Hindawi
Journal of Mathematics
Volume 2022, Article ID 9546205, 14 pages
https://doi.org/10.1155/2022/9546205

Research Article

Pattern Formation of a Bubbly Fluid Mixture under the Effect of Thermodynamics via Kudryashov-Sinelshchikov Model

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Received 30 April 2022; Revised 9 June 2022; Accepted 23 July 2022; Published 16 September 2022

Academic Editor: Francisco J. Garcia Pacheco

In this paper, new explicit wave solutions via liquid-gas bubbles are obtained for the fractional Kudryashov-Sinelshchikov (KS) equation under thermodynamic assumptions. A new fractional definition is applied to get these solutions that are utilized to represent the phenomenon of pressure waves under thermodynamic conditions. Two analytical techniques are used to explore the model which is sinh-Gorden equation expansion and Riccati-Bernoulli Sub-ODE methods. These approaches provide complex hyperbolic, hyperbolic, complex trigonometric, and trigonometric solutions for the fractional KS equation, particularly singular, combined singular, dark, bright, combined dark-bright, and other soliton solutions. Furthermore, acquired results are illustrated by 3D graphs for suitable parametric values that highlight the physical importance and dynamical behaviors of the equation. It is also demonstrated that the purposed approaches are powerful strategies for developing exact traveling wave solutions for a wide range of problems found in mathematical sciences.

1. Introduction

In recent years, exact solutions to nonlinear partial differential equations (NLPDEs) have received the attention of many researchers and they have used a variety of approaches. Some of them are listed below. Wazwaz used the tanh approach to develop periodic soliton solutions to the Dodd-Bullough-Mikhailov and Tzitzeica-Dodd-Bullough equations. Sun et al. used the Hirota bilinear approach to find the M-lump solutions B-Kadomtsev-Petviashvili equation. Durur et al. applied the KdV6 subequation approach. Hosseini et al. used the linear superposition technique to investigate soliton solutions of the Hirota-Satsuma-Ito problem. Raza et al. used the Painleve method to solve a nonlinear Kudryashov problem. The modified simple equation approach was used by Akbulut et al. to solve the fifth-order KdV equation. Ma et al. used the first integral approach to extract dark and bright solitons from the Hirota-Maccari system. Kumar developed the generalized exponential rational function approach to discover traveling waves, kink waves, rational function, lump-type solitons, multi-solitons, hyperbolic function, and trigonometric solutions. Using the simplest equation approach, Inc et al. discovered accurate analytic solutions for the $(2+1)$-dimensional Ito problem [1–10].

Many natural phenomena are determined from NLPDEs of integer order. These models are used in numerous disciplines of research such as bio-sciences, engineering, and economics [11–13]. However, these integer-order models are insufficient without the nonlocal property. To overcome this problem, fractional nonlinear partial differential equations (FNPDEs) are introduced. Studying these models reveals the unique aspects while employing computational and numerical approaches. Fractional operators are employed to transform fractional expressions to nonlinear ordinary differential equations (NLODEs) of integer order such as Laplacian, Caputo-Fabrizio, Yang-Abdel-Aty-Cattani, Weyl, and Riesz derivatives are used to see the memory effects in a number of physical disciplines [14–18].

One of the most significant research fields in ocean engineering and geo physics is the solitons theory. Several soliton solutions have been determined in the last two decades utilizing numerical and analytical techniques. Researchers are looking for new and general solitons from various NLPDEs, such as the generalized Burgers, Broer-Kaup-Kupershmidt, modified Kawahara, Dullin-Gottwald-Holm, modified Zakharov-Kuznetsov, fractional wazwaz-benjamin-bona-mahony, Camassa-Holm, $(3+1)$-dimensional extended Jimbo-Miva equation, and nonlinear Schrödinger dynamical models. Furthermore, they have recently proposed and implemented integration strategies to analyze NLPDEs to find the soliton solutions [19–27].

In recent decades, utilizing fractional derivatives to investigate soliton solutions has become a fascinating research topic. As a subcategory of classical calculus, fractional calculus (FC) has been shown to be a convenient field in the study of a wide range of nonlinear processes. Fractional differential equations (FDEs) are the best methods of identifying nonlinear systems. FC can also be used to develop physical models that are based on both the time moment and the time history [28]. The fractional-order strategies can be applied to those models which have more degrees of freedom. Furthermore, integer-order equations have several limitations that are resolved by FDEs. A fractional operator acts as a classical derivative when it approaches unity while the standard calculus includes the function composition, chain rule, linearity, quotient, and product rules. Solitary wave solutions to FDEs have been investigated by many researchers.

A classical nonlinear media can be thought of as a combination of liquid and gas bubbles of the same size. The study of pressure wave propagation in a liquid with gas bubbles is a challenging subject in mathematics and physics. In such mixes, there exist solitary and periodic waves that can be explained using nonlinear partial differential equations. In 2010, Kudryashov and Sinelshchikov developed a more generic nonlinear partial differential equation which is Kudryashov-Sinelshchikov model to explain pressure waves in a combination of liquid and gas bubbles by taking into account liquid viscosity and heat transport [29, 30]. Finding accurate solutions to nonlinear evolution equations originating in mathematical physics has been more significant in the study of nonlinear physical events in recent decades. Many mathematicians and physicists are interested in traveling wave solutions, which are important solutions to nonlinear evolution equations. Finding novel solutions is important because they can give additional information for understanding physical processes, whether they are exact or numerical approximation solutions.

In this work, the Atangana-Baleanu (AB) fractional derivative is used to locate solitary wave solutions of NLPDE, namely, the fractional Kudryashov-Sinelshikov (FKS) model. We will use the extended sinh-Gordon equation expansion method (EShGEEM) and the Riccati-Bernoulli (R-B) Sub-ODE method to find some new complex hyperbolic and complex trigonometric function solutions, particularly dark, bright, combined dark-bright, singular, combined singular soliton, and other soliton solutions from the FKS model [31–34]. This paper is organized as follows: the mathematical analysis of the considered model is presented in Section 2. Then, description of utilized techniques, namely, the EShGEE and the *R-B* sub-ODE techniques is given in Section 3. The extraction of soliton solutions for the proposed model is given in Section 4. Section 5 gives graphical illustrations of the obtained results. Finally, Section 6 gives the conclusion of the complete work.

2. Governing Model

The nonlinear fractional Kudryashov-Sinelshikov (FKS) equation [35] is expressed as

$$\begin{aligned} &D_x^{\xi}\Theta_t + \lambda_1\Theta\Theta_x + \Theta_{xxx} + \alpha_1\left(\Theta\Theta_{xx}\right)_x - \mu_1\Theta_x\Theta_{xx} - \beta_1\Theta_{xx} \\ &\quad - \sigma_1\left(\Theta\Theta_x\right)_x = 0, \end{aligned} \tag{1}$$

where the dynamical behavior of nonlinear wave processes in a liquid containing gas bubbles is described by the function $\Theta = \Theta(x,t)$ and $\lambda_1, \alpha_1, \mu_1, \beta_1$, and σ_1 are constants with $\xi \in [0,1]$. This equation was presented by Kudryashov and Sinelshchikov to describe the wave dynamics in liquids. This equation is also named as Korteweg-de Vries (KdV) and KdV-Burger models under assumptions given as follows:

(i) For $\mu_1 = \alpha_1 = \sigma_1 = \beta_1 = 0$, equation (1) matches the well-known KdV equation

(ii) For $\alpha_1 = \mu_1 = \sigma_1 = 0$, equation (1) matches the well-known KdV Burgers equation

(iii) For $\lambda_1 = \alpha_1 = 1, \beta_1 = \sigma_1 = 0$, equation (1) matches the generalized KdV equation

2.1. $\mathscr{ABR}$ Fractional Derivative. The $\mathscr{ABR}$ fractional operator is given by

$$(\mathscr{ABR})D_{b+}^{\xi}S(t) = \frac{\mathscr{B}(\xi)}{1-\xi}\frac{\mathrm{d}}{\mathrm{d}t}\int_b^t S(\xi)G_{\xi}\left(\frac{-\xi(t-\xi)^{\xi}}{1-\xi}\right)\mathrm{d}x, \tag{2}$$

where the defining formula for the Mittage-Leffler function G_{ξ} is given as follows:

$$G_{\xi}\left(\frac{-\xi(t-\xi)^{\xi}}{1-\xi}\right) = \sum_{m=0}^{\infty}\frac{(-\xi/(1-\xi))^m(t-\xi)^{\xi m}}{\Gamma(\xi m+1)}, \tag{3}$$

where the normalization function is $\mathscr{B}(\xi)$, $\mathscr{B}(0) = 1 = \mathscr{B}(1)$, and $\mathscr{B}(\xi) = (1 - \xi + (\xi/(\Gamma(\xi))))$ for $\xi \in (0, 1)$. Thus,

$$(\mathscr{A}\mathscr{B}\mathscr{R})D_{b+}^{\xi}S(x) = \frac{\mathscr{B}(\xi)}{1-\xi}\sum_{m=0}^{\infty}\left(\frac{-\xi}{1-\xi}\right)^{\xi}(\mathscr{R}\mathscr{L})I_{b+}^{\xi m}(Sx). \tag{4}$$

As a result,

$$\Theta(x,t) = \Theta(\Psi),$$
$$\Psi = x + \frac{w(1-\xi t)t^{-\xi m}}{(\mathscr{B})(\xi)\sum_{m=0}^{\infty}(-\xi/1-\xi)^m\Gamma(1-\xi m)}, \tag{5}$$

where w is an unknown constant. This transformation turns equation (1) to an ODE. Once acquired ODE is integrated using constant of integration to be zero, the result is

$$w\Theta + \frac{\lambda_1}{2}\Theta^2 + \Theta'' - \alpha_1\Theta\Theta'' - \frac{\mu_1}{2}\Theta'^2 - \beta_1\Theta' - \sigma_1\Theta\Theta' = 0. \tag{6}$$

Balancing the higher-order nonlinear term Θ^2 and dispersive term Θ'', that is, $2M = M + 2$, gives $M = 2$ of the above equation.

3. Description of Applied Methods

In this section, we will explain the methodologies extended sinh-Gordon equation expansion and Riccati-Bernoulli Sub-ODE to find soliton solution of the governing model.

3.1. The Algorithm of the EShGEEM. Consider the following NLPDE:

$$F_1\left(\Theta,\Theta_x,\Theta_y,\Theta_z,\mathscr{D}_t^{\xi}\Theta,\Theta_{xx},\Theta_{yy},\Theta_{zz},\Theta_{xy},\mathscr{D}_t^{2\xi}\Theta,\ldots\right) = 0, \tag{7}$$

where F_1 is the polynomial in Θ with its partial derivatives. By applying the transformation given by equation (5), equation (7) is turned into ODE.

$$G_1(\Theta,\Theta',\Theta'',\Theta''',\ldots), \tag{8}$$

where prime shows derivative with respect to Ψ. The methodology of EShGEEM to find the solution of ODE is given as follows.

Consider the following sinh-Gordon equation:

$$\Theta_{xt} = c_1 \sin h(\Theta), \tag{9}$$

where c_1 is the speed and Θ is the amplitude of traveling wave. Using the transformation $\Theta(x,t) = \Theta(\Psi)$, given in equation (5), reduces the above equation to the following ODE:

$$\Theta'' = \frac{c_1}{w}\sin h(\Theta). \tag{10}$$

Multiplying Θ' on both sides of the above equation and integrating it once give

$$\left[\left(\frac{\Theta}{2}\right)'\right]^2 = \frac{c_1}{w}\sin h^2\left(\frac{\Theta}{2}\right) + q, \tag{11}$$

where q is the constant of integration. By inserting $(\Theta/2) = \nu(\xi)$ and $(c_1/w) = p$ in the above equation, we obtain

$$\nu' = \sqrt{q + p\sin h^2(\nu)}. \tag{12}$$

The above equation has the following sets of solutions for different parametric values of q and p.

Case 1. When $q = 0$ and $p = 0$, equation (12) takes the form as

$$\nu' = \sin h(\nu). \tag{13}$$

When equation (13) is simplified, it gives the following results:

$$\begin{aligned}\sin h(\nu) &= \pm\text{sec } h(\Psi),\\ \cos h(\nu) &= -\tan h(\Psi),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\sin h(\nu) &= \pm\csc h(\Psi),\\ \cos h(\nu) &= -\cot h(\Psi).\end{aligned} \tag{15}$$

Case 2. When $q = 1$ and $p = 1$, equation (12) becomes

$$\nu' = \cos h(\nu), \tag{16}$$

and when equation (13) is simplified, we get the following results:

$$\begin{aligned}\sin h(\nu) &= \tan(\Psi),\\ \cos h(\nu) &= \pm\sec(\Psi),\end{aligned} \tag{17}$$

and

$$\begin{aligned}\sin h(\nu) &= -\cot(\Psi),\\ \cos h(\nu) &= \pm\csc(\Psi).\end{aligned} \tag{18}$$

Now, to find the solution of equation (6), we use the following expressions:

$$\Theta(\nu) = \sum_{i=1}^{M}\cos h^{i-1}(\nu)\left[E_i \sin h(\nu) + P_i\cos h(\nu)\right] + P_0. \tag{19}$$

It is considered that the solution $\Theta(\nu)$ of the above equation as well as equations (13)–(15) can be expressed as follows:

$$\Theta(\Psi) = \sum_{i=1}^{M}(-\tan h(\Psi))^{i-1}\left[\pm\iota E_i \text{sec } h(\Psi) - P_i\tan h(\Psi)\right] + P_0, \tag{20}$$

and

$$\Theta(\Psi)=\sum_{i=1}^{M}(-\cot h(\Psi))^{i-1}\left[\pm E_i \csc h(\Psi)-P_i\cot h(\Psi)\right]+P_0. \tag{21}$$

Similarly, it is supposed that the solution $\Theta(\Psi)$ of equation (6) along with equations (13)–(15) can be expressed as follows:

$$\Theta(\Psi)=\sum_{i=1}^{M}(\pm\sec(\Psi))^{i-1}\left[E_i\tan(\Psi)\pm P_i\sec(\Psi)\right]+P_0, \tag{22}$$

and

$$\Theta(\Psi)=\sum_{i=1}^{M}(\pm\csc(\Psi))^{i-1}\left[-E_i\cot(\Psi)\pm P_i\csc(\Psi)\right]+P_0. \tag{23}$$

Calculate the value of M using the homogeneous balancing rule which is done by balancing the higher order nonlinear and dispersive term. Then, substituting this value in equations (13) and (19) yields a system of equations. By equating the coefficients of $\sin h^j(v)\cos h^k(v)$ to zero and solving the obtained system, we get the values of P_i, E_i, and w. Then, put these values in equations (20) and (21) to obtain equation (1) solutions (for example, Case 1). Similarly, we can apply the same steps in Case 2 to obtain the solutions.

3.2. The Fundamental Aspects of the R-B Sub-ODE Method.

This section describes the *R-B* sub-ODE methodology.

Step 1. Consider the NLPDE along with the transformation equation and ODE given in equations (5), (7), and (8), respectively.

Step 2. We assume that equation (8) has the solution of the form given as follows:

$$\Theta' = a_1\Theta^{2-m}+b_1\Theta+c_1\Theta^m, \tag{24}$$

where a_1, b_1, c_1, and m are unknowns. From equation (24), we get

$$\begin{aligned}\Theta'' &= a_1b_1(3-m)\Theta^{2-m}+a_1^2(2-m)\Theta^{3-2m}+mc_1^2\Theta^{2m-1}\\&\quad+b_1c_1(m+1)\Theta^m+\left(2a_1c_1+b_1^2\right)\Theta,\\ \Theta''' &=\left(a_1b_1(3-m)(2-m)\Theta^{1-m}+a_1^2(3-2m)(2-m)\Theta^{2-2m}\right)\\&\quad\left(a_1\Theta^{2-m}+\dot{b}_1\Theta+c_1\Theta^m\right)\\&\quad+\left(m(2m-1)c_1^2\Theta^{2m-2}+b_1c_1m(m+1)\Theta^{m-1}+2a_1c_1+b_1^2\right)\\&\quad\left(a_1\Theta^{2-m}+b_1\Theta+c_1\Theta^m\right).\end{aligned} \tag{25}$$

Remark 1. Equation (24) is known as the R-B equation. At $a_1\neq 0$, $c_1=0$, and m not equal to zero, it is referred as Bernoulli equation. At $a_1c_1\neq 0$ and $m=0$, this model is regarded as Riccati equation.

The solutions for *R-B* equation (24) are explained as follows.

Case 3. When $m=1$, the solution is

$$\Theta(\Psi)=\Omega\exp^{(a_1+b_1+c_1)\Psi}. \tag{26}$$

Case 4. When $m\neq 1$, $b_1=0$, and $c_1=0$, the solution is

$$\Theta(\Psi)=\left(a_1(m-1)(\Psi+\Omega)\right)^{1/(m-1)}. \tag{27}$$

Case 5. When $m\neq 1$, $b_1\neq 0$, and $c_1=0$, the solution is

$$\Theta(\Psi)=\left(\frac{a_1}{b_1}+\Omega\exp^{(b_1(m-1))\Psi}\right)^{1/(m-1)}. \tag{28}$$

Case 6. When $m\neq 1$, $a_1\neq 0$, and $b_1^2-4a_1c_1<0$, the solution is

$$\Theta(\Psi)=\left(\frac{-b_1}{2a_1}+\frac{\sqrt{-b_1^2+4a_1c_1}}{2a_1}\cdot\tan\left(\frac{(1-m)\sqrt{-b_1^2+4a_1c_1}(\Omega+\Psi)}{2}\right)\right)^{1/(m-1)}, \tag{29}$$

and

$$\Theta(\Psi)=\left(\frac{-b_1}{2a_1}-\frac{\sqrt{-b_1^2+4a_1c_1}}{2a_1}\cdot\cot\left(\frac{(1-m)\sqrt{-b_1^2+4a_1c_1}(\Omega+\Psi)}{2}\right)\right)^{1/(m-1)}. \tag{30}$$

Case 7. When $m\neq 1$, $a_1\neq 0$, and $b_1^2-4a_1c_1>0$, the solution is

$$\Theta(\Psi)=\left(\frac{-b_1}{2a_1}-\frac{\sqrt{b_1^2-4a_1c_1}}{2a_1}\cdot\coth\left(\frac{(1-m)\sqrt{b_1^2-4a_1c_1}(\Omega+\Psi)}{2}\right)\right)^{1/(m-1)} \tag{31}$$

and

$$\Theta(\Psi) = \left(\frac{-b_1}{2a_1} + \frac{\sqrt{b_1^2 - 4a_1c_1}}{2a_1} \cdot \tanh\left(\frac{(1-m)\sqrt{b_1^2 - 4a_1c_1}\,(\Omega + \Psi)}{2}\right)\right)^{1/(m-1)}. \tag{32}$$

Case 8. When $m \neq 1$, $a_1 \neq 0$, and $b_1^2 - 4a_1c_1 = 0$, the solution is

$$\Theta(\Psi) = \left(\frac{-b_1}{2a_1} + \frac{1}{a_1(m-1)(\Omega + \Psi)}\right)^{1/(m-1)}. \tag{33}$$

Here, Ω is a constant.

Step 3. The algebraic equations of Θ are obtained by inserting the derivatives in equation (8). m can be calculated using the symmetry of the right-hand elements in equation (24) and setting the greatest power exponents of Θ to equivalence in equation (8). When the coefficients of Θ^j ($j = 1, 2, 3, \ldots$) are compared, algebraic equations of a_1, b_1, c_1, and w are obtained. By solving these equations and inserting m, a_1, b_1, c_1, w, and Ψ into equations (26)–(33), the traveling wave solutions of equation (6) are obtained.

4. Soliton Solutions of Fractional KS Equation

This section will analyze the EShGEE and *R-B* sub-ODE models. The algorithms for the proposed techniques are explained in the following subsections. As a consequence, we will obtain certain solutions, namely, dark, bright, combined dark-bright, singular, combined singular, and other soliton solutions.

4.1. Applying First Method. In this part, a novel form of EShGEEM is used to find innovative soliton solutions of the given model.

4.1.1. Case 1: $v' = \sin h(v)$. With the help of equations (19)–(21), the solution of equation (6) has the form as follows:

$$\Theta(\Psi) = \pm \iota E_1 \sec h(\Psi) - P_1 \tan h(\Psi) + P_0 \mp \iota E_2 \sec h(\Psi) \tan h(\Psi) + P_2 \tan h^2(\Psi), \tag{34}$$

and

$$\Theta(\Psi) = \pm E_1 \csc h(\Psi) - P_1 \cot h(\Psi) + P_0 \mp E_2 \cot h(\Psi). \cdot \csc h(\Psi) + P_2 \cot h^2(\Psi), \tag{35}$$

and so

$$\Theta(v) = E_1 \sin h(v) + P_1 \cos h(v) + P_0 + E_2 \cos h(v) \sin h(v) + P_2 \cos h^2(\Psi), \tag{36}$$

where either E_1 or E_2 and P_1 or P_2 can be zero but both cannot be zero at the same time. By putting equation (36) into (6), we get nonlinear algebraic system that yields the solution sets shown as follows:

Set 1-1:

$$\beta_1 = \pm(w+1), \quad \mu_1 = -4\alpha_1, \quad \sigma_1 = \pm\frac{\alpha_1 w + \lambda_1}{w},$$
$$E_1 = \frac{\pm w}{\lambda_1}, \quad E_2 = 0, \quad P_0 = \frac{-w}{\lambda_1}, \tag{37}$$
$$P_1 = \frac{\pm w}{\lambda_1}, \quad P_2 = 0.$$

The new solitary wave solutions for the FKS equation are obtained by inserting the aforementioned values into equations (34) and (35) which are given as follows:

$$\Theta_{1-1-1}(\Psi) = \frac{-w}{\lambda_1} \pm \frac{\iota w \sec h(\Psi)}{\lambda_1} \mp \frac{w \tan h(\Psi)}{\lambda_1}, \tag{38}$$

and

$$\Theta_{1-1-2}(\Psi) = \frac{-w}{\lambda_1} \pm \frac{w \csc h(\Psi)}{\lambda_1} \mp \frac{w \cot h(\Psi)}{\lambda_1}. \tag{39}$$

Set 1-2:

$$\beta_1 = 0, \quad \lambda_1 = \pm\frac{12w^2}{E_2(3w-1)}, \quad \mu_1 = \mp\frac{18w(w-1)}{E_2(3w-1)},$$
$$\sigma_1 = 0, \quad E_1 = 0, \quad P_0 = \pm\frac{(3w+1)E_2}{6w},$$
$$P_1 = 0, \quad P_2 = \mp E_2, \quad \alpha_1 = \pm\frac{6w(w-1)}{E_2(3w-1)}. \tag{40}$$

By putting equations (34) and (35) into (6), we get nonlinear algebraic system that yields the solution sets shown as follows:

$$\Theta_{1-2-1}(\Psi) = \pm\frac{(3w+1)}{6w} - \iota \sec h(\Psi) \tan h(\Psi) \mp \tan h^2(\Psi), \tag{41}$$

and

$$\Theta_{1-2-2}(\Psi) = \pm\frac{(3w+1)}{6w} \mp \cot h(\Psi) \csc h(\Psi) \mp \cot h^2(\Psi). \tag{42}$$

Set 1–3:

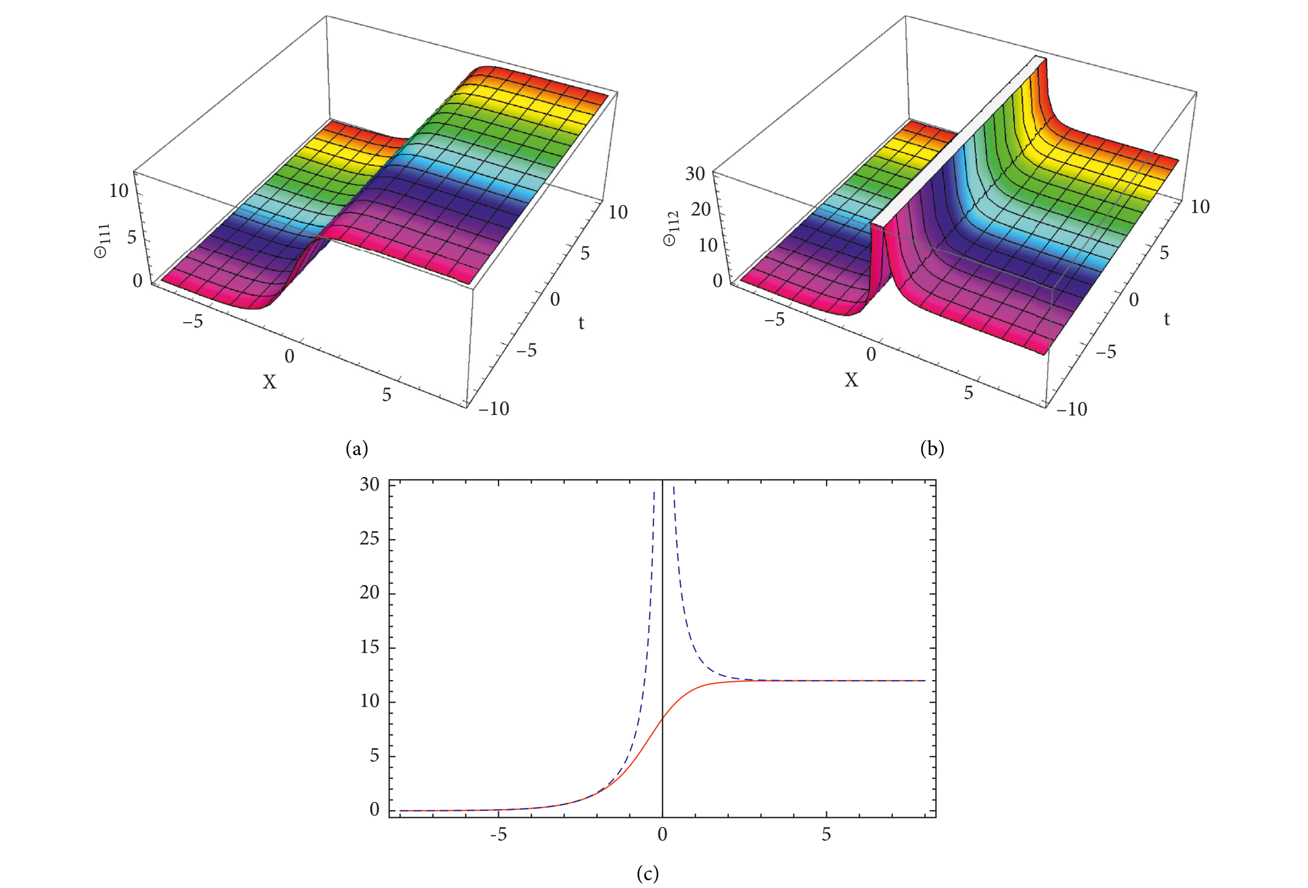


Figure 1: Plots of equations (38) and (39) for $m = 2$, $\xi = 0.3$, $\lambda_1 = 0.5$, and $w = 3$. (a) Θ_{111}, (b) Θ_{112}, (c) Θ_{111}, Θ_{112}.

$$\begin{aligned}
&\beta_1 = \pm(w+1), \quad \lambda_1 = \frac{-2w}{P_0}, \quad \mu_1 = -\frac{w+6}{2P_0},\\
&\sigma_1 = \pm\frac{w+6}{6P_0}, \quad E_1 = \pm\frac{P_0}{2}, P_1 = \pm\frac{P_0}{2},\\
&P_2 = -\frac{P_0}{2}, \quad \alpha_1 = \pm\left(\frac{6+w}{6P_0}\right).
\end{aligned} \tag{43}$$

By putting equations (34) and (35) into (6), we get nonlinear algebraic system that yields the solution sets shown as follows:

$$\begin{aligned}
\Theta_{1-3-1}(\Psi) = 1 &\pm \frac{\imath \sec h(\Psi)}{2} \pm \frac{\imath \sec h(\Psi)\tan h(\Psi)}{2}\\
&\mp \frac{\tan h(\Psi)}{2} - \frac{\tan h^2(\Psi)}{2},
\end{aligned} \tag{44}$$

and

$$\Theta_{1-3-2}(\Psi) = \pm\frac{(3w+1)}{6w} \mp \cot h(\Psi)\csc h(\Psi) \mp \cot h^2(\Psi). \tag{45}$$

Set 1–4:

$$\begin{aligned}
&\beta_1 = \pm(w+1), \quad \lambda_1 = \frac{-w}{E_2(w-1)},\\
&\mu_1 = \frac{3}{E_2(w-1)(w-2)}, \quad \sigma_1 = \mp\frac{1}{(w-2)E_2},\\
&E_1 = \mp E_2(w-1), \quad P_1 = \mp E_2(w-1), \quad P_2 = -E_2,\\
&\alpha_1 = \frac{-1}{E_2(w-1)(w-2)}.
\end{aligned} \tag{46}$$

By inserting equations (34) and (35) into (6), we get nonlinear algebraic system that yields the solution sets shown as follows:

$$\begin{aligned}
\Theta_{1-4-1}(\Psi) = (w-2) &\pm \imath(w-1)\sec h(\Psi)\\
&\mp\imath \sec h(\Psi)\tan h(\Psi) - (1-w)\tan h(\Psi)\\
&+ \tan h^2(\Psi),
\end{aligned} \tag{47}$$

and

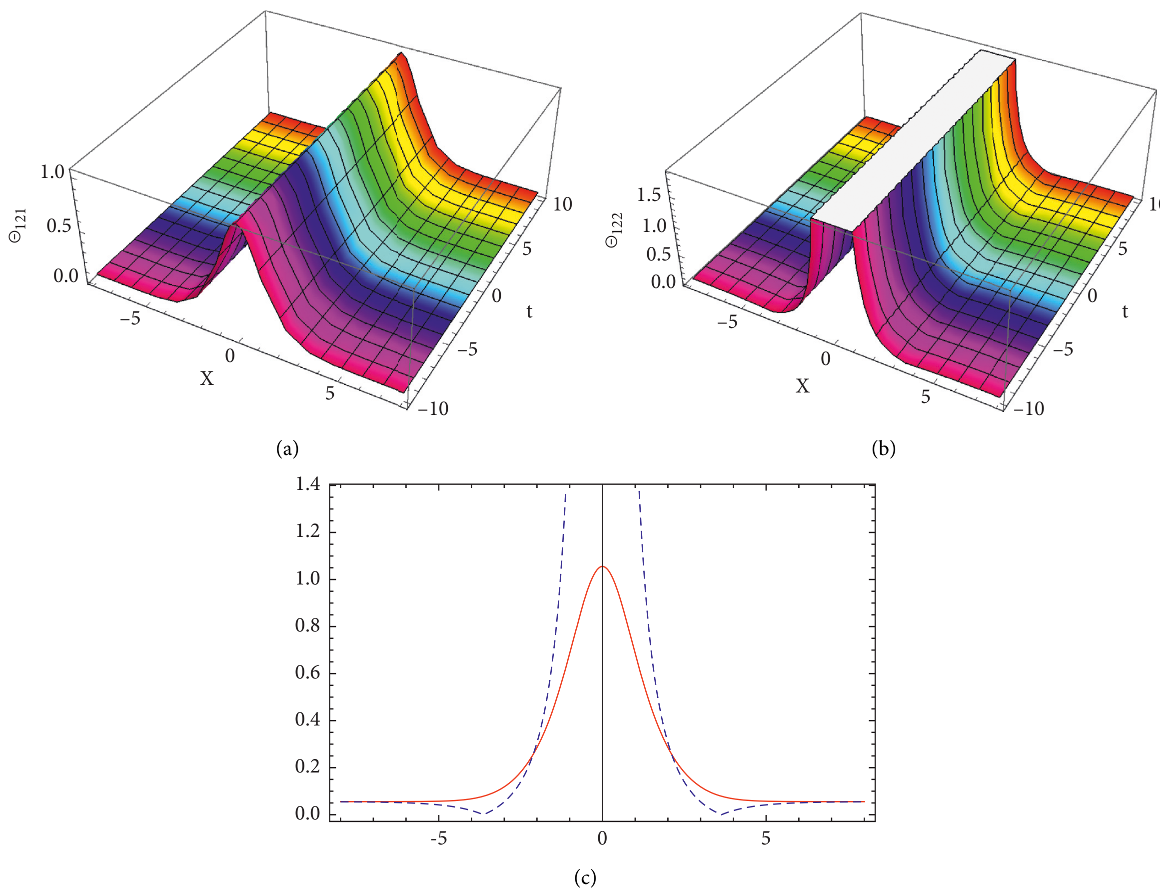


Figure 2: Plots of equations (41) and (42) for $m = 2$, $\xi = 0.3$, $\lambda_1 = 0.5$, and $w = 3$. (a) Θ_{121}, (b) Θ_{122}, (c) Θ_{121}, Θ_{122}.

$$\begin{aligned}\Theta_{1-4-2}(\Psi) = {} & (w-2) \pm (w-1)\operatorname{csc} h(\Psi) \\ & \mp \operatorname{csc} h(\Psi)\cot h(\Psi) - (1-w)\cot h(\Psi) \\ & + \cot h^2(\Psi).\end{aligned} \tag{48}$$

Set 1–5:

$$\begin{aligned} & \beta_1 = \pm(w+1), \quad \lambda_1 = \frac{w}{E_2(w-1)}, \\ & \mu_1 = \frac{-3}{E_2(w-1)(w-2)}, \quad \sigma_1 = \mp\frac{1}{(w-2)E_2}, \\ & E_1 = \mp E_2(w-1), \quad P_1 = \mp E_2(w-1), \quad P_2 = -E_2, \\ & \alpha_1 = \frac{1}{E_2(w-1)(w-2)}. \end{aligned} \tag{49}$$

By using equations (34) and (35) into (6), we get nonlinear algebraic system that yields the solution sets shown as follows:

$$\begin{aligned}\Theta_{1-5-1}(\Psi) = {} & (-w+2) \pm \iota(w-1)\sec h(\Psi) \mp \iota \sec h(\Psi)\tan h(\Psi) \\ & - (-1+w)\tan h(\Psi) - \tan h^2(\Psi),\end{aligned} \tag{50}$$

and

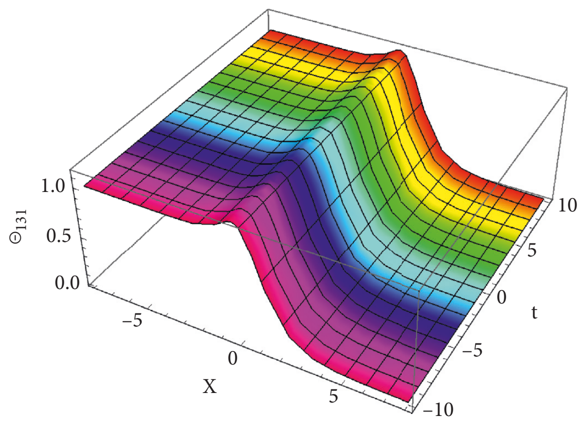
$$\begin{aligned}\Theta_{1-5-2}(\Psi) = {} & -(w-2) \mp (w-1)\operatorname{csc} h(\Psi) \mp \operatorname{csc} h(\Psi)\cot h(\Psi) \\ & \mp(-1+w)\cot h(\Psi) - \cot h^2(\Psi).\end{aligned} \tag{51}$$

4.1.2. Case 2: $v' = \cos h(v)$. With the help of equations (19), (22), and (23), the solution of equation (6) has the form as follows:

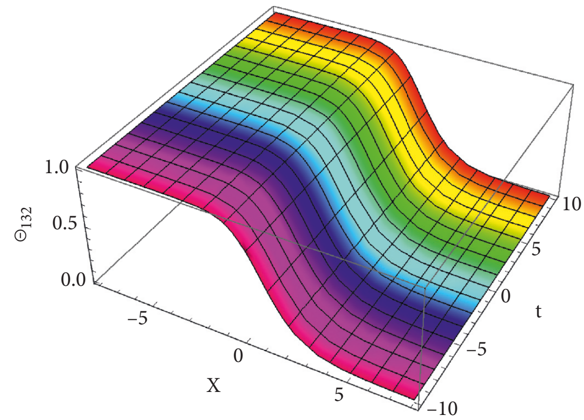
$$\begin{aligned}\Theta(\Psi) = {} & E_1 \tan(\Psi) \pm P_1 \sec(\Psi) + P_0 \pm E_2 \sec(\Psi)\tan(\Psi) \\ & \pm P_2 \sec^2(\Psi),\end{aligned} \tag{52}$$

and

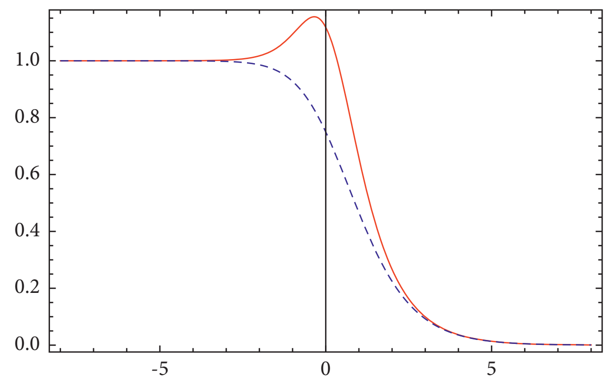
$$\begin{aligned}\Theta(\Psi) = {} & -E_1 \cot(\Psi) \pm P_1 \csc(\Psi) + P_0 \mp E_2 \cot(\Psi)\csc(\Psi) \\ & \pm P_2 \csc^2(\Psi),\end{aligned} \tag{53}$$



(a)



(b)



(c)

FIGURE 3: Plots of equation (44) and (45) for $m = 2$, $\xi = 0.3$, $\lambda_1 = 0.5$, and $w = 0.3$. (a) Θ_{131}, (b) Θ_{132}, (c) Θ_{131}, Θ_{132}.

and so

$$\Theta(\nu) = E_1 \sin h(\nu) + P_1 \cos h(\nu) + P_0 + E_2 \cos h(\nu) \sin h(\nu) + P_2 \cos h^2(\Psi), \tag{54}$$

where either E_1 or E_2 and P_1 or P_2 may be zero but both cannot be zero at the same time. By substituting equation (54) into (6), we arrive at a nonlinear algebraic system which gives following solution sets.
Set 2-1:

$$\beta_1 = 0,\quad \mu_1 = \frac{18w(w+1)}{P_2(3w+1)},\quad \sigma_1 = 0,\quad E_1 = 0,\quad E_2 = P_2,\quad P_0 = \frac{1}{6},\quad P_1 = 0. \tag{55}$$

By plugging the above values in equations (52) and (53), we obtain the new solitary wave solutions for the fractional KS equation as

$$\Theta_{2-1-1}(\Psi) = \frac{-(3w+1)}{6w} \mp \iota \sec(\Psi)\tan(\Psi) \pm \sec^2(\Psi), \tag{56}$$

and

$$\Theta_{2-1-2}(\Psi) = \frac{-(3w+1)}{6w} \pm \csc(\Psi)\cot(\Psi) \pm \csc^2(\Psi). \tag{57}$$

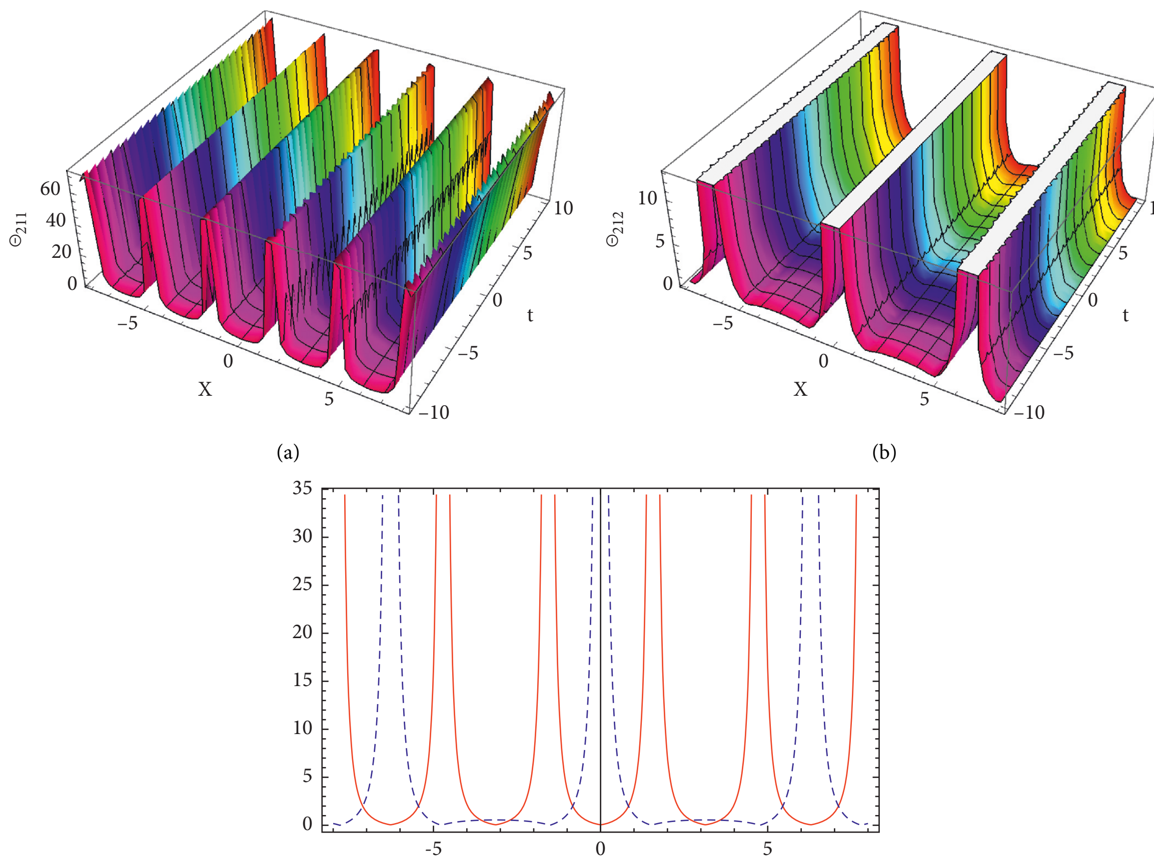
Set 2-2:

$$\beta_1 = \iota(w-1),\quad \lambda_1 = \mp\frac{\iota w}{P_1},\quad \mu_1 = \mp\frac{3\iota}{P_1(w+2)},$$
$$\sigma_1 = \mp\frac{w+1}{(w+2)P_1},\quad E_1 = \pm P_1,\quad P_0 = -\iota P_1, \tag{58}$$
$$P_2 = \mp\frac{\iota P_1}{w+1},\quad E_2 = \frac{\iota P_1}{w+1},\quad \alpha_1 = \pm\frac{1}{P_1(w+2)}.$$

By plugging the above values in equations (52) and (53), we get new solitary wave solutions for the fractional KS model as

$$\Theta_{2-2-1}(\Psi) = \mp\iota\mp\iota \tan(\Psi) \pm \frac{\sec(\Psi)\tan(\Psi)}{w+1} \pm \sec(\Psi) \mp \frac{\iota \sec^2(\Psi)}{w+1}, \tag{59}$$

and



(a) (b) (c)

Figure 4: Plots of equations (56) and (57) for $m = 2$, $\xi = 0.3$, $\lambda_1 = 0.5$, and $w = 0.3$. (a) Θ_{211}, (b) Θ_{212}, (c) Θ_{211}, Θ_{212}.

$$\Theta_{2-2-2}(\Psi) = \mp\iota\mp\iota\cot(\Psi) \pm \frac{\csc(\Psi)\cot(\Psi)}{w+1} \pm \csc(\Psi) \mp \frac{\iota\csc^2(\Psi)}{w+1}. \tag{60}$$

Set 2-3:

$$\beta_1 = \pm(-w+1),\quad \lambda_1 = \frac{-w}{P_0},\quad \mu_1 = -\frac{w-6}{4P_0},$$
$$\sigma_1 = \pm\frac{\iota(w-6)}{12P_0},\quad E_1 = \pm\iota P_0,\quad P_1 = \iota P_0, \tag{61}$$
$$P_2 = P_0,\quad \alpha_1 = \frac{6-w}{12P_0}.$$

By plugging the above values in equations (52) and (53), we obtain following new solitary wave solutions for the fractional KS model:

$$\Theta_{2-3-1}(\Psi) = 1 \pm \sec(\Psi)\tan(\Psi) \pm \iota\tan(\Psi) \pm \iota\sec(\Psi) \pm \sec^2(\Psi), \tag{62}$$

and

$$\Theta_{2-3-2}(\Psi) = 1\mp\csc(\Psi)\cot(\Psi)\mp\iota\cot(\Psi) \pm \iota\csc(\Psi) \pm \csc^2(\Psi). \tag{63}$$

Set 2–4:

$$\beta_1 = \frac{-\iota(w-4)}{2},\quad \lambda_1 = \frac{-w}{2P_2},\quad \mu_1 = \frac{w-24}{32P_2},$$
$$\sigma_1 = \frac{\iota(w-24)}{48P_2},\quad E_2 = 0,$$
$$E_1 = -2\iota P_2,\quad P_1 = 0,\quad P_0 = 2P_2,\quad \alpha_1 = \frac{-(w-24)}{96P_2}. \tag{64}$$

By plugging the above values in equations (52) and (53), we obtain the new solitary wave solutions for the fractional KS model as

$$\Theta_{2-4-1}(\Psi) = 2 + 2\iota\tan(\Psi) \pm \sec^2(\Psi), \tag{65}$$

and

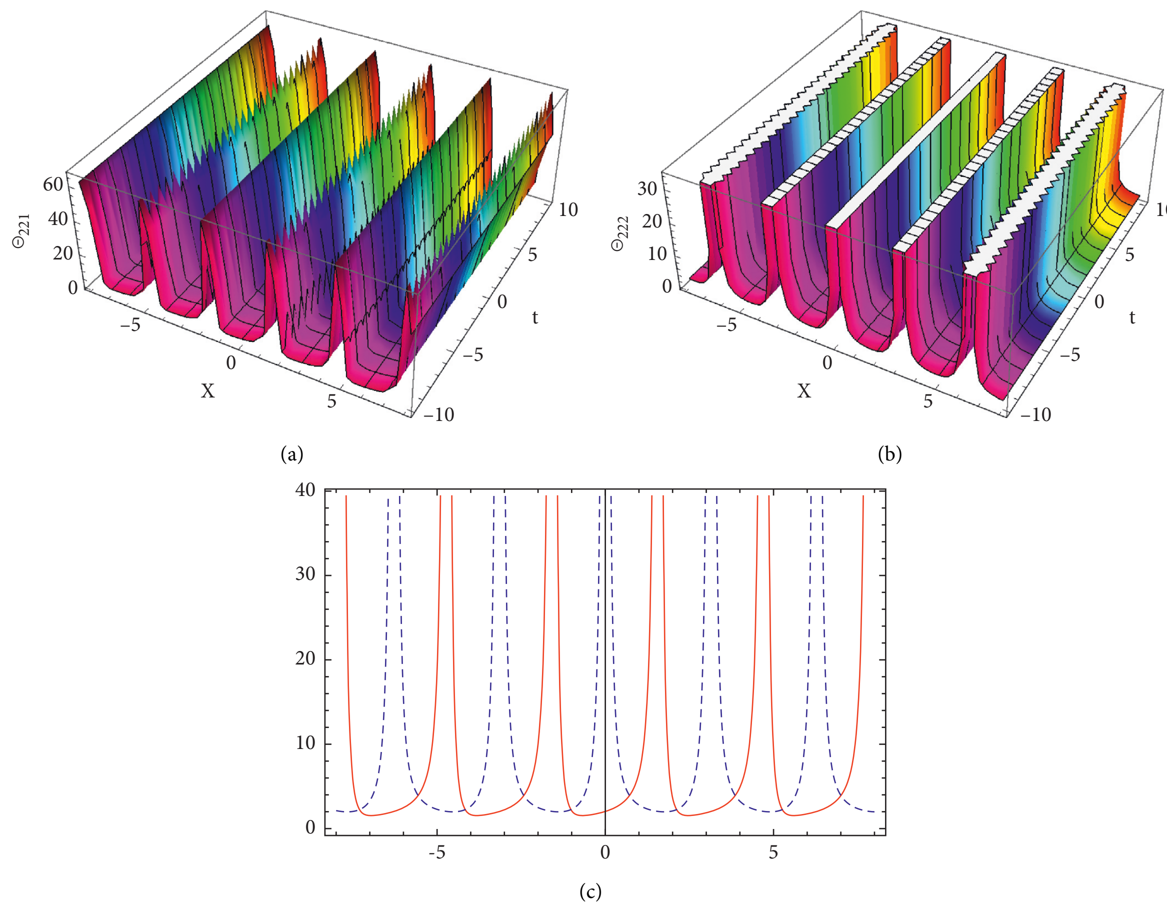


FIGURE 5: Plots of equations (59) and (60) for $m = 2$, $\xi = 0.3$, $\lambda_1 = 0.5$, and $w = 0.3$. (a) Θ_{221}, (b) Θ_{222}, (c) Θ_{221}, Θ_{222}.

$$\Theta_{2-4-2}(\Psi) = 2 - 2\iota \cot(\Psi) \pm \csc^2(\Psi). \tag{66}$$

Set 2–5:

$$\beta_1 = \frac{-\iota(w-4)}{2}, \quad \lambda_1 = \frac{-2w}{P_2(w+4)},$$
$$\mu_1 = \frac{24}{P_2(w+4)(w+8)}, \quad \sigma_1 = \frac{-4\iota}{P_2(w+8)}, \quad E_2 = 0,$$
$$E_1 = \frac{\iota(w+4)P_2}{2}, \quad P_1 = 0, \quad P_0 = \frac{P_2(w+4)}{2},$$
$$\alpha_1 = \frac{8}{(w+4)(w+8)P_2}. \tag{67}$$

By plugging the above values in equations (52) and (53), we get the following new solitary wave solutions for the fractional KS equation:

$$\Theta_{2-5-1}(\Psi) = \frac{(w+4)}{2} + \frac{\iota \tan(\Psi)(w+4)}{2} \pm \sec^2(\Psi), \tag{68}$$

and

$$\Theta_{2-5-2}(\Psi) = \frac{(w+4)}{2} - \frac{\iota \cot(\Psi)(w+4)}{2} \pm \csc^2(\Psi). \tag{69}$$

Remark 2. To the best of our knowledge, certain solutions have not yet been documented in the literature by other researchers [35–37]. All answers are verified by re-entering them into the given model using maple software and proved to be accurate and new.

4.2. Application of R-B Sub-ODE Method. This part explains the solutions using *R-B* sub-ODE method. For this purpose, consider the ODE given in equation (6) and assume the solution in the following form:

$$\Theta' = a_1\Theta^{2-m} + b_1\Theta + c_1\Theta^m. \tag{70}$$

Substituting the above equation in equation (6) gives the system of equations as follows:

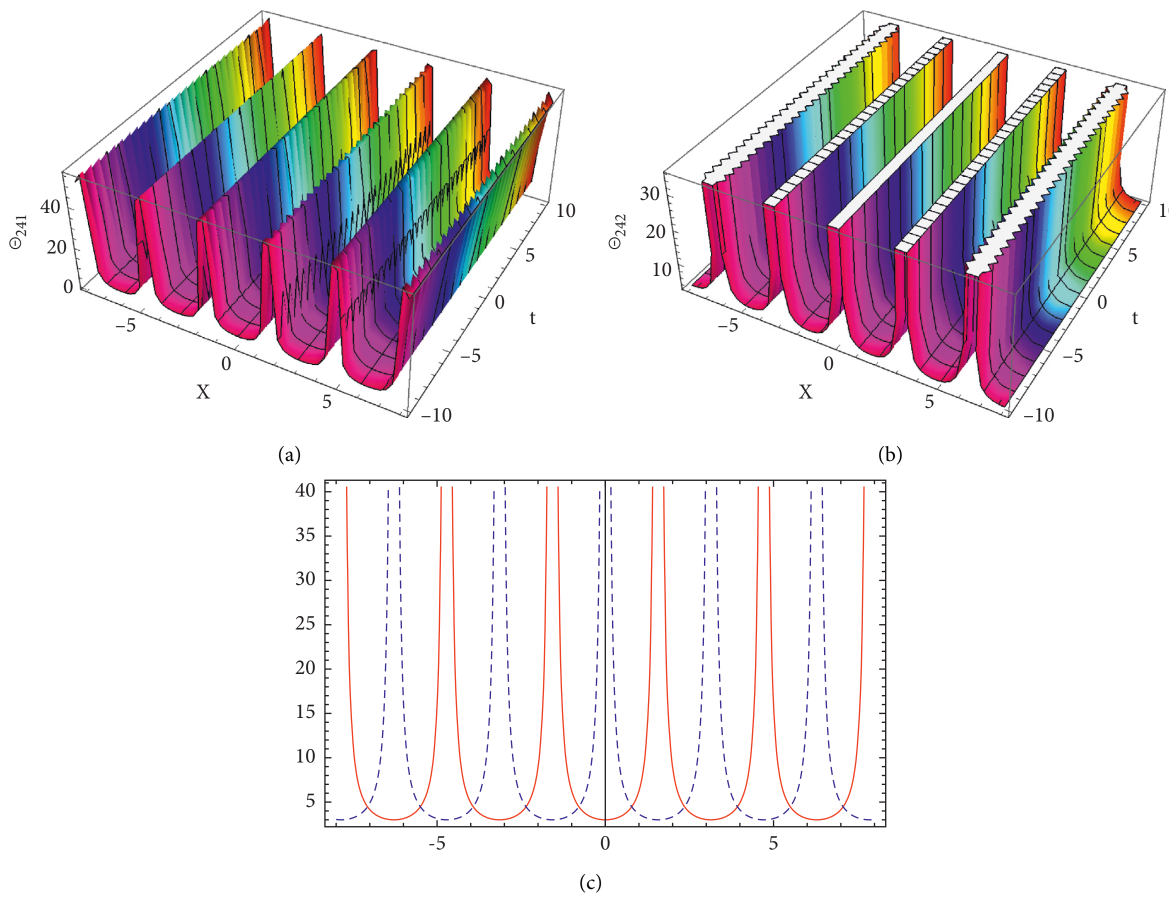


FIGURE 6: Plots of equations (65) and (66) for $m = 2$, $\xi = 0.3$, $\lambda_1 = 0.3$, and $w = 0.3$. (a) Θ_{241}, (b) Θ_{242}, (c) Θ_{241}, Θ_{242}.

$$
\begin{aligned}
&-2\alpha_1 a_1^2-\frac{1}{2}\mu_1 a_1^2=0,\\
&-3a_1\alpha_1 b_1-a_1 b_1\mu_1+2a_1^2-a_1\sigma_1=0,\\
&\frac{\lambda_1}{2}+3a_1 b_1-\alpha_1\left(2a_1c_1+b_1^2\right)-\frac{1}{2}\mu_1\left(2a_1c_1+b_1^2\right)-\beta_1 a_1-\sigma_1 b_1=0,\\
&-\alpha_1 b_1 c_1-b_1 c_1\mu_1+2a_1c_1+b_1^2-b_1\beta_1-c_1\sigma_1+w=0,\\
&b_1c_1-\frac{1}{2}\mu_1 c_1^2-\beta_1 c_1=0.
\end{aligned}
\tag{71}
$$

Solving the system gives the solution set as follows. Set

$$
\begin{aligned}
&a_1=\frac{2\lambda_1\sigma_1}{(-\mu_1 w+4\lambda_1)},\ \alpha_1=-\frac{1}{4}\mu_1,\ b_1=\frac{4\sigma_1 w}{(-\mu_1 w+4\lambda_1)},\ c_1=0,\\
&\beta_1=\frac{\left(\mu_1^2w^2-8\lambda_1\mu_1 w+16\sigma_1^2 w+16\lambda_1^2\right)}{4\sigma_1(-\mu_1 w+4\lambda_1)}.
\end{aligned}
\tag{72}
$$

Now, we will look at the cases of solutions.

(1) When $m\neq 1$, $b_1\neq 0$, and $c_1=0$, the solution of equation (6) using equation (28) is

$$
\Theta_1(\Psi)=\frac{-\lambda_1}{2w}+\Omega\ \exp\left(\frac{-4w\Psi\sigma_1}{-\mu_1 w+4\lambda_1}\right). \tag{73}
$$

(2) When $m\neq 1$, $a_1\neq 0$, and $b_1^2-4a_1c_1>0$, the solution of equation (6) using equations (31) and (32) is

$$
\begin{aligned}
\Theta_2(\Psi)=&\frac{-w}{\lambda_1}-\frac{1}{4\lambda_1\sigma_1}\sqrt{\frac{16\sigma_1^2w^2}{(-\mu_1 w+4\lambda_1)^2}}\\
&\cdot\tanh\left(\frac{1}{2}\sqrt{\frac{16\sigma_1^2w^2}{(-\mu_1 w+4\lambda_1)^2}}(\Psi+\Omega)\right)(-\mu_1 w+4\lambda_1),
\end{aligned}
\tag{74}
$$

and

$$
\begin{aligned}
\Theta_3(\Psi)=&\frac{-w}{\lambda_1}-\frac{1}{4\lambda_1\sigma_1}\sqrt{\frac{16\sigma_1^2w^2}{(-\mu_1 w+4\lambda_1)^2}}\\
&\cdot\coth\left(\frac{1}{2}\sqrt{\frac{16\sigma_1^2w^2}{(-\mu_1 w+4\lambda_1)^2}}(\Psi+\Omega)\right)(-\mu_1 w+4\lambda_1),
\end{aligned}
\tag{75}
$$

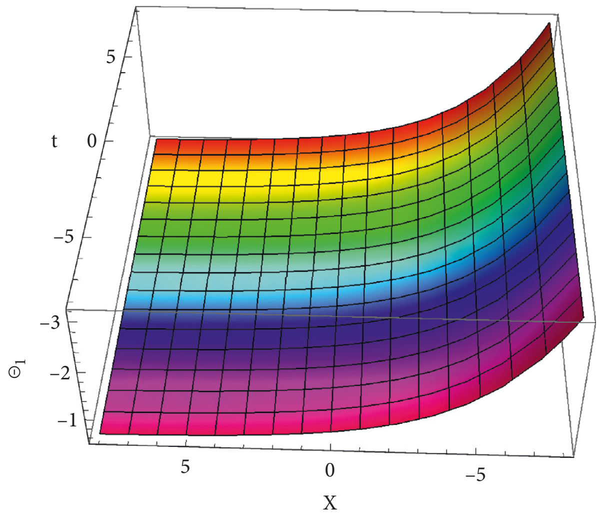


Figure 7: Plots of equations (73), for $m = 0$, $\xi = 0.2$, $\lambda_1 = 0.2$, $\Omega_1 = 0.2$, $\sigma_1 = 0.3$, $\mu_1 = 0.3$, and $w = 0.2$.

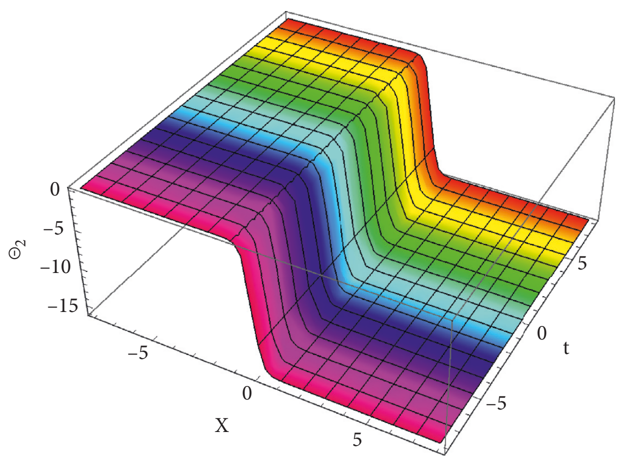


Figure 8: Plots of equations (74), for $m = 0$, $\xi = 0.2$, $\lambda_1 = 0.25$, $\Omega_1 = 0.2$, $\sigma_1 = 0.3$, $\mu_1 = 0.3$, and $w = 2$.

where λ_1, Ψ, Ω, σ_1, and μ_1 are the arbitrary constants.

5. Graphical Representations

In this part, we show graphical representations of a few of the determined solutions. It is worth noting that explicit and consistent solutions are obtained by employing two distinct reliable approaches. From the first scheme, Figures 1–3 represent the graphs of solution sets 1, 2, and 3 of Case 1 and Figures 4–6 represent the graphs of solution sets 1, 2, and 4 of Case 2 using appropriate parameters values. Similarly, Figures 7–9 represent the graphs of sets 1 and 2 of the second method for suitable parametric values.

5.1. Results and Discussion. Graphs are the important tool to visualize data. In this part, we will discuss graphical visualization of obtained results. Results are obtained in form of trigonometric and hyperbolic functions. The absolute 3D and line graphs of positive functions are plotted of FKS

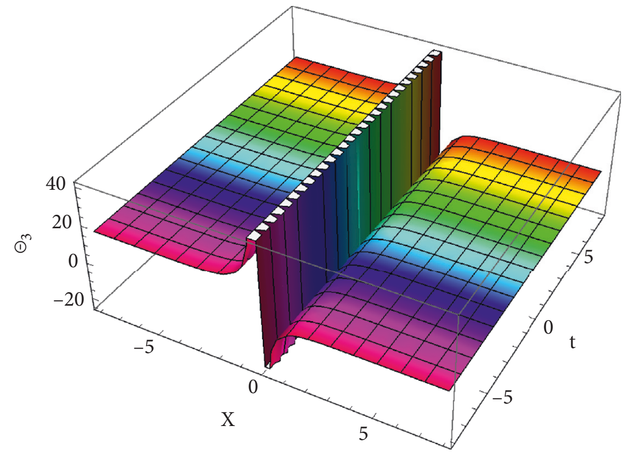


Figure 9: Plots of equations (75), for $m = 0$, $\xi = 0.2$, $\lambda_1 = 0.25$, $\Omega_1 = 0.3$, $\sigma_1 = 0.3$, $\mu_1 = 0.3$, and $w = 2$.

model using EShGEE technique. In Case 1 and Case 2, singular soliton and solitary wave solutions are obtained by taking suitable parametric values $m = 2$, $\xi = 0.3$, $\lambda_1 = 0.5$, and $w = 3$ shown in Figures (1)–(6). If negative term values are plotted, the shape of graph is reversed. By employing *R-B* technique, exponential, trigonometric and hyperbolic solutions are obtained. Their graphs are shown in Figures (7)–(9) which are solitary and singular solitons by taking these parametric $m = 0$, $\xi = 0.2$, $\lambda_1 = 0.25$, $\Omega_1 = 0.3$, $\sigma_1 = 0.3$, $\mu_1 = 0.3$, and $w = 2$ values. Under thermodynamic assumptions, obtained solutions have many applications in the liquids containing gas bubbles.

6. Conclusion

This study successfully employed the extended sinh-Gorden equation expansion and Riccati-Bernoulli sub-ODE techniques with a novel fractional operator to the nonlinear fractional Kudryashov-Sinelshchikov equation that occurs in nonlinear wave processes in a liquid containing gas bubbles. The methodologies adopted gave rise to new hyperbolic, complex hyperbolic, trigonometric, and complex trigonometric solutions for the model, namely, dark, brilliant, combined dark-bright, singular, combined singular, and other solitary wave solutions. Furthermore, the 3D graphics for suitable parametric values have been displayed that highlight the physical importance and dynamical behaviors of the governed model. Therefore, the results obtained illustrate that the implemented approaches are highly efficient and resilient for solving many nonlinear problems in mathematical physics.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors express their appreciation to “The Research Center for Advanced Materials Science (RCAMS)” at King Khalid University, Saudi Arabia, for funding this work under the grant number RCAMS/KKU/004-21. Emad E. Mahmoud acknowledges Taif University Researchers Supporting Project number (TURSP-2020/20), Taif University, Taif, Saudi Arabia.

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