\frac{z \partial_q R^1_q f(z)}{R^1_q f_{ij}(z)} \sim (\mathcal{X} (1 + q) - (-q + 3)) p_1(z) - (\mathcal{X} (1 + q) - (-q + 3))
\frac{1}{(\mathcal{Y} (1 + q) + (-q + 3)) p_1(z) - (\mathcal{Y} (1 + q) - (-q + 3))}.
\]
(131)

where \( p_1(z) \) is given in (15), thus from (131), we have the following equation:
\[
\frac{z \partial_q R^1_q f(z)}{R^1_q f_{ij}(z)} = (\mathcal{X} (1 + q) - (-q + 3)) p_1(w(z)) - (\mathcal{X} (1 + q) - (-q + 3))
\frac{1}{(\mathcal{Y} (1 + q) + (-q + 3)) p_1(w(z)) - (\mathcal{Y} (1 + q) - (-q + 3))}.
\]
(132)

From (130), we have the following equation:
\[
p_1\left(w(z)\right) = 1 + c_1 w(z) + c_{21} \left(w(z)\right)^2 + c_{33} \left(w(z)\right)^3 + \cdots
\]
\[
= 1 + c_1 \left( \frac{c_1}{2} z + \left( \frac{1}{2} \frac{c_1^2}{4} \right) z^2 + \left( \frac{1}{2} \frac{c_1^3 c_2}{8} + \frac{c_1^3}{8} \right) z^3 \cdots \right)
\]
\[
+ c_{21} \left( \frac{c_1}{2} z + \left( \frac{1}{2} \frac{c_1^2}{4} \right) z^2 + \left( \frac{1}{2} \frac{c_1^3 c_2}{8} + \frac{c_1^3}{8} \right) z^3 \cdots \right)^2
\]
\[
+ c_{33} \left( \frac{c_1}{2} z + \left( \frac{1}{2} \frac{c_1^2}{4} \right) z^2 + \left( \frac{1}{2} \frac{c_1^3 c_2}{8} + \frac{c_1^3}{8} \right) z^3 \cdots \right)^3 \cdots .
\]
(133)

After some simplification we have the following equation:
\[
p_1\left(w(z)\right) = 1 + \frac{c_1}{2} c_1 z + \left( \frac{c_{21} c_1^2}{4} + \frac{1}{2} \left( c_2 - \frac{c_1}{2} \right) c_1 \right) z^2
\]
\[
+ \left( \frac{c_1}{8} - \frac{c_{21}}{4} \right) c_1 c_2 + \frac{c_1 c_3}{2} \right) z^3 + \cdots .
\]
(134)
Using (134) and (132), we have the following equation:

\[
\frac{z\partial_1 R^{j_1}_q f(z)}{R^{j_1}_q f(z)} = 1 + \left( \frac{(X - \Psi)(1 + q)c_1}{4(3 - q)} \right) c_1 z
\]

\[
+ \left( \frac{(X - \Psi)(1 + q)}{2(3 - q)} \right) \left( \frac{S_{12}^1}{4} \frac{S_{12}^1}{4} - \frac{(\Psi + \Psi q + 3 - q)c_1^2}{8(3 - q)} \right) c_1 \left\{ \frac{S_{12}^1}{2} + \frac{S_{12}^1}{2} \right\} c_2 z^2
\]

\[
+ \left( \frac{(X - \Psi)(1 + q)}{2(-q + 3)} \right) \left( \frac{S_{12}^1}{8} \frac{S_{12}^1}{8} - \frac{(\Psi + \Psi q + 3 - q)c_1 S_{12}^1}{8(3 - q)} \right) + \left( \frac{\Psi + \Psi q + 3 - q}{8(3 - q)} \right) c_1^2 - \frac{(\Psi + \Psi q + 3 - q)^2}{32(3 - q)} c_1^2 \right\} c_3^2 + \cdots.
\]

(135)

From (47) and (48), we have the following equation:

\[
\frac{z\partial_1 R^{j_1}_q f(z)}{R^{j_1}_q f(z)} = 1 + ((1 + q) - \Psi_2)\psi_{q,1} a_2 z
\]

\[
+ \left[ ((1 + q + q^2) - \Psi_1)\psi_{q,2} a_3 - ((1 + q) - \Psi_2)(\psi_{q,1}^2 \psi_{q,2}^2) \right] z^2
\]

\[
+ \left[ (1 + q)(1 + q^2) - \Psi_1 \psi_{q,1} a_4 - ((1 + q) - \Psi_2)\psi_{q,2} + ((1 + q + q^2) - \Psi_1)\psi_{q,2} \psi_{q,3} \psi_{q,4} a_3 + (1 + q) - \Psi_2)(\psi_{q,1}^3 \psi_{q,2} \psi_{q,3}) \right] z^3 + \cdots.
\]

(136)

From (135) and (136), equating coefficients of \(z\) and \(z^2\) and gives us the following equation:

\[
a_2 = \left( \frac{(X - \Psi)(1 + q)c_1}{4(3 - q)((1 + q) - \Psi_2)\psi_{q,1}} \right) c_1,
\]

(137)

\[
a_3 = \frac{(X - \Psi)(1 + q)}{2(3 - q)((1 + q + q^2) - \Psi_1)\psi_{q,2}}
\]

\[
\left\{ \frac{S_{12}^1}{4} - \frac{S_{12}^1}{4} - \frac{(\Psi + \Psi q + 3 - q)}{8(3 - q)} \right\} + \left( \frac{(X - \Psi)(1 + q)c_1^2}{8(3 - q)((1 + q) - \Psi_2)} \right) c_1^2 + \frac{c_1}{2c_2} \right\}.
\]

(138)

By using Lemma 3, on (137) and (138) we obtain the result asserted by Theorem 3.

\[\square\]

**Theorem 5.** Let \(-1 \leq \Psi < X \leq 1\) and \(0 \leq k < \infty\) be fixed and let \(f(z) \in k - \Psi^{[-1,k]}(\lambda, X, \Psi)\), then for a complex number \(\mu\).
where $v$, $\mathbf{x}_1$, $\mathbf{y}_1$, and $C_1$ are given by (141), (142), (143), and (144), respectively.

**Proof.** Using (137) and (138), we have

$$|a_3 - \mu a_2|^2 = \frac{\mathbf{x}_1}{2} \left| c_2 - \frac{2}{\mathbf{x}_1 c_1} (\mu c_2^2 - \mathbf{x}_1 \mathbf{y}_1) c_1 \right|^2,$$  \hspace{1cm} (140)

where

$$v = \frac{2}{\mathbf{x}_1 c_1} (\mu c_2^2 - \mathbf{x}_1 \mathbf{y}_1),$$  \hspace{1cm} (141)

$$\mathbf{x}_1 = \frac{(\mathbf{x} - \mathbf{y})(1 + q)}{2(-q + 3)((1 + q + q^2) - \psi_3)\psi_{q,2}},$$  \hspace{1cm} (142)

$$\mathbf{y}_1 = \frac{\mathbf{x}_1}{4} \mathbf{y}_1 - \frac{(\mathbf{x} + \mathbf{y} q + 3 - q)}{8(3 - q)} \frac{(\mathbf{x} - \mathbf{y})(1 + q)(\mathbf{c}_i)^2 \psi_2}{8(3 - q)((1 + q) - \psi_2)},$$  \hspace{1cm} (143)

and

$$C_1 = \frac{(\mathbf{x} - \mathbf{y})(1 + q)\mathbf{c}_1}{4(3 - q)((1 + q) - \psi_2)\psi_{q,1}}.$$  \hspace{1cm} (144)

Apply Lemma 2 on (140) we have. \hfill \square

**Theorem 6.** Let $0 \leq k \leq q$ and let $f(z) \in k^{(j)}_q(\lambda_1, \mathbf{x}, \mathbf{y})$ of the form (3), then for a complex number $\mu$

$$|a_3 - \mu a_2|^2 \leq \frac{\mathbf{x}_1}{2} \delta_k \max\{1, |2v - 1|\}, \quad \forall \mu \in \mathbb{C}.$$  \hspace{1cm} (145)

where $v$, $\mathbf{x}_1$, $\mathbf{y}_1$, $C_1$ given by (141), (142), (143), and (144).

**Proof.** From (140), we have the following equation:

$$|a_3 - \mu a_2|^2 = \frac{\mathbf{x}_1}{2} \left| c_2 - \frac{2}{\mathbf{x}_1 c_1} (\mu c_2^2 - \mathbf{x}_1 \mathbf{y}_1) c_1 \right|^2,$$  \hspace{1cm} (146)

Apply Lemma 4 in conjunction with (146), we obtain the result asserted by Theorem 5. \hfill \square

## 5. Conclusion

We have successfully studied the uses of certain Ruscheweyh-type $q$-differential operator to a new subclass of $q$-starlike symmetric functions, which involving both the conic domains and the well-known celebrated Janowski functions in the open unit disk $\mathbb{D}$. We then investigated many properties for the newly defined functions class, including for example coefficients inequalities, the Fekete–Szegö Problems, and a sufficient condition. We have also emphasized certain known results of our key findings.

The interested readers should be advised not to be misled to believe that the so-called $k$-Gamma function provides a “generalization” of the classical (Euler’s) Gamma function. Similar remarks will apply also to all of the usages of the so-called $k$-Gamma function including (for example) the so-called $(k, s)$-extensions of the Riemann–Liouville and other operators of fractional integral and fractional derivatives.

## Data Availability

No data were used in the paper.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors’ Contributions

All authors contributed equally to this manuscript and approved its final version.

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