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Research Article

A Note on q-Analogues of Degenerate Catalan-Daehee Numbers and Polynomials

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Recently, Yuankui et al. (Filomat J. 35 (5):17, 2022) studied q-analogues of Catalan-Daehee numbers and polynomials by making use of p-adic q-integrals on $\mathbb{Z}_p$. Motivated by this study, we consider q-analogues of degenerate Catalan-Daehee numbers and polynomials with the help of p-adic q-integrals on $\mathbb{Z}_p$. By using their generating function, we derive some new relations including the degenerate Stirling numbers of the first and second kinds. Moreover, we also derive some new identities and properties of this type of polynomials and numbers.

1. Introduction

Numerous exceptional numbers and polynomials have been concentrated by utilizing different techniques, including producing capacities, p-adic investigation, combinatorial techniques, umbral math, differential conditions, likelihood hypothesis, and scientific number hypothesis. In [1], Kuculoglu et al. developed producing capacities for new classes of Catalan-type numbers and polynomials. Utilizing these capacities and their useful conditions, they gave different personalities and relations, including these numbers and polynomials, and different classes of extraordinary numbers, polynomials, and capacities. Some endless series portrayals, including Catalan-type numbers and combinatorial numbers, were examined. In addition, a few repeat relations and computational calculations in the Python programming language represented the Catalan-type numbers and polynomials with their plots under the extraordinary circumstances. They likewise gave a few subordinate equations for these polynomials. Above, polynomials and numbers can be determined by utilizing the Riemann indispensable, shape fundamental, Volkenborn vital, and fermionic p-adic essential.

Catalan-Daehee numbers and polynomials were presented in [2], and a few properties and personalities related with those numbers and polynomials were inferred by using umbral analytic procedures. The group of straight differential conditions emerging from the creating capacity of Catalan-Daehee numbers was thought of as in [3]. In [4], a few properties and personalities related with Catalan numbers and polynomials were inferred by using umbral analytic procedures. Dolgy et al. [5] gave a few new characters for those numbers and polynomials got from p-adic Volkenborn vital on $\mathbb{Z}_p$. Recently, Yuankui et al. [6] presented and contemplated q-analogues of the Catalan-Daehee numbers and polynomials with the help of p-adic q-integrals on $\mathbb{Z}_p$. The point of this study is to present q-analogues of the ruffian Catalan-Daehee numbers and polynomials by utilizing p-adic q-essential on $\mathbb{Z}_p$ and infer a few unequivocal characters for those numbers and polynomials connected with different exceptional numbers and polynomials. For the remainder of this segment, we review the important realities that are required throughout this study.

Let p be a fixed odd prime number. Throughout this study, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote, respectively, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as using $|p|_p = 1/p$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-(1/(p-1))}$. The q-analogue of x is defined through $[x]_q = (1-q^x)/(1-q)$. Note that

$\lim_{q\longrightarrow 1}[x]_q = x$. Let f be a uniformly differentiable function on $\mathbb{Z}_p$. Then, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as [4, 7]

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu_q(x) &= \lim_{N\longrightarrow\infty}\sum_{x=0}^{p^N-1} f(x)\mu_q\left(x+p^N\mathbb{Z}_p\right)\\ &= \lim_{N\longrightarrow\infty}\frac{1}{\left[p^N\right]_q}\sum_{x=0}^{p^N-1} f(x)q^x,\ (\text{see}[7,8]).\end{aligned} \tag{1}$$

From (1), we note that

$$\begin{aligned}q\int_{\mathbb{Z}_p} f(x+1)\mathrm{d}\mu_q(x) &= \int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu_q(x) + (q-1)f(0)\\ &\quad+\frac{q-1}{\log q}f'(0),\end{aligned} \tag{2}$$

where $f'(0) = (\mathrm{d}f/\mathrm{d}x)|_{x=0}$ ([7–12]).

Let us take $f(x) = e^{xt}$. By (1), we get

$$\frac{(q-1)+((q-1)/\log q)t}{qe^t-1} = \int_{\mathbb{Z}_p} e^{xt}\mathrm{d}\mu_q(x). \tag{3}$$

The q-Bernoulli numbers are defined by ([6])

$$\frac{(q-1)+((q-1)/\log q)t}{qe^t-1} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{4}$$

From (4), we note that

$$q\left(B_q+1\right)^n - B_{n,q} = \begin{cases} q-1, & \text{if } n=0,\\ \dfrac{q-1}{\log q}, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{5}$$

with the usual convention about replacing B_q^n by $B_{n,q}$.

The Catalan numbers are defined by the generating function as follows ([1, 4, 6, 13, 14]):

$$\frac{2}{1+\sqrt{1-4t}} = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} C_n t^n, \tag{6}$$

where $t\in\mathbb{C}_p$ with $|t|_p < p^{-(1/p-1)}$ and $C_n = \binom{2n}{n}(1/(n+1))$, $(n\geq 0)$.

The Catalan polynomials are defined by the generating function as follows ([4, 13]):

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-4t)^{(x+y)/2}\mathrm{d}\mu_{-1}(y) &= \frac{2}{1+\sqrt{1-4t}}(1-4t)^{x/2}\\ &= \sum_{n=0}^{\infty} C_n(x)t^n,\end{aligned} \tag{7}$$

where $t\in\mathbb{C}_p$ with $|t|_p < p^{-(1/(p-1))}$.

When $x=0$, $C_n = C_n(0)$ are called the Catalan numbers.

Thus, by (6) and (7), we have

$$C_n(x) = \sum_{m=0}^{n}\sum_{j=0}^{m}\left(\frac{x}{2}\right)^j S_1(m,j)(-4)^m\frac{C_{n-m}}{m!}. \tag{8}$$

Kim-Kim [2, 3] introduced the Catalan-Daehee polynomials defined by

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-4t)^{(x+y)/2}\mathrm{d}\mu(y) &= \frac{(1/2)\log(1-4t)}{\sqrt{1-4t}-1}(1-4t)^{x/2}\\ &= \sum_{n=0}^{\infty} d_n(x)t^n.\end{aligned} \tag{9}$$

When $x=0$, $d_n = d_n(0)$ are called the Catalan-Daehee numbers.

From (6) and (9), we get

$$d_n = \begin{cases} 1, & \text{if } n=0,\\ \dfrac{4^n}{n+1} - \displaystyle\sum_{m=0}^{n-1}\frac{4^{n-m-1}}{n-m}C_m, & \text{if } n\geq 1.\end{cases} \tag{10}$$

By using (2), the q-analogues of Catalan-Daehee numbers are defined by ([6])

$$\begin{aligned}\int_{\mathbb{Z}_p}(1-4t)^{x/2}\mathrm{d}\mu_q(x) &= \frac{q-1+((q-1)/\log q)(1/2)\log(1-4t)}{q\sqrt{1-4t}-1}\\ &= \sum_{n=0}^{\infty} d_{n,q}t^n.\end{aligned} \tag{11}$$

Note that $\lim_{q\longrightarrow 1} d_{n,q} = d_n$, $(n\geq 0)$.

Jeong et al. [15] introduced the degenerate q-Daehee polynomials defined by

$$\begin{aligned}&\int_{\mathbb{Z}_p}\left(1+\log\ (1+\lambda t)^{1/\lambda}\right)^{x+y} d\mu_q(y)\\ &= \frac{q-1+((q-1)/\log q)\log\left(1+\log\ (1+\lambda t)^{1/\lambda}\right)}{q-1+q\log\ (1+\lambda t)^{1/\lambda}}\\ &\quad\cdot\left(1+\log\ (1+\lambda t)^{1/\lambda}\right)^x\\ &= \sum_{n=0}^{\infty} D_{n,q}(x|\lambda)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

In the case when $x=0$, $D_{n,q}(\lambda) = D_{n,q}(0|\lambda)$ are called the degenerate q-Daehee numbers.

Note that

$$\lim_{\lambda\longrightarrow 0} D_{n,q}(x|\lambda) = D_{n,q}(x),\quad (n\geq 0). \tag{13}$$

For $n\geq 0$, the Stirling numbers of the first kind are defined by ([4, 5, 8, 9, 13, 15, 16])

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{14}$$

where $(x)_0 = 1$, and $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$. From (14), it is easy to see that ([1–3, 7, 10, 14, 17, 18])

$$\frac{1}{l!}(\log(1+t))^l = \sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!}. \tag{15}$$

For $n \geq 0$, the Stirling numbers of the second kind are defined by ([6, 11, 12, 18–22])

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l. \tag{16}$$

From (16), we see that

$$\frac{1}{l!}\left(e^t - 1\right)^l = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}. \tag{17}$$

2. The q-Analogues of Degenerate Catalan-Daehee Numbers and Polynomials

In this section, we introduce q-analogues of degenerate Catalan-Daehee numbers which are derived from the fermionic p-adic q-integral on $\mathbb{Z}_p$. First, we present the following definition.

For $\lambda, t, q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $|\lambda t| < p^{-(1/(p-1))}$, $(1+\lambda t)^{x/\lambda} = e^{(x/\lambda)\log(1+\lambda t)}$. Now, we define the q-analogue of degenerate Catalan-Daehee numbers which are given by the generating function

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(1 - 4\log\,(1+\lambda t)^{1/\lambda}\right)^{x/2} d\mu_q(x) &= \frac{q - 1 + ((q-1)/\log q)(1/2)\log\left(1 - 4\log\,(1+\lambda t)^{1/\lambda}\right)}{q\sqrt{1 - 4\log\,(1+\lambda t)^{1/\lambda}} - 1} \\ &= \sum_{n=0}^{\infty} d_{n,\lambda,q} t^n.\end{aligned} \tag{18}$$

Note that [13]

$$\lim_{\lambda \longrightarrow 0} d_{n,\lambda,q} = d_{n,q}, \quad (n \geq 0)\,(\text{see}\,[6]). \tag{19}$$

From (6) and (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,\lambda,q} t^n &= \frac{1}{2}\left(\frac{2(q-1) + ((q-1)/\log q)\log\left(1 - 4\log\,(1+\lambda t)^{1/\lambda}\right)}{q^2\left(1 - 4\log\,(1+\lambda t)^{1/\lambda}\right) - 1}\right)\left(q\sqrt{\left(1 - 4\log\,(1+\lambda t)^{1/\lambda}\right)} + 1\right) \\ &= \frac{1}{2}\left(\sum_{l=0}^{\infty}(-4)^l D_{l,q}(0|\lambda)\frac{t^l}{l!}\right)\left(1 + q - 2q\sum_{m=0}^{\infty} C_{m,\lambda} t^{m+1}\right) \\ &= \frac{[2]_q}{2}\sum_{n=0}^{\infty}(-4)^n\frac{D_{n,q}(0|\lambda)}{n!}t^n - q\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\frac{(-4)^{n-m-1}}{(n-m-1)!}D_{n-m-1,q}(0|\lambda)C_{m,\lambda}\right)t^n \\ &= 1 + \sum_{n=1}^{\infty}\frac{[2]_q}{2}\frac{(-4)^n}{n!}D_{n,q}(0|\lambda)t^n - q\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\frac{(-4)^{n-m-1}}{(n-m-1)!}D_{n-m-1,q}(0|\lambda)C_{m,\lambda}\right)t^n \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{[2]_q}{2}\frac{(-4)^n}{n!}D_{n,q}(0|\lambda) - q\sum_{m=0}^{n-1}\frac{(-4)^{n-m-1}}{(n-m-1)!}D_{n-m-1,q}(0|\lambda)C_{m,\lambda}\right)t^n.\end{aligned} \tag{20}$$

Therefore, by comparing the coefficients on both sides of (20), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$d_{n,\lambda,q} = \begin{cases} 1, & \text{if } n = 0, \\ \dfrac{[2]_q}{2}\dfrac{(-4)^n}{n!}D_{n,q}(0|\lambda) - q\displaystyle\sum_{m=0}^{n-1}\dfrac{(-4)^{n-m-1}}{(n-m-1)!}D_{n-m-1,q}(0|\lambda)C_{m,\lambda}, & \text{if } n \geq 1. \end{cases} \tag{21}$$

By (18), we note that

$$\begin{aligned} \int_{\mathbb{Z}_p}\left(1 - 4\log(1+\lambda t)^{1/\lambda}\right)^{x/2}d\mu_q(x) &= \sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}x^m d\mu_q(x)\frac{1}{2^m}\frac{1}{m!}\left(-4\log(1+\lambda t)^{1/\lambda}\right)^m \\ &= \sum_{m=0}^{\infty}B_{m,q}(-2)^m\lambda^{n-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_{m,q}(-2)^m\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{22}$$

Therefore, by (18) and (22), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$d_{n,\lambda,q} = \frac{(-1)^m}{n!}\sum_{m=0}^{n}B_{m,q}2^m\lambda^{n-m}S_1(n,m). \tag{23}$$

On replacing t by $(1/\lambda)(e^{\lambda t}-1)$ in (18), we have

$$\begin{aligned} \int_{\mathbb{Z}_p}(1-4t)^{x/2}d\mu_q(x) &= \sum_{m=0}^{\infty}d_{m,\lambda,q}m!\frac{\left((1/\lambda)\left[e^{\lambda t}-1\right]\right)^m}{m!} \\ &= \sum_{m=0}^{\infty}d_{m,\lambda,q}\lambda^{-m}m!\sum_{n=m}^{\infty}S_2(n,m)\lambda^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}d_{m,\lambda,q}\lambda^{n-m}m!S_2(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{24}$$

On the other hand,

$$\begin{aligned} &\int_{\mathbb{Z}_p}(1-4t)^{x/2}d\mu_q(x) \\ &= \frac{q-1+((q-1)/\log q)(1/2)\log(1-4t)}{q\sqrt{1-4t}-1} = \sum_{n=0}^{\infty}d_{n,q}t^n. \end{aligned} \tag{25}$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$\begin{aligned} d_{n,q} &= \sum_{m=0}^{n}d_{m,\lambda,q}\lambda^{n-m}S_2(n,m)\frac{m!}{n!}, \\ \sum_{m=0}^{n}(-1)^n 2^{2n-m}B_{m,q}S_1(n,m) &= \sum_{m=0}^{n}d_{m,\lambda,q}\lambda^{n-m}S_2(n,m)m!. \end{aligned} \tag{26}$$

From (18), we observe that

$$\begin{aligned} \int_{\mathbb{Z}_p}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{x/2}d\mu_q(x) &= \sum_{m=0}^{\infty}(-1)^m 4^m\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{m}d\mu_q(x)\frac{\left[\log(1+\lambda t)^{1/\lambda}\right]^m}{m!} \\ &= \sum_{m=0}^{\infty}d_{m,q}(-1)^m 4^m\lambda^{n-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}d_{m,q}(-1)^m 4^m\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{27}$$

Therefore, by (18) and (27), we get the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$d_{n,\lambda,q} = \frac{1}{n!}\sum_{m=0}^{n} d_m (-1)^m 4^m \lambda^{n-m} S_1(n,m). \tag{28}$$

For $\lambda, t, q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $|\lambda t| < p^{-(1/(p-1))}$. The q-analogue of λ-Daehee polynomials $D_{n,\lambda,q}(x)$ are defined by the following generating function ([6]):

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+t)^{\lambda y + x} \mathrm{d}\mu_q(y) &= \frac{q-1+\lambda((q-1)/\log q)\log(1+t)}{q(1+t)^{\lambda}-1} \\ &\quad \cdot (1+t)^x \\ &= \sum_{n=0}^{\infty} D_{n,\lambda,q}(x)\frac{t^n}{n!}.\end{aligned} \tag{29}$$

When $x=0$, $D_{n,\lambda,q} = D_{n,\lambda,q}(0)$ are called the q-analogue of λ-Daehee numbers.

On setting $\lambda = 1/2$ and $t \longrightarrow -4\log(1+\lambda t)^{1/\lambda}$ in (29), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{x/2}\mathrm{d}\mu_q(x) \\ &= \sum_{m=0}^{\infty} D_{m,(1/2),q}\frac{\left(-4\log(1+\lambda t)^{1/\lambda}\right)^m}{m!} \\ &= \sum_{m=0}^{\infty} D_{m,(1/2),q}(-4)^m\lambda^{-m}\sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} D_{m,(1/2),q}\lambda^{n-m}S_1(n,m)\frac{(-4)^m}{n!}\right)t^n.\end{aligned} \tag{30}$$

Therefore, by (18) and (30), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$\begin{aligned}d_{n,\lambda,q} &= \sum_{m=0}^{n} D_{m,(1/2),q}\lambda^{n-m}S_1(n,m)\frac{(-4)^m}{n!}, \\ \sum_{m=0}^{n}(-1)^m B_{m,q}2^m\lambda^{n-m}S_1(n,m) &= \sum_{m=0}^{n} D_{m,(1/2),q}\lambda^{n-m}S_1(n,m)(-4)^m.\end{aligned} \tag{31}$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (11), we get

$$\begin{aligned}\frac{q-1+((q-1)/\log q)(1/2)\log\left(1-4\log(1+\lambda t)^{1/\lambda}\right)}{q\sqrt{1-4\log(1+\lambda t)^{1/\lambda}}-1} &= \sum_{m=0}^{\infty} d_{m,q}m!\frac{\left[\log(1+\lambda t)^{1/\lambda}\right]^m}{m!} \\ &= \sum_{m=0}^{\infty} d_{m,q}\lambda^{-m}m!\frac{(\log(1+\lambda t))^m}{m!} \\ &= \sum_{m=0}^{\infty} d_{m,q}\lambda^{-m}m!\sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} d_{m,q}\lambda^{n-m}m!S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{32}$$

Therefore, by (18) and (32), we get the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$d_{n,\lambda,q} = \sum_{m=0}^{n} d_{m,q}\lambda^{n-m}m!S_1(n,m)\frac{m!}{n!}. \tag{33}$$

Now, we observe that

$$\begin{aligned}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{x/2} &= \sum_{m=0}^{\infty}\left(\frac{x}{2}\right)^{m}(-1)^{m}4^{m}\frac{\left[\log(1+\lambda t)^{1/\lambda}\right]^{m}}{m!}\\ &= \sum_{m=0}^{\infty}\left(\frac{x}{2}\right)^{m}\lambda^{-m}(-1)^{m}4^{m}\sum_{n=m}^{\infty}S_{1}(n,m)\frac{\lambda^{n}t^{n}}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{x}{2}\right)^{m}\lambda^{n-m}(-1)^{m}4^{m}S_{1}(n,m)\right)\frac{t^{n}}{n!}.\end{aligned} \tag{34}$$

Now, we consider the q-analogue of degenerate Catalan-Daehee polynomials which are given by the following generating function:

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{(x+y)/2}d\mu_q(y) &= \frac{q-1+((q-1)/\log q)(1/2)\log\left(1-4\log(1+\lambda t)^{1/\lambda}\right)}{q\sqrt{1-4\log(1+\lambda t)^{1/\lambda}}-1}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{x/2}\\ &= \sum_{n=0}^{\infty}d_{n,\lambda,q}(x)t^{n}.\end{aligned} \tag{35}$$

When $x=0$, $d_{n,\lambda,q}=d_{n,\lambda,q}(0)$ are called the q-analogue of degenerate Catalan-Daehee numbers.

From (35), we note that

$$\begin{aligned}&\frac{q-1+((q-1)/\log q)(1/2)\log\left(1-4\log(1+\lambda t)^{1/\lambda}\right)}{q\sqrt{1-4\log(1+\lambda t)^{1/\lambda}}-1}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{x/2}\\ &= \left(\sum_{n=0}^{\infty}d_{n,\lambda,q}t^{n}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{x}{2}}{m}(-1)^{m}2^{2m}m!\frac{\left(\log(1+\lambda t)^{1/\lambda}\right)^{m}}{m!}\right)\\ &= \left(\sum_{n=0}^{\infty}d_{n,\lambda,q}t^{n}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{x}{2}}{m}(-1)^{m}2^{2m}\lambda^{-m}m!\sum_{l=m}^{\infty}S_{1}(l,m)\frac{\lambda^{l}t^{l}}{l!}\right)\\ &= \left(\sum_{n=0}^{\infty}d_{n,\lambda,q}t^{n}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{\frac{x}{2}}{m}(-1)^{m}2^{2m}\lambda^{l-m}m!S_{1}(l,m)\frac{t^{l}}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{\frac{x}{2}}{m}(-1)^{m}2^{2m}\lambda^{l-m}S_{1}(l,m)d_{n-l,\lambda,q}\frac{m!}{l!}\right)t^{n}.\end{aligned} \tag{36}$$

By (35) and (36), we obtain the following theorem.

Theorem 7. *For $n\geq 0$, we have*

$$d_{n,\lambda,q}(x)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{\frac{x}{2}}{m}(-1)^m 2^{2m}\lambda^{l-m}S_1(l,m)d_{n-l,\lambda,q}\frac{m!}{l!}. \tag{37}$$

From (35), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}d_{n,\lambda,q}(x)t^n&=\frac{q-1+((q-1)/\log q)(1/2)\log\left(1-4\log(1+\lambda t)^{1/\lambda}\right)}{q\sqrt{1-4\log(1+\lambda t)^{1/\lambda}}-1}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{x/2}\\&=\left(\sum_{n=0}^{\infty}d_{n,\lambda,q}t^n\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\left(\frac{x}{2}\right)^m\lambda^{k-m}(-1)^m4^mS_1(k,m)\right)\frac{t^k}{k!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\left(\frac{x}{2}\right)^m\lambda^{k-m}S_1(k,m)(-1)^m4^md_{n-k,\lambda,q}\frac{1}{k!}\right)t^n.\end{aligned} \tag{38}$$

Therefore, by (38), we obtain the following theorem.

Theorem 8. *For $n\geq 0$, we have*

$$d_{n,q,\lambda}(x)=\sum_{k=0}^{n}\sum_{m=0}^{k}\left(\frac{x}{2}\right)^m\lambda^{k-m}(-1)^m4^mS_1(k,m)d_{n-k,\lambda,q}\frac{1}{k!}. \tag{39}$$

On replacing t by $(1/\lambda)e^{\lambda t}-1$ in (35), we define q-analogue Catalan polynomials are given by

$$\int_{\mathbb{Z}_p}(1-4t)^{(x+y)/2}d\mu_q(y)=\frac{q-1+((q-1)/\log q)(1/2)\log(1-4t)}{q\sqrt{1-4t}-1}\sqrt{(1-4t)^x}=\sum_{n=0}^{\infty}d_{n,q}(x)t^n. \tag{40}$$

On the other hand,

$$\begin{aligned}\sum_{m=0}^{\infty}d_{m,\lambda,q}(x)m!\frac{\left((1/\lambda)e^{\lambda t}-1\right)^m}{m!}&=\sum_{m=0}^{\infty}d_{m,\lambda,q}(x)m!\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^nt^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}d_{m,\lambda,q}(x)\lambda^{n-m}S_1(n,m)m!\right)\frac{t^n}{n!}.\end{aligned} \tag{41}$$

Therefore, by (40) and (41), we state the following theorem.

Theorem 9. *For $n\geq 0$, we have*

$$d_{n,q}(x)=\sum_{m=0}^{n}d_{m,\lambda,q}(x)\lambda^{n-m}S_1(n,m)\frac{m!}{n!}. \tag{42}$$

From (35), we have

$$\frac{q-1+((q-1)/\log q)(1/2)\log\left(1-4\log(1+\lambda t)^{1/\lambda}\right)}{q\sqrt{1-4\log(1+\lambda t)^{1/\lambda}}-1}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{\frac{x}{2}}$$
$$=\int_{\mathbb{Z}_p}\left(1-4\log(1+\lambda t)^{1/\lambda}\right)^{(x+y)/2}\mathrm{d}\mu_q(y)$$
$$=\sum_{m=0}^{\infty}2^{-m}\frac{1}{m!}\left(-4\log(1+\lambda t)^{1/\lambda}\right)^m\int_{\mathbb{Z}_p}(x+y)^m\mathrm{d}\mu_q(y) \tag{43}$$
$$=\sum_{m=0}^{\infty}2^m(-1)^mB_{m,q}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}2^m(-1)^mB_{m,q}(x)\lambda^{n-m}S_1(n,m)\frac{1}{n!}\right)t^n.$$

Thus, by (35) and (43), we get the following theorem.

Theorem 10. *For $n\geq 0$, we have*

$$d_{n,\lambda,q}(x)=\sum_{m=0}^{n}2^m(-1)^mB_m(x)\lambda^{n-m}S_1(n,m)\frac{1}{n!}. \tag{44}$$

Replacing t by $(e^{-(\lambda t/4)}-1)/\lambda$ in (35), we get

$$\int_{\mathbb{Z}_p}(1+t)^{(x+y)/2}\mathrm{d}\mu_q(y)=\sum_{n=0}^{\infty}2^{-n}\int_{\mathbb{Z}_p}(x+y)^n\mathrm{d}\mu_q(y)\frac{t^n}{n!}. \tag{45}$$

On the other hand, we have

$$\sum_{m=0}^{\infty}d_{m,\lambda,q}(x)m!\frac{\left(\left(e^{-(\lambda t/4)}-1\right)/\lambda\right)^m}{m!}=\sum_{m=0}^{\infty}d_{m,q,\lambda}(x)m!\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{(-4)^n\lambda^n t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}d_{m,\lambda,q}(x)m!\lambda^{n-m}(-1)^n4^nS_2(n,m)\frac{t^n}{n!}. \tag{46}$$

Therefore, by (45) and (46), we obtain the following theorem.

Theorem 11. *For $n\geq 0$, we have*

$$\int_{\mathbb{Z}_p}(x+y)^n\mathrm{d}\mu_q(y)=(-2)^n\sum_{m=0}^{n}d_{m,\lambda,q}(x)m!\lambda^{n-m}S_2(n,m). \tag{47}$$

3. Conclusion

By utilizing various apparatuses, many unique numbers and polynomials have been concentrated. Beforehand, the Catalan-Daehee numbers and polynomials were presented through p-adic Volkenborn, and a few intriguing outcomes for them were obtained by utilizing creating capacities, differential conditions, umbral math, and p-adic Volkenborn integrals. In this study, we presented degenerate q-analogues of the Catalan-Daehee numbers and polynomials and acquired a few unequivocal articulations and personalities connected with them. In more detail, we communicated the Catalan-Daehee numbers as far as ruffian q-Daehee numbers and the q-Bernoulli polynomials and Stirling quantities of the primary kind. We acquired a personality, including q-Bernoulli numbers, degenerate q-Catalan-Daehee numbers, and Stirling quantities of the subsequent kind. Likewise, we got an unequivocal articulation for the ruffian q-analogues of Catalan-Daehee polynomials, which include the savage q-analogues of Catalan-Daehee numbers and Stirling quantities of the primary kind.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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