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Research Article

Coefficient Inequalities for a Subclass of Symmetric q -Starlike Functions Involving Certain Conic Domains

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In this paper, we make use of a certain Ruscheweyh-type q -differential operator to introduce and study a new subclass of q -starlike symmetric functions, which are associated with conic domains and the well-known celebrated Janowski functions in $\mathbb{D}$. We then investigate many properties for the newly defined functions class, including for example coefficients inequalities, the Fekete–Szegö Problems, and a sufficient condition. There are also relevant connections between the results provided in this study and those in a number of other published articles on this subject.

1. Introduction, and Preliminaries

Let $\mathscr{H}(\mathbb{D})$ be a class of analytic functions where $\mathbb{D}$ is the open unit disk and is given by the following equation:

$$\mathbb{D} = \left\{ \begin{array}{c} z\colon z \in \mathbb{C}, \\ |z| < 1 \end{array} \right\}, \tag{1}$$

and let $f \in \mathscr{A}$ be those functions in the open unit disk $\mathbb{D}$ which are normalized by the following equation:

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1, \end{aligned} \tag{2}$$

thus, we have the following series form for $f \in \mathscr{A}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{3}$$

Moreover, all normalized univalent functions in $\mathbb{D}$ are contained in the set $\mathscr{A} \subset \mathscr{S}$. For two given functions $g_1, g_2 \in \mathscr{A}$, we say that g_1 is subordinate to g_2, written symbolically as $g_1 \prec g_2$, if there exist a Schwarz function w, which is holomorphic in $\mathbb{D}$ with

$$\begin{aligned} w(0) &= 0, \\ |w(z)| &< 1, \end{aligned} \tag{4}$$

so that

$$g_1(z) = g_2(w(z)) \quad (z \in \mathbb{D}). \tag{5}$$

Moreover, if the function g_2 is univalent in $\mathbb{D}$, then the following equivalence hold true:

$$\begin{aligned} g_1 \prec g_2 \Leftrightarrow g_1(0) &= g_2(0), \\ g_1(\mathbb{D}) &\subset g_2(\mathbb{D}). \end{aligned} \tag{6}$$

Let $\mathcal{P}$ be the class of Carathedory function, an analytic function $\chi \in \mathcal{P}$ if

$$\chi(z) = 1 + \sum_{n=1}^{\infty} \chi_n z^n, \tag{7}$$

such that

$$\begin{aligned} &\chi(0) = 1, \\ &\Re\{\chi(z)\} > 0 \quad (\forall\, z \in \mathbb{D}). \end{aligned} \tag{8}$$

Definition 1. A given function p is said to be in the class $\mathcal{P}(\mathfrak{X}, \mathfrak{Y})$ if

$$p(z) \prec \frac{1 + \mathfrak{X}z}{1 + \mathfrak{Y}z} \quad (-1 \le \mathfrak{Y} < \mathfrak{X} \le 1). \tag{9}$$

Janowski [1] investigated the class of functions $\mathcal{P}(\mathfrak{X}, \mathfrak{Y})$ and found that $p(z) \in \mathcal{P}(\mathfrak{X}, \mathfrak{Y})$ if and only if there exist a function $\chi \in \mathcal{P}$ such that

$$p(z) = \frac{(\mathfrak{X} + 1)\chi(z) - (\mathfrak{X} - 1)}{(\mathfrak{Y} + 1)\chi(z) - (\mathfrak{Y} - 1)} \quad (-1 \le \mathfrak{Y} < \mathfrak{X} \le 1). \tag{10}$$

Definition 2. A function f of the form (3) be in the functions class $\mathcal{S}^*(\mathfrak{X}, \mathfrak{Y})$ if and only if

$$\frac{zf'(z)}{f(z)} = \frac{(\mathfrak{X} + 1)\chi(z) - (\mathfrak{X} - 1)}{(\mathfrak{Y} + 1)\chi(z) - (\mathfrak{Y} - 1)} \quad (-1 \le \mathfrak{Y} < \mathfrak{X} \le 1). \tag{11}$$

Historically speaking, Kanas and Wiśniowska were the first (see [2], [3]), (see also [4]) who introduced and defined the class of λ-uniformly convex functions $(\lambda - \mathcal{UCV})$ and λ-starlike functions $(\lambda - \mathcal{ST})$ subject to the conic domain Ω_λ, where

$$\Omega_\lambda = \left\{u + iv\colon\, u > \lambda\sqrt{(u-1)^2 + v^2},\ u > 0\right\}, \lambda \ge 0. \tag{12}$$

Moreover if λ, is fixed and $(\lambda = 0)$ then Ω_λ denote the conic region bounded by the imaginary axis, if $\lambda = 1$, we have a parabola, if $0 < \lambda < 1$ this domain represents the right branch of hyperbola and for $\lambda > 1$ an ellipse.

For these conic areas, the following functions serve as extremal functions

$$p_\lambda(z) = \begin{cases} \dfrac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots, & (\lambda = 0), \\[2ex] 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (\lambda = 1), \\[2ex] 1 + \dfrac{2}{1-\lambda^2}\sinh^2\left\{\left(\dfrac{2}{\pi}\arccos\,\lambda\right)\arctan\, p\sqrt{z}\right\}, & (0 \le \lambda < 1), \\[2ex] 1 + \dfrac{1}{\lambda^2 - 1}\sin\left(\dfrac{\pi}{2K(\kappa)}\displaystyle\int_0^{u(z)/\sqrt{\kappa}} \dfrac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) + \dfrac{1}{\lambda^2 - 1}, & (\lambda > 1), \end{cases} \tag{13}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa z}} \quad (\forall\, z \in \mathbb{D}), \tag{14}$$

and $\kappa \in (0, 1)$ is chosen such that $\lambda = \cos h(\pi K'(\kappa)/(4K(\kappa)))$. Here, $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1-\kappa^2})$, that is $K'(t)$ is the complementary integral of $K(t)$.

The function $p_\lambda(z)$ in [5] be given as follows:

$$p_\lambda(z) = 1 + \varsigma_\lambda z + \varsigma_{2\lambda} z^2 + \varsigma_{3\lambda} z^3 + \cdots, \tag{15}$$

where

$$\varsigma_\lambda = \begin{cases} \dfrac{8(\arccos\kappa)^2}{\pi^2(1-\kappa^2)} & 0 \le \lambda < 1, \\[2ex] \dfrac{8}{\pi^2} & \lambda = 1, \\[2ex] \dfrac{\pi^2}{4(\kappa^2 - 1)R^2(t)\sqrt{t}\,(1+t)} & \lambda > 1, \end{cases} \tag{16}$$

$$\varsigma_{2\lambda} = \varsigma_1(\varsigma_\lambda), \tag{17}$$

$$\varsigma_1 = \begin{cases} \dfrac{((2/\pi)\arccos\ \kappa)^2 + 2}{3} & 0 \le \lambda < 1, \\ \dfrac{2}{3} & \lambda = 1, \quad \dfrac{n!}{r!(n-r)!}, \\ \dfrac{4R^2(t)\left(t^2 + 6t + 1\right) - \pi^2}{24R^2(t)(1+t)\sqrt{t}} & \lambda > 1, \end{cases} \tag{18}$$

and $t \in (0, 1)$.

The following is defined by Noor et al. [6], who combine the ideas of Janowski functions and conic regions.

Definition 3. A function p from the functions class $\mathscr{P}$ be in the functions class $\lambda - \mathscr{P}(\mathfrak{X}, \mathfrak{Y})$, if

$$p(z) \prec \frac{(\mathfrak{X}+1)p_\lambda(z) - (\mathfrak{X}-1)}{(\mathfrak{Y}+1)p_\lambda(z) - (\mathfrak{Y}-1)}, \quad \begin{pmatrix} -1 \le \mathfrak{Y} < \mathfrak{X} \le 1, \\ \lambda \ge 0 \end{pmatrix}, \tag{19}$$

where $p_\lambda(z)$ is defined by (13).

Geometrically for the domain $\Omega_\lambda[\mathfrak{X}, \mathfrak{Y}]$, we have the following

$$\Omega_\lambda[\mathfrak{X}, \mathfrak{Y}] = \left\{ w\colon \Re\left(\frac{(\mathfrak{Y}-1)w - (\mathfrak{X}-1)}{(\mathfrak{Y}+1))w - (\mathfrak{X}+1)}\right) > \lambda\left|\frac{(\mathfrak{Y}-1)w - (\mathfrak{X}-1)}{(\mathfrak{Y}+1))w - (\mathfrak{X}+1)}\right| \right\}, \tag{20}$$

or equivalently for $w = u + iv$, we have the following equation:

$$\left[\left(\mathfrak{Y}^2 - 1\right)\left(u^2 + v^2\right) - 2(\mathfrak{X}\mathfrak{Y} - 1)u + \left(\mathfrak{X}^2 - 1\right)\right]^2 > \lambda\left[\left(-2(\mathfrak{Y}+1))\left(u^2 + v^2\right) + 2(\mathfrak{X} + \mathfrak{Y} + 2)u - 2(\mathfrak{X}+1)\right)^2 + 4(\mathfrak{X} - \mathfrak{Y})^2 v^2\right]. \tag{21}$$

The domain $\Omega_\lambda[\mathfrak{X}, \mathfrak{Y}]$ denote the conic type regions (see [6]).

Definition 4 (see [6]). An analytic function f be in the class $\lambda - \mathcal{ST}(\mathfrak{X}, \mathfrak{Y})$, if

$$\frac{zf'(z)}{f(z)} \in \lambda 0\mathscr{P}(\mathfrak{X}, \mathfrak{Y}) \quad (\forall z \in \mathbb{D} \text{ and } \lambda \ge 0). \tag{22}$$

The angle $2\pi/i$ at which domain D rotates around the origin maps D onto itself The domain D is thus said to as i-fold symmetric. In $\mathbb{D}$ a function f is said to be i-fold symmetric if for every z in $\mathbb{D}$

$$f\left(e^{2\pi I/i}z\right) = e^{2\pi I/i}f(z),\ i \in \mathbb{Z}^+. \tag{23}$$

Here, $\mathcal{S}^{(i)}$ denote the i-fold symmetric functions and i symbolize the imaginary unit. For $(j = 0, 1, 2, \ldots, i-1;\quad i = 2, 3, \ldots;)$, the (j, i)-symmetric functions are an extension of the concept of even, odd, and i-symmetric functions. Several applications of the theory of (j, i) -symmetric functions may be found in [7]. For $e = e^{(2\pi I/i)}$, the functions $f\colon \mathbb{D} \longrightarrow \mathbb{C}$ is known to be (j, i) -symmetric if

$$f\left(e^j z\right) = e^j f(z). \tag{24}$$

The set of all (j, i)-symmetric functions is denote by $\mathcal{S}^{(j,i)}$. First of all, if $j = 0$ and $i = 2$, then $\mathcal{S}^{(j,i)}$ is called even symmetric functions. Secondly, if $j = 1$ and $i = 2$, then $\mathcal{S}^{(j,i)}$ is called odd symmetric functions. Thirdly, if $j = 1$, then $\mathcal{S}^{(j,i)}$ is called i-symmetric functions.

Theorem 1 (see [7]). *For mapping $f\colon \mathbb{D} \longrightarrow \mathbb{C}$, there is just one series of (j, i) -symmetrical functions $f_{j,i}(z)$ exists that given as follows:*

$$f(z) = \sum_{j=0}^{i-1} f_{j,i}(z). \tag{25}$$

From equation (25), we can get the following equation:

$$f_{j,i}(z) = \frac{1}{i}\sum_{v=0}^{i-1} e^{-vj} f(e^v z) = \frac{1}{i}\sum_{v=0}^{i-1} e^{-vj}\left(\sum_{n=1}^{\infty} a_n (e^v z)^n\right), \tag{26}$$

then

$$f_{j,i}(z) = \sum_{n=1}^{\infty} \psi_n a_n z^n, \quad a_1 = 1, \ \psi_1 = 1, \tag{27}$$

where $f \in \mathscr{A}$ and

$$e = e^{2\pi I/i} \quad (\mathbb{N}_0 = \{1, 2, 3, \ldots\}), \tag{28}$$

$$\psi_n = \frac{1}{i} \sum_{v=0}^{i-1} e^{(n-j)v} = \left\{ \begin{array}{l} 1, \ n = li + j \\ 0, \ n \neq li + j \end{array} \right\}. \tag{29}$$

2. Some Basic Concepts of q-Calculus

The usage of the quantum (or q -) calculus in many diverse areas of mathematics and physics are quite significant. In the theory of Univalent functions, Srivastava [8] firstly apply the q-calculus in order to put the foundation of a new direction for other researchers. Motivated by [8] of Srivastava, many researchers have worked on this direction. For example, the convolution theory, enable us to investigate various properties of analytic functions. Due to the large range of applications of q-calculus and the importance of q-operators instead of regular operators, many researchers have explored q-calculus in depth, such as, Kanas and Reducanu [9], Muhammad and Sokol [10] and Noor et al. [11–15]. Also in [1–5, 9, 16–22], Ahmad et al. see also [21], have used the q-derivative operator to define a new subclass q-meromorphic starlike functions. They also developed some remarkable results for their defined classes of analytic functions. In addition, Srivastava [23] see also [8, 12–15, 23–26] recently published survey-cum-expository review paper this might be useful for researchers and scholars working on these subjects. For some recent and related study about q-series, we may refer the interested readers to see [27–29].

Definition 5 (see [30]). Let $q \in (0, 1)$ and q-integer n, be defined as follows:

$$[n, \mathfrak{q}] = \frac{1 - \mathfrak{q}^n}{1 - \mathfrak{q}} = 1 + \mathfrak{q} + \cdots + \mathfrak{q}^{n-1}, \quad [0, \mathfrak{q}] = 0. \tag{30}$$

Definition 6. We define the $\mathfrak{q}$-shifted factorial as follows:

$$\begin{aligned} &[0, \mathfrak{q}]! = 1, \\ &[n, \mathfrak{q}]! = [1, \mathfrak{q}][2, \mathfrak{q}] \ldots [n, \mathfrak{q}]. \end{aligned} \tag{31}$$

It could be seen that

$$\begin{aligned} &\lim_{\mathfrak{q} \longrightarrow 1^-} [n, \mathfrak{q}] = n, \\ &\lim_{\mathfrak{q} \longrightarrow 1^-} [n, \mathfrak{q}]! = n!. \end{aligned} \tag{32}$$

In general $[t, \mathfrak{q}] = 1 - \mathfrak{q}^t / 1 - \mathfrak{q}$.

Definition 7 (see [20]). For an analytic function f, the $\mathfrak{q}$-deformation or $\mathfrak{q}$-generalization of derivative is defined by the following equation:

$$\partial_{\mathfrak{q}} f(z) = \frac{f(z) - f(\mathfrak{q}z)}{(1 - \mathfrak{q})z} \quad (z \in \mathbb{D}), \tag{33}$$

$$\partial_{\mathfrak{q}} f(z) = 1 + \sum_{n=2}^{\infty} [n, \mathfrak{q}] a_n z^{n-1}, \tag{34}$$

and

$$\partial_{\mathfrak{q}} z^n = [n, \mathfrak{q}] z^{n-1}. \tag{35}$$

Definition 8. The generalized $\mathfrak{q}$-Pochhammer symbol is given by the following equation:

$$[t, \mathfrak{q}]_n = [t, \mathfrak{q}][t + 1, \mathfrak{q}][t + 2, \mathfrak{q}] \ldots [t + n - 1, \mathfrak{q}]. \tag{36}$$

and $\mathfrak{q}$-gamma function be given as follows:

$$\begin{aligned} &\Gamma_{\mathfrak{q}}(t + 1) = [t, \mathfrak{q}] \Gamma_{\mathfrak{q}}(t), \\ &\Gamma_{\mathfrak{q}}(1) = 1. \end{aligned} \tag{37}$$

Definition 9 (see [31]). A function $p \in \lambda\text{-}\mathscr{P}_{\mathfrak{q}}(\mathfrak{X}, \mathfrak{Y})$ if and only if

$$p(z) \prec p_{\lambda,\mathfrak{q}}(\mathfrak{X}, \mathfrak{Y}, z), \tag{38}$$

where

$$p_{\lambda,\mathfrak{q}}(\mathfrak{X}, \mathfrak{Y}, z) = \frac{(\mathfrak{X}(1 + \mathfrak{q}) + (-\mathfrak{q} + 3) p_\lambda(z) - (\mathfrak{X}(1 + \mathfrak{q} - (-\mathfrak{q} + 3))}{(\mathfrak{Y}(1 + \mathfrak{q}) + (-\mathfrak{q} + 3)) p_\lambda(z) - (\mathfrak{Y}(1 + \mathfrak{q}) - (-\mathfrak{q} + 3))}, \quad \lambda \geq 0. \tag{39}$$

and $p_\lambda(z)$ is defined by (13).

Geometrically, we have the following equation:

$$\Omega_{\lambda,\mathfrak{q}}(\mathfrak{X}, \mathfrak{Y}) = \{w = u + iv: \mathfrak{R}(\Psi) > \lambda|\Psi - 1|\}, \tag{40}$$

where

$$\Psi = \frac{(\mathfrak{Y}(1 + \mathfrak{q} - (-\mathfrak{q} + 3))w(z) - (\mathfrak{X}(1 + \mathfrak{q} - (-\mathfrak{q} + 3))}{(\mathfrak{Y}(1 + \mathfrak{q}) + (-\mathfrak{q} + 3))w(z) - (\mathfrak{X}(1 + \mathfrak{q}) + (-\mathfrak{q} + 3))}. \tag{41}$$

For more detail (see [31]).

Definition 10 (see [9]). For $f \in \mathscr{A}$, the Rusceweyh $\mathfrak{q}$ -differential operator be defined as follows:

$$R_{\mathfrak{q}}^{\lambda_1} f(z) = f(z) * \mathscr{H}_{\mathfrak{q},\lambda_1+1}(z), \quad (z \in \mathbb{D}, \lambda_1 > -1), \tag{42}$$

where

$$\begin{aligned}\mathscr{H}_{\mathfrak{q},\lambda_1+1}(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_{\mathfrak{q}}(\lambda_1+n)}{[n-1,\mathfrak{q}]!\Gamma_{\mathfrak{q}}(1+\lambda_1)} z^n \\ &= z + \sum_{n=2}^{\infty} \frac{[\lambda_1+1,\mathfrak{q}]_{n-1}}{[n-1,\mathfrak{q}]!} z^n \\ &= z + \sum_{n=2}^{\infty} \varphi_{\mathfrak{q},n-1} z^n,\end{aligned} \tag{43}$$

with

$$\varphi_{\mathfrak{q},n-1} = \frac{[\lambda_1+1,\mathfrak{q}]_{n-1}}{[n-1,\mathfrak{q}]!}. \tag{44}$$

By " $*$ " we mean convolution (or Hadamard product). Moreover from (42), we have the following equation:

$$R_{\mathfrak{q}}^{0} f(z) = f(z), \quad R_{\mathfrak{q}}^{1} f(z) = z\partial_{\mathfrak{q}} f(z), \tag{45}$$

and

$$R_{\mathfrak{q}}^{m} f(z) = \frac{z\partial_{\mathfrak{q}}^{m}\left(z^{m-1} f(z)\right)}{[m,\mathfrak{q}]!}, \quad (m \in \mathbb{N}). \tag{46}$$

Making use of (42) and (43), the power series of $R_{\mathfrak{q}}^{\lambda_1} f(z)$ is given by the following equation:

$$\begin{aligned}R_{\mathfrak{q}}^{\lambda_1} f(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_{\mathfrak{q}}(\lambda_1+n)}{[n-1,\mathfrak{q}]!\Gamma_{\mathfrak{q}}(1+\lambda_1)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \frac{[\lambda_1+1,\mathfrak{q}]_{n-1}}{[n-1,\mathfrak{q}]!} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \varphi_{\mathfrak{q},n-1} a_n z^n.\end{aligned} \tag{47}$$

Similarly,

$$\begin{aligned}R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z) &= f_{j,i}(z) * \mathscr{H}_{\mathfrak{q},\lambda_1+1}(z), \ (z \in \mathbb{D},\ \lambda_1 > -1), \\ &= z + \sum_{n=2}^{\infty} \frac{[\lambda_1+1,\mathfrak{q}]_{n-1}}{[n-1,\mathfrak{q}]!} \psi_n a_n z^n, \\ &= z + \sum_{n=2}^{\infty} \varphi_{\mathfrak{q},n-1} \psi_n a_n z^n, \quad a_1 = 1,\ \psi_1 = 1,\ \varphi_{\mathfrak{q},0} = 1,\end{aligned} \tag{48}$$

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where $\varphi_{\mathfrak{q},n-1}$ is given by (44) and ψ_n is given by (49).

Note that,

$$\lim_{\mathfrak{q}\longrightarrow 1^-} \mathscr{H}_{\mathfrak{q},\lambda_1+1}(z) = \frac{z}{(1-z)^{\lambda_1+1}}, \tag{49}$$

$$\lim_{\mathfrak{q}\longrightarrow 1^-} R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z) = f_{j,i}(z) * \frac{z}{(1-z)^{\lambda_1+1}}, \tag{50}$$

and

$$\lim_{\mathfrak{q}\longrightarrow 1^-} R_{\mathfrak{q}}^{\lambda_1} f(z) = f(z) * \frac{z}{(1-z)^{\lambda_1+1}}. \tag{51}$$

When $\mathfrak{q} \longrightarrow 1-$, Ruscheweyh $\mathfrak{q}$-differential operator reduce to Ruscheweyh differential operator [14]. Also, note that

$$\begin{aligned}z\partial_{\mathfrak{q}}\left(\mathscr{H}_{\mathfrak{q},\lambda_1+1}(z)\right) &= \left(1 + \frac{[\lambda_1,\mathfrak{q}]}{\mathfrak{q}^{\lambda_1}}\right)\mathscr{H}_{\mathfrak{q},\lambda_1+2}(z) \\ &\quad - \frac{[\lambda_1.\mathfrak{q}]}{\mathfrak{q}^{\lambda_1}}\mathscr{H}_{\mathfrak{q},\lambda_1+1}(z),\end{aligned} \tag{52}$$

and

$$z\partial_{\mathfrak{q}}\left(R_{\mathfrak{q}}^{\lambda_1} f(z)\right) = \left(1 + \frac{[\lambda_1,\mathfrak{q}]}{\mathfrak{q}^{\lambda_1}}\right) R_{\mathfrak{q}}^{\lambda_1+1} f(z) - \frac{[\lambda_1,\mathfrak{q}]}{\mathfrak{q}^{\lambda_1}} R_{\mathfrak{q}}^{\lambda_1} f(z). \tag{53}$$

Definition 11. A given function f is said to be in the functions class $\lambda\text{-}\mathscr{V}_{\mathfrak{q}}^{(j,i)}(\lambda_1, \mathfrak{X}, \mathfrak{Y})$, $\lambda \geq 0$, $-1 \leq \mathfrak{Y} < \mathfrak{X} \leq 1$, if

$$\Re\left(\varphi(\mathfrak{X},\mathfrak{Y},\mathfrak{q},\lambda_1)\right) > \lambda\left|\varphi(\mathfrak{X},\mathfrak{Y},\mathfrak{q},\lambda_1) - 1\right|, \tag{54}$$

or equivalently

$$\frac{z\partial_{\mathfrak{q}} R_{\mathfrak{q}}^{\lambda_1} f(z)}{R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z)} \in \lambda - \mathscr{P}_{\mathfrak{q}}(\mathfrak{X},\mathfrak{Y}), \tag{55}$$

where

$$\varphi(\mathfrak{X},\mathfrak{Y},\mathfrak{q},\lambda_1) = \frac{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))\left(z\partial_{\mathfrak{q}} R_{\mathfrak{q}}^{\lambda_1} f(z)/R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z)\right) - (\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))\left(z\partial_{\mathfrak{q}} R_{\mathfrak{q}}^{\lambda_1} f(z)/R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z)\right) - (\mathfrak{X}(1+\mathfrak{q}) + (-\mathfrak{q}+3))}. \tag{56}$$

Each of the following special case of the above-defined functions class $\lambda\text{-}\mathscr{V}_{\mathfrak{q}}^{(j,i)}(\lambda_1, \mathfrak{X}, \mathfrak{Y})$ is worthy of note.

(i) One can easily seen that

$$\lim_{\mathfrak{q}\longrightarrow 1-}\lambda - \mathscr{V}_{\mathfrak{q}}^{(j,i)}(0, \mathfrak{X}, \mathfrak{Y}) = \lambda - \mathscr{V}^{(j,i)}(\mathfrak{X}, \mathfrak{Y}), \quad (57)$$

where $\lambda\text{-}\mathscr{V}^{(j,i)}(\mathfrak{X}, \mathfrak{Y})$ is the functions classes studied by Al-Sarari and Latha (see [18]).

(ii) If we set

$$\begin{aligned} \mathfrak{q} &\longrightarrow 1-, \\ \lambda_1 &= 0, \\ j &= 1. \end{aligned} \quad (58)$$

in Definition 11, we have class $\lambda\text{-}\mathcal{ST}(\mathfrak{X}, \mathfrak{Y}, i)$ introduced in [17].

(iii) If we put

$$\begin{aligned} \mathfrak{q} &\longrightarrow 1-, \\ \lambda_1 &= 0, \\ j &= 1 = i. \end{aligned} \quad (59)$$

we get the functions class $\lambda\text{-}\mathcal{ST}(\mathfrak{X}, \mathfrak{Y})$ (see [30]).

(iv) We see that

$$\lim_{\mathfrak{q}\longrightarrow 1-}\lambda 0\mathscr{V}_{\mathfrak{q}}^{(1,1)}(0, 1, -1) = \lambda 0\mathcal{ST}, \quad (60)$$

where Kanas and Wisniowska [3] have studied the class $\lambda\text{-}\mathcal{ST}$.

(v) If we set

$$\begin{aligned} \mathfrak{q} &\longrightarrow 1-, \\ \mathfrak{Y} &= 1 = i = j = \lambda_1 + 1, \\ \mathfrak{X} &= 1 - 2\alpha. \end{aligned} \quad (61)$$

in Definition 11, we get the class $\mathcal{SD}(\lambda, \alpha)$ (see [24]).

(vi) It can be easily seen that

$$\lim_{\mathfrak{q}\longrightarrow 1-}00\mathscr{V}_{\mathfrak{q}}^{(1,1)}(0, \mathfrak{X}, \mathfrak{Y}) = \mathcal{S}^*(\mathfrak{X}, \mathfrak{Y}), \quad (62)$$

where the functions class $\mathcal{S}^*(\mathfrak{X}, \mathfrak{Y})$ is studied in [1].

3. A Set of Lemmas

Lemma 1 (see [13]). *Let $\chi(z) = 1 + \sum_{n=1}^{\infty} \chi_n z^n \prec \mathcal{F}(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$. If $\mathcal{F}(z)$ is convex univalent in $\mathbb{D}$, then*

$$\begin{aligned} |\chi_n| &\le |C_1|, \\ n &\ge 1. \end{aligned} \quad (63)$$

Lemma 2 (see [32]). *If $\chi(z) = 1 + \chi_1 z + \chi_2 z^2 + \cdots$ is an analytic function with positive real part in $\mathbb{D}$, then*

$$|\chi_2 - v\chi_1^2| \le \begin{cases} -4v + 2 & \text{if} \quad v < 0, \\ 2 & \text{if} \quad 0 \le v \le 1, \\ 4v - 2 & \text{if} \quad v > 1. \end{cases} \quad (64)$$

The equality holds for

$$\chi(z) = \frac{1 + z}{1 - z}, \quad \text{when } v < 0 \text{ or } v > 1, \quad (65)$$

or one of its relations. The equality holds for

$$\chi(z) = \frac{1 + z^2}{1 - z^2}, \quad \text{when } 0 < v < 1. \quad (66)$$

or one of its relations. If $v = 0$, the equality holds if and only if

$$\chi(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1 + z}{1 - z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1 - z}{1 + z}, \quad (0 \le \lambda \le 1), \quad (67)$$

or one of rotation of it. The above upper bound is sharp as well, when

$$|\chi_2 - v\chi_1^2| + v|\chi_1|^2 \le 2, \left(0 < v \le \frac{1}{2}\right). \quad (68)$$

and

$$|\chi_2 - v\chi_1^2| + (1 - v)|\chi_1|^2 \le 2\left(\frac{1}{2} < v \le 1\right). \quad (69)$$

Lemma 3 (see [32]). *If the function $\chi(z)$ given by (8) belongs to the class $\mathcal{P}$ of analytic functions and have positive real part in $\mathbb{D}$, then*

$$|\chi_n| \le 2, \quad n \in \mathbb{N}. \quad (70)$$

The equality holds for

$$f(z) = \frac{1 + z}{1 - z}. \quad (71)$$

Lemma 4 (see [32]). *If the function $\chi(z)$ given by (8) belongs to the class $\mathcal{P}$ of analytic functions and have positive real part in $\mathbb{D}$, then*

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2\mu - 1|\}, \quad \forall \mu \in \mathbb{C}. \quad (72)$$

Lemma 5. *Let $\lambda \in [0, \infty)$ be fixed and*

$$p_{\lambda,\mathfrak{q}}(\mathfrak{X}, \mathfrak{Y}, z) = \frac{(\mathfrak{X}(1 + \mathfrak{q}) + (-\mathfrak{q} + 3))p_\lambda(z) - (\mathfrak{X}(1 + \mathfrak{q}) - (-\mathfrak{q} + 3))}{(\mathfrak{Y}(1 + \mathfrak{q}) + (-\mathfrak{q} + 3))p_\lambda(z) - (\mathfrak{Y}(1 + \mathfrak{q}) - (-\mathfrak{q} + 3))}, \quad (73)$$

then

$$p_{\lambda,\mathfrak{q}}(\mathfrak{X},\mathfrak{Y},z) = 1 + \Upsilon_5(\mathfrak{q})\varsigma_\lambda z + \left\{\Upsilon_5(\mathfrak{q})\varsigma_{2\lambda} - \Upsilon_6(\mathfrak{q})\varsigma_\lambda^2\right\}z^2 + \cdots, \tag{74}$$

where $\Upsilon_5(\mathfrak{q})$, $\Upsilon_6(\mathfrak{q})$, ς_λ *and* $\varsigma_{2\lambda}$ *given by (83), (84), (16), and (17).*

Proof. From (39), we have the following equation:

$$\begin{aligned}
p_{\lambda,\mathfrak{q}}(\mathfrak{X},\mathfrak{Y},z) &= \frac{(\mathfrak{X}(1+\mathfrak{q}) + (-\mathfrak{q}+3))p_\lambda(z) - (\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))p_\lambda(z) - (\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}\\
&= \left[(\mathfrak{X}(1+\mathfrak{q}) + (-\mathfrak{q}+3))p_\lambda(z) - (\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))\right]\\
&\quad \cdot \left[(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))p_\lambda(z) - (\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))\right]^{-1}\\
&= \frac{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}\left[1 - \frac{(\mathfrak{X}(1+\mathfrak{q}) + (-\mathfrak{q}+3)}{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3)}p_\lambda(z)\right]\\
&\quad \cdot \left[1 + \sum_{n=1}^{\infty}\left(\frac{(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}p_\lambda(z)\right)^n\right]\\
&= \frac{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))} + \Upsilon_1(\mathfrak{q})p_\lambda(z) + \Upsilon_2(\mathfrak{q})\left(p_\lambda(z)\right)^2 + \Upsilon_3(\mathfrak{q})\left(p_\lambda(z)\right)^3 + \cdots,
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
\Upsilon_1(\mathfrak{q}) &= \frac{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))^2} - \frac{(\mathfrak{X}(1+\mathfrak{q}) + (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))},\\
\Upsilon_2(\mathfrak{q}) &= \frac{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))^2}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))^3} - \frac{(\mathfrak{X}(1+\mathfrak{q}) + (-\mathfrak{q}+3))(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))^2},
\end{aligned} \tag{76}$$

and

$$\Upsilon_3(\mathfrak{q}) = \frac{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))^3}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))^4} - \frac{(\mathfrak{X}(1+\mathfrak{q}) + (-\mathfrak{q}+3))(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))^2}{(\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))^3}. \tag{77}$$

By using (15) and (75), we have the following equation:

$$p_{\lambda,\mathfrak{q}}(\mathfrak{X},\mathfrak{Y},z)=\sum_{n=1}^{\infty}\frac{-2(\mathfrak{Y}(1+\mathfrak{q})+(-\mathfrak{q}+3))^{n-1}}{((\mathfrak{Y}(1+\mathfrak{q})-(-\mathfrak{q}+3)))^{n}}\cdot\frac{2n((\mathfrak{X}(1+\mathfrak{q})-\mathfrak{Y}(1+\mathfrak{q}))(\mathfrak{Y}(1+\mathfrak{q})+(-\mathfrak{q}+3))^{n-1}}{(\mathfrak{Y}(1+\mathfrak{q})-(-\mathfrak{q}+3))^{n+1}}+\sum_{n=1}^{\infty}\varsigma_{\lambda}z+\left\{\sum_{n=1}^{\infty}\frac{2n((\mathfrak{X}(1+\mathfrak{q})-\mathfrak{Y}(1+\mathfrak{q}))(\mathfrak{Y}(1+\mathfrak{q})+(-\mathfrak{q}+3))^{n-1}}{(\mathfrak{Y}(1+\mathfrak{q})-(-\mathfrak{q}+3))^{n+1}}\varsigma_{2\lambda}+\sum_{n=1}^{\infty}\Upsilon_4(\mathfrak{q})\varsigma_{\lambda}^{2}\right\}z^2+\cdots, \tag{78}$$

where

$$\Upsilon_4(\mathfrak{q})=\frac{2n(\mathfrak{Y}(1+\mathfrak{q})+(-\mathfrak{q}+3))((\mathfrak{X}(1+\mathfrak{q})-\mathfrak{Y}(1+\mathfrak{q}))((3-\mathfrak{q})+\mathfrak{Y}(1+\mathfrak{q}))^{n}}{(\mathfrak{Y}(1+\mathfrak{q})-(3-\mathfrak{q}))^{n+2}}. \tag{79}$$

The series

$$\sum_{n=1}^{\infty}\frac{-2((3-\mathfrak{q})+\mathfrak{Y}(1+\mathfrak{q}))^{n-1}}{((\mathfrak{Y}(1+\mathfrak{q})-(-\mathfrak{q}+3)))^{n}},\quad \sum_{n=1}^{\infty}\frac{2n((\mathfrak{X}(1+\mathfrak{q}))-\mathfrak{Y}(1+\mathfrak{q}))(\mathfrak{Y}(1+\mathfrak{q})+(-\mathfrak{q}+3))^{n-1}}{(\mathfrak{Y}(1+\mathfrak{q})-(-\mathfrak{q}+3))^{n+1}}, \tag{80}$$

and

$$\sum_{n=1}^{\infty}\frac{2n(\mathfrak{Y}(1+\mathfrak{q})+(-\mathfrak{q}+3))((\mathfrak{X}(1+\mathfrak{q})-\mathfrak{Y}(1+\mathfrak{q}))((3-\mathfrak{q})+\mathfrak{Y}(1+\mathfrak{q}))^{n}}{(\mathfrak{Y}(1+\mathfrak{q})-(3-\mathfrak{q}))^{n+2}}. \tag{81}$$

are convergent and convergent to 1, $(1+\mathfrak{q})(\mathfrak{X}-\mathfrak{Y})/4$ and $(\mathfrak{Y}(1+\mathfrak{q})+(-\mathfrak{q}+3))(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})/8$, respectively.

Therefore (78) becomes

$$p_{\lambda,\mathfrak{q}}(\mathfrak{X},\mathfrak{Y},z)=1+\Upsilon_5(\mathfrak{q})\varsigma_{\lambda}z+\left\{\Upsilon_5(\mathfrak{q})\varsigma_{2\lambda}-\Upsilon_6(\mathfrak{q})\varsigma_{\lambda}^{2}\right\}z^2+\cdots, \tag{82}$$

where

$$\Upsilon_5(\mathfrak{q})=\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{4}. \tag{83}$$

and

$$\Upsilon_6(\mathfrak{q})=\frac{((3-\mathfrak{q})+\mathfrak{Y}(1+\mathfrak{q}))(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{8}. \tag{84}$$

which is our required result. □

Remark 1. If we set $A=1$, $B=-1$ and let $q\longrightarrow 1-$, in Lemma 5, we will arrive it that of a result given by Sim et al. [25].

Lemma 6. *Let $\chi(z)=1+\sum_{n=1}^{\infty}\chi_n z^n\in\lambda\text{-}\mathscr{P}_{\mathfrak{q}}(\mathfrak{X},\mathfrak{Y})$, then*

$$|\chi_n|\le\Upsilon_5(\mathfrak{q})|\varsigma_{\lambda}|=\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{4}|\varsigma_{\lambda}|,\quad n\ge 1. \tag{85}$$

Proof. By Definition 9 and $\chi(z)\in\lambda\text{-}\mathscr{P}_{\mathfrak{q}}(\mathfrak{X},\mathfrak{Y})$ if and only if

$$\chi(z)\prec p_{\lambda,\mathfrak{q}}(\mathfrak{X},\mathfrak{Y},z)\,(\lambda\ge 0) \tag{86}$$

$p_{\lambda,\mathfrak{q}}(\mathfrak{X},\mathfrak{Y},z)$ is given by (39).

By using (82) and (86), we have the following equation:

$$\chi(z)\prec 1+\Upsilon_5(\mathfrak{q})|\varsigma_{\lambda}|z+\left\{\Upsilon_5(\mathfrak{q})|\varsigma_{2\lambda}|-\Upsilon_5(\mathfrak{q})|\varsigma_{\lambda}^{2}|\right\}z^2+\cdots. \tag{87}$$

Now by using Lemma 1 on (87), we have the following equation:

$$|\chi_n|\le\Upsilon_5(\mathfrak{q})|\varsigma_{\lambda}|=\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{4}|\varsigma_{\lambda}|, \tag{88}$$

which is our required result. □

4. Main Results

Theorem 2. *An analytic function f of the form (3) belongs to the class $k\text{-}\mathscr{V}_q^{(j,i)}(\lambda_1, \mathfrak{X}, \mathfrak{Y})$, if the following condition is satisfied:*

$$\sum_{n=2}^{\infty}(\mathscr{F}_1+\mathscr{F}_2)\varphi_{q,n-1}|a_n| \le |\mathfrak{Y}-\mathfrak{X}|(1+q), \tag{89}$$

where

$$\mathscr{F}_1 = 2(k+1)(3-q)\left|\psi_n - [n,q]\right|, \tag{90}$$

ψ_n is given by (29) and

$$\mathscr{F}_2 = \left|(\mathfrak{Y}(q+1)-(3-q))[n,q] - (\mathfrak{X}(q+1)+(3-q))\psi_n\right|. \tag{91}$$

Proof. Suppose that (89) holds, then it is enough to show that

$$\lambda\left|\varphi(\mathfrak{X},\mathfrak{Y},\mathfrak{q},\lambda_1)-1\right| - \Re\{\varphi(\mathfrak{X},\mathfrak{Y},\mathfrak{q},\lambda_1)-1\} < 1. \tag{92}$$

where $\varphi(\mathfrak{X},\mathfrak{Y},\mathfrak{q},\lambda_1)$ is given by (56).

Now, we have the following equation:

$$\begin{aligned}(1+\lambda)&\left|\varphi(\mathfrak{X},\mathfrak{Y},\mathfrak{q},\lambda_1)-1\right| \\ &= \left|\frac{2(1+\lambda)(3-\mathfrak{q})\left(R_{\mathfrak{q}}^{\lambda_1}f_{j,i}(z) - z\partial_{\mathfrak{q}}R_{\mathfrak{q}}^{\lambda_1}f(z)\right)}{((3-\mathfrak{q})+\mathfrak{Y}(1+\mathfrak{q}))z\partial_{\mathfrak{q}}R_{\mathfrak{q}}^{\lambda_1}f(z) - (\mathfrak{X}(1+\mathfrak{q})+(-\mathfrak{q}+3))R_{\mathfrak{q}}^{\lambda_1}f_{j,i}(z)}\right| \\ &\le \frac{2(1+\lambda)(3-\mathfrak{q})\sum_{n=2}^{\infty}\left\{\left|\psi_n-[n,\mathfrak{q}]\right|\right\}\varphi_{\mathfrak{q},n-1}|a_n|}{|(\mathfrak{Y}-\mathfrak{X})|(1+\mathfrak{q}) - \sum_{n=2}^{\infty}\Lambda_{\mathfrak{q}}\varphi_{\mathfrak{q},n-1}|a_n|}.\end{aligned} \tag{93}$$

where

$$\% \ \Lambda_{\mathfrak{q}} = \left|((3-\mathfrak{q})+\mathfrak{Y}(1+\mathfrak{q}))[n,\mathfrak{q}]) - (\mathfrak{X}(1+\mathfrak{q})+(-\mathfrak{q}+3))\psi_n\right|. \tag{94}$$

Expression in (93) is bounded above by 1 if

$$\sum_{n=2}^{\infty}(\mathscr{F}_1+\mathscr{F}_2)\varphi_{\mathfrak{q},n-1},|a_n| \le |\mathfrak{Y}-\mathfrak{X}|(1+\mathfrak{q}), \tag{95}$$

where $\mathscr{F}_1$ and $\mathscr{F}_2$ are defined by (90) and (91) respectively. Thus, we have completed the proof of our Theorem.

If in Theorem 1 we put

$$\begin{aligned}\mathfrak{q} \longrightarrow 1-\lambda_1 &= 0,\\ i &= 1\end{aligned} \tag{96}$$

then the result was reduced to the following known one. □

Corollary 1 (see [30]). *An analytic function f of the form (3) is in the class $\lambda\text{-}\mathscr{ST}(\mathfrak{X},\mathfrak{Y})$, if it hold the condition*

$$\sum_{n=2}^{\infty}\{2(1+\lambda)(n-1) + |n(\mathfrak{Y}+1)-(\mathfrak{X}+1)|\}|a_n| \le |\mathfrak{Y}-\mathfrak{X}|. \tag{97}$$

If in Theorem 1, we set

$$\lambda_1 + 1 = j = 1 = i = \mathfrak{X} = \mathfrak{Y} + 2. \tag{98}$$

and let $\mathfrak{q} \longrightarrow 1-$, then we get the following well-known consequence.

Corollary 2 (see [2]). *An analytic function f of the form (3) is in the class $\lambda\text{-}\mathscr{ST}$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n+\lambda(n-1)\}|a_n| \le 1. \tag{99}$$

If we put

$$\begin{aligned}\lambda_1 + 1 = j = 1 = i &= \mathfrak{Y}+2,\\ \mathfrak{X} &= 1-2\alpha\,(0\le\alpha<1),\end{aligned} \tag{100}$$

and let $\mathfrak{q} \longrightarrow 1-$, in Theorem 2, then we have the following known result.

Corollary 3 (see [24]). *An analytic $f \in \mathscr{A}$ and of the form (3) is in the class $\lambda\text{-}\mathscr{SD}(\lambda,\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n(1+\lambda)-(\lambda+\alpha)\}|a_n| \le 1-\alpha\begin{pmatrix}0\le\alpha<1,\\ \lambda\ge 0\end{pmatrix}. \tag{101}$$

Theorem 3. *If $f(z) \in k\text{-}\mathscr{V}_q^{(j,i)}(\lambda_1,\mathfrak{X},\mathfrak{Y})$, then*

$$\left|a_n\right| \leq \prod_{h=1}^{n-1}\left(\frac{\left|\delta_k(\mathfrak{X}-\mathfrak{Y})(1+q)-4\left|[h,q]-\psi_h\right|\varphi_{q,h-1}\mathfrak{Y}\right|}{4\left|[h+1,q]-\psi_{h+1}\right|\varphi_{q,h}}\right) \quad (\mathrm{n} \geq 2), \tag{102}$$

where $k \geq 0$, $-1 \leq \mathfrak{Y} < \mathfrak{X} \leq 1$ *and* δ_k, ψ_n, $\varphi_{q,n-1}$ *are defined by (16), (29) and (44).*

Proof. Let $f(z) \in \lambda\text{-}\mathscr{V}_{\mathfrak{q}}^{(j,i)}(\lambda_1, \mathfrak{X}, \mathfrak{Y})$ and

$$\frac{z\partial_{\mathfrak{q}} R_{\mathfrak{q}}^{\lambda_1} f(z)}{R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z)} = \chi(z), \tag{103}$$

which implies that

$$z\partial_{\mathfrak{q}} R_{\mathfrak{q}}^{\lambda_1} f(z) = R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z)\chi(z). \tag{104}$$

By using (47) and (48) on (104), we have the following equation:

$$z + \sum_{n=2}^{\infty}[n,\mathfrak{q}]\varphi_{\mathfrak{q},n-1}a_n z^n = \left(z + \sum_{n=2}^{\infty}\varphi_{\mathfrak{q},n-1}\psi_n a_n z^n\right)\left(1 + \sum_{n=1}^{\infty}c_n z^n\right). \tag{105}$$

By making use of the well-known Cauchy Product formula on Right hand side of (105), we get the following equation:

$$\sum_{n=2}^{\infty}\left([n,\mathfrak{q}]-\psi_n\right)\varphi_{\mathfrak{q},n-1}a_n z^n = \sum_{n=2}^{\infty}\left\{\sum_{p=1}^{n-1}\varphi_{\mathfrak{q},n-1}\psi_p c_{n-p}a_p\right\}z^n. \tag{106}$$

On both sides of equation (106), comparing the coefficients of z^n, we have the following equation:

$$\left([n,\mathfrak{q}]-\psi_n\right)\varphi_{\mathfrak{q},n-1}a_n = \sum_{p=1}^{n-1}\varphi_{\mathfrak{q},n-1}\psi_p c_{n-p}a_p, \quad a_1 = 1,\ \psi_1 = 1,\ \varphi_0 = 1, \tag{107}$$

which implies that

$$\left|a_n\right| \leq \frac{1}{\left|\left([n,\mathfrak{q}]-\psi_n\right)\varphi_{\mathfrak{q},n-1}\right|}\sum_{p=1}^{n-1}\left|\varphi_{\mathfrak{q},n-1}\psi_p\right|\left|a_p\right|\left|c_{n-p}\right|. \tag{108}$$

By using Lemma 6 together with (108), we have the following equation:

$$\left|a_n\right| \leq \frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\left|\varsigma_\lambda\right|}{4\left|\left([n,\mathfrak{q}]-\psi_n\right)\varphi_{\mathfrak{q},n-1}\right|}\sum_{p=1}^{n-1}\left|\varphi_{p-1}\psi_p\right|\left|a_p\right|. \tag{109}$$

We claim that

$$\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\left|\varsigma_\lambda\right|}{4\left|\left([n,\mathfrak{q}]-\psi_n\right)\varphi_{\mathfrak{q},n-1}\right|}\sum_{p=1}^{n-1}\left|\varphi_{\mathfrak{q},p-1}\psi_p\right|\left|a_p\right| \leq \prod_{p=1}^{n-1}\left(\frac{\left|\varsigma_\lambda(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})-4\left([p,\mathfrak{q}]-\psi_p\right)\varphi_{\mathfrak{q},p-1}\mathfrak{Y}\right|}{4\left([p+1,\mathfrak{q}]-\psi_{p+1}\right)\varphi_{\mathfrak{q},p}}\right). \tag{110}$$

We prove (110) by using the principle of the mathematical induction method.

For $n = 2$, from (109), we have the following equation:

$$\left|a_2\right| \leq \frac{\left|\varsigma_\lambda\right|(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{4\left|\left([2,\mathfrak{q}]-\psi_2\right)\varphi_{\mathfrak{q},1}\right|}. \tag{111}$$

Again from (110), we have the following equation:

$$\left|a_2\right| \leq \frac{\left|\varsigma_\lambda\right|(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{4\left|\left([2,\mathfrak{q}]-\psi_2\right)\varphi_{\mathfrak{q},1}\right|}. \tag{112}$$

Let the hypothesis be true for $n = m$, then from (109), we have the following equation:

$$\left|a_m\right| \leq \left\{\frac{\left|\varsigma_\lambda(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left|\left\{[m,\mathfrak{q}]-\psi_m\right\}\varphi_{\mathfrak{q},m-1}\right|}\times\sum_{p=1}^{m-1}\left|\varphi_{\mathfrak{q},p-1}\psi_p\right|\left|a_p\right|\right\}. \tag{113}$$

From (110), we have the following equation:

$$\left|a_m\right| \leq \prod_{p=1}^{m-1}\left(\frac{\left|\varsigma_\lambda(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})-4\left\{[p,\mathfrak{q}]-\psi_p\right\}\varphi_{\mathfrak{q},p-1}\mathfrak{Y}\right|}{4\left([p+1,\mathfrak{q}]-\psi_{p+1}\right)\varphi_{\mathfrak{q},p}}\right). \tag{114}$$

Using modulus properties we get the following equation:

$$\left|a_m\right| \leq \prod_{p=1}^{m-1}\left(\frac{\left|\varsigma_\lambda(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[p,\mathfrak{q}]-\psi_p\right|\varphi_{\mathfrak{q},p-1}}{4\left|\left([p+1,\mathfrak{q}]-\psi_{p+1}\right)\right|\varphi_{\mathfrak{q},p}}\right). \tag{115}$$

By the induction hypothesis, we have the following equation:

$$\frac{\left|\varsigma_\lambda(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left|\left\{[m,\mathfrak{q}]-\psi_m\right\}\varphi_{\mathfrak{q},m-1}\right|}\sum_{p=1}^{m-1}\left|\varphi_{\mathfrak{q},p-1}\psi_p\right|\left|a_p\right| \leq \prod_{p=1}^{m-1}\left(\frac{\left|\varsigma_\lambda(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[p,\mathfrak{q}]-\psi_p\right|\varphi_{\mathfrak{q},p-1}}{4\left|\left([p+1,\mathfrak{q}]-\psi_{p+1}\right)\right|\varphi_{\mathfrak{q},p}}\right). \tag{116}$$

Multiplying both sides of (116) by the following equation:

$$\frac{\left|\varsigma_\lambda(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[m,\mathfrak{q}]-\psi_m\right|\varphi_{\mathfrak{q},m-1}}{4\left|[m+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},m}}, \tag{117}$$

we have

$$\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[m,\mathfrak{q}]-\psi_{m}\right|\varphi_{\mathfrak{q},m-1}}{4\left|[m+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},m}}$$
$$\prod_{p=1}^{m-1}\left(\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[p,\mathfrak{q}]-\psi_{p}\right|\varphi_{\mathfrak{q},p-1}}{4\left|[p+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},p}}\right)$$
$$\geq\left\{\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[m,\mathfrak{q}]-\psi_{m}\right|\varphi_{\mathfrak{q},m-1}}{4\left|[p+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},m}}\right\}$$
$$\cdot\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left([m,\mathfrak{q}]-\psi_{m}\right)\varphi_{\mathfrak{q},m-1}}\sum_{p=1}^{m-1}\left|\varphi_{\mathfrak{q},p-1}\psi_{p}\right|\left|a_{p}\right|. \tag{118}$$

Furthermore,

$$\prod_{p=1}^{m}\left(\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[p,\mathfrak{q}]-\psi_{p}\right|\varphi_{\mathfrak{q},p-1}}{4\left|[p+1,\mathfrak{q}]-\psi_{p+1}\right|\varphi_{\mathfrak{q},p}}\right)$$
$$\geq\left\{\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left|[m+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},m}}\left\{\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left|[m,\mathfrak{q}]-\psi_{m}\right|\varphi_{\mathfrak{q},m-1}}\sum_{p=1}^{m-1}\left|\varphi_{\mathfrak{q},p-1}\psi_{p}\right|\left|a_{p}\right|\right.\right.$$
$$\left.\left.+\sum_{p=1}^{m-1}\left|\varphi_{\mathfrak{q},p-1}\psi_{p}\right|\left|a_{p}\right|\right\}\right\}$$
$$\geq\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left|[m+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},m}}\left\{\left|a_{m}\right|+\sum_{p=1}^{m-1}\left|\varphi_{\mathfrak{q},p-1}\psi_{p}\right|\left|a_{p}\right|\right\}$$
$$=\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left|[m+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},m}}\sum_{p=1}^{m}\left|\varphi_{\mathfrak{q},p-1}\psi_{p}\right|\left|a_{p}\right|. \tag{119}$$

That is,

$$\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|}{4\left|[m+1,\mathfrak{q}]-\psi_{m+1}\right|\varphi_{\mathfrak{q},m}}\sum_{p=1}^{m}\left|\varphi_{\mathfrak{q},p-1}\psi_{p}\right|\left|a_{p}\right|$$
$$\leq\prod_{p=1}^{m}\left(\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\right|+4\left|[p,\mathfrak{q}]-\psi_{p}\right|\varphi_{\mathfrak{q},p-1}}{4\left|[p+1,\mathfrak{q}]-\psi_{p+1}\right|\varphi_{\mathfrak{q},p}}\right). \tag{120}$$

Hence inequality is true for $n=m+1$. Which completes the proof. □

If in Theorem 2, we set

$$\lambda_1=0,\quad j=1=i, \tag{121}$$

and let $\mathfrak{q}\longrightarrow 1-$, we get the following known result. □

Corollary 4 (see [30]). *An analytic function f of the form)(3) belongs to the class λ-$\mathcal{ST}(\mathfrak{X},\mathfrak{Y})$, if it satisfies*

$$\left|a_n\right|\leq\prod_{j=0}^{n-2}\left(\frac{\left|\varsigma_{\lambda}(\mathfrak{X}-\mathfrak{Y})-2j\mathfrak{Y}\right|}{2(j+1)}\right). \tag{122}$$

If we set

$$\lambda_1=0,\quad j=1=i=\mathfrak{X}=-\mathfrak{Y}, \tag{123}$$

and let $\mathfrak{q}\longrightarrow 1-$, in Theorem 2, we have the following result.

Corollary 5 (see [3]). *An analytic function f of the form (3) belongs to the class λ-$\mathcal{ST}$, if it satisfies*

$$\left|a_n\right|\leq\prod_{j=0}^{n-2}\left(\frac{\left|\varsigma_{\lambda}+j\right|}{(j+1)}\right). \tag{124}$$

If we set

$$\lambda_1=0,,\quad j=1=i=\mathfrak{X}=-\mathfrak{Y} \tag{125}$$

and let $\mathfrak{q}\longrightarrow 1-$, also by taking $\lambda=0$, then $\varsigma_{\lambda}=2$, in Theorem 2, we have the following result.

Corollary 6 (see [1]). *An analytic function f of the form (3) is in the class $\mathcal{S}^{*}(\mathfrak{X},\mathfrak{Y})$, if it satisfies*

$$\left|a_n\right|\leq\prod_{j=0}^{n-2}\left(\frac{\left|(\mathfrak{X}-\mathfrak{Y})-j\mathfrak{Y}\right|}{(j+1)}\right),\quad -1\leq\mathfrak{Y}<\mathfrak{X}\leq 1. \tag{126}$$

Theorem 4. *If the function $f(z)$ given by (3) belongs to the class k-$\mathcal{V}_q^{(j,i)}(\lambda_1,\mathfrak{X},\mathfrak{Y})$, then*

$$\left|a_2\right|\leq\frac{(\mathfrak{X}-\mathfrak{Y})(1+q)\delta_k}{2(3-q)\left((1+q)-\psi_2\right)\varphi_{q,1}}, \tag{127}$$

and

$$\left|a_3\right|\leq\frac{(\mathfrak{X}-\mathfrak{Y})(1+q)}{2(3-q)\left(\left(1+q+q^2\right)-\psi_3\right)\varphi_{q,2}}$$
$$\left\{\left|\frac{\delta_{2k}}{1}-\frac{\delta_k}{1}-\frac{(\mathfrak{Y}+\mathfrak{Y}q+3-q)}{2(3-q)}+\frac{(\mathfrak{X}-\mathfrak{Y})(1+q)(\delta_k)^2\psi_2}{2(3-q)\left((1+q)-\psi_2\right)}\right|+\left|\delta_k\right|\right\}. \tag{128}$$

Proof. Let $\chi \in \mathscr{P}$ and defined by the following equation:

$$\chi(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots, \tag{129}$$

where $w(z)$ is a Schwarz function, such that $w(0)=0$ and $|w(z)|<1$. It follows seasily that

$$w(z) = \frac{\chi(z)-1}{\chi(z)+1} = \frac{c_1}{2} z + \left(\frac{1}{2}c_2 - \frac{c_1^2}{4}\right) z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_1 c_2 + \frac{c_1^3}{8}\right) z^3 \ldots. \tag{130}$$

If $f(z) \in \lambda\text{-}\mathscr{V}_{\mathfrak{q}}^{(j,i)}(\lambda_1, \mathfrak{X}, \mathfrak{Y})$, thus from (55), we have the following equation:

$$\frac{z\partial_{\mathfrak{q}} R_{\mathfrak{q}}^{\lambda_1} f(z)}{R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z)} \prec \frac{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))p_\lambda(z) - (\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))p_\lambda(z) - (\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}, \tag{131}$$

where $p_\lambda(z)$ is given in (15), thus from (131), we have the following equation:

$$\frac{z\partial_{\mathfrak{q}} R_{\mathfrak{q}}^{\lambda_1} f(z)}{R_{\mathfrak{q}}^{\lambda_1} f_{j,i}(z)} = \frac{(\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))p_\lambda(w(z)) - (\mathfrak{X}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}{(\mathfrak{Y}(1+\mathfrak{q}) + (-\mathfrak{q}+3))p_\lambda(w(z)) - (\mathfrak{Y}(1+\mathfrak{q}) - (-\mathfrak{q}+3))}. \tag{132}$$

From (130), we have the following equation:

$$\begin{aligned} p_\lambda(w(z)) &= 1 + \varsigma_\lambda w(z) + \varsigma_{2\lambda}(w(z))^2 + \varsigma_{3\lambda}(w(z))^3 + \cdots \\ &= 1 + \varsigma_\lambda\left(\frac{c_1}{2} z + \left(\frac{1}{2}c_2 - \frac{c_1^2}{4}\right) z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_1 c_2 + \frac{c_1^3}{8}\right) z^3 \ldots\right) \\ &\quad + \varsigma_{2\lambda}\left(\frac{c_1}{2} z + \left(\frac{1}{2}c_2 - \frac{c_1^2}{4}\right) z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_1 c_2 + \frac{c_1^3}{8}\right) z^3 \ldots\right)^2 \\ &\quad + \varsigma_{3\lambda}\left(\frac{c_1}{2} z + \left(\frac{1}{2}c_2 - \frac{c_1^2}{4}\right) z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_1 c_2 + \frac{c_1^3}{8}\right) z^3 \ldots\right)^3 + \cdots. \end{aligned} \tag{133}$$

After some simplification we have the following equation:

$$\begin{aligned} p_\lambda(w(z)) &= 1 + \frac{\varsigma_\lambda}{2} c_1 z + \left(\frac{\varsigma_{2\lambda} c_1^2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\varsigma_\lambda\right) z^2 \\ &+ \left(\left(\frac{\varsigma_\lambda}{8} - \frac{\varsigma_{2\lambda}}{4} + \frac{\varsigma_{3\lambda}}{8}\right) c_1^3 + \left(\frac{\varsigma_{2\lambda}}{2} - \frac{\varsigma_\lambda}{2}\right) c_1 c_2 + \frac{\varsigma_\lambda}{2} c_3\right) z^3 + \cdots. \end{aligned} \tag{134}$$

Using (134) and (132), we have the following equation:

$$\frac{z\partial_{\mathfrak{q}}R_{\mathfrak{q}}^{\lambda_1}f(z)}{R_{\mathfrak{q}}^{\lambda_1}f_{j,i}(z)}=1+\left(\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\varsigma_\lambda}{4(3-\mathfrak{q})}\right)c_1z+\left\{\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{2(3-\mathfrak{q})}\left(\frac{\varsigma_{2\lambda}}{4}-\frac{\varsigma_\lambda}{4}-\frac{(\mathfrak{Y}+\mathfrak{Y}\mathfrak{q}+3-\mathfrak{q})\varsigma_\lambda^2}{8(3-\mathfrak{q})}\right)c_1^2+\left(\frac{\varsigma_\lambda}{2}\right)c_2\right\}z^2+\left(\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{2(-\mathfrak{q}+3)}\right)\left\{\begin{array}{c}\frac{\varsigma_\lambda}{2}c_3+\left(\frac{\varsigma_{2\lambda}}{2}-\frac{\varsigma_\lambda}{2}-\frac{(\mathfrak{Y}+\mathfrak{Y}\mathfrak{q}+3-\mathfrak{q})}{4(3-\mathfrak{q})}\right)c_1c_2\\+\left(\frac{\varsigma_\lambda}{8}-\frac{\varsigma_{2\lambda}}{4}+\frac{\varsigma_{3\lambda}}{8}-\frac{(\mathfrak{Y}+\mathfrak{Y}\mathfrak{q}+3-\mathfrak{q})\varsigma_\lambda\varsigma_{2\lambda}}{8(3-\mathfrak{q})}+\frac{(\mathfrak{Y}+\mathfrak{Y}\mathfrak{q}+3-\mathfrak{q})}{8(3-\mathfrak{q})}\varsigma_\lambda^2-\frac{(\mathfrak{Y}+\mathfrak{Y}\mathfrak{q}+3-\mathfrak{q})^2}{32(3-\mathfrak{q})^2}\varsigma_\lambda^3\right)\\c_1^3\end{array}\right\}z^3+\cdots. \tag{135}$$

From (47) and (48), we have the following equation:

$$\frac{z\partial_{\mathfrak{q}}R_{\mathfrak{q}}^{\lambda_1}f(z)}{R_{\mathfrak{q}}^{\lambda_1}f_{j,i}(z)}=1+((1+\mathfrak{q})-\psi_2)\varphi_{\mathfrak{q},1}a_2z+\left\{\left(\left(1+\mathfrak{q}+\mathfrak{q}^2\right)-\psi_3\right)\varphi_{\mathfrak{q},2}a_3-((1+\mathfrak{q})-\psi_2)\left(\varphi_{\mathfrak{q},1}\right)^2\psi_2a_2^2\right\}z^2+\left\{\left((1+\mathfrak{q})\left(1+\mathfrak{q}^2\right)-\psi_4\right)\varphi_{\mathfrak{q},3}a_4-\left\{((1+\mathfrak{q})-\psi_2)\psi_3+\left(\left(1+\mathfrak{q}+\mathfrak{q}^2\right)-\psi_3\right)\varphi_{\mathfrak{q},2}\psi_2\right\}\varphi_{\mathfrak{q},1}a_2a_3+(1+(\mathfrak{q})-\psi_2)\left(\varphi_{\mathfrak{q},1}\right)^3\psi_2^2a_2^3\right\}z^3+\cdots. \tag{136}$$

From (135) and (136), equating coefficients of z and z^2 and gives us the following equation:

$$a_2=\left(\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\varsigma_\lambda}{4(3-\mathfrak{q})((1+\mathfrak{q})-\psi_2)\varphi_{\mathfrak{q},1}}\right)c_1, \tag{137}$$

$$a_3=\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{2(3-\mathfrak{q})\left(\left(1+\mathfrak{q}+\mathfrak{q}^2\right)-\psi_3\right)\varphi_{\mathfrak{q},2}}\left\{\left(\frac{\varsigma_{2\lambda}}{4}-\frac{\varsigma_\lambda}{4}-\frac{(\mathfrak{Y}+\mathfrak{Y}\mathfrak{q}+3-\mathfrak{q})}{8(3-\mathfrak{q})}+\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})(\varsigma_\lambda)^2\psi_2}{8(3-\mathfrak{q})((1+\mathfrak{q})-\psi_2)}\right)c_1^2+\frac{\varsigma_\lambda}{2}c_2\right\}. \tag{138}$$

By using Lemma 3, on (137) and (138) we obtain the result asserted by Theorem 3. □

Theorem 5. *Let* $-1\le\mathfrak{Y}<\mathfrak{X}\le 1$ *and* $0\le k<\infty$ *be fixed and let* $f(z)\in k-\mathscr{V}_q^{(j,i)}(\lambda_1,\mathfrak{X},\mathfrak{Y})$, *then for a complex number* μ.

$$\left|a_3-\mu a_2^2\right|\leq\frac{\mathfrak{X}_1\delta_k}{2}\begin{cases}\dfrac{-8}{\mathfrak{X}_1\delta_k}\left(\mu C_1^2-\mathfrak{X}_1\mathfrak{Y}_1\right)+2 & \left(\mu<\dfrac{\mathfrak{X}_1\mathfrak{Y}_1}{2C_1^2}\right),\\ 2 & \left(\dfrac{\mathfrak{X}_1\mathfrak{Y}_1}{2C_1^2}\leq\ \mu\leq\dfrac{\mathfrak{X}_1\left(\delta_k+\mathfrak{Y}_1\right)}{2C_1^2}\right),\\ \dfrac{8}{\mathfrak{X}_1\delta_k}\left(\mu C_1^2-\mathfrak{X}_1\mathfrak{Y}_1\right)-2 & \left(\mu>\dfrac{\mathfrak{X}_1\left(\delta_k+\mathfrak{Y}_1\right)}{2C_1^2}\right),\end{cases}\tag{139}$$

where v, $\mathfrak{X}_1, \mathfrak{Y}_1$ and C_1 are given by (141), (142), (143), and (144), respectively.

Proof. Using (137) and (138), we have

$$\left|a_3-\mu a_2^2\right|=\frac{\mathfrak{X}_1\varsigma_\lambda}{2}\left|c_2-\frac{2}{\mathfrak{X}_1\varsigma_\lambda}\left(\mu C_1^2-\mathfrak{X}_1\mathfrak{Y}_1\right)c_1^2\right|,\tag{140}$$

where

$$v=\frac{2}{\mathfrak{X}_1\varsigma_\lambda}\left(\mu C_1^2-\mathfrak{X}_1\mathfrak{Y}_1\right),\tag{141}$$

$$\mathfrak{X}_1=\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})}{2(-\mathfrak{q}+3)\left(\left(1+\mathfrak{q}+\mathfrak{q}^2\right)-\psi_3\right)\varphi_{\mathfrak{q},2}},\tag{142}$$

$$\mathfrak{Y}_1=\frac{\varsigma_{2\lambda}}{4}-\frac{\varsigma_\lambda}{4}-\frac{(\mathfrak{Y}+\mathfrak{Y}\mathfrak{q}+3-\mathfrak{q})}{8(3-\mathfrak{q})}+\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})(\varsigma_\lambda)^2\psi_2}{8(3-\mathfrak{q})((1+\mathfrak{q})-\psi_2)},\tag{143}$$

and

$$C_1=\frac{(\mathfrak{X}-\mathfrak{Y})(1+\mathfrak{q})\varsigma_\lambda}{4(3-\mathfrak{q})((1+\mathfrak{q})-\psi_2)\varphi_{\mathfrak{q},1}}.\tag{144}$$

Apply Lemma 2 on (140) we have. □

Theorem 6. *Let $0\leq k<\infty$ and let $f(z)\in k\text{-}\mathscr{V}_q^{(j,i)}(\lambda_1,\mathfrak{X},\mathfrak{Y})$ of the form (3), then for a complex number μ*

$$\left|a_3-\mu a_2^2\right|\leq\frac{\mathfrak{X}_1\delta_k}{2}\max\{1,|2v-1|\},\quad\forall\mu\in\mathbb{C}.\tag{145}$$

where v, $\mathfrak{X}_1, \mathfrak{Y}_1, C_1$ given by (141), (142), (143), and (144).

Proof. From (140), we have the following equation:

$$\left|a_3-\mu a_2^2\right|=\frac{\mathfrak{X}_1\varsigma_\lambda}{2}\left|c_2-\frac{2}{\mathfrak{X}_1\varsigma_\lambda}\left(\mu C_1^2-\mathfrak{X}_1\mathfrak{Y}_1\right)c_1^2\right|,\tag{146}$$

Apply Lemma 4 in conjunction with (146), we obtain the result asserted by Theorem 5. □

5. Conclusion

We have successfully studied the uses of certain Ruscheweyh-type $\mathfrak{q}$-differential operator to a new subclass of $\mathfrak{q}$-starlike symmetric functions, which involving both the conic domains and the well-known celebrated Janowski functions in the open unit disk $\mathbb{D}$. We then investigated many properties for the newly defined functions class, including for example coefficients inequalities, the Fekete–Szegö Problems, and a sufficient condition. We have also emphasized certain known results of our key findings.

The interested readers should be advised not to be misled to believe that the so-called k-Gamma function provides a "generalization" of the classical (Euler's) Gamma function. Similar remarks will apply also to all of the usages of the so-called k-Gamma function including (for example) the so-called (k, s)-extensions of the Riemann–Liouville and other operators of fractional integral and fractional derivatives.

Data Availability

No data were used in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this manuscript and approved its final version.

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