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Research Article

On the Bounded Partition Dimension of Some Generalised Graph Structures

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Consider λ to be a connected graph with a vertex set $V(\lambda)$ that may be partitioned into any partition set S. If each vertex in λ has a separate representation with regard to S and is an ordered k partition, then the set with S is a resolving partition of λ.. A partition dimension of λ, represented by pd, is the minimal cardinality of resolving k partitions of $V(\lambda)$. The partition dimension of various generalised families of graphs, such as the Harary graph, Cayley graph, and Pendent graph, is given as a sharp upper bound in this article.

1. Introduction and Preliminaries

Computation is the process of analysing anything using mathematical or logical principles. An algorithm's implementation is a well-known example of computing. An issue is deemed inherently difficult if solving it requires significant resources regardless of the algorithm used. The problem discussed in this research work has computational cost nondeterministic polynomial time, and therefore, it is called as NP-hard problem in short. The partition dimension is introduced in [1] as a natural generalization of the metric dimension.

The concept of resolving set was introduced by the authors of [2] and later introduced by the authors of [3]. In [4], the authors worked on the star partition dimension of the cycle and complete graph by using a comb product. Authors worked on the partition dimension of different classes of circulant graph [5]. The authors worked on the cartesian product of different graphs and wrote a note on the partition dimension [6]. Recently [7, 8], the partition dimensions of the various families of convex polytopes and their types of graphs, such as with pendant edges, have been calculated.

The partition dimension is used in a variety of fields, including robot navigation, strategies for the mastermind game, network discovery and verification, pattern recognition, and image processing, the Djokovic-Winkler relation, chemistry for representing, chemical compounds, all of which involve the use of hierarchical data structures [9–17].

Literature on partition dimension and its bounds on the specific families of graphs are found in [18, 19]. The antivirus properties of COVID structures are discussed in terms of partition dimension and bounds in [20], novel chemical structures of diamonded structures are discovered in [21] in terms of partition dimension bounds, carbon nanotube, and its partition dimension bounds are described in [22], hexagonal Mobius ladder networks are discussed in terms of bounds on the partition dimension in [23], Harary graphs are found in terms of partition dimension bounds in [24], alpha-boron nanotubes are available in [25], a novel chemical structure is found in terms of metric-based parameters in [26], and different generalised families of graphs are discussed in terms of resolving set and partition dimension in [27–29].

Definition 1. "Let $\xi(\lambda)$ and $\eta(\lambda)$ be the vertex set and edge set (respectively) of a simple, connected graph (λ). The least count of the edges between two vertices $x_1, x_2 \in \xi(\lambda)$ is called the distance $d(x_1, x_2)$ between x_1 and x_2."

Definition 2. "Let $\rho = \{\rho_1, \rho_2, \ldots, \rho_p\}$ be an p-ordered set of $\xi(\lambda)$. The distance between a vertex $x_1 \in \xi(\lambda)$ and a $\rho_j \subset \xi(\lambda)$ is defined as $d(x_1, \rho_j) = \min_{y \in \rho_j}\{d(x_1, y)\}, 1 \le j \le p$. A planar graph is one that can be embedded in the plane; i.e., it can be drawn on the plane so that no two edges overlap. Let $Q = \{a_1, a_2, \ldots, a_k\}$ ordered set of vertices of λ. A vertex $a \in \xi(\lambda)$, the representations denoted by $r(a|Q)$, is the p -tuple distances as $d(a, a_1), d(a, a_2), \ldots, d(a, a_p))$. If every vertices have the different representations with respect to Q, then Q is called the resolving set of vertices λ."

Definition 3. The least number of p in the resolving set is known as the metric dimension $\dim(\lambda)$ of λ.

Definition 4. "Let χ be the k-ordered partition set and $r(a|\chi) = \{d(a, \chi_1), d(a, \chi_2), \ldots, d(a, \chi_k)\}, d(a, \chi k)$ be the k -tuple distance representations of a vertex a to χ. If the representations of a differ with respect to χ, then χ is the resolving partition set of a graph's vertex set."

Definition 5. The partition dimension $(pd(\lambda))$ of λ is defined as the smallest count of sections in the resolving partition set of $\xi(\lambda)$ [1].

The $\dim(\lambda)$ and $pd(\lambda)$ can be related for any nontrivial connected graph λ in [1]

$$pd(\lambda) \le \dim(\lambda) + 1. \tag{1}$$

The theorems that follow are quite useful in determining the partition dimension of a graph.

Theorem 1 (see[1]). *"Let χ be a resolving partition of $\xi(\lambda)$ and $x_1, x_2 \in \xi(\lambda)$. If $d(x_1, x_3) = d(x_2, x_3)$ for all vertices $x_3 \in \xi(\lambda)/(x_1, x_2)$, Then x_1, x_2 belong to different classes of χ. "*

Theorem 2 (see[1]). *"Let λ be a simple and connected graph. Then,*

(i) $pd(\lambda)$ is 2 iff λ is a path graph

(ii) $pd(\lambda)$ is n iff λ is a complete graph."

2. Harary Graph $H_{x,y}$

Harary$H_{x,y}$ is an x-regular graph with order y the vertex set $\xi(H_{x,y}) = \{\beta_\zeta : \zeta = 1, 2, \ldots, y\}$, if x is even, then $x = 2\mu \le y - 1$ for some integer $\mu \le (y - 1/2)$. For each $\zeta (1 \le \zeta \le y)$, we join β_ζ to $\beta_{\zeta+1}, \beta_{\zeta+2}, \ldots, \beta_{\zeta+\mu}$ and to $\beta_{\zeta-1}, \beta_{\zeta-2}., \ldots, \beta_{\zeta-\mu}$.

The arrangement of the vertices is made like $\beta_1, \beta_2, \ldots, \beta_y$ developed cycle, and then, its every vertex β_ζ is attached to the μ-vertices that come right after β_ζ and the μ vertices that move right away β_ζ. Here are some limits for the Harary graph's $(H_{x,y})$ partition dimension.

Theorem 3. *With parametric values $y \ge 5, y \equiv 0, 2, 3 \pmod{x}$ and $x = 4$. Let $H_{x,y}$ is a generalised Harary graph. Then, the partition dimension of $H_{x,y}$ is ≤ 4.*

Proof. In order to prove $pd(H_{x,y}) \le 4$, we sorted the proof into the given below categories:

Case 1. $y \equiv 0 \pmod 4, y = 4\mu, \mu(\ge 2) \in \mathbb{Z}^+$.

Suppose you have a resolving partition set $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{\beta_1\}, \chi_2 = \{\beta_3\}, \chi_3 = \{\beta_{y-2}\}, \chi_4 = V(H_{x,y})/\{\beta_1, \beta_3, \beta_{y-2}\}$, the unique and distinct codes of the whole vertex set of $H_{x,y}$ with regard to χ are as follows:

$$r(\beta_{2x}|\chi) = \begin{cases} (x, x-1, x-1, 0) & \text{if } x = 2, 3, \ldots, \mu 0.2\text{cm}; \\ (\mu, \mu, \mu, 0) & \text{if } x = \mu + 1 0.2\text{cm}; \\ (2\mu - x + 1, 2\mu - x + 1, 2\mu - x + 1, 0) & \text{if } x = \mu + 2, \ldots, 2\mu 0.2\text{cm}, \end{cases} \tag{2}$$

$$r(\beta_{2x+1}|\chi) = \begin{cases} (x, x, x-1, 0) & \text{if } x = 2, 3, \ldots, \mu 0.2\text{cm}; \\ (2\mu - x, 2\mu - x + 1, 2\mu - x + 1, 0) & \text{if } x = \mu + 1, \ldots, 2\mu - 1 0.2\text{cm}. \end{cases}$$

Case 2. $y \equiv 2 (\text{mod}4), y = 4\mu + 2, \mu \in \mathbb{Z}^+$.

Suppose the resolving partition set $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{\beta_1\}, \chi_2 = \{\beta_2\}, \chi_3 = \{\beta_3\}, \chi_4 = V(H_{x,y})/\{\beta_1, \beta_2, \beta_3\}$, the unique and distinct codes of the whole vertex set of $H_{x,y}$ with regard to χ are as follows:

$$r(\beta_{2x}|\chi) = \begin{cases} (x, x-1, x-1, 0) & \text{if } x = 2, 3, \ldots, \mu + 10.2\text{cm}; \\ (2\mu - x + 2, 2\mu - x + 2, 2\mu - x + 3, 0) & \text{if } x = \mu + 2, \ldots, 2\mu + 10.2\text{cm}, \end{cases}$$
$$r(\beta_{2x+1}|\chi) = \begin{cases} (x, x, x-1, 0) & \text{if } x = 2, 3, \ldots, \mu 0.2\text{cm}; \\ (\mu, \mu+1, \mu, 0) & \text{if } x = \mu + 10.2\text{cm}; \\ (2\mu - x + 1, 2\mu - x + 2, 2\mu - x + 2, 0) & \text{if } x = \mu + 2, \ldots, 2\mu 0.2\text{cm}. \end{cases} \quad (3)$$

Case 3. $y \equiv 3(\text{mod}4), y = 4\mu + 3, \mu \in \mathbb{Z}^+$.

Suppose the resolving partition set $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{\beta_1\}, \chi_2 = \{\beta_2\}, \chi_3 = \{\beta_3\}, \chi_4 = V(H_{x,y})/\{\beta_1, \beta_2, \beta_3\}$, the unique and distinct codes of the whole vertex set of $H_{x,y}$ with regard to χ are as follows:

$$r(\beta_{2x}|\chi) = \begin{cases} (x, x-1, x-1, 0) & \text{if } x = 2, 3, \ldots, \mu + 10.2\text{cm}; \\ (2\mu - x + 2, 2\mu - x + 3, 2\mu - x + 3, 0) & \text{if } x = \mu + 2, \ldots, 2\mu + 10.2\text{cm}, \end{cases}$$
$$r(\beta_{2x+1}|\chi) = \begin{cases} (x, x, x-1, 0) & \text{if } x = 2, 3, \ldots, \mu + 10.2\text{cm}; \\ (2\mu - x + 2, 2\mu - x + 2, 2\mu - x + 3, 0) & \text{if } \mu = \mu + 2, \ldots, 2\mu + 10.2\text{cm}. \end{cases} \quad (4)$$

There are various representations for the complete vertex set of $H_{x,y}$ in relation to the resolving partition set χ. Hence,

$$pd(H_{x,y}) \leq 4. \quad (5)$$

□

3. The Partition Dimension of Barycentric Subdivision of Cayley Graphs $S(\text{Cay}(Z_n \oplus Z_2))_n$

The Cayley graphs $\text{Cay}(Z_n \oplus Z_2), n \geq 3$, is a cubic graph that can be obtained as the Cartesian product of a path on two vertices and cycle on n vertices. The Cayley graphs $S(\text{Cay}(Z_n \oplus Z_2))_n$ consist of an outer n-cycle $\{y_1, y_2, \ldots y_n\}$ and inner n-cycle $\{x_1, x_2, \ldots x_n\}$ and set of n-spokes $\{x_\mu y_\mu : \mu = 1, 2, \ldots, n\}$ with order, size, and faces $2n, 3n$, and $n + 2$, respectively.

The barycentric Cayley graph $S(\text{Cay}(Z_n \oplus Z_2))$ can be obtained by adding new vertices u_μ between x_μ and $x_{1+\mu}$, v_μ between x_μ and y_μ, and w_μ between y_μ and $y_{1+\mu}$, modulo n, which is shown in the Figure 1. Its order and size become $5n$ and $6n$, respectively. The cycles' arrangements in the graph like, $\{x_\mu : 1 \leq \mu \leq n\}$ called the inner cycle, $\{y_\mu : 1 \leq \mu \leq n\} \bigcup \{w_\mu : 1 \leq \mu \leq n\}$ named as outer cycle, $\{v_\mu : 1 \leq \mu \leq n\}$ set of interior vertices.

Theorem 4. *Let $S(Cay(Z_n \oplus Z_2))_n$ denote the barycentric subdivisions of Cayley graphs. Then, $pd(Cay(Z_n \oplus Z_2)) \leq 4$ for every $n \geq 4$.*

Proof.

Case 1. When n taken as an even,

For this particular case, we assume $n = 2\zeta, \zeta \geq 2, \zeta \in Z^+$. Let $\chi = \{\chi_1, \chi_2, \chi_3 \chi_4\}$ be a resolving partition set where $\chi_1 = \{x_1\}, \chi_2 = \{x_2\}, \chi_3 = \{x_{1+\zeta}\}$, and $\chi_4 = \{\forall V(G) | \notin \{\chi_1, \chi_2, \chi_3\}\}$. Representations of all vertices $V(\text{Cay}(Z_n \oplus Z_2))/\{\chi_1 \bigcup \chi_2 \bigcup \chi_3\}$ regarding the resolving partition set χ.

The vertices on the inner cycle are represented as follows:

$$r(x_\mu|\chi) = \begin{cases} (2 - 2 + \mu, 2\mu - 4, 2 - 2 + \zeta 2 + \mu, 0), & \text{if } 3 \leq \mu \leq \zeta; \\ (4 - 2 + \zeta 2 + \mu, 4 - 2 + \zeta\mu + 4, 2 - 2 + \mu - 2 + \zeta, 0), & \text{if } 2 + \zeta \leq \mu \leq 2\zeta, \end{cases}$$
$$r(u_\mu|\chi) = \begin{cases} (1, 1, 2 - 1 + \zeta, 0), & \text{if } \mu = 1; \\ (2 - 1 + \mu, 2\mu - 3, 2 - 2 + \zeta 1 + \mu, 0), & \text{if } 2 \leq \mu \leq \zeta; \\ (2 - 1 + \zeta, 2 - 1 + \zeta, 1, 0), & \text{if } \mu = 1 + \zeta; \\ (4 - 2 + \zeta 1 + \mu, 4 - 2 + \zeta\mu + 3, 2 - 2 + \mu - 1 + \zeta, 0), & \text{if } 2 + \zeta \leq \mu \leq 2\zeta. \end{cases} \quad (6)$$

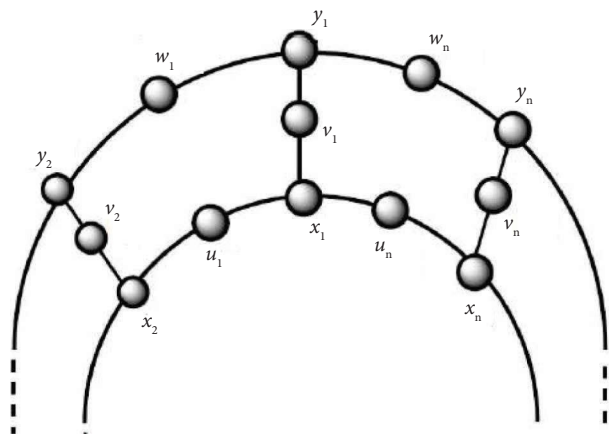

Figure 1: Barycentric subdivision of cayley graphs $S(\mathrm{Cay}(Z_n \oplus Z_2))_n$.

The internal cycle vertices are represented as follows:

$$r(v_\mu|\chi) = \begin{cases} (1,\ 3,\ 21+\zeta,\ 0), & \text{if} \quad \mu = 1; \\ (2-1+\mu,\ 2\mu-3,\ 2-2+\zeta\mu+3, 0), & \text{if} \quad 2 \le \mu \le 1+\zeta; \\ (4-2+\zeta\mu+3,\ 4-2+\zeta\mu+5,\ 2-2+\mu-1+\zeta,\ 0), & \text{if} \quad 2+\zeta \le \mu \le 2\zeta. \end{cases} \tag{7}$$

The following is vertices' representations on the outer cycle:

$$\begin{aligned} r(y_\mu|\chi) &= \begin{cases} (2, 4, 22+\zeta, 0), & \text{if } \mu = 1; \\ (2\mu, 2-2+\mu, 2-2+\zeta\mu+4, 0), & \text{if } 2 \le \mu \le 1+\zeta; \\ (4-2+\zeta\mu+4, 4-2+\zeta\mu+6, 2-2+\mu\zeta, 0), & \text{if } 2+\zeta \le \mu \le 2\zeta, \end{cases} \\ r(w_\mu|\chi) &= \begin{cases} (3, 3, 21+\zeta, 0), & \text{if } \mu = 1; \\ (21+\mu, 2-1+\mu, 2-2+\zeta\mu+3, 0), & \text{if } 2 \le \mu \le \zeta; \\ (21+\zeta, 21+\zeta, 3, 0), & \text{if } \mu = 1+\zeta; \\ (4-2+\zeta\mu+3, 4-2+\zeta\mu+5, 2-2+\mu 1+\zeta, 0), & \text{if } 2+\zeta \le \mu \le 2\zeta. \end{cases} \end{aligned} \tag{8}$$

Case 2. When n consider as an odd,

For this particular case, we assume $n = 2\zeta + 1, \zeta \ge 2, \zeta \in Z$. Let $\chi = \{\chi_1, \chi_2, \chi_3 \chi_4\}$ be a resolving set where $\chi_1 = \{x_1\}, \chi_2 = \{x_2\}, \chi_3 = \{u_{1+\zeta}\}$, and $\chi_4 = \{\forall V(G) | \notin \{\chi_1, \chi_2, \chi_3\}\}$. Representations of all vertices $V(\mathrm{Cay}(Z_n \oplus Z_2))/\{\chi_1 \bigcup \chi_2 \bigcup \chi_3\}$ regarding the resolving partition set χ.

The vertices on the inner cycle are represented as follows:

$$r(x_\mu|\chi) = \begin{cases} (2-2+\mu, 2\mu-4, 2-2+\zeta\mu+3, 0), & \text{if } 3 \le \mu \le 1+\zeta; \\ (2\zeta, 2\zeta, 1, 0), & \text{if } \mu = 2+\zeta; \\ (4-2+\zeta\mu+4, 4-2+\zeta\mu+6, 2-2+\mu-3+\zeta, 0), & \text{if } 3+\zeta \le \mu \le 21+\zeta, \end{cases}$$
$$r(u_\mu|\chi) = \begin{cases} (1, 1, 2\zeta, 0), & \text{if } \mu = 1; \\ (2-1+\mu, 2\mu-3, 2-2+\zeta 2+\mu, 0), & \text{if } 2 \le \mu \le \zeta; \\ (4-2+\zeta\mu+3, 4-2+\zeta\mu+5, 2-2+\mu-2+\zeta, 0), & \text{if } 2+\zeta \le \mu \le 21+\zeta. \end{cases} \tag{9}$$

The internal cycle vertices are represented as follows:

$$r(v_\mu|\chi) = \begin{cases} (1, 3, 2-2+\zeta, 0), & \text{if } \mu = 1; \\ (2-1+\mu, 2\mu-3, 2-2+\zeta\mu+4, 0), & \text{if } 2 \le \mu \le 1+\zeta; \\ (2-3+\zeta, 2-1+\zeta, 2, 0), & \text{if } \mu = 2+\zeta; \\ (4-2+\zeta 1+\mu, 4-2+\zeta\mu+3, 2-2+\mu-2+\zeta, 0), & \text{if } 3+\zeta \le \mu \le 21+\zeta; \\ (2, 3, 1+\zeta, 0), & \text{if } \mu = 21+\zeta. \end{cases} \tag{10}$$

The followings are vertices' representations on the outer cycle:

$$r(y_\mu|\chi) = \begin{cases} (2, 4, 2-1+\zeta, 0), & \text{if } \mu = 1; \\ (2\mu, 2-2+\mu, 2-2+\zeta\mu+5, 0), & \text{if } 2 \le \mu \le 1+\zeta; \\ (2-2+\zeta, 2\zeta, 3, 0), & \text{if } \mu = 2+\zeta; \\ (4-2+\zeta 2+\mu, 4-2+\zeta\mu+4, 2-2+\mu-1+\zeta, 0), & \text{if } 3+\zeta \le \mu \le 21+\zeta, \end{cases}$$
$$r(w_\mu|\chi) = \begin{cases} (3, 3, 2\zeta, 0), & \text{if } \mu = 1; \\ (21+\mu, 2-1+\mu, 2-2+\zeta\mu+4, 0), & \text{if } 2 \le \mu \le \zeta; \\ (2-1+\zeta, 21+\zeta, 3, 0), & \text{if } \mu = 1+\zeta; \\ (4-2+\zeta 1+\mu, 4-2+\zeta\mu+3, 2-2+\mu\zeta, 0), & \text{if } 2+\zeta \le \mu \le 21+\zeta. \end{cases} \tag{11}$$

With respect to the resolving partition set χ, the complete vertex set of $S(\mathrm{Cay}(Z_n \oplus Z_2))_n$ has unique representations.

$$pd(S(\mathrm{Cay}(Z_n \oplus Z_2))_n) \le 4. \tag{12}$$

□

4. The Partition Dimension of Pendent Graph $ext^{(1)}(G)$

Theorem 5. *Let $ext^{(1)}(G)$ denote the Pendent graph. Then, $pd(ext^{(1)}(G)) \le 4$ for every $n \equiv 0 \pmod 4$, $n \ge 16$.*

Proof. Let $\chi = \{\chi_1, \chi_2, \chi_3\chi_4\}$ be a resolving partition set of the graph shown in Figure 2, where $\chi_1 = \{u_1\}, \chi_2 = \{u_7\}, \chi_3 = \{v_{23+\zeta}\}$, and $\chi_4 = \{\forall V(G) | \notin \{\chi_1, \chi_2, \chi_3\}\}$. Representations of all vertices $V(\text{ext}^1(G))/\{\chi_1 \bigcup \chi_2 \bigcup \chi_3\}$ regarding the resolving partition set χ.

Case 1. When even vertices

$$r\left(w_{2\mu}|\chi\right) = \begin{cases} (3, 5, 3+\zeta, 0), & \text{if } \mu = 1; \\ (4, 4, 3+\zeta, 0), & \text{if } \mu = 2; \\ (5, 3, 2+\zeta, 0), & \text{if } \mu = 3; \\ (2+\mu, -1\mu, \zeta+5-\mu, 0), & \text{if } 4 \le \mu \le -1+\zeta; \\ (2+\zeta, -1+\zeta, 4, 0), & \text{if } \mu = \zeta; \\ (2+\zeta, \zeta, 2, 0), & \text{if } \mu = 1+\zeta; \\ (1+\zeta, 1+\zeta, 2, 0), & \text{if } \mu = 2+\zeta; \\ (\zeta, 2+\zeta, 4, 0), & \text{if } \mu = 3+\zeta; \\ (23+\zeta-\mu, 6+2\zeta-\mu, 2+\mu-\zeta, 0), & \text{if } \zeta+4 \le \mu \le 2\zeta. \end{cases} \tag{13}$$

Case 2. When odd vertices

$$r\left(w_{2-1+\mu}|\chi\right) = \begin{cases} (2, 5, 2+\zeta, 0), & \text{if } \mu = 1; \\ (3, 4, 3+\zeta, 0), & \text{if } \mu = 2; \\ (4, 3, 2+\zeta, 0), & \text{if } \mu = 3; \\ (1+\mu, -2+\mu, \zeta-\mu+5, 0), & \text{if } 4 \le \mu \le \zeta; \\ (2+\zeta, -1+\zeta, 3, 0), & \text{if } \mu = 1+\zeta; \\ (1+\zeta, \zeta, 1, 0), & \text{if } \mu = 2+\zeta; \\ (\zeta, 1+\zeta, 3, 0), & \text{if } \mu = 3+\zeta; \\ (2\zeta-\mu+3, 2\zeta-\mu+6, -1+\mu+\zeta, 0), & \text{if } \zeta+4 \le \mu \le 2\zeta. \end{cases} \tag{14}$$

With respect to the resolving partition set χ, the complete vertex set of $\text{ext}^{(1)}(G)$ has unique representations. Hence,

$$pd\left(\text{ext}^{(1)}(G)\right) \le 4. \tag{15}$$

□

Theorem 6. *Let $ext^{(1)}(G)$ denote the Pendent graph. Then, $pd(ext^{(1)}(G)) \le 5$ for every $n \equiv 1 (mod\, 4) n \ge 17$.*

Proof. Let $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$ be a resolving partition set where $\chi_1 = \{u_1\}, \chi_2 = \{u_2\}, \chi_3 = \{u_7\}, \chi_4 = [v_{21+\zeta}]$, and $\chi_5 = \{\forall V(G) | \notin \{\chi_1, \chi_2, \chi_3, \chi_4\}\}$. Representations of all vertices $V(\text{ext}^1(G))/\{\chi_1 \bigcup \chi_2 \bigcup \chi_3 \bigcup \chi_4\}$ regarding the resolving partition set χ. The representations when $n = 13$ are shown in Tables 1–3.

Case 1. When even vertices

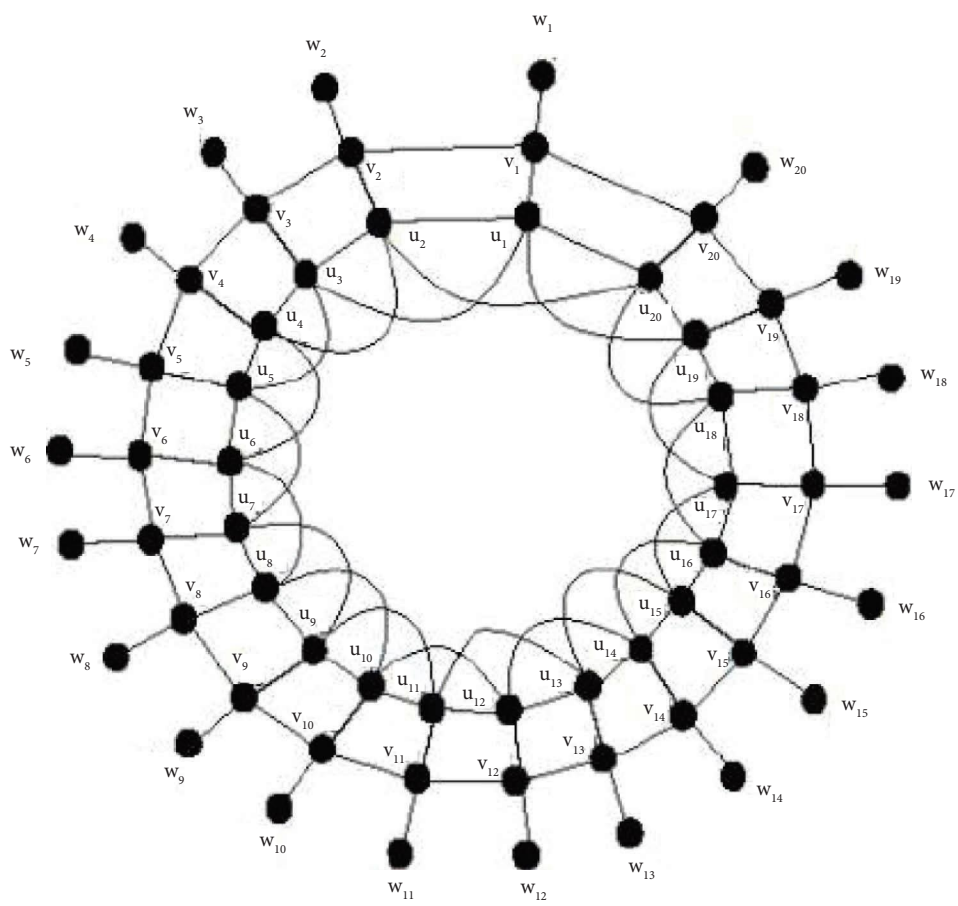

Figure 2: Pendent graph $\mathrm{ext}^{(1)}(G)$ for $n = 20$.

Table 1: Representations of the vertices when $n = 13$ w.r.t χ.

$r(w_i\|\chi)$	χ_1	χ_2	χ_3	χ_4	χ_5
w_1	2	3	6	0	0
w_2	3	2	5	6	0
w_3	3	3	4	6	0
w_4	4	3	4	5	0
w_5	4	4	3	4	0
w_6	5	4	3	3	0
w_7	5	5	2	2	0
w_8	5	5	3	1	0
w_9	5	5	3	2	0
w_{10}	4	5	4	3	0
w_{11}	4	4	4	4	0
w_{12}	2	3	4	4	0
w_{13}	3	3	5	6	0

Table 2: Representations of the vertices when $n = 13$ w.r.t χ.

$r(w_i\|\chi)$	χ_1	χ_2	χ_3	χ_4
w_1	2	4	7	0
w_2	3	3	7	0
w_3	3	3	8	0
w_4	4	2	7	0
w_5	4	3	7	0
w_6	5	3	6	0
w_7	5	4	6	0
w_8	6	4	5	0
w_9	6	5	4	0
w_{10}	7	5	3	0
w_{11}	6	6	2	0
w_{12}	6	6	1	0
w_{13}	5	7	2	0
w_{14}	5	6	3	0
w_{15}	4	6	4	0
w_{16}	4	5	5	0
w_{17}	3	5	6	0
w_{18}	3	4	6	0

Table 3: Representations of the vertices when $n = 13$ with regards to χ.

$r(w_i\|\chi)$	χ_1	χ_2	χ_3	χ_4
w_1	2	4	6	0
w_2	3	3	6	0
w_3	3	3	5	0
w_4	4	2	4	0
w_5	4	3	3	0
w_6	5	3	2	0
w_7	5	4	1	0
w_8	4	4	2	0
w_9	4	5	3	0
w_{10}	3	5	4	0
w_{11}	3	4	5	0

$$r\left(w_{2\mu}|\chi\right) = \begin{cases} (3,2,5,3+\zeta,0), & \text{if } \mu = 1; \\ (4,3,4,2+\zeta,0), & \text{if } \mu = 2; \\ (5,4,3,1+\zeta,0), & \text{if } \mu = 3; \\ (2+\mu,1+\mu,-1+\mu,\zeta-\mu+4,0), & \text{if } 4 \le \mu \le -1+\zeta; \\ (2+\zeta,1+\zeta,-1+\zeta,3,0), & \text{if } \mu = \zeta; \\ (2+\zeta,2+\zeta,\zeta,1,0), & \text{if } \mu = 1+\zeta; \\ (1+\zeta,2+\zeta,1+\zeta,3,0), & \text{if } \mu = 2+\zeta; \\ (\zeta,1+\zeta,2+\zeta,5,0), & \text{if } \mu = 3+\zeta; \\ (2\zeta-\mu+3,2\zeta-\mu+4,2\zeta-\mu+6,-2+\mu+\zeta,0), & \text{if } \zeta+4 \le \mu \le 2\zeta. \end{cases} \tag{16}$$

Case 2. When odd vertices

$$r\left(w_{2-1+\mu}|\chi\right) = \begin{cases} (2,\ 3,\ 5,\ 3+\zeta,\ 0), & if \quad \mu = 1; \\ (3,3,4,3+\zeta,0), & \text{if} \quad \mu = 2; \\ (4,4,3,2+\zeta,0), & \text{if} \quad \mu = 3; \\ (1+\mu,1+\mu,-2+\mu,\zeta-\mu+5,0), & \text{if} \quad 4 \le \mu \le -1+\zeta; \\ (1+\zeta,1+\zeta,-2+\zeta,4,0), & \text{if} \quad \mu = \zeta; \\ (2+\zeta,2+\zeta,-1+\zeta,2,0), & \text{if} \quad \mu = 1+\zeta; \\ (2+\zeta,2+\zeta,\zeta,2,0), & \text{if} \quad \mu = 2+\zeta; \\ (1+\zeta,1+\zeta,1+\zeta,4,0), & \text{if} \quad \mu = 3+\zeta; \\ (\zeta,\zeta,2+\zeta,6,0), & \text{if} \quad \mu = \zeta+4; \\ (2\zeta-\mu+4,2\zeta-\mu+4,2\zeta-\mu+7,-2+\mu+\zeta,0), & \text{if} \quad \zeta+5 \le \mu \le 21+\zeta. \end{cases} \tag{17}$$

Every vertex of$\text{ext}^{(1)}(G)$gains the unique representation with respect to the resolving setχ; hence,

$$pd\left(\text{ext}^{(1)}(G)\right) \le 5. \tag{18}$$

□

Theorem 7. *Let $ext^{(1)}(G)$ denote the Pendent graph. Then, $pd(ext^{(1)}(G)) \le 4$ for every $n \equiv 2 \pmod 4 n \ge 18$.*

Proof. Let $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ be a resolving partition set where $\chi_1 = \{u_1\}, \chi_2 = \{u_4\}, \chi_3 = \{v_{2\zeta+4}\}$, and $\chi_4 = \{\forall V(G) | \notin \{\chi_1, \chi_2, \chi_3\}\}$. Representations of all vertices

$V(\text{ext}^1(G))/\{\chi_1 \bigcup \chi_2 \bigcup \chi_3\}$ regarding the resolving partition set χ.

Case 1. When even vertices

$$r\left(w_{2\mu}|\chi\right) = \begin{cases} (3,3,3+\zeta,0), & \text{if } \mu = 1; \\ (2+\mu,\mu,\zeta-\mu+5,0), & \text{if } 2 \le \mu \le \zeta; \\ (3+\zeta,1+\zeta,3,0), & \text{if } \mu = 1+\zeta; \\ (2+\zeta,2+\zeta,1,0), & \text{if } \mu = 2+\zeta; \\ (1+\zeta,1+\zeta,3,0), & \text{if } \mu = 3+\zeta; \\ (2\zeta-\mu+4,2\zeta-\mu+5,-1+\mu+\zeta,0), & \text{if } \zeta+4 \le \mu \le 21+\zeta. \end{cases} \tag{19}$$

Case 2. When odd vertices

$$r\left(w_{2-1+\mu}|\chi\right) = \begin{cases} (2,4,3+\zeta,0), & \text{if } \mu = 1; \\ (3,3,\zeta+4,0), & \text{if } \mu = 2; \\ (1+\mu,\mu,\zeta-\mu+6,0), & \text{if } 3 \le \mu \le \zeta; \\ (2+\zeta,1+\zeta,4,0), & \text{if } \mu = 1+\zeta; \\ (2+\zeta,2+\zeta,2,0), & \text{if } \mu = 2+\zeta; \\ (1+\zeta,3+\zeta,2,0), & \text{if } \mu = 3+\zeta; \\ (\zeta,2+\zeta,4,0), & \text{if } \mu = \zeta+4; \\ (2\zeta-\mu+4,2\zeta-\mu+6,-1+\mu+\zeta,0), & \text{if } \zeta+5 \le \mu \le 2\zeta. \end{cases} \tag{20}$$

Every vertex of$\text{ext}^{(1)}(G)$gains the unique representation with respect to the resolving setχ; hence,

$$pd\left(\text{ext}^{(1)}(G)\right) \le 4. \tag{21}$$

□

Theorem 8. *Let $ext^{(1)}(G)$ denote the Pendent graph. Then, $pd(ext^{(1)}(G)) \le 4$ for every $n \equiv 3 \pmod 4 n \ge 11$.*

Proof. Let $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ be a resolving partition set where $\chi_1 = \{u_1\}, \chi_2 = \{u_4\}, \chi_3 = \{v_{23+\zeta}\}$, and $\chi_4 = \{\forall V(G) | \notin \{\chi_1, \chi_2, \chi_3\}\}$. Representations of all vertices $V(\text{ext}^1(G))/\{\chi_1 \bigcup \chi_2 \bigcup \chi_3\}$ regarding the resolving partition set χ.

Case 1. When even vertices,

$$r\left(w_{2\mu}|\chi\right) = \begin{cases} (3,3,\zeta+4,0), & \text{if } \mu = 1; \\ (2+\mu,\mu,\zeta-\mu+5,0), & \text{if } 2 \le \mu \le -1+\zeta; \\ (2+\zeta,\zeta,4,0), & \text{if } \mu = \zeta; \\ (3+\zeta,1+\zeta,2,0), & \text{if } \mu = 1+\zeta; \\ (2+\zeta,2+\zeta,2,0), & \text{if } \mu = 2+\zeta; \\ (1+\zeta,3+\zeta,4,0), & \text{if } \mu = 3+\zeta; \\ (2\zeta-\mu+4,2\zeta-\mu+6,-2+\mu+\zeta,0), & \text{if } \zeta+4 \le \mu \le 21+\zeta. \end{cases} \tag{22}$$

Case 2. When odd vertices,

$$r\left(w_{2-1+\mu}|\chi\right)=\begin{cases}(2,4,\zeta+4,0), & \text{if } \mu=1;\\ (3,3,3+\zeta,0), & \text{if } \mu=2;\\ (1+\mu,\mu,\zeta-\mu+5,0), & \text{if } 3\le\mu\le\zeta;\\ (2+\zeta,1+\zeta,3,0), & \text{if } \mu=1+\zeta;\\ (3+\zeta,2+\zeta,1,0), & \text{if } \mu=2+\zeta;\\ (2+\zeta,3+\zeta,3,0), & \text{if } \mu=3+\zeta;\\ (2\zeta-\mu+5,2\zeta-\mu+6,-1+\mu+\zeta,0), & \text{if } \zeta+4\le\mu\le 22+\zeta.\end{cases} \tag{23}$$

Every vertex of$\text{ext}^{(1)}(G)$gains the unique representation with respect to the resolving setχ; hence,

$$pd\left(\text{ext}^{(1)}(G)\right)\le 4. \tag{24}$$

□

5. Conclusion and Discussion

This article investigates the generalised Harary $(H_{x,y})$ graph's partition dimension's precise boundaries, and we concluded that

$$pd\left(H_{x,y}\right)\le 4. \tag{25}$$

For the Barycentric subdivision of Cayley Graphs $S(\text{Cay}(Z_n\oplus Z_2))_n$,pt

$$pd\left(\text{Cay}\left(Z_n\oplus Z_2\right)\right)\le 4. \tag{26}$$

Furthermore, we also discussed the graph named as pendent Graph $\text{ext}^{(1)}(G)$, and this also gave the same bounds of partition dimension which is

$$pd\left(\text{ext}^{(1)}(G)\right)\le 4. \tag{27}$$

We proved all the chosen graphs have upper bounds of partition dimension 4.

Data Availability

There are no data associated with this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] G. Chartrand, E. Salehi, and P. Zhang, "The partition dimension of a graph," *Aequationes Mathematicae*, vol. 59, no. 1, pp. 45–54, 2000.

[2] P. J. Slater, "Leaves of trees," *Congress Numerentium*, vol. 14, pp. 549–559, 1975.

[3] F. Harary and R.A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, pp. 191–195, 1976.

[4] R. Alfarisi and D. Darmaji, "On the star partition dimension of comb product of cycle and complete graph," *Journal of Physics: Conference Series*, vol. 855, Article ID 12005, 2017.

[5] C. Grigorious, S. Stephen, B. Rajan, M. Miller, and A. William, "On the partition dimension of a class of circulant graphs," *Information Processing Letters*, vol. 114, no. 7, pp. 353–356, 2014.

[6] I. G. Yero and J. A. Rodruez-Velàzquez, "A note on the partition dimension of Cartesian product graphs," *Applied Mathematics and Computation*, vol. 217, 2018.

[7] Y. M. Chu, M. F. Nadeem, M. Azeem, and M. K. Siddiqui, "On sharp bounds on partition dimension of convex polytopes," *IEEE Access*, vol. 8, Article ID 224781, 2020.

[8] A. Khalil, S. K. S. Husain, and M. F. Nadeem, "On bounded partition dimension of different families of convex polytopes with pendant edges," *AIMS Mathematics*, vol. 7, no. 3, pp. 4405–4415, 2021.

[9] Z. Beerliova, F. Eberhard, T. Erlebach et al., "Network discovery and verification," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 12, pp. 2168–2181, 2006.

[10] J. Caceres, C. Hernando, M. Mora et al., "On the Metric Dimension of Cartesian Products of Graphs," *SIAM Journal on Discrete Mathematics*, vol. 21, no. 2, pp. 423–441, 2007.

[11] G. Chartrand, L. Eroh, M.A. Johnson, and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph," *Discrete Applied Mathematics*, vol. 105, no. 1-3, pp. 99–113, 2000.

[12] V. Chvatal, "Mastermind," *Combinatorica*, vol. 3, no. 3-4, pp. 325–329, 1983.

[13] M. A. Johnson, "Structure-activity maps for visualizing the graph variables arising in drug design," *Journal of Biopharmaceutical Statistics*, vol. 3, no. 2, pp. 203–236, 1993.

[14] M. A. Johnson, "Browsable structure-activity datasets," *Advances in Molecular Similarity*, vol. 2, pp. 153–170, 1998.

[15] S. Khuller, B. Raghavachari, and A. Rosenfeld, "Landmarks in graphs," *Discrete Applied Mathematics*, vol. 70, no. 3, pp. 217–229, 1996.

[16] R. A. Melter and I. Tomescu, "Metric bases in digital geometry," *Computer Vision Graphics and Image Processing*, vol. 25, no. 1, pp. 113–121, 1984.
[17] P. J. Slater, "Dominating and reference sets in graphs," *Journal of Mathematical Physics*, vol. 22, pp. 445–455, 1998.
[18] J. A. Rodríguez-Velázquez, I. González Yero, and M. Lemanska, "On the partition dimension of trees," *Discrete Applied Mathematics*, vol. 166, pp. 204–209, 2014.
[19] J. A. Rodruez-Velàzquez, I. G. Yero, and H. Fernau, "On the partition dimension of unicyclic graphs," *Bulletin of Mathematical Society of Science and Mathematis Roumanie*, vol. 57, pp. 381–391, 2014.
[20] A. A. Khabyah, M. K. Jamil, A. N. A. Koam, A. Javed, and M. Azeem, "Partition dimension of COVID antiviral drug structures," *Mathematical Biosciences and Engineering*, vol. 19, 2022.
[21] H. Alshehri, A. Ahmad, Y. Alqahtani, and M. Azeem, "Vertex metric-based dimension of generalized perimantanes diamondoid structure," *IEEE Access*, vol. 10, 2022.
[22] A. N. A. Koam, A. Ahmad, M. Azeem, and M. F. Nadeem, "Bounds on the Partition Dimension of One Pentagonal Carbon Nanocone Structure," *Arabian Journal of Chemistry*, vol. 15, no. 7, Article ID 103923, 2022.
[23] M. Azeem, M. Imran, and M. F. Nadeem, "Sharp bounds on partition dimension of hexagonal Möbius ladder," *Journal of King Saud University-Science*, vol. 34, Dec. 2021.
[24] H. Raza, J. B. Liu, M. Azeem, and M. F Nadeem, "Partition Dimension of Generalized Petersen Graph," *Complexity*, vol. 2021, Article ID 5592476, 14 pages, 2021.
[25] A. Shabbir and M. Azeem, "On the partition dimension of trihexagonal alpha-boron nanotube," *IEEE Access*, vol. 9, Article ID 55644, 2021.
[26] M. Azeem and M. F. Nadeem, "Metric-based resolvability of polycyclic aromatic hydrocarbons," *The European Physical Journal Plus*, vol. 136, no. 4, p. 395, 2021.
[27] J. B. Liu, M. F. Nadeem, and M. Azeem, "Bounds on the partition dimension of convex polytopes," *Combinatorial Chemistry & High Throughput Screening*, vol. 25, no. 3, pp. 547–553, 2022.
[28] C. Wei, M. F. Nadeem, H. M. Afzal Siddiqui, M. Azeem, J. B. Liu, and A Khalil, "On partition dimension of some cycle related graphs," *Mathematical Problems in Engineering*, vol. 2021, pp. 1–8, 2021.
[29] M. Azeem, M. F. Nadeem, A. Khalil, and A. Ahmad, "On the bounded partition dimension of some classes of convex polytopes," *Journal of Discrete Mathematical Sciences and Cryptography*, pp. 1–14, 2021.