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Research Article

Maximum Colored Cuts in Edge-Colored Complete Graphs

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Max-Cut problem is one of the classical problems in graph theory and has been widely studied in recent years. Maximum colored cut problem is a more general problem, which is to find a bipartition of a given edge-colored graph maximizing the number of colors in edges going across the bipartition. In this work, we gave some lower bounds on maximum colored cuts in edge-colored complete graphs containing no rainbow triangles or properly colored 4-cycles.

1. Introduction

One of the classical partition problems is the well-known Max-Cut problem: given a graph G, find a partition (V_1, V_2) of $V(G)$ maximizing the number of edges going across V_1 and V_2. It has been a very active research subject in both combinatorics and computer science in the last 30 years. For a graph G, let $f(G)$ be the maximum number of edges in a cut of G. For an integer m, let $f(m)$ denote the minimum value of $f(G)$, as G ranges over all graphs with m edges. A simple probabilistic argument or a greedy algorithm shows that $f(m) \geq (m/2)$. Answering a question of Erdos, Edwards [1, 2] in 1973 proved that

$$f(m) \geq (m/2) + (\sqrt{8m+1}/8) - (1/8), \tag{1}$$

and this is right as evidenced by complete graph with an odd number of vertices.

In this work, we pay our attention to maximum colored cuts in edge-colored graphs: given a simple graph $G(V, E)$ with an edge coloring $c: E \longrightarrow \{1, 2, \ldots, p\}$, and for each $i \in [p]$, $c^{-1}(i) \neq \varnothing$, find a partition (V_1, V_2) of $V(G)$ maximizing the number of colors of edges going across V_1 and V_2. Faria et al. [3] proposed this problem and studied its complexity. Let $m = |E(G)|$. This problem converts to Max-Cut problem when $p = m$.

For an edge-colored graph $G(V, E)$, we use G^i to denote the spanning subgraph of G induced by the edges of color i. We use $\mathrm{col}(G)$ to denote the set of colors appearing on $E(G)$. For each $e \in E(G)$, we also use $\mathrm{col}(e)$ to denote the color appearing on e. For any two disjoint subgraphs F and H of G, we use $\mathrm{col}(F, H)$ to denote the set of colors appearing on the edges between F and H. If $V(F) = \{v\}$, then we write $\mathrm{col}(v, H)$ instead of $\mathrm{col}(F, H)$. For any two disjoint subsets S and T of $V(G)$, we use $\mathrm{col}(S, T)$ as shorthand for $\mathrm{col}(G[S], G[T])$, where $G[S]$ denote the subgraph of G induced by S. We say that G is properly colored if each pair of adjacent edges in G are assigned distinct colors. We say that G is monochromatic if $|\mathrm{col}(G)| = 1$. And we say that G is rainbow if all edges of G are assigned different colors. Let K_n be a complete graph with n vertices. A rainbow K_3 is called a rainbow triangle. For a vertex $v \in V(G)$, the color degree of v, denoted by $d_G^c(v)$, is the number of different colors appearing on the edges incident with v.

For an edge-colored graph G with p colors, let $f^c(G)$ be the maximum number of colors in a cut of G. Let $f^c(p)$ denote the minimum value of $f^c(G)$, as G ranges over all edge-colored graphs with p colors. To distinguish from Max-Cut in graphs, we first need to forbid the following situation: p is close to m. If $m - p = o(m)$, then $f^c(p)$ is close to $f(m)$. In addition, let $G = K_n$ be an edge-colored complete graph with $p = \binom{k}{2} + 1$ colors such that a subgraph K_k is rainbow and the remaining edges are monochromatic. Then, it is easy to verify that $f^c(G) = f(p-1) + 1$ although $p \ll m$. In these cases, we can give the lower bounds of the maximum colored cuts with the help of the results of the maximum cuts in graphs. To avoid these situations, we in this paper just discuss maximum colored cuts in edge-colored complete graphs, which contains no rainbow triangles or properly colored 4-cycles.

Using the probabilistic approach, we first give a lower bound of $f^c(G)$ by constraining $\min_{i\in[p]}\{\Delta(G^i)\}$.

Theorem 1. *Let G be an edge-colored graph with* $|col(G)| = p$. *For each color* $i \in [p]$, *if* $\Delta(G^i) \geq k \geq 1$, *then* $f^c(G) \geq (1-(1/2^k))p$.

The following four results discuss the lower bound of $f^c(G)$, as G is an edge-colored complete graph containing no rainbow triangles, properly colored 4-cycles, or monochromatic 4-cycles.

Theorem 2. *Let G be an edge-colored complete graph with* $|col(G)| = p$. *If G contains no rainbow triangles or properly colored 4-cycles, then* $f^c(G) = p$.

Theorem 3. *Let G be an edge-colored complete graph with* $|col(G)| = p$. *If GG contains no properly colored 4-cycles, then* $f^c(G) \geq (3p/4)$.

Theorem 4. *Let G be an edge-colored complete graph with n vertices and p colors. If G contains no rainbow triangles, then there is a constant* $\delta > 0$, *such that* $f^c(G) \geq (p/2) + (\delta p/\sqrt{2\log_2 n})$.

Theorem 5. *Let G be an edge-colored complete graph with* $|col(G)| = p$. *If G contains no rainbow triangles or monochromatic 4-cycles, then* $f^c(G) \geq (3(p-1)/4)$.

In the remainder of this paper, some useful lemmas are given in Section 2, the proofs of Theorems 1–3 are given in Section 3, and the proofs of Theorem 4 and Theorem 5 are given in Section 4. Finally, we enumerate some open problems.

2. Preliminaries

We first state a useful lemma on the structure of edge-colored complete graphs containing no properly colored 4-cycle.

Lemma 1 (see Martin et al. [4]). *Let G be an edge-colored complete graph containing no properly colored 4-cycle. Then for each color i,* G^i *contains a dominating vertex.*

The following fundamental result on the existence of properly colored cycles in colored graphs plays a key role in our proofs.

Lemma 2 (see Grossman and Häggkvist [5] and Yeo [6]). *Let G be an edge-colored graph containing no properly colored cycles. Then G contains a vertex v such that no component of* $G-v$ *is joined to v with edges of more than one color.*

For an edge-colored graph G, a partition $(V_1, V_2, \ldots, V_t)$ $(t \geq 2)$ of $V(G)$ is called a Gallai partition if $|\cup_{1\leq i<j\leq t} \mathrm{col}(V_i, V_j)| \leq 2$ and $|\mathrm{col}(V_i, V_j)| = 1$ for $1 \leq i < j \leq t$.

Lemma 3 (see Gallai [7]). *Let G be an edge-colored* K_n. *If G contains no rainbow triangles, then G has a Gallai partition.*

A graph G has vertices $v_1, v_2, \ldots, v_n$, let $d(v_i)$ be the degree of v_i for $1 \leq i \leq n$. The sequence $d(v_1), d(v_2), \ldots, d(v_n)$ is called a degree sequence of G. The following lemma, proved by Alon et al. [8], gives a lower bound of $f(G)$ with respect to degree sequence of a sparse graph G.

Lemma 4 (see Alon et al. [8]). *There exists an absolute constant* $\varepsilon > 0$ *such that, for every constant* $C > 0$, *there is a constant* $\delta = \delta(C) > 0$ *with the following property. Let G be a graph with n vertices, m edges and degree sequence* $d(v_1), d(v_2), \ldots, d(v_n)$. *Suppose that the induced subgraph on any set of* $d \geq C$ *vertices all of which have a common neighbor contains at most* $\varepsilon d^{3/2}$ *edges. Then,*

$$f(G) \geq \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d(v_i)}. \quad (2)$$

3. Proofs of Theorems 1–3

Proof of Theorem 1. Let $V(G) = V_1 \cup V_2$ be a random partition of $V(G)$ by placing each $v \in V(G)$ into V_1 or V_2, independently, with probability 1/2. For each color $i \in [p]$, let I_i be the indicator random variable of the event that there exists an edge $e = uv$ colored i such that $u \in V_1$, $v \in V_2$ or $v \in V_1$, $u \in V_2$. Recall that $\Delta(G^i) \geq k \geq 1$. Then, there is a vertex $v \in V(G)$, which incidents with at least k edges-colored i. Let $\mathrm{col}(V_1, V_2)$ denote the set of colors appearing on the edges between V_1 and V_2. Then,

$$\Pr(i \in \mathrm{col}(V_1, V_2)) = E[I_i] \geq 1 - (1/2^k). \quad (3)$$

By the linearity of expectation,

$$E[|\mathrm{col}(V_1, V_2)|] = \sum_{i\in[p]} E[I_i] \geq \left(1 - \left(1/2^k\right)\right)p. \quad (4)$$

Thus, there exists a partition $V(G) = V_1 \cup V_2$ such that $|\mathrm{col}(V_1, V_2)| \geq (1-(1/2^k))p$. □

Proof of Theorem 2. Let G be an edge-colored complete graph containing no rainbow triangles or properly colored 4-cycles. Li et al [9] gave an observation, which shows that this graph G contains no properly colored cycles. For the sake of completeness, we recall the proof. By contradiction, suppose that $C = v_1v_2, \ldots, v_tv_1$ $(t \geq 5)$ is a properly colored cycle of minimal length in G. Without loss of generality, assume that $\mathrm{col}(v_1v_2) = 1$ and $\mathrm{col}(v_1v_t) = 2$, then $\mathrm{col}(v_3v_4) \neq 1$. Since otherwise, $C_1 = v_1v_4, \ldots, v_{t-1}v_tv_1$ or $C_2 = v_1v_2v_3v_4v_1$ would be a properly colored cycle shorter than C, a contradiction. Similarly, $\mathrm{col}(v_3v_4) \neq 2$. Since otherwise, $C_3 = v_1v_3, \ldots, v_{t-1}v_tv_1$ or $C_4 = v_1v_2v_3v_1$ would be a properly colored cycle shorter than C, a contradiction too. Assume that $\mathrm{col}(v_3v_4) = 3$. Recall that $C_2 = v_1v_2v_3v_4v_1$ is not a properly colored cycle. Hence, $\mathrm{col}(v_1v_4) = 1$ or 3. If $\mathrm{col}(v_1v_4) = 3$, then $C_1 = v_1v_4, \ldots, v_{t-1}v_tv_1$ would be a properly colored cycle, a contradiction, so $\mathrm{col}(v_1v_4) = 1$. Similarly, since $C_3 = v_1v_3, \ldots, v_{t-1}v_tv_1$ is not a properly colored cycle. Hence, $\mathrm{col}(v_1v_3) = 2$ or 3. If $\mathrm{col}(v_1v_3) = 3$, then $C_4 = v_1v_2v_3v_1$ would be a properly colored cycle, So $\mathrm{col}(v_1v_3) = 2$. Thus, $v_1v_3v_4v_1$ would be a properly colored cycle.

Note that G contains no properly colored cycle. Then, by Lemma 2, there is a $v_n \in V(G)$ such that $|\mathrm{col}(v_n, G_1 = G - v_n)| = 1$. Note that G_1 contains no properly colored cycle. There is a $v_{n-1} \in V(G_1)$ such that $|\mathrm{col}(v_{n-1}, G_2 = G_1 - v_{n-1})| = 1$. Keep on doing this, we obtain a vertex sequence $v_1, v_2, \ldots, v_n$ of $V(G)$ and a graph sequence $G_n, G_{n-1}, \ldots, G_1$, such that $|\mathrm{col}(v_i, G_{n+1-i})| = 1$ for $i \in [n]$.

Now we place v_n into V_1 and place v_{n-1} into V_2. Thus, $\mathrm{col}(v_n, G_1) \subseteq \mathrm{col}(V_1, V_2)$. Then, we place v_{n-2} into V_1 and place v_{n-3} into V_2, Thus, $\mathrm{col}(v_{n-1}, G_2) \subseteq \mathrm{col}(V_1, V_2)$ and $\mathrm{col}(v_{n-2}, G_3) \subseteq \mathrm{col}(V_1, V_2)$. The rest vertices can be placed in this way. Then, we have $\mathrm{col}(v_i, G_{n+1-i}) \subseteq \mathrm{col}(V_1, V_2)$ for $1 \leq i \leq n$. Note that $|\cup_{1 \leq i \leq n} \mathrm{col}(v_i, G_{n+1-i})| = |\mathrm{col}(G)| = p$. Hence, we can obtain a partition $V(G) = V_1 \cup V_2$ such that each edges colored fall into $E(V_1, V_2)$. □

Proof of Theorem 3. Let G be an edge-colored complete graph containing no properly colored 4-cycle. We get the result easily if $p \leq 2$. So, we suppose that $p \geq 3$. If for each color $i \in [p]$, $\Delta(G^i) \geq 2$, then by Theorem 1, we have $f^c(G) \geq (3p/4)$. We are done. Now, we suppose that there exists a color $j \in [p]$ such that G^j is a match. By Lemma 1, G^j contains a dominating vertex. So G^j is an edge. Assume that $E(G^j) = e = uv$. If $|\mathrm{col}(\{u, v\}, V(G) \backslash \{u, v\})| = p - 1$, then we are done. So we suppose that $|\mathrm{col}(\{u, v\}, V(G) \backslash \{u, v\})| \leq p - 2$, which implies that there is an edge in $G \backslash \{u, v\}$, say $f = u'v'$, colored k and $k \notin \mathrm{col}(\{u, v\}, V(G) \backslash \{u, v\})$. Thus, $uvv'u'u$ would be a properly colored 4-cycle, a contradiction. □

4. Proofs of Theorems 4 and 5

Proof of Theorem 4. Let G be an edge-colored K_n containing no rainbow triangles. By Lemma 3, G has a Gallai partition $(V_1^1, V_2^1, \ldots, V_t^1)$ $(t \geq 2)$. Without loss of generality, suppose that $|V_1^1| \leq (n/2)$. Let $G[V_1]$ be a subgraph of G induced by V_1. Then, $G[V_1]$ has a Gallai partition $(V_1^2, V_2^2, \ldots, V_s^2)$ $(s \geq 2)$. Continuing this process, without loss of generality, we can get a V_1^l that $|V_1^l| = 1$ and $l \leq \log_2 n$. Let $V_1^l = \{v_n\}$. By the definition of Gallai partition, we have $d_G^c(v_n) \leq 2\log_2 n$. Let $G_1 = G \backslash \{v_n\}$. Then, there exists a vertex, say $v_{n-1} \in V(G_1)$, such that $d_{G_1}^c(v_{n-1}) \leq 2\log_2(n-1)$. Let $G_2 = G_1 \backslash \{v_{n-1}\}$. Similarly, there exists a vertex, say $v_{n-2} \in V(G_2)$, such that $d_{G_2}^c(v_{n-2}) \leq 2\log_2(n-2)$. Continuing this process, for each $1 \leq i \leq n$, there exists $v_i \in V(G_{n-i})$ such that $d_{G_{n-i}}^c(v_i) \leq 2\log_2 i$. We get a vertex sequence $v_1, v_2, \ldots, v_n$. Now we construct a rainbow subgraph G' of G such that $|\mathrm{col}(G')| = |\mathrm{col}(G)| = p$. Let $G'G'$ be a graph obtained from G by keeping one arbitrary edge from each color. Then, we have $f^c(G) \geq f^c(G') = f(G')$. Note that $G'G'$ contains no triangles. Let $d_1, d_2, \ldots, d_n$ be a degree sequence of G'. Let d_i^+ denotes the number of neighbors v_j of v_i with $j < i$. Note that $\sum_{i=1}^n d_i^+ = p$. Then,

$$\sum_{i=1}^n \sqrt{d_i} \geq \sum_{i=1}^n \sqrt{d_i^+} \geq \frac{\sum_{i=1}^n d_i^+}{\sqrt{2\log_2 n}} = \frac{p}{\sqrt{2\log_2 n}}. \quad (5)$$

By Lemma 4, we complete the proof. □

Proof of Theorem 5. Let G be an edge-colored complete graph containing no rainbow triangles and monochromatic 4-cycle. We claim that there is at most 1 color $i \in [p]$, such that $\Delta(G^i) = 1$. Otherwise, let $j, k \in [p]$, such that $\Delta(G^j) = 1$ and $\Delta(G^k) = 1$. Let $e = uv$ be colored j, and let $f = xy$ be colored k. Obviously, e and f are not incident, as G contains no rainbow triangles, which implies that $|\mathrm{col}(\{u, v\}, \{x, y\})| = 1$. Thus, $uxvyu$ would be a monochromatic 4-cycle, a contradiction. So, there is at most 1 color in $[p]$ that the edges colored and the color is a match. Let G' be the subgraph of G by deleting the match. By Theorem 1, we have $f^c(G) \geq f^c(G') \geq (3(p-1)/4)$. □

5. Open Problems

In this section, we naturally proposed the following two problems on maximum colored cuts in edge-colored complete graphs.

Problem 1. What is the smallest $c(p)$ such that each edge-colored complete graph G with p colors containing no rainbow triangles admits a partition (V_1, V_2) satisfying $|\mathrm{col}(V_1, V_2)| \geq c(p)$?

Problem 2. What is the smallest $c(p)$ such that each edge-colored complete graph G with p colors containing no properly colored 4-cycles admits a partition (V_1, V_2) satisfying $|\mathrm{col}(V_1, V_2)| \geq c(p)$?

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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