Research Article

On the Bounded Partition Dimension of Some Generalised Graph Structures

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Consider λ to be a connected graph with a vertex set V (λ) that may be partitioned into any partition set S. If each vertex in λ has a separate representation with regard to S and is an ordered k partition, then the set with S is a resolving partition of λ. A partition dimension of λ, represented by pd, is the minimal cardinality of resolving k partitions of V (λ). The partition dimension of various generalised families of graphs, such as the Harary graph, Cayley graph, and Pendant graph, is given as a sharp upper bound in this article.

1. Introduction and Preliminaries

Computation is the process of analysing anything using mathematical or logical principles. An algorithm’s implementation is a well-known example of computing. An issue is deemed inherently difficult if solving it requires significant resources regardless of the algorithm used. The problem discussed in this research work has computational cost nondeterministic polynomial time, and therefore, it is called as NP-hard problem in short. The partition dimension is introduced in [1] as a natural generalization of the metric dimension.

The concept of resolving set was introduced by the authors of [2] and later introduced by the authors of [3]. In [4], the authors worked on the star partition dimension of the cycle and complete graph by using a comb product. Authors worked on the partition dimension of different classes of circulant graph [5]. The authors worked on the cartesian product of different graphs and wrote a note on the partition dimension [6]. Recently [7, 8], the partition dimensions of the various families of convex polytopes and their types of graphs, such as with pendant edges, have been calculated.

The partition dimension is used in a variety of fields, including robot navigation, strategies for the mastermind game, network discovery and verification, pattern recognition, and image processing, the Djokovic-Winkler relation, chemistry for representing, chemical compounds, all of which involve the use of hierarchical data structures [9–17].

Literature on partition dimension and its bounds on the specific families of graphs are found in [18, 19]. The antivirus properties of COVID structures are discussed in terms of partition dimension and bounds in [20], novel chemical structures of diamonded structures are discovered in [21] in terms of partition dimension bounds, carbon nanotube, and its partition dimension bounds are described in [22], hexagonal Mobius ladder networks are discussed in terms of bounds on the partition dimension in [23], Harary graphs are found in terms of partition dimension bounds in [24], alpha-boron nanotubes are available in [25], a novel chemical structure is found in terms of metric-based parameters in [26], and different generalised families of graphs are discussed in terms of resolving set and partition dimension in [27–29].
Definition 1. “Let $\xi(\lambda)$ and $\eta(\lambda)$ be the vertex set and edge set (respectively) of a simple, connected graph $\lambda$. The least count of the edges between two vertices $x_1, x_2 \in \xi(\lambda)$ is called the distance $d(x_1, x_2)$ between $x_1$ and $x_2$."

Definition 2. “Let $\rho = \{\rho_1, \rho_2, \ldots, \rho_p\}$ be an $p$-ordered set of $\xi(\lambda)$. The distance between a vertex $x_i \in \xi(\lambda)$ and a $\rho_j \in \xi(\lambda)$ is defined as $d(x_i, \rho_j) = \min_{y \in \rho_j} d(x_i, y), 1 \leq j \leq p$. A planar graph is one that can be embedded in the plane; i.e., it can be drawn on the plane so that no two edges overlap. Let $Q = \{a_1, a_2, \ldots, a_k\}$ ordered set of vertices of $\lambda$. A vertex $a \in \xi(\lambda)$, the representations denoted by $r(a\{Q\})$, is the $p$-tuple distances as $d(a, a_1), d(a, a_2),\ldots, d(a, a_p)$. If every vertices have the different representations with respect to $Q$, then $Q$ is called the resolving set of vertices $\lambda$.”

Definition 3. The least number of $p$ in the resolving set is known as the metric dimension $\text{dim}(\lambda)$ of $\lambda$.

Definition 4. “Let $\chi$ be the $k$-ordered partition set and $r(a\chi) = \{d(a, \chi_1), d(a, \chi_2),\ldots, d(a, \chi_k)\}$ be the $k$-tuple distance representations of a vertex $a$ to $\chi$. If the representations of $a$ differ with respect to $\chi$, then $\chi$ is the resolving partition set of a graph’s vertex set.”

Definition 5. The partition dimension ($pd(\lambda)$) of $\lambda$ is defined as the smallest count of sections in the resolving partition set of $\xi(\lambda)$ [1].

The $\text{dim}(\lambda)$ and $pd(\lambda)$ can be related for any non trivial connected graph $\lambda$ in [1]

$$pd(\lambda) \leq \text{dim}(\lambda) + 1. \quad (1)$$

The theorems that follow are quite useful in determining the partition dimension of a graph.

**Theorem 1** (see[1]). “Let $\chi$ be a resolving partition of $\xi(\lambda)$ and $x_1, x_2 \in \xi(\lambda)$. If $d(x_1, x_2) = d(x_2, x_3)$ for all vertices $x_3 \in \xi(\lambda)'(x_1, x_2)$, then $x_1, x_2$ belong to different classes of $\chi$."

**Theorem 2** (see[1]). “Let $\lambda$ be a simple and connected graph. Then,

(i) $pd(\lambda)$ is 2 iff $\lambda$ is a path graph

(ii) $pd(\lambda)$ is $n$ iff $\lambda$ is a complete graph.”

**2. Harary Graph $H_{x,y}$**

Harary$H_{x,y}$ is an $x$-regular graph with order $y$ the vertex set $\xi(H_{x,y}) = \{\beta_1, \beta_2, \ldots, \beta_y\}$, if $x$ is even, then $x = 2\mu = y - 1$ for some integer $\mu \leq (y - 1)/2$. For each $\beta (1 \leq \beta \leq y)$, we join $\beta_i$ to $\beta_{\beta_i+1}, \beta_{\beta_i+2}, \ldots, \beta_{\beta_i+\mu}$, and to $\beta_{\beta_i-1}, \beta_{\beta_i-2}, \ldots, \beta_{\beta_i}$. The arrangement of the vertices is made like $\beta_1, \beta_2, \ldots, \beta_y$, developed cycle, and then, its every vertex $\beta_i$ is attached to the $\mu$-vertices that come right after $\beta_i$ and the $\mu$ vertices that move right away $\beta_i$. Here are some limits for the Harary graph’s ($H_{x,y}$) partition dimension.

**Theorem 3.** With parametric values $y \geq 5, y \equiv 0, 2, 3 (\text{mod} x)$ and $x = 4$. Let $H_{x,y}$ is a generalized Harary graph. Then, the partition dimension of $H_{x,y}$ is $\leq 4$.

**Proof.** In order to prove $pd(H_{x,y}) \leq 4$, we sorted the proof into the given below categories:

Case 1. $y \equiv 0 (\text{mod} 4), y = 4\mu, \mu \geq 2 \in \mathbb{Z}^+$.

Suppose you have a resolving partition set $\chi = \{x_1, x_2, x_3, x_4\}$ where $x_1 = \{\beta_1\}, x_2 = \{\beta_1, \beta_3\}, x_3 = \{\beta_1\}, x_4 = V (H_{x,y})/\{\beta_1, \beta_2, \beta_3\} \big\}$, the unique and distinct codes of the whole vertex set of $H_{x,y}$, with regard to $\chi$ are as follows:

$$r(\beta_{2x+1}) = \begin{cases} (x, x - 1, x - 1, 0) & \text{if } x = 2, 3, \ldots, \mu/0.2cm; \\ (\mu, \mu, \mu, 0) & \text{if } x = \mu + 10.2cm; \\ (2\mu - x + 1, 2\mu - x + 1, 2\mu - x + 1, 0) & \text{if } x = \mu + 2, \ldots, 2\mu/0.2cm, \end{cases} \quad (2)$$

$$r(\beta_{2x+1}) = \begin{cases} (x, x - 1, 0) & \text{if } x = 2, 3, \ldots, \mu/0.2cm; \\ (2\mu - x, 2\mu - x + 1, 2\mu - x + 1, 0) & \text{if } x = \mu + 1, \ldots, 2\mu - 10.2cm. \end{cases}$$

Case 2. $y \equiv 2 (\text{mod} 4), y = 4\mu + 2, \mu \in \mathbb{Z}^+$.

Suppose the resolving partition set $\chi = \{x_1, x_2, x_3, x_4\}$ where $x_1 = \{\beta_1\}, x_2 = \{\beta_2\}, x_3 = \{\beta_3\}, x_4 = V (H_{x,y})/\{\beta_1, \beta_2, \beta_3\}$, the unique and distinct codes of the whole vertex set of $H_{x,y}$, with regard to $\chi$ are as follows:
Set of vertices and cycle on $n$ can be obtained as the Cartesian product of a path on two
which is shown in the Figure 1. Its order and size become $5n$.

Case 3. $y \equiv 3 \pmod{4}$, $y = 4\mu + 3, \mu \in \mathbb{Z}^+$.

$\chi = \{\beta_1, \beta_2, \beta_3\}$, the unique and distinct codes of
the whole vertex set of $H_{x,y}$ with regard to $\chi$ are as follows:

$$r(\beta_2 | \chi) = \begin{cases} (x, x - 1, x - 1, 0) & \text{if } x = 2, 3, \ldots, \mu + 10.2c; \\ (2\mu - x + 2, 2\mu - x + 2, 2\mu - x + 3, 0) & \text{if } x = \mu + 2, \ldots, 2\mu + 10.2c, \\ (x, x, x - 1, 0) & \text{if } x = 2, 3, \ldots, 0.2c; \\ (\mu, \mu + 1, 0, \mu) & \text{if } x = \mu + 10.2c; \\ (2\mu - x + 1, 2\mu - x + 2, 2\mu - x + 2, 0) & \text{if } x = \mu + 2, \ldots, 2\mu 0.2c. \end{cases}$$ (3)

$\chi = \{\beta_1, \beta_2, \beta_3\}$, the unique and distinct codes of
the whole vertex set of $H_{x,y}$ with regard to $\chi$ are as follows:

$$r(\theta_1 | \chi) = \begin{cases} (x, x - 1, x - 1, 0) & \text{if } x = 2, 3, \ldots, \mu + 10.2c; \\ (2\mu - x + 2, 2\mu - x + 3, 2\mu - x + 3, 0) & \text{if } x = \mu + 2, \ldots, 2\mu + 10.2c, \\ (x, x, x - 1, 0) & \text{if } x = 2, 3, \ldots, \mu + 10.2c; \\ (2\mu - x + 2, 2\mu - x + 2, 2\mu - x + 3, 0) & \text{if } x = \mu + 2, \ldots, 2\mu + 10.2c. \end{cases}$$ (4)

There are various representations for the complete vertex
set of $H_{x,y}$ in relation to the resolving partition set $\chi$. Hence
$pd(H_{x,y}) \leq 4$. (5)

3. The Partition Dimension of Barycentric
Subdivision of Cayley
Graphs $S(Cay(Z_n \oplus Z_2))_n$

The Cayley graphs $Cay(Z_n \oplus Z_2), n \geq 3$, is a cubic graph that
can be obtained as the Cartesian product of a path on two
vertices and cycle on $n$ vertices. The Cayley graphs
$S(Cay(Z_n \oplus Z_2))_n$ consist of an outer $n$-cycle $\{y_1, y_2, \ldots, y_n\}$
and inner $n$-cycle $\{x_1, x_2, \ldots, x_n\}$ and set of $n$-spokes
$\{x_{\mu}, y_{\mu} : 1 \leq \mu \leq n\}$ with order, size, and faces $2n, 3n, n + 2$, respectively.

The barycentric Cayley graph $S(Cay(Z_n \oplus Z_2))$ can be obtained
by adding new vertices $u_{\mu}$ between $x_{\mu}$ and $x_{1+\mu}$, $v_{\mu}$ between $x_{\mu}$ and $y_{\mu}$, and $w_{\mu}$ between $y_{\mu}$ and $y_{1+\mu}$, modulo $n$, which is shown in the Figure 1. Its order and size become $5n$ and $6n$, respectively. The cycles’ arrangements in the graph

$$r(\theta_{x_{\mu}} | \chi) = \begin{cases} (2 - 2 + \mu, 2\mu - 4, 2 - 2 + \zeta, 2 + \mu, 0), & \text{if } 3 \geq \mu \leq \zeta; \\ (4 - 2 + (2 + \mu, 4 - 2 + \zeta + 2 + \mu, 4 - 2 - 2 + \mu - 2 + \zeta, 0), & \text{if } 2 + \zeta \geq \mu \leq 2\zeta; \\ (1, 1, 2 - 1 + \zeta, 0), & \text{if } \mu = 1; \\ (2 - 1 + \mu, 2\mu - 3, 2 - 2 + \zeta + 1 + \mu, 0), & \text{if } 2 \geq \mu \leq \zeta; \\ (2 - 1 + \zeta, 2 - 1 + \zeta, 1, 0), & \text{if } \mu = 1 + \zeta; \\ (4 - 2 + (1 + \mu, 4 - 2 + \zeta + 3, 2 - 2 + \mu - 1 + \zeta, 0), & \text{if } 2 + \zeta \geq \mu \leq 2\zeta. \end{cases}$$ (6)

Theorem 4. Let $S(Cay(Z_n \oplus Z_2))_n$ denote the barycentric
subdivisions of Cayley graphs. Then, $pd(Cay(Z_n \oplus Z_2)) \leq 4$ for
every $n \geq 4$.

Proof. Case 1. When $n$ taken as an even,

For this particular case, we assume $n = 2\zeta, \zeta \geq 2, \zeta \in \mathbb{Z}^+$.
Let $\chi = \{x_1, x_2, x_3, x_4\}$ be a resolving partition set where
$\chi_1 = \{x_1, x_2, x_3, x_4\}$, $\chi_2 = \{x_1, x_2, x_3\}$, $\chi_3 = \{x_4\}$, and
$\chi_4 = \{x_1, x_2, x_3, x_4\}$. Representations of all vertices $V(Cay(Z_n \oplus Z_2))_n$ regarding the
resolving partition set $\chi$. The vertices on the inner cycle are represented as follows:

$$r(\theta_{x_{\mu}} | \chi) = \begin{cases} (2 - 2 + \mu, 2\mu - 4, 2 - 2 + \zeta, 2 + \mu, 0), & \text{if } 3 \geq \mu \leq \zeta; \\ (4 - 2 + (2 + \mu, 4 - 2 + \zeta + 2 + \mu, 4 - 2 - 2 + \mu - 2 + \zeta, 0), & \text{if } 2 + \zeta \geq \mu \leq 2\zeta; \\ (1, 1, 2 - 1 + \zeta, 0), & \text{if } \mu = 1; \\ (2 - 1 + \mu, 2\mu - 3, 2 - 2 + \zeta + 1 + \mu, 0), & \text{if } 2 \geq \mu \leq \zeta; \\ (2 - 1 + \zeta, 2 - 1 + \zeta, 1, 0), & \text{if } \mu = 1 + \zeta; \\ (4 - 2 + (1 + \mu, 4 - 2 + \zeta + 3, 2 - 2 + \mu - 1 + \zeta, 0), & \text{if } 2 + \zeta \geq \mu \leq 2\zeta. \end{cases}$$ (6)
The internal cycle vertices are represented as follows:

\[ r(v_\mu | \chi) = \begin{cases} 
(1, 3, 21 + \zeta, 0), & \text{if } \mu = 1; \\
(2 - 1 + \mu, 2\mu - 3, 2 - 2 + \zeta\mu + 3, 0), & \text{if } 2 \leq \mu \leq 1 + \zeta; \\
(4 - 2 + \zeta\mu + 3, 4 - 2 + \zeta\mu + 5, 2 - 2 + \mu - 1 + \zeta, 0), & \text{if } 2 + \zeta \leq \mu \leq 2\zeta.
\end{cases} \]  

The following is vertices’ representations on the outer cycle:

\[ r(y_\mu | \chi) = \begin{cases} 
(2, 4, 22 + \zeta, 0), & \text{if } \mu = 1; \\
(2\mu, 2 - 2 + \mu, 2 - 2 + \zeta\mu + 4, 0), & \text{if } 2 \leq \mu \leq 1 + \zeta; \\
(4 - 2 + \zeta\mu + 4, 4 - 2 + \zeta\mu + 6, 2 - 2 + \mu\zeta, 0), & \text{if } 2 + \zeta \leq \mu \leq 2\zeta,
\end{cases} \]

\[ r(w_\mu | \chi) = \begin{cases} 
(3, 3, 21 + \zeta, 0), & \text{if } \mu = 1; \\
(21 + \mu, 2 - 1 + \mu, 2 - 2 + \zeta\mu + 3, 0), & \text{if } 2 \leq \mu \leq \zeta; \\
(21 + \zeta, 21 + \zeta, 3, 0), & \text{if } \mu = 1 + \zeta; \\
(4 - 2 + \zeta\mu + 3, 4 - 2 + \zeta\mu + 5, 2 - 2 + \mu 1 + \zeta, 0), & \text{if } 2 + \zeta \leq \mu \leq 2\zeta.
\end{cases} \]  

Case 2. When \( n \) consider as an odd, for this particular case, we assume \( n = 2\zeta + 1, \zeta \geq 2, \zeta \in \mathbb{Z} \). Let \( \chi = \{x_1, x_2, x_3, x_4\} \) be a resolving set where \( x_4 = \{\forall V(G) \neq \{x_1, x_2, x_3\} \} \). Representations of all vertices \( V(Cay(Z_n \oplus \mathbb{Z}_2)) \setminus \{x_1 \cup x_2 \cup x_3\} \) regarding the resolving partition set \( \chi \).
The vertices on the inner cycle are represented as follows:

\[
\begin{align*}
   r(x_{\mu}|\chi) &= \begin{cases} 
      (2 - 2 + \mu, 2\mu - 4, 2 - 2 + \zeta\mu + 3, 0), & \text{if } 3 \leq \mu \leq 1 + \zeta; \\
      (2\zeta, 2\zeta, 1, 0), & \text{if } \mu = 2 + \zeta; \\
      (4 - 2 + \zeta\mu + 4, 4 - 2 + \zeta\mu + 6, 2 - 2 + \mu - 3 + \zeta, 0), & \text{if } 3 + \zeta \leq \mu \leq 21 + \zeta,
   \end{cases} \\
r(u_{\mu}|\chi) &= \begin{cases} 
      (1, 1, 2\zeta, 0), & \text{if } \mu = 1; \\
      (2 - 1 + \mu, 2\mu - 3, 2 - 2 + \zeta 2 + \mu, 0), & \text{if } 2 \leq \mu \leq \zeta; \\
      (4 - 2 + \zeta\mu + 3, 4 - 2 + \zeta\mu + 5, 2 - 2 + \mu - 2 + \zeta, 0), & \text{if } 2 + \zeta \leq \mu \leq 21 + \zeta.
   \end{cases}
\end{align*}
\]

The internal cycle vertices are represented as follows:

\[
\begin{align*}
   r(v_{\mu}|\chi) &= \begin{cases} 
      (1, 3, 2 - 2 + \zeta, 0), & \text{if } \mu = 1; \\
      (2 - 1 + \mu, 2\mu - 3, 2 - 2 + \zeta\mu + 4, 0), & \text{if } 2 \leq \mu \leq 1 + \zeta; \\
      (2 - 3 + \zeta, 2 - 1 + \zeta, 2, 0), & \text{if } \mu = 2 + \zeta; \\
      (4 - 2 + (1 + \mu, 4 - 2 + \zeta\mu + 3, 2 - 2 + \mu - 2 + \zeta, 0), & \text{if } 3 + \zeta \leq \mu \leq 21 + \zeta; \\
      (2, 3, 1 + \zeta, 0), & \text{if } \mu = 21 + \zeta.
   \end{cases}
\end{align*}
\]

The followings are vertices' representations on the outer cycle:

\[
\begin{align*}
   r(y_{\mu}|\chi) &= \begin{cases} 
      (2, 4, 2 - 1 + \zeta, 0), & \text{if } \mu = 1; \\
      (2\mu, 2 - 2 + \mu, 2 - 2 + \zeta\mu + 5, 0), & \text{if } 2 \leq \mu \leq 1 + \zeta; \\
      (2 - 2 + \zeta, 2\zeta, 3, 0), & \text{if } \mu = 2 + \zeta; \\
      (4 - 2 + (2 + \mu, 4 - 2 + \zeta\mu + 4, 2 - 2 + \mu - 1 + \zeta, 0), & \text{if } 3 + \zeta \leq \mu \leq 21 + \zeta,
   \end{cases} \\
r(w_{\mu}|\chi) &= \begin{cases} 
      (3, 3, 2\zeta, 0), & \text{if } \mu = 1; \\
      (21 + \mu, 2 - 1 + \mu, 2 - 2 + \zeta\mu + 4, 0), & \text{if } 2 \leq \mu \leq \zeta; \\
      (2 - 1 + \zeta, 21 + \zeta, 3, 0), & \text{if } \mu = 1 + \zeta; \\
      (4 - 2 + (1 + \mu, 4 - 2 + \zeta\mu + 3, 2 - 2 + \mu\zeta, 0), & \text{if } 2 + \zeta \leq \mu \leq 21 + \zeta.
   \end{cases}
\end{align*}
\]

With respect to the resolving partition set \( \chi \), the complete vertex set of \( S(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))_n \) has unique representations.

\[
\text{pd} (S(\text{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_2))_n) \leq 4. \tag{12}
\]

\textbf{Theorem 5.} Let \( \text{ext}^{(1)}(G) \) denote the Pendent graph. Then, \( \text{pd}(\text{ext}^{(1)}(G)) \leq 4 \) for every \( n \equiv 0 \pmod{4}, n \geq 16. \)
Proof. Let $\chi = \{x_1, x_2, x_3, x_4\}$ be a resolving partition set of the graph shown in Figure 2, where $x_1 = \{u_1\}, x_2 = \{u_2\}, x_3 = \{v_{23+1}\}$, and $x_4 = \{G \setminus \{x_1, x_2, x_3\}\}$. Representations of all vertices $V(\text{ext}_1(G))\setminus \{x_1, x_2, x_3\}$ regarding the resolving partition set $\chi$.

Case 1. When even vertices

$$r(\omega_{2\mu}|\chi) = \begin{cases} (3, 5, 3 + \zeta, 0), & \text{if } \mu = 1; \\ (4, 4, 3 + \zeta, 0), & \text{if } \mu = 2; \\ (5, 3, 2 + \zeta, 0), & \text{if } \mu = 3; \\ (2 + \mu_1 - 1\mu, \zeta + 5 - \mu, 0), & \text{if } 4 \leq \mu \leq 1 + \zeta; \\ (2 + \zeta_1 - 1 + \zeta, 4, 0), & \text{if } \mu = \zeta; \\ (2 + \zeta, 2, 0), & \text{if } \mu = 1 + \zeta; \\ (1 + \zeta, 1 + \zeta, 2, 0), & \text{if } \mu = 2 + \zeta; \\ (\zeta, 2 + \zeta, 4, 0), & \text{if } \mu = 3 + \zeta; \\ (23 + \zeta - \mu, 6 + 2\zeta - \mu, 2 \mu - \zeta, 0), & \text{if } \zeta + 4 \leq \mu \leq 2\zeta. \end{cases}$$

Case 2. When odd vertices

$$r(\omega_{2\mu}|\chi) = \begin{cases} (2, 5, 2 + \zeta, 0), & \text{if } \mu = 1; \\ (3, 4, 3 + \zeta, 0), & \text{if } \mu = 2; \\ (4, 3, 2 + \zeta, 0), & \text{if } \mu = 3; \\ (1 + \mu_1 - 2 + \mu, \zeta - \mu + 5, 0), & \text{if } 4 \leq \mu \leq 1 + \zeta; \\ (2 + \zeta_1 - 1 + \zeta, 3, 0), & \text{if } \mu = 1 + \zeta; \\ (1 + \zeta, 1, 0), & \text{if } \mu = 2 + \zeta; \\ (\zeta, 1 + \zeta, 3, 0), & \text{if } \mu = 3 + \zeta; \\ (2\zeta - \mu + 3, 2\zeta - \mu + 6, 1 + \mu + \zeta, 0), & \text{if } \zeta + 4 \leq \mu \leq 2\zeta. \end{cases}$$

With respect to the resolving partition set $\chi$, the complete vertex set of $\text{ext}_1(G)$ has unique representations. Hence,

$$pd(\text{ext}_1(G)) \leq 4. \tag{15}$$

**Theorem 6.** Let $\text{ext}_1(G)$ denote the Pendant graph. Then, $pd(\text{ext}_1(G)) \leq 5$ for every $n \equiv 1 (\mod 4)$, $n \geq 17$.

Proof. Let $\chi = \{x_1, x_2, x_3, x_4, x_5\}$ be a resolving partition set where $x_1 = \{u_1\}, x_2 = \{u_2\}, x_3 = \{u_3\}, x_4 = \{u_4\},$ and $x_5 = \{G \setminus \{x_2, x_3, x_4\}\}$. Representations of all vertices $V(\text{ext}_1(G))\setminus \{x_1, x_2, x_3, x_4\}$ regarding the resolving partition set $\chi$. The representations when $n = 13$ are shown in Tables 1-3.

Case 1. When even vertices
Figure 2: Pendent graph $\text{ext}^{(1)}(G)$ for $n = 20$.

| $r(w_i|\chi)$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|----------------|-------|-------|-------|-------|-------|
| $w_1$          | 2     | 3     | 6     | 0     | 0     |
| $w_2$          | 3     | 2     | 5     | 6     | 0     |
| $w_3$          | 3     | 3     | 4     | 6     | 0     |
| $w_4$          | 4     | 3     | 4     | 5     | 0     |
| $w_5$          | 4     | 4     | 3     | 4     | 0     |
| $w_6$          | 5     | 4     | 3     | 3     | 0     |
| $w_7$          | 5     | 5     | 2     | 2     | 0     |
| $w_8$          | 5     | 5     | 3     | 1     | 0     |
| $w_9$          | 5     | 5     | 3     | 2     | 0     |
| $w_{10}$       | 4     | 5     | 4     | 3     | 0     |
| $w_{11}$       | 4     | 4     | 4     | 4     | 0     |
| $w_{12}$       | 2     | 3     | 4     | 4     | 0     |
| $w_{13}$       | 3     | 3     | 5     | 6     | 0     |

Table 2: Representations of the vertices when $n = 13$ w.r.t $\chi$.

| $r(w_i|\chi)$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|----------------|-------|-------|-------|-------|
| $w_1$          | 2     | 4     | 7     | 0     |
| $w_2$          | 3     | 3     | 7     | 0     |
| $w_3$          | 3     | 3     | 8     | 0     |
| $w_4$          | 4     | 2     | 7     | 0     |
| $w_5$          | 4     | 3     | 7     | 0     |
| $w_6$          | 5     | 3     | 6     | 0     |
| $w_7$          | 5     | 4     | 6     | 0     |
| $w_8$          | 6     | 4     | 5     | 0     |
| $w_9$          | 6     | 5     | 4     | 0     |
| $w_{10}$       | 7     | 5     | 3     | 0     |
| $w_{11}$       | 6     | 6     | 2     | 0     |
| $w_{12}$       | 6     | 6     | 1     | 0     |
| $w_{13}$       | 5     | 7     | 2     | 0     |
| $w_{14}$       | 5     | 6     | 3     | 0     |
| $w_{15}$       | 4     | 6     | 4     | 0     |
| $w_{16}$       | 4     | 5     | 5     | 0     |
| $w_{17}$       | 3     | 5     | 6     | 0     |
| $w_{18}$       | 3     | 4     | 6     | 0     |
Table 3: Representations of the vertices when \( n = 13 \) with regards to \( \chi \).

| \( r(w_i|\chi) \) | \( \chi_1 \) | \( \chi_2 \) | \( \chi_3 \) | \( \chi_4 \) |
|-----------------|------|------|------|------|
| \( w_1 \)       | 2    | 4    | 6    | 0    |
| \( w_2 \)       | 3    | 3    | 6    | 0    |
| \( w_3 \)       | 3    | 3    | 0    | 0    |
| \( w_4 \)       | 4    | 2    | 4    | 0    |
| \( w_5 \)       | 4    | 3    | 3    | 0    |
| \( w_6 \)       | 5    | 2    | 2    | 0    |
| \( w_7 \)       | 5    | 0    | 1    | 0    |
| \( w_8 \)       | 4    | 4    | 3    | 0    |
| \( w_9 \)       | 4    | 5    | 0    | 0    |
| \( w_{10} \)    | 3    | 5    | 0    | 0    |
| \( w_{11} \)    | 3    | 4    | 5    | 0    |

Every vertex of \( \text{ext}^{(1)}(G) \) gains the unique representation with respect to the resolving set \( \chi \); hence,

\[
\text{pd}(\text{ext}^{(1)}(G)) \leq 5. \tag{18}
\]

**Theorem 7.** Let \( \text{ext}^{(1)}(G) \) denote the Pendant graph. Then, \( \text{pd}(\text{ext}^{(1)}(G)) \leq 4 \) for every \( n \equiv 2 \mod 4 \) \( n \geq 18 \).

**Proof.** Let \( \chi = \{\chi_1, \chi_2, \chi_3, \chi_4\} \) be a resolving partition set where \( \chi_1 = \{u_1\}, \chi_2 = \{u_2\}, \chi_3 = \{v_{2x}\}, \) and \( \chi_4 = \{V(G) \notin \{\chi_1, \chi_2, \chi_3\}\} \). Representations of all vertices...
Theorem 8. Let $\text{ext}^1(G)$ denote the Pendent graph. Then, $pd(\text{ext}^1(G)) \leq 4$ for every $n \equiv 3 \pmod{4} n \geq 11$.
Case 2. When odd vertices,

\[
r(\mu_{2-1+\mu}) = \begin{cases} 
(2, 4, \zeta + 4, 0), & \text{if } \mu = 1; \\
(3, 3, 3 + \zeta, 0), & \text{if } \mu = 2; \\
(1 + \mu, \mu, \zeta - \mu + 5, 0), & \text{if } 3 \leq \mu \leq \zeta; \\
(2 + \zeta, 1 + \zeta, 3, 0), & \text{if } \mu = 1 + \zeta; \\
(3 + \zeta, 2 + \zeta, 1, 0), & \text{if } \mu = 2 + \zeta; \\
(2 + \zeta, 3 + \zeta, 3, 0), & \text{if } \mu = 3 + \zeta; \\
(2\zeta - \mu + 5, 2\zeta - \mu + 6, -1 + \mu + \zeta, 0), & \text{if } \zeta + 4 \leq \mu \leq 22 + \zeta. 
\end{cases}
\]

Every vertex of ext\(^{(1)}\) (G) gains the unique representation with respect to the resolving set \(S\); hence,

\[
\text{pd}(\text{ext\(^{(1)}\)} (G)) \leq 4. \tag{24}
\]

5. Conclusion and Discussion

This article investigates the generalised Harary \((H_{x,y})\) graph’s partition dimension’s precise boundaries, and we concluded that

\[
\text{pd}(H_{x,y}) \leq 4. \tag{25}
\]

For the Barycentric subdivision of Cayley Graphs \(S(\text{Cay}(Z_n \oplus Z_2))_{\text{nppt}}\)

\[
\text{pd}(\text{Cay}(Z_n \oplus Z_2)) \leq 4. \tag{26}
\]

Furthermore, we also discussed the graph named as pendant Graph ext\(^{(1)}\) (G), and this also gave the same bounds of partition dimension which is

\[
\text{pd}(\text{ext\(^{(1)}\)} (G)) \leq 4. \tag{27}
\]

We proved all the chosen graphs have upper bounds of partition dimension 4.

Data Availability

There are no data associated with this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References


