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Research Article

A Valuation Formula for Chained Options with *n*-Barriers

Won Choi,[1] Doobae Jun ,[2] and Hyejin Ku[3]

[1]*Department of Mathematics, Incheon National University, Incheon 22012, Republic of Korea*
[2]*Department of Mathematics and Research Institute of Natural Science, Gyeongsang National University, Jinju 52828, Republic of Korea*
[3]*Department of Mathematics and Statistics, York University, 4700 Keele St., Toronto, ON, Canada*

Correspondence should be addressed to Doobae Jun; dbjun@gnu.ac.kr

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This study examines chained options that are connected in the sense that another barrier option becomes active continuously after the underlying asset price crosses a primary barrier. These barrier options have several advantages. First, they preserve the merit of regular barrier options, but demand far lower option premiums, which appeal to option traders. Second, they reduce the higher risk of loss of double barrier options, making option strategies more profitable in certain cases. Third, they have closed-form pricing formulas, unlike double-barrier options, and, thus, avoid the complexity of option pricing. Therefore, they help to enlarge the range of trader's choice according to a variety of demand of buyers. The values of chained options are compared to those of similar single- and double-barrier options. This study extends the chained option with two barriers to a generalized chained option with *n*-barriers. In addition, this paper proves the closed formulas of generalized chained options with *n*-barriers using mathematical induction.

1. Introduction

The rapid spread of the novel coronavirus (COVID-19) has dramatically impacted financial markets worldwide. It has created an unprecedented level of risk, causing investors to suffer significant losses in a very short period of time [1]. In a market facing sudden crisis, financial derivatives, such as barrier options, can be considered to reduce the risk of volatility and save money.

Barrier options are a widely used class of path-dependent financial derivatives because they are flexible and less expensive than vanilla options. Merton [2], Rubinstein and Reiner [3], and Rich [4] provided a mathematical framework and derived closed-form pricing formulas for various types of single-barrier options. These studies assume that the underlying asset price is monitored with respect to a single constant barrier for the entire life of the option. Owing to their popularity in the market, more complicated structures of barrier options have been studied. Kunitomo and Ikeda [5], Geman and Yor [6], and Pelsser [7] provided double-barrier options with two barriers.

Additional modifications of the barrier options include window barrier (partial barrier) options. In the case of a window barrier option, the trigger is valid only within a certain period. Heynen and Kat [8] studied partial barrier options, where the underlying price was monitored during only part of the option's lifetime. By solving the Black–Scholes partial differential eqnarray under the appropriate boundary conditions, Hui [9] priced front-end and rear-end double barrier options, featuring early-ending and forward-starting monitoring, respectively. Guillaume [10] provided expressions of standard and partial window double-barrier option values as the infinite sums of three-dimensional and seven-dimensional normal distribution functions.

All the papers described above are concerned with barrier options, where the monitoring of the barrier starts at a predetermined date. There is another class of barrier options with two barriers. The option is a chained option in which another barrier option is activated when a primary barrier is hit. This option has become popular in over-the-counter equity and foreign exchange derivative markets. Jun and Ku [11, 12] derived closed-form valuation formulas for chained barrier options of various types. Furthermore, Jun [13] explored the continuity correction approach for the approximate pricing of discretely monitored chained options.

An international pattern of exotic derivatives, whose value is determined by the price or volatility of the underlying asset, appeared to have helped to transmit the financial crisis of 2008 from the United States and European Union to many different emerging market economies. The direct cost of these derivatives losses on nonfinancial firms is approximately \$530 billion, based on the sum of national estimates [14]. The complexity of pricing exotic derivatives may put investors at risk if they cannot independently evaluate the risks and price the derivatives.

In this study, the features of the chained option provide several desirable properties. The chained options can provide an initial protection period during which the option cannot be knocked out so that it reduces the risk of loss when a trader chooses to buy a double barrier option and is still cheaper than a single barrier option. In addition, chained options are the least expensive in certain cases among similar single and double barrier options.

In Section 2, the values of chained options are compared to those of similar single- and double-barrier options, and graphs are shown to illustrate that chained options are less expensive and more effective in various circumstances. Section 3 gives the valuation formula of the generalized chained option, which is an extended version of the chained option with two barriers. Finally, conclusions are presented in Section 4.

2. Comparison of Chained Options with Other Barrier Options

The valuation formulas for chained barrier options of various types are given in [11] for the constant barrier cases and [12] for general curved barriers. In chained options, a regular barrier option is activated when a primary barrier is hit. For example, a down-and-in chained call (DIC_u) is a down-and-in call option activated when the underlying asset price hits an upper barrier level. This option gives the option holder the payoff of a call if the price of the underlying asset rises above an upper barrier and then falls below a lower barrier before expiry, and it pays off zero otherwise. An up-and-in doubly chained call (UIC_{ud}) is an up-and-in call option which is activated at a time when the asset price crosses two different barrier levels (an up barrier followed by a down barrier).

Chained options differ from double-barrier options in the sense that the other barrier does not exist until one barrier is hit according to their predetermined order. In other words, crossing a barrier leads to the creation or cancelation of a barrier option rather than a call or put.

Let r be the risk-free interest rate and σ be a constant. Assume that the price of the underlying asset S follows a geometric Brownian motion:

$$S_t = S_0 e^{\overline{\mu}t+\sigma W_t}, \tag{1}$$

where $\overline{\mu} = r - (\sigma^2/2)$ and W_t is a standard Brownian motion under a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \overline{P})$. Here, $\overline{P}$ is the risk-neutral probability measure.

Let us denote with τ_u and τ_d the first time that the underlying asset price S_t at time t hits the up barrier U and down barrier D, respectively, and with τ_{ud} the first time after τ_u that the underlying asset price touches the down barrier D:

$$\begin{aligned} \tau_u &:= \min\{t: S_t = U, U > S_0\}, \\ \tau_d &:= \min\{t: S_t = D, D < S_0\}, \\ \tau_{ud} &:= \min\{t > \tau_u: S_t = D\}. \end{aligned} \tag{2}$$

Let $X_t = (1/\sigma)\ln(S_t/S_0), k = (1/\sigma)\ln(K/S_0), u = (1/\sigma)\ln(U/S_0)$ and $d = (1/\sigma)\ln(D/S_0)$. The minimum and maximum for X_t are

$$\begin{aligned} m_a^b &= \inf_{t\in[a,b]} (X_t), \\ M_a^b &= \sup_{t\in[a,b]} (X_t), \end{aligned} \tag{3}$$

and are denoted by $E^{\overline{P}}$ the expectation operator under the $\overline{P}$-measure.

The chained options $(\mathrm{DOC}_u, \mathrm{DIC}_u)$ and double-barrier options (UIDOC, UIDIC) can be defined as follows:

$$\begin{aligned} \mathrm{DOC}_u &= e^{-rT}E^{\overline{P}}\left[(S_T - K)^+ 1_{\{m_{\tau_u}^T > d, \tau_u \le T\}} \right], \\ \mathrm{DIC}_u &= e^{-rT}E^{\overline{P}}\left[(S_T - K)^+ 1_{\{m_{\tau_u}^T \le d, \tau_u \le T\}}, \right] \\ \mathrm{UIDOC} &= e^{-rT}E^{\overline{P}}\left[(S_T - K)^+ 1_{\{\tau_u \le T < \tau_d\}} \right], \\ \mathrm{UIDIC} &= e^{-rT}E^{\overline{P}}\left[(S_T - K)^+ 1_{\{\tau_u \le T, \tau_d \le T\}} \right], \end{aligned} \tag{4}$$

where 1 is an indicator function, K is the strike price, and T is the maturity of the options.

The down-and-out chained call option (DOC_u) is similar to a double-barrier option with upper "in" and lower "out" barriers (UIDOC) in the sense that they pay off a call if the up barrier is touched and the down barrier is not before expiry. However, they differ in the case of $\tau_d < \tau_u < T < \tau_{ud}$; the payoff of UIDOC is zero because the down barrier has been touched, while DOC_u pays off a call $(S_T - K)^+$. In addition, the down-and-in call option DIC_u and up-and-in and down-and-in double-barrier option UIDIC are similar, but have different payoffs when $\tau_d < \tau_u < T < \tau_{ud}$. Because DIC_u ignores the movement of the underlying asset price before reaching the up barrier, even if the down barrier has been hit before τ_u, it pays off 0 unless the down barrier is breached again after τ_u.

If one expects the underlying stock to rally strongly, an up-and-in call option can result in a higher profitability than the vanilla call option, and it is cheaper. If the trader wants to maximize their profits, they may bet on a double barrier option with an upper knock-in and a lower knock-out barrier (UIDOC) at an even cheaper price. However, the trader faces a higher risk of loss if the underlying asset moves downward and crosses the lower barrier at any time. What if the trader holds a down-and-out chained option (DOC_u)? In this case, a lower knock-out barrier does not appear until the upper barrier is touched. Once the underlying asset price reaches the upper-barrier level, the probability of hitting the down barrier afterward is much smaller than before. The chained option has a much lower risk of loss than a double-barrier option (UIDOC) and is still cheaper than an up-and-in call option (UIC), which results in higher profits.

Figure 1 shows the graphs of the option values UIC, UIDOC, and DOC_u at time $t\,(<\tau_u)$ and $t\,(\tau_d<\tau_u<t<\tau_{ud})$. The parameter values in the left figure are $U=105, D=90, K=100, r=0.05, \sigma=0.3,$ and $T=1$. The closer the current asset price is to the up barrier, the greater the likelihood of the option being knocked in and thus the higher the option values.

The parameter values that we used in the right figure are $U=110, D=95, K=100, r=0.05, \sigma=0.3,$ and $T=1$. Once the down barrier was breached, the UIDOC was 0. If the underlying asset price then rises to the up barrier, UIC equals the vanilla call price, while DOC_u is equivalent to the regular down-and-out call price. From Figure 1, it is observed that the chained down-and-out call option DOC_u is always in between UIC and UIDOC. Furthermore, when the volatility is higher, DOC_u is farther from the UIC because the likelihood of the option being knocked out is greater.

As the down barrier decreases, the values of the three options become close. However, if the down barrier approaches the initial spot S_0, the difference between DOC_u and UIDOC becomes larger. Because they pay off a call option, barrier criteria are satisfied, and the values naturally increase as the strike price K increases. When S_0 approaches the up barrier, the value UIDOC approaches DOC_u.

Next, we consider chained options with knock-in barriers. A down-and-in chained call option (DIC_u) is a down-and-in call option activated at a time when the underlying asset price hits an upper-barrier level. DIC_u demands a far lower premium than both the single up-and-in call option (UIC) and double touch knock-in option (UIDIC). Figure 2 shows that DIC_u is lower than UIC and UIDIC at different times. The parameter values are $U=110, D=90, K=100, r=0.05, \sigma=0.3,$ and $T=1$. UIDIC is the same as UIC if the current spot price is below the down barrier 90 and decreases as the asset price S_t increases otherwise. When $\tau_d<\tau_u<t<\tau_{ud}$, both UIC and UIDIC are equal to the vanilla call price. However, DIC_u is the same as the vanilla call option if S_t is below the down barrier, while it is equal to the regular down-and-in barrier option price otherwise.

3. Pricing Formulas for General Chained Option

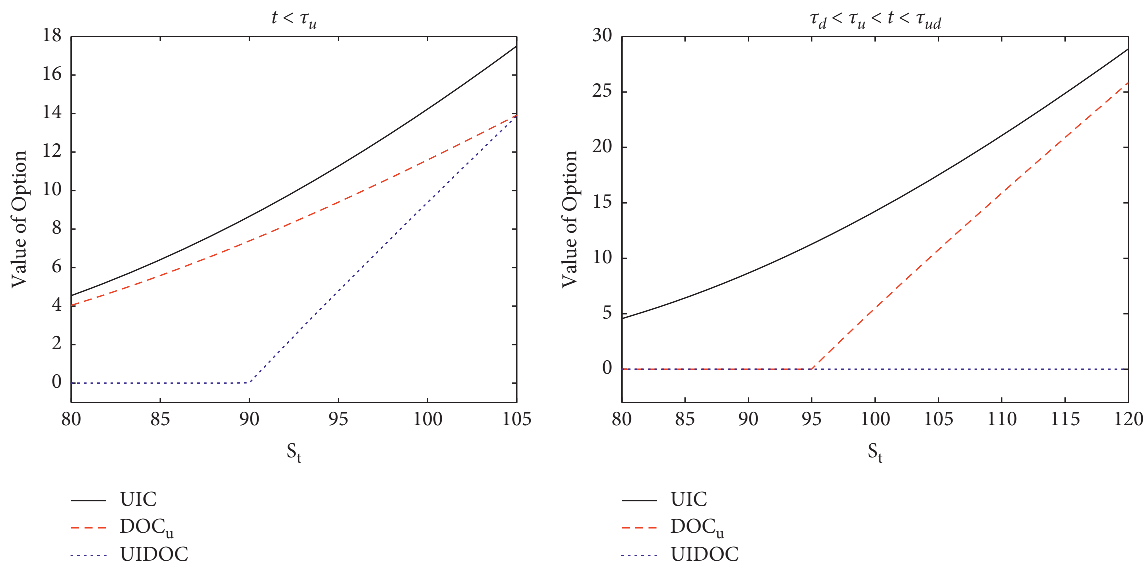
The chained option pricing formula with only two barriers is provided in the work of Jun and Ku [11]. In this section, the valuation formula of the chained option with $n\,(\geq 2)$-barriers is described.

Consider European options expiring at T with strike price K, up barriers $U_i\,(i=1,\ldots,n)\,(n\geq 2)$, and down barriers $D_i\,(i=1,\ldots,n)\,(n\geq 2)$. These options are chained options, where a barrier option is given when the underlying asset price hits up and down barriers in the following four orders:

$$\begin{aligned}
&A_1\colon U_1D_1\cdots U_{n-1}D_{n-1}U_n\,\left[D_i<U_i, D_i<U_{i+1}\,(i=1,\ldots,n-1)\right],\\
&A_2\colon D_1U_1\cdots D_nU_n\,\left[D_i<U_i\,(i=1,\ldots,n), U_i>D_{i+1}\,(i=1,\ldots,n-1)\right],\\
&A_3\colon U_1D_1\cdots U_nD_n\,\left[D_i<U_i\,(i=1,\ldots,n), D_i<U_{i+1}\,(i=1,\ldots,n-1)\right],\\
&A_4\colon D_1U_1\cdots D_{n-1}U_{n-1}D_n\,\left[D_i<U_i, U_i>D_{i+1}\,(i=1,\ldots,n-1)\right].
\end{aligned}\tag{5}$$

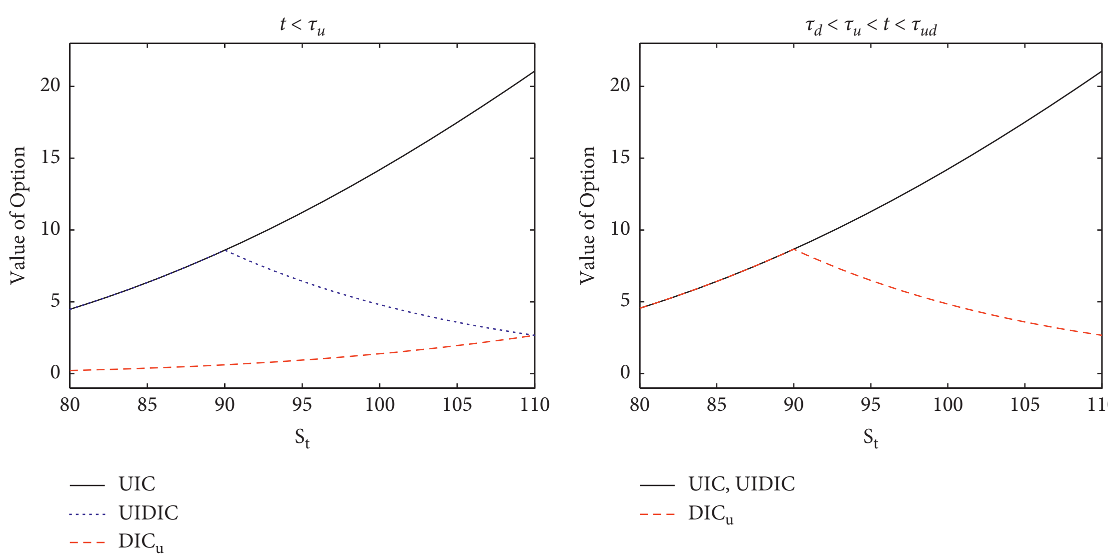
The following theorem presents the valuation formula of the general chained option, where the down-and-in call option is given when the underlying asset price hits the barriers in the order A_1 or A_2 with down barrier $D\,(<U_n)$.

Theorem 1. *The valuation formula for the down-and-in call option commencing at a time the underlying asset hits the barriers in the order A_1 or A_2 and across the last up barrier U_n when the strike price K is greater than the down barrier D is*



(a) (b)

Figure 1: Comparison of UIC, DOC_u, UIDOC when (a) $t < \tau_u$ and when (b) $\tau_d < \tau_u < t < \tau_{ud}$.



(a) (b)

Figure 2: Comparison of UIC, UIDIC, DIC_u when (a) $t < \tau_u$ and (b) $\tau_d < \tau_u < t < \tau_{ud}$.

$$\text{(I) } \mathrm{DIC}_{U_1 D_1 \cdots U_{n-1} D_{n-1} U_n} = S_0 \left(\frac{D}{U_n} \prod_{i=1}^{n-1} \frac{D_i}{U_i} \right)^{2\tilde{\mu}/\sigma^2} N(z_1) - e^{-rT} K \left(\frac{D}{U_n} \prod_{i=1}^{n-1} \frac{D_i}{U_i} \right)^{2\overline{\mu}/\sigma^2} N\left(z_1 - \sigma\sqrt{T}\right),$$

$$\text{(II) } \mathrm{DIC}_{D_1 U_1 \cdots D_n U_n} = S_0 \left(\frac{D}{S_0} \prod_{i=1}^{n} \frac{D_i}{U_i} \right)^{2\tilde{\mu}/\sigma^2} N(z_2) - e^{-rT} K \left(\frac{D}{S_0} \prod_{i=1}^{n} \frac{D_i}{U_i} \right)^{2\overline{\mu}/\sigma^2} N\left(z_2 - \sigma\sqrt{T}\right), \quad (6)$$

where

$$z_1 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U_n^2 K}\prod_{i=1}^{n-1}\left(\frac{D_i}{U_i}\right)^2\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T},$$
$$\widetilde{\mu} = r + \frac{\sigma^2}{2}, z_2 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2}{K S_0}\prod_{i=1}^{n}\left(\frac{D_i}{U_i}\right)^2\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T}, \tag{7}$$

and $N(\cdot)$ is the cumulative standard normal distribution function.

Proof. Mathematical induction can be used to prove the closed formula of chained option $\mathrm{DIC}_{U_1 D_1 \cdots U_{n-1} D_{n-1} U_n}$.

When $n = 1$ in (I), then

$$\mathrm{DIC}_{U_1} = S_0\left(\frac{D}{U_1}\right)^{2\widetilde{\mu}/\sigma^2} N(z_1) - e^{-rT}K\left(\frac{D}{U_1}\right)^{2\widetilde{\mu}/\sigma^2} \cdot N\left(z_1 - \sigma\sqrt{T}\right) \tag{8}$$

where

$$z_1 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U_1^2 K}\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T}, \widetilde{\mu} = r + \frac{\sigma^2}{2} \tag{9}$$

This formula is the same as Theorem 2.1 in [11]. Thus, (I) holds for $n = 1$.

Suppose formula (I) holds for $n = k$:

$$\mathrm{DIC}_{U_1 D_1 \cdots U_{k-1} D_{k-1} U_k} = S_0\left(\frac{D}{U_k}\prod_{i=1}^{k-1}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2} N(z_1) - e^{-rT}K\left(\frac{D}{U_k}\prod_{i=1}^{k-1}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2} N\left(z_1 - \sigma\sqrt{T}\right), \tag{10}$$

where

$$z_1 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U_n^2 K}\prod_{i=1}^{k-1}\left(\frac{D_i}{U_i}\right)^2\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T}, \widetilde{\mu} = r + \frac{\sigma^2}{2}. \tag{11}$$

Then, we show that (I) holds for $n = k + 1$.

Under the risk-neutral probability $\overline{P}$, the chained option is

$$\begin{aligned}\mathrm{DIC}_{U_1 D_1 \cdots U_k D_k U_{k+1}} &= e^{-rT}E^{\overline{P}}\left[(S_T - K)^+ 1_{\left\{m_{\tau_{2k+1}}^T < d, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T, S_{\tau_1} = U_1, S_{\tau_2} = D_1, \ldots, S_{\tau_{2k+1}} = U_{k+1}\right\}}\right] \\ &= e^{-rT}E^{\overline{P}}\left[(S_T - K) 1_{\left\{m_{\tau_{2k+1}}^T < d, S_T > K, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T, S_{\tau_1} = U_1, S_{\tau_2} = D_1, \ldots, S_{\tau_{2k+1}} = U_{k+1}\right\}}\right],\end{aligned} \tag{12}$$

where $m_{\tau_{2k+1}}^T = \inf_{t \in [\tau_{2k+1}, T]}(S_t)$. Let us define a new measure $\widetilde{P}$ such that

$$\frac{\mathrm{d}\widetilde{P}}{\mathrm{d}\overline{P}} = e^{-(1/2)\sigma^2 T + \sigma W_T}. \tag{13}$$

Then, we have

$$\begin{aligned}&\mathrm{DIC}_{U_1 D_1 \cdots U_k D_k U_{k+1}} \\ &= S_0\widetilde{P}\left(m_{\tau_{2k+1}}^T < d, S_T > K, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T,\ S_{\tau_1} = U_1, S_{\tau_2} = D_1, \ldots, S_{\tau_{2k+1}} = U_{k+1}\right) \\ &\quad - e^{-rT}K\overline{P}\left(m_{\tau_{2k+1}}^T < d, S_T > K, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T,\ S_{\tau_1} = U_1, S_{\tau_2} = D_1, \ldots, S_{\tau_{2k+1}} = U_{k+1}\right).\end{aligned} \tag{14}$$

We calculate the required probability only under the $\overline{P}$-measure. Note that

$$\begin{aligned}&\overline{P}\left(m^T_{\tau_{2k+1}} < d, S_T > K, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T,\ S_{\tau_1} = U_1, S_{\tau_2} = D_1, \ldots, S_{\tau_{2k+1}} = U_{k+1}\right)\\ &\quad = \overline{P}\left(m^T_{\tau_{2k+1}} < d, X_T > k, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T,\ X_{\tau_1} = u_1, X_{\tau_2} = d_1, \ldots, X_{\tau_{2k+1}} = u_{k+1}\right),\end{aligned} \tag{15}$$

where $X_t = W_t + (\overline{\mu}/\sigma)t$ is a standard Brownian motion under the equivalent probability measure Q, defined by

$$\frac{dQ}{d\overline{P}} = \exp\left[-\frac{\overline{\mu}}{\sigma}W_T - \frac{1}{2}\left(\frac{\overline{\mu}}{\sigma}\right)^2 T\right]. \tag{16}$$

Then,

$$\begin{aligned}&\overline{P}\left(m^T_{\tau_{2k+1}} < d, X_T > k, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T,\ X_{\tau_1} = u_1, X_{\tau_2} = d_1, \ldots, X_{\tau_{2k+1}} = u_{k+1}\right)\\ &\quad = E^Q\left[\frac{d\overline{P}}{dQ} 1_{\left\{m^T_{\tau_{2k+1}} < d, X_T > k, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T, X_{\tau_1} = u_1, X_{\tau_2} = d_1, \ldots, X_{\tau_{2k+1}} = u_{k+1}\right\}}\right]\\ &\quad = E^Q\left[e^{(\overline{\mu}/\sigma)X_T - (1/2)\left(\overline{\mu}^2/\sigma^2\right)T} 1_{\left\{m^T_{\tau_{2k+1}} < d, X_T > k, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T, X_{\tau_1} = u_1, X_{\tau_2} = d_1, \ldots, X_{\tau_{2k+1}} = u_{k+1}\right\}}\right].\end{aligned} \tag{17}$$

Let us introduce a process $\widetilde{X}_t, t \in [0, T]$, defined by the formula

$$\widetilde{X}_t = \begin{cases} X_t, & (t \le \tau_{2k}),\\ 2d_k - X_t, & (t > \tau_{2k}).\end{cases} \tag{18}$$

By virtue of the reflection principle, process $\widetilde{X}_t$ follows a standard Brownian motion under Q, and

$$\begin{aligned}&E^Q\left[e^{(\overline{\mu}/\sigma)X_T - (1/2)\left(\overline{\mu}^2/\sigma^2\right)T} 1_{B_1}\right]\\ &\quad = E^Q\left[e^{(\overline{\mu}/\sigma)\left(2d_k - \widetilde{X}_T\right) - (1/2)\left(\overline{\mu}^2/\sigma^2\right)T} 1_{B_2}\right]\end{aligned} \tag{19}$$

where

$$\begin{aligned}B_1 &= \left\{m^T_{\tau_{2k+1}} < d, X_T > k, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T, X_{\tau_1} = u_1, X_{\tau_2} = d_1, \ldots, X_{\tau_{2k+1}} = u_{k+1}\right\},\\ B_2 &= \left\{\widetilde{M}^T_{\tau_{2k+1}} \ge 2d_k - d, \widetilde{X}_T < 2d_k - k, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T, X_{\tau_1} = u_1, X_{\tau_2} = d_1, \ldots, X_{\tau_{2k-1}} = u_{k-1}, X_{\tau_{2k+1}} = 2d_k - u_{k+1}\right\}.\end{aligned} \tag{20}$$

Here, $\widetilde{M}^T_{\tau_{2k+1}} = \sup_{t \in [\tau_{2k+1}, T]}(\widetilde{X}_t)$. We apply the reflection principle again. Let us introduce a process $\widehat{X}_t, t \in [0, T]$ defined by the formula

$$\widehat{X}_t = \begin{cases} \widetilde{X}_t, & (t \le \tau_{2k+1}),\\ 2\left(2d_k - u_{k+1}\right) - \widetilde{X}_t, & (t > \tau_{2k+1}).\end{cases} \tag{21}$$

Then, the process $\widehat{X}_t$ also follows a standard Brownian motion under Q, and

$$\begin{aligned}&E^Q\left[e^{(\overline{\mu}/\sigma)\left(2d_k - \widetilde{X}_T\right) - (1/2)\left(\overline{\mu}^2/\sigma^2\right)T} 1_{B_2}\right]\\ &\quad = E^Q\left[e^{(\overline{\mu}/\sigma)\left(-2d_k + 2u_{k+1} + \widehat{X}_T\right) - (1/2)\left(\overline{\mu}^2/\sigma^2\right)T} 1_{B_3}\right]\end{aligned} \tag{22}$$

where

$$B_3 = \left\{\widehat{m}^T_{\tau_{2k+1}} \le 2d_k - 2u_{k+1} + d, \widehat{X}_T > 2d_k - 2u_{k+1} + k, \tau_1 < \tau_2 < \cdots < \tau_{2k+1} \le T, X_{\tau_1} = u_1, X_{\tau_2} = d_1, \ldots, X_{\tau_{2k-1}} = u_{k-1}\right\}, \tag{23}$$

and $\widehat{m}^T_{\tau_{2k+1}} = \inf_{t \in [\tau_{2k+1}, T]}(\widehat{X}_t)$.

Since $u_{k+1} > d_k$, $2d_k - 2u_{k+1} + d = d - 2\left(u_{k+1} - d_k\right) < d$, and

$$\begin{aligned}&\left\{\widehat{m}^T_{\tau_{2k+1}} \le 2d_k - 2u_{k+1} + d, \tau_{2k+1} \le T\right\}\\ &\quad = \left\{\widehat{m}^T_{\tau_{2k-1}} \le 2d_k - 2u_{k+1} + d\right\}\end{aligned} \tag{24}$$

thus,

$$\begin{aligned} &E^{Q}\left[e^{(\overline{\mu}/\sigma)\left(-2d_k+2u_{k+1}+\widehat{X}_T\right)-(1/2)\left(\overline{\mu}^2/\sigma^2\right)T}1_{B_3}\right] \\ &\quad= e^{(\overline{\mu}/\sigma)\left(-2d_k+2u_{k+1}\right)}E^{Q}\left[e^{(\overline{\mu}/\sigma)\widehat{X}_T-(1/2)\left(\overline{\mu}^2/\sigma^2\right)T}1_{B_4}\right] \\ &\quad= \left(\frac{U_{k+1}}{D_k}\right)^{2\widetilde{\mu}/\sigma^2}E^{Q}\left[e^{(\overline{\mu}/\sigma)\widehat{X}_T-(1/2)\left(\overline{\mu}^2/\sigma^2\right)T}1_{B_4}\right], \end{aligned} \tag{25}$$

where

$$B_4=\left\{\widehat{m}^T_{\tau_{2k-1}}\leq 2d_k-2u_{k+1}+d, \widehat{X}_T>2d_k-2u_{k+1}+k, \tau_1<\tau_2<\cdots<\tau_{2k-1}\leq T, X_{\tau_1}=u_1, X_{\tau_2}=d_1,\ldots,X_{\tau_{2k-1}}=u_{k-1}\right\}. \tag{26}$$

Then,

$$\begin{aligned} &\overline{P}\left(m^T_{\tau_{2k+1}}<d, X_T>k, \tau_1<\tau_2<\cdots<\tau_{2k+1}\leq T, X_{\tau_1}=u_1, X_{\tau_2}=d_1,\ldots,X_{\tau_{2k+1}}=u_{k+1}\right) \\ &\quad=\left(\frac{U_{k+1}}{D_k}\right)^{2\widetilde{\mu}/\sigma^2}\overline{P}\left(m^T_{\tau_{2k-1}}\leq 2d_k-2u_{k+1}+d, X_T>2d_k-2u_{k+1}+k,\ \tau_1<\tau_2<\cdots<\tau_{2k-1}\leq T, X_{\tau_1}=u_1, X_{\tau_2}=d_1,\ldots,X_{\tau_{2k-1}}=u_{k-1}\right). \end{aligned} \tag{27}$$

By combining the chained option formula for $n = k$, we complete the proof of (I). The proof of (II) is similar to the proof process of (I).

The following theorem presents the valuation formula of the general chained option, where the up-and-in call option is given when the underlying asset price hits the barriers in the order A_3 or A_4 with down barrier $U(>D_n)$. □

Theorem 2. *The valuation formula for the up-and-in call option commencing at a time the underlying asset hits the barriers in the order A_3 or A_4 and across the last up-barrier D_n when the strike price K is lower than the up barrier $U(>D_n)$ is*

$$\begin{aligned} \text{(I) }\mathrm{UIC}_{U_1D_1\cdots U_nD_n} &= S_0\left[\left(\prod_{i=1}^{n}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2}N(z_3)+\left(\frac{U}{S_0}\prod_{i=1}^{n}\frac{U_i}{D_i}\right)^{2\widetilde{\mu}/\sigma^2}\left\{N(z_4)-N(z_5)\right\}\right] \\ &\quad- e^{-rT}K\left[\left(\prod_{i=1}^{n}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2}N\left(z_3-\sigma\sqrt{T}\right)+\left(\frac{U}{S_0}\prod_{i=1}^{n}\frac{U_i}{D_i}\right)^{2\widetilde{\mu}/\sigma^2}\left\{N\left(z_4-\sigma\sqrt{T}\right)-N\left(z_5-\sigma\sqrt{T}\right)\right\}\right], \\ \text{(II) }\mathrm{UIC}_{D_1U_1\cdots D_{n-1}U_{n-1}D_n} &= S_0\left[\left(\frac{D_n}{S_0}\prod_{i=1}^{n-1}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2}N(z_6)+\left(\frac{D_n}{U}\prod_{i=1}^{n-1}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2}\left\{N(z_7)-N(z_8)\right\}\right] \\ &\quad- e^{-rT}K\left[\left(\frac{D_n}{S_0}\prod_{i=1}^{n-1}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2}N\left(z_6-\sigma\sqrt{T}\right)+\left(\frac{D_n}{U}\prod_{i=1}^{n-1}\frac{D_i}{U_i}\right)^{2\widetilde{\mu}/\sigma^2}\left\{N\left(z_7-\sigma\sqrt{T}\right)-N\left(z_8-\sigma\sqrt{T}\right)\right\}\right], \end{aligned} \tag{28}$$

where

$$z_3 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{S_0}{U}\prod_{i=1}^{n}\left(\frac{D_i}{U_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T},$$

$$z_4 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{U^2}{KS_0}\prod_{i=1}^{n}\left(\frac{U_i}{D_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T},$$

$$z_5 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{U}{S_0}\prod_{i=1}^{n}\left(\frac{U_i}{D_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T},$$

$$z_6 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D_n^2}{US_0}\prod_{i=1}^{n-1}\left(\frac{D_i}{U_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T}, \tag{29}$$

$$z_7 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{U^2S_0}{D_n^2K}\prod_{i=1}^{n-1}\left(\frac{U_i}{D_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T},$$

$$z_8 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{US_0}{D_n^2}\prod_{i=1}^{n-1}\left(\frac{U_i}{D_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T}.$$

Remark 1. Suppose that $n = 1, U_1 = U$, and $D_1 = D$ in Theorem 2 (I). Then, $\mathrm{UIC}_{U_1D_1}$ is equal to UIC_{ud} in Theorem 3.1 of [11].

The following theorem presents the valuation formula of the general chained option, where the down-and-in put option is given when the underlying asset price hits the barriers in the order A_1 or A_2 with down barrier $D(<U_n)$.

Theorem 3. *The valuation formula for the down-and-in put option commencing at a time the underlying asset hits the barriers in the order A_1 or A_2 and across the last up barrier U_n when the strike price K is greater than the down barrier $D(<U_n)$ is*

$$\begin{aligned}
\text{(I) } \mathrm{DIP}_{U_1D_1\cdots U_{n-1}D_{n-1}U_n} &= S_0\left[\left(\frac{U_n}{S_0}\prod_{i=1}^{n-1}\frac{U_i}{D_i}\right)^{2\tilde{\mu}/\sigma^2} N(z_9) + \left(\frac{D}{U_n}\prod_{i=1}^{n-1}\frac{D_i}{U_i}\right)^{2\tilde{\mu}/\sigma^2}\{N(z_{10}) - N(z_1)\}\right] \\
&\quad - e^{-rT}K\left[\left(\frac{U_n}{S_0}\prod_{i=1}^{n-1}\frac{U_i}{D_i}\right)^{2\overline{\mu}/\sigma^2} N\left(z_9 + \sigma\sqrt{T}\right)\right. \\
&\quad \left. + \left(\frac{D}{U_n}\prod_{i=1}^{n-1}\frac{D_i}{U_i}\right)^{\frac{2\tilde{\mu}}{\sigma^2}}\left\{N\left(z_{10} - \sigma\sqrt{T}\right) - N\left(z_1 - \sigma\sqrt{T}\right)\right\}\right], \\
\text{(II) } \mathrm{DIP}_{D_1U_1\cdots D_nU_n} &= S_0\left[\left(\prod_{i=1}^{n}\frac{U_i}{D_i}\right)^{2\tilde{\mu}/\sigma^2} N(z_{11}) + \left(\frac{D}{S_0}\prod_{i=1}^{n}\frac{D_i}{U_i}\right)^{2\tilde{\mu}/\sigma^2}\{N(z_{12}) - N(z_2)\}\right] \\
&\quad - e^{-rT}K\left[\left(\prod_{i=1}^{n}\frac{U_i}{D_i}\right)^{2\overline{\mu}/\sigma^2} N\left(z_{11} + \sigma\sqrt{T}\right)\right. \\
&\quad \left. + \left(\frac{D}{S_0}\prod_{i=1}^{n}\frac{D_i}{U_i}\right)^{2\tilde{\mu}/\sigma^2}\left\{N\left(z_{12} - \sigma\sqrt{T}\right) - N\left(z_2 - \sigma\sqrt{T}\right)\right\}\right],
\end{aligned} \tag{30}$$

where

$$
\begin{aligned}
z_9 &= \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{DS_0}{U_n^2}\prod_{i=1}^{n-1}\left(\frac{D_i}{U_i}\right)^2\right) - \frac{\tilde{\mu}}{\sigma}\sqrt{T},\\
z_{10} &= \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{DS_0}{U_n^2}\prod_{i=1}^{n-1}\left(\frac{D_i}{U_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T},\\
z_{11} &= \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D}{S_0}\prod_{i=1}^{n}\left(\frac{D_i}{U_i}\right)^2\right) - \frac{\tilde{\mu}}{\sigma}\sqrt{T},\\
z_{12} &= \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D}{S_0}\prod_{i=1}^{n}\left(\frac{D_i}{U_i}\right)^2\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T}.
\end{aligned}
\tag{31}
$$

The following theorem presents the valuation formula of the general chained option, where the up-and-in put option is given when the underlying asset price hits the barriers in the order A_3 or A_4 with down barrier $U(>D_n)$.

Theorem 4. *The valuation formula for the up-and-in put option commencing at a time the underlying asset hits the barriers in the order A_3 or A_4 and across the last down barrier D_n when the strike price K is lower than the up barrier $U(>D_n)$ is*

$$
\begin{aligned}
&\text{(I) } \mathrm{UIP}_{U_1D_1\cdots U_nD_n}\\
&\quad = S_0\left(\frac{U}{S_0}\prod_{i=1}^{n}\frac{U_i}{D_i}\right)^{2\tilde{\mu}/\sigma^2} N(-z_4) - e^{-rT}K\left(\frac{U}{S_0}\prod_{i=1}^{n}\frac{U_i}{D_i}\right)^{2\overline{\mu}/\sigma^2} N\left(-z_4+\sigma\sqrt{T}\right),\\
&\text{(II) } \mathrm{UIP}_{D_1U_1\cdots D_{n-1}U_{n-1}D_n}\\
&\quad = S_0\left(\frac{U}{D_n}\prod_{i=1}^{n-1}\frac{U_i}{D_i}\right)^{2\tilde{\mu}/\sigma^2} N(-z_7) - e^{-rT}K\left(\frac{U}{D_n}\prod_{i=1}^{n-1}\frac{U_i}{D_i}\right)^{2\overline{\mu}/\sigma^2} N\left(-z_7+\sigma\sqrt{T}\right).
\end{aligned}
\tag{32}
$$

The closed formulas of the generalized chained option of the knock-out type can be obtained as follows. The down-and-out option can be represented by the difference between the up-and-in option and the down-and-in option. The up-and-out option can be expressed as the difference between the down-and-in option and the up-and-in option:

$$
\begin{aligned}
&(1)\ \mathrm{DOC}_{U_1D_1\cdots U_{n-1}D_{n-1}U_n} = \mathrm{UIC}_{U_1D_1\cdots U_{n-1}D_{n-1}}(U=U_n) - \mathrm{DIC}_{U_1D_1\cdots U_{n-1}D_{n-1}U_n},\\
&(2)\ \mathrm{DOC}_{D_1U_1\cdots D_nU_n} = \mathrm{UIC}_{D_1U_1\cdots U_{n-1}D_n}(U=U_n) - \mathrm{DIC}_{D_1U_1\cdots D_nU_n},\\
&(3)\ \mathrm{UOC}_{U_1D_1\cdots U_nD_n} = \mathrm{DIC}_{U_1D_1\cdots D_{n-1}U_n}(D=D_n) - \mathrm{UIC}_{U_1D_1\cdots U_nD_n},\\
&(4)\ \mathrm{UOC}_{D_1U_1\cdots D_{n-1}U_{n-1}D_n} = \mathrm{DIC}_{D_1U_1\cdots D_{n-1}U_{n-1}}(D=D_n) - \mathrm{UIC}_{D_1U_1\cdots D_{n-1}U_{n-1}D_n},\\
&(5)\ \mathrm{DOP}_{U_1D_1\cdots U_{n-1}D_{n-1}U_n} = \mathrm{UIP}_{U_1D_1\cdots U_{n-1}D_{n-1}}(U=U_n) - \mathrm{DIP}_{U_1D_1\cdots U_{n-1}D_{n-1}U_n},\\
&(6)\ \mathrm{DOP}_{D_1U_1\cdots D_nU_n} = \mathrm{UIP}_{D_1U_1\cdots U_{n-1}D_n}(U=U_n) - \mathrm{DIP}_{D_1U_1\cdots D_nU_n},\\
&(7)\ \mathrm{UOP}_{U_1D_1\cdots U_nD_n} = \mathrm{DIP}_{U_1D_1\cdots D_{n-1}U_n}(D=D_n) - \mathrm{UIP}_{U_1D_1\cdots U_nD_n},\\
&(8)\ \mathrm{UOP}_{D_1U_1\cdots D_{n-1}U_{n-1}D_n} = \mathrm{DIP}_{D_1U_1\cdots D_{n-1}U_{n-1}}(D=D_n) - \mathrm{UIP}_{D_1U_1\cdots D_{n-1}U_{n-1}D_n}.
\end{aligned}
\tag{33}
$$

Remark 2. The generalized chained option of the knock-out type is obtained from the knock-in and knock-out properties of the chained option. The property is that the sum of the knock-in chained option and knock-out chained option is equal to the opposite side knock-in chained option of the previous step.

4. Conclusion

This paper reviews chained options and compares chained options with similar single- and double-barrier options. A chained option is viewed as a barrier option that is chained together, each with a payoff contingent on a specified barrier.

When the underlying asset price hits a primary barrier, a secondary barrier option is given to the primary barrier option holder. If the asset price hits another barrier, a third barrier option is given, and so on. This paper proves the closed formulas of generalized chained options with n-barriers using mathematical induction. These formulas include valuations of down-and-in call options when the underlying asset price hits the barriers in the order A_1 or A_2 and the up-and-in call option commencing at a time the underlying asset hits the barriers in the order A_3 or A_4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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