Hindawi
Journal of Mathematics
Volume 2022, Article ID 9595398, 15 pages
https://doi.org/10.1155/2022/9595398

Research Article

A Study on ψ-Caputo-Type Hybrid Multifractional Differential Equations with Hybrid Boundary Conditions

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Received 18 November 2021; Accepted 13 December 2021; Published 19 January 2022

Academic Editor: Melike Kaplan

In this research paper, we investigate the existence, uniqueness, and Ulam–Hyers stability of hybrid sequential fractional differential equations with multiple fractional derivatives of ψ-Caputo with different orders. Using an advantageous generalization of Krasnoselskii's fixed point theorem, we establish results of at least one solution, whereas the uniqueness of solution is derived via Banach's fixed point theorem. Besides, the Ulam–Hyers stability for the proposed problem is investigated by applying the techniques of nonlinear functional analysis. In the end, we provide an example to illustrate the applicability of our results.

1. Introduction

Over the last couple of years, the concept of fractional differential equations (FDEs) has been validated as being an effective and powerful gadget to model complex and real world phenomena due to their wide range of applications in several fields of sciences and engineering; see [1–5] and the references therein. Recently, FDEs gained the attention of mathematicians and researchers working in different disciplines of science and technology, which resulted in plenty of research papers that have been carried out on FDEs. That made a valuable contribution ranging from the qualitative theory of the solutions of FDEs, such as existence, uniqueness, stability, and controllability to the numerical analysis. Speaking in this context, the stability analysis of functional and differential equations is important in many applications, such as optimization and numerical analysis, where computing the exact solution is rather hard. There are various kinds of stability; one of those types has recently received considerable attention from many mathematicians, so-called Ulam–Hyers (U-H) stability. The source of U-H stability goes back to 1940 by Ulam [6], next by Hyers [7]. A variety of works have been done by many authors in regard of the U-H stability of FDEs; for example, the authors in [8] studied the existence and stability results for implicit FDE. Some recent developments in Ulam's type stability are discussed by Belluot et al. [9]. Ibrahim, in [10], obtained the generalized U-H stability for FDEs. Some approximate analytical methods for solving FDEs can be found in [11–14]; also, computational analyses of some fractional dynamical and biological models were investigated recently, see [15, 16].

On the contrary, quadratic perturbation of nonlinear differentials, also known as the hybrid differential equations, had rapid progress over the last years; this is due to its importance, which lies in the fact that they include perturbations that facilitate the study of such equations by using the perturbation techniques. These equations are also considered as a particular case in dynamic systems. The starting point for this field is when Dhage and Lakshemikantham

[17] formulated a hybrid differential equation, where they investigated the existence and uniqueness of the solutions to the following hybrid equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\varkappa(\theta)}{g(\theta,\varkappa(\theta)}\right) = f(\theta,\varkappa(\theta)), & \text{a.e. } \theta \in [\theta_0, \theta_0 + \mho], \theta_0 \in \mathbb{R}, \mho > 0, \\ \varkappa(\theta_0) = \varkappa_0, & \varkappa_0 \in \mathbb{R}. \end{cases} \tag{1}$$

Their results were based on the fixed point theorem (FPT) for the product of two operators in Banach algebra.

In 2011, Zhao et al. [18] extended Dhage's work [17] to fractional order and studied the existence of solutions to the following Riemann–Liouville (RL)-type hybrid FDEs:

$$\begin{cases} D^p\left(\frac{\varkappa(\theta)}{h(\theta,\varkappa(\theta))}\right) = f(\theta,\varkappa(\theta)), & \text{a.e. } \theta \in [0,\mho], 0 < p \leq 1, \\ \varkappa(0) = 0. \end{cases} \tag{2}$$

After several years, Sitho et al. [19] derived a new existence result for the following hybrid sequential integro-differential equations:

$$\begin{cases} D^p\left[\frac{D^q\varkappa(\theta) - \sum_{i=1}^n I^{\eta_i} g_i(\theta,\varkappa(\theta))}{h(\theta,\varkappa(\theta))}\right] = f(\theta,\varkappa(\theta), I^{\gamma}\varkappa(\theta)), & \theta \in [0,\mho], 0 < p, q \leq 1, \\ \varkappa(0) = 0, \\ D^q\varkappa(0) = 0. \end{cases} \tag{3}$$

In [20], Baitiche et al. studied the existence of solutions for the following hybrid sequential FDEs:

$$\begin{cases} \left({}^cD^{p+1} + \lambda {}^cD^p\right)\left(\frac{\varkappa(\theta)}{h(\theta,\varkappa(\theta))}\right) = f(\theta,\varkappa(\theta)), & \text{a.e. } \theta \in [0,1], 0 < p \leq 2, \\ \varkappa(0) = \varkappa(\eta) = \varkappa(1) = 0, & 0 < \eta < 1. \end{cases} \tag{4}$$

They generalized Darbo's FPT for the product of two operators associated with measures of noncompactness.

Some existence results for ψ-Caputo-type hybrid FDEs are obtained in [21, 22]. For recent analysis techniques in FDEs involving generalized Caputo FD, we refer to [23, 24]. Just recently, Boutiara et al. [25] discussed some qualitative analyses to the following fractional hybrid system:

$$\begin{cases} D^{p_i,\psi}\left[\dfrac{\varkappa_i(\theta)-\sum_{k=1}^{n} I^{\eta_{i_k},\psi} g_{i_k}(\theta,\varkappa(\theta))}{h_i(\theta,\varkappa(\theta))}\right] = f_i(\theta,\varkappa_1(\theta),\varkappa_2(\theta)), \quad \theta\in[a,b], 0<p_i\leq 1, \eta_i>0, \\ \varkappa_i(a)=0, \quad i=1,2. \end{cases} \tag{5}$$

The above findings motivated us to study the existence, uniqueness, and U-H stability of solutions for the following ψ-Caputo hybrid fractional sequential integro-differential equation (for short, ψ-Caputo HFSIDE):

$$L_{\psi}^{p}\left[\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m} I^{\eta_i;\psi} g_i(\theta,\varkappa(\theta))}{h(\theta,\varkappa(\theta))}\right] = f(\theta,\varkappa(\theta),\delta I^{\gamma;\psi}\varkappa(\theta)), \quad \theta\in\mathfrak{J}=[0,\mho], \tag{6}$$

endowed with the hybrid fractional integral boundary conditions:

$$\begin{cases} \varkappa(0)=0, \quad {}^{c}D^{q;\psi}\varkappa(0)=0, \\ \left[\dfrac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m} I^{\eta_i;\psi} g_i(\theta,\varkappa(\theta))}{h(\theta,\varkappa(\theta))}\right]_{\theta=\mho} = \rho\left[\dfrac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m} I^{\eta_i;\psi} g_i(\theta,\varkappa(\theta))}{h(\theta,\varkappa(\theta))}\right]_{\theta=\xi}, \quad 0<\rho,\xi<\mho, \end{cases} \tag{7}$$

with

$$L_{\psi}^{p} = {}^{c}D^{p;\psi} + \lambda\, {}^{c}D^{p-1;\psi}, \tag{8}$$

where $1<p\leq 2$, $0<q\leq 1$, ${}^{c}D^{p;\psi}$ and ${}^{c}D^{q;\psi}$ denote the ψ-Caputo FD of order p, q, respectively, and $I^{\eta_i;\psi}$ and $I^{\gamma;\psi}$ are the ψ-RL fractional integral of order $\eta_i>0$, $(i=1,\ldots,m)$ and $\gamma>0$, respectively, $h\in C(\mathfrak{J}\times\mathbb{R},\mathbb{R}\setminus 0)$, $f\in C(\mathfrak{J}\times\mathbb{R}^2,\mathbb{R})$, $g_i\in C(\mathfrak{J}\times\mathbb{R},\mathbb{R})$ with $g_i(0,0)=0$, and λ is appropriate positive real constants.

The modernity of our proposed problem in contrast with past problems is that, in this work, we consider a kind of general case of boundary value problems in a setup of a ψ-Caputo HFSIDE delineated by (6) and (7). To be sure, the advantage of this work is that the applied FD has the freedom of choice of the kernel ψ which makes it conceivable to bring together and cover most of the preceding results on hybrid FDEs.

The research paper is organized as follows. Section 2 presents some basic mathematical pieces of knowledge required throughout the paper. The main results for ψ-Caputo HFSIDE (6), (7) are proved in Section 3. In Section 4, we give an illustrative numerical example, and Section 5 is related to a brief conclusion.

2. Preliminaries

Let $\mathbf{E}=C(\mathfrak{J},R)$ and $\mathbf{E}^*=C(\mathfrak{J},\mathbb{R}\setminus\{0\})$ be Banach spaces of continuous real-valued functions defined on $\mathfrak{J}$. Clearly, E is a Banach algebra with the norm

$$\|\varkappa\| = \sup_{\theta\in\mathfrak{J}} |\varkappa(\theta)|, \tag{9}$$

and multiplication

$$(\varkappa\overline{\varkappa})(\theta) = \varkappa(\theta)\overline{\varkappa}(\theta), \quad \forall\theta\in\mathfrak{J}. \tag{10}$$

Let an increasing function $\psi\colon \mathfrak{J}\longrightarrow\mathbb{R}$ satisfy $\psi'(\theta)\neq 0$, for all $\theta\in\mathfrak{J}$. For effortlessness, we set $\Psi^r(\theta,s) := \psi'(s)(\psi(\theta)-\psi(s))^r$ and $\Psi_0^r(\theta)=(\psi(\theta)-\psi(0))^r$.

Definition 1 (see [3]). *The ψ -RL fractional integral of order $p(>0)$ of an integrable function $\varkappa\colon [a,\infty)\longrightarrow\mathbb{R}$ is defined by*

$$I^{p;\psi}\varkappa(\theta) = \frac{1}{\Gamma(p)}\int_a^{\theta} \Psi^{p-1}(\theta,s)\varkappa(s)\mathrm{d}s, \quad a<\theta. \tag{11}$$

Definition 2 (see [26]). *The ψ -Caputo FD of order $p(n-1<p<n\in\mathbb{N})$ of a function $\varkappa\in C^n[0,\infty)$ is defined by*

$${}^{c}D^{p;\psi}\varkappa(\theta) = \frac{1}{\Gamma(p-n)}\int_a^{\theta} \Psi^{p-n-1}(\theta,s)D_{\psi}^{n}\varkappa(s)\mathrm{d}s, \quad a<\theta, \tag{12}$$

where $n=[p]+1$ and $D_{\psi}^{n} = ((1/\psi'(\theta))(\mathrm{d}/\mathrm{d}\theta))^n$.

In case, if $1<p\leq 2$, we have

$${}^{c}D^{p;\psi}\varkappa(\theta) = \frac{1}{\Gamma(p-2)}\int_a^{\theta} \Psi^{p-3}(\theta,s)D_{\psi}^{2}\varkappa(s)\mathrm{d}s, \quad a<\theta. \tag{13}$$

As a special case, if $\psi(\theta)=\theta$, then the above definitions reduce to the well-known classical fractional definitions; for more details, see [1,3].

Lemma 1 (see [26]). *Let $p > 0$. The following holds:*

(i) If $\varkappa \in \mathbf{E}$, then

$$ {}^{c}D^{p;\psi} I^{p;\psi} \varkappa(\theta) = \varkappa(\theta), \quad \theta \in \mathfrak{J}. \tag{14} $$

(ii) If $\varkappa \in \mathbf{E}^{n}$, $n-1 < p < n$, then

$$ I^{p;\psi c} D^{p;\psi} \varkappa(\theta) = \varkappa(\theta) - \sum_{k=0}^{n-1} c_k \Psi_0^k(\theta), \quad \theta \in \mathfrak{J}, \tag{15} $$

where $c_k = (D_{\psi}^{k} \varkappa(0)/k!)$.

Concerning the applied FPTs, we will suffice with indication to [27, 28].

3. Main Results

Lemma 2. *Let $1 < p \leq 2$ and $0 < q \leq 1$. For any functions $F \in \mathbf{E}$, $H \in \mathbf{E}^{*}$, and $G_i \in \mathbf{E}$ with $G_i(0) = 0, i = 1, \ldots, 2$, the following linear fractional BVP,*

$$ L_{\psi}^{p} \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right] = F(\theta), \quad \theta \in \mathfrak{J}, \tag{16} $$

supplemented with the conditions,

$$ \begin{cases} \varkappa(0) = 0, \\ {}^{c}D^{q;\psi} \varkappa(0) = 0, \\ \left[\dfrac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right]_{\theta=\varpi} = \rho \left[\dfrac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right]_{\theta=\xi}, \quad 0 < \rho, \xi < \varpi, \end{cases} \tag{17} $$

has a unique solution, that is,

$$ \begin{aligned} \varkappa(\theta) = {} & I^{q;\psi} \left(H(s) \int_0^s I^{p-1;\psi} F(\tau) \psi'(\tau) e^{-\lambda\left(\Psi_0(s) - \Psi_0(\tau)\right)} \mathrm{d}\tau \right)(\theta) \\ & + \left(\int_0^{\varpi} I^{p-1;\psi} F(\tau) \psi'(\tau) e^{-\lambda\left(\Psi_0(\varpi) - \Psi_0(\tau)\right)} \mathrm{d}\tau - \rho \int_0^{\xi} I^{p-1;\psi} F(\tau) \psi'(\tau) e^{-\lambda\left(\Psi_0(\xi) - \Psi_0(\tau)\right)} \mathrm{d}\tau \right) \\ & \times \frac{1}{\Delta} I^{q;\psi} \left(H(s) \left(1 - e^{-\lambda \Psi_0(s)} \right) \right)(\theta) + \sum_{i=1}^{m} I^{\eta_i + q;\psi} G_i(\theta), \end{aligned} \tag{18} $$

where $\Delta = (1 + \rho e^{-\lambda \Psi_0(\xi)} - e^{-\lambda \Psi_0(\varpi)} - \rho) \neq 0$, $\lambda \in \mathbb{R}^{+}$.

proof. Applying the ψ-RL fractional integral of order $p-1$ to both sides of (16) and using Lemma 1, we obtain

$$ \begin{aligned} & {}^{c}D^{1;\psi} \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right] \\ & + \lambda \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right] \\ & = I^{p-1;\psi} F(\theta) + c_0, \quad c_0 \in \mathbb{R}. \end{aligned} \tag{19} $$

By multiplying $\psi'(\theta) e^{\lambda \Psi_0(\theta)}$ to both sides of (16), we find that

$$ \begin{aligned} & e^{\lambda \Psi_0(\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta} \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right] \\ & + \lambda \psi'(\theta) e^{\lambda \Psi_0(\theta)} \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right] \\ & = \psi'(\theta) e^{\lambda \Psi_0(\theta)} I^{p-1;\psi} F(\theta) + c_0 \psi'(\theta) e^{\lambda \Psi_0(\theta)}, \quad c_0 \in \mathbb{R}. \end{aligned} \tag{20} $$

On the contrary, we have

$$ \begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}\theta} \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} . e^{\lambda \Psi_0(\theta)} \right] \\ & = e^{\lambda \Psi_0(\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta} \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right] \\ & + \lambda \psi'(\theta) e^{\lambda \Psi_0(\theta)} \left[\frac{{}^{c}D^{q;\psi} \varkappa(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} G_i(\theta)}{H(\theta)} \right]. \end{aligned} \tag{21} $$

From (20) and (21), we find that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(\theta)}{H(\theta)}.e^{\lambda\Psi_0(\theta)}\right] = I^{p-1;\psi}F(\theta)\psi'(\theta)e^{\lambda\Psi_0(\theta)}+c_0\psi'(\theta)e^{\lambda\Psi_0(\theta)}. \tag{22}$$

Integrating from 0 to θ and using the fact that $G_i(0)=0$, $i=1,\ldots,m$, and from the condition ${}^{c}D^{q;\psi}\varkappa(0)=0$ in (17), we have

$$\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(\theta)}{H(\theta)}.e^{\lambda\Psi_0(\theta)} = \int_0^{\theta}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{\lambda\Psi_0(\tau)}\mathrm{d}\tau+\frac{c_0}{\lambda}\left(e^{-\lambda\Psi_0(\theta)-1}\right)+c_1, \tag{23}$$

where $c_1\in\mathbb{R}$; by multiplying $e^{-\lambda\Psi_0(\theta)}$ to both sides, we obtain

$$\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(\theta)}{H(\theta)} = \int_0^{\theta}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\theta)-\Psi_0(\tau)\right)}\mathrm{d}\tau + \frac{c_0}{\lambda\left(1-e^{-\lambda\Psi_0(\theta)}\right)+c_1e^{-\lambda\Psi_0(\theta)}}. \tag{24}$$

Next, applying ψ-RL fractional integral of order q to both sides of (24) and using Lemma 1, we obtain

$$\begin{aligned}\varkappa(\theta) &= I^{q;\psi}\left(H(s)\left(\int_0^{s}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau+\frac{c_0}{\lambda}\left(1-e^{-\lambda\Psi_0(s)}\right)+c_1e^{-\lambda\Psi_0(s)}\right)\right)(\theta)\\ &\quad+\sum_{i=1}^{m}I^{\eta_i+q;\psi}G_i(s)(\theta)+c_2,\end{aligned} \tag{25}$$

where $c_2\in\mathbb{R}$ with the help of conditions ${}^{c}D^{q;\psi}\varkappa(0)=0$, $\varkappa(0)=0$, and $G_i(0)=0$, $i=1,\ldots,m$; we find $c_1=c_2=0$. Then, we apply the third condition of (17) in (25), and we obtain

$$\begin{aligned}&\int_0^{\varpi}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau+\frac{c_0}{\lambda}\left(1-e^{-\lambda\Psi_0(\varpi)}\right)\\ &=\rho\int_0^{\xi}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau+\frac{c_0}{\lambda}\left(1-e^{-\lambda\Psi_0(\xi)}\right).\end{aligned} \tag{26}$$

Some computations give us

$$\begin{aligned}c_0 = \frac{\lambda}{\Delta}\Bigg(&\int_0^{\varpi}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\\ &-\rho\int_0^{\xi}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau).\end{aligned} \tag{27}$$

Inserting c_0, c_1, and c_2 in (25), leads to solution (18).

Conversely, by Lemma 1 and by taking ${}^{c}D^{q;\psi}$ on both sides of (25), we obtain

$$\begin{aligned}\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(\theta)}{H(\theta)} &= \int_0^{\theta}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\theta)-\Psi_0(\tau)\right)}\mathrm{d}\tau\\ &\quad+\left(\int_0^{\varpi}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau-\rho\int_0^{\xi}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right)\\ &\quad\frac{1-e^{-\lambda\Psi_0(\theta)}}{\Delta}.\end{aligned} \tag{28}$$

Next, operating ${}^{c}D^{p;\psi}+\lambda{}^{c}D^{p-1;\psi}$ on both sides of the above equation, with the help of Lemma 1, we obtain

$$\begin{aligned}
&\left[{}^{c}D^{p;\psi}+\lambda{}^{c}D^{p-1;\psi}\right]\left[\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(\theta)}{H(\theta)}\right]\\
&=\left[{}^{c}D^{p;\psi}+\lambda{}^{c}D^{p-1;\psi}\right]\left(\int_0^{\theta}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\theta)-\Psi_0(\tau)\right)}\mathrm{d}\tau+\left(\int_0^{\varpi}I^{p-1;\psi}F(\tau)\psi'(\tau)\right.\right.\\
&\left.\left.\times e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau-\rho\int_0^{\xi}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right)\frac{1-e^{-\lambda\Psi_0(\theta)}}{\Delta}\right)\\
&=\left[{}^{c}D^{p;\psi}+\lambda{}^{c}D^{p-1;\psi}\right]\int_0^{\theta}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\theta)-\Psi_0(\tau)\right)}\mathrm{d}\tau-\frac{c_0}{\lambda}\\
&={}^{c}D^{p-1;\psi}\left[\frac{1}{\psi'(\theta)}\frac{\mathrm{d}}{\mathrm{d}\theta}+\lambda\right]\int_0^{\theta}I^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\theta)-\Psi_0(\tau)\right)}\mathrm{d}\tau\\
&={}^{c}D^{p-1;\psi}I^{p-1;\psi}F(\theta)\\
&=F(\theta).
\end{aligned}\tag{29}$$

Now, it remains to review the boundary conditions (17) of our problem (16). Substituting $\theta=0$ in (18) with the fact that $g_i(0)=0, i=1,\ldots,m$, leads to $u(0)=0$. Next, we apply ${}^{c}D^{q;\psi}$ on (18); then, we substitute $\theta=0$; it follows that ${}^{c}D^{q;\psi}\varkappa(0)=0$. Substituting $\theta=\varpi$ and $\theta=\xi$, we find that the two resulting equations are equal, and from it, we get that

$$\begin{aligned}
&\left[\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(\theta)}{H(\theta)}\right]_{\theta=\varpi}\\
&=\rho\left[\frac{{}^{c}D^{q;\psi}\varkappa(\theta)-\sum_{i=1}^{m}I^{\eta_i;\psi}G_i(\theta)}{H(\theta)}\right]_{\theta=\xi}.
\end{aligned}\tag{30}$$

This means that $\varkappa(\theta)$ satisfies (16) and (17). Therefore, $\varkappa(\theta)$ is solution of problem (16) and (17). □

Lemma 3. *For $F,H\in\mathbf{E}$,we have*

(1) $\int_0^{\theta}(\Psi^{q-1}(\theta,s)/\Gamma(q))H(s)\int_0^{s}\int_0^{\tau}(\Psi^{p-2}(\tau,\sigma)/\Gamma(p-1))F(\sigma)d\sigma\psi'(\tau)e^{-\lambda}(\Psi_0(s)-\Psi_0(\tau))d\tau ds\le(\Psi_0^{p+q-1}(\varpi)/\lambda\Gamma(p+q)\Gamma(q))(1-e^{-\lambda\Psi_0(\theta)})\|H\|\|F\|$

(2) $|\int_0^{\varpi}\int_0^{\tau}(\Psi^{p-2}(\theta,\sigma)/\Gamma(p-1))F(\sigma)d\sigma\psi'(\tau)e^{-\lambda(\Psi_0(\varpi)-\Psi_0(\tau))}d\tau|\le(\Psi_0^{p-1}(\varpi)/\lambda\Gamma(p))(1-e^{-\lambda\Psi_0(\varpi)})\|F\|$

(3) $|\int_0^{\xi}\int_0^{\tau}(\psi'^{p-2}(\theta,\sigma)/\Gamma(p-1))F(\sigma)d\sigma\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau|\le(\Psi_0^{p-1}(\xi)/\lambda\Gamma(p))(1-e^{-\lambda\Psi_0(\xi)})\|F\|$

proof. To prove property *(1), we have*

$$\begin{aligned}
&\int_0^{\tau}\frac{\Psi^{p-2}(\theta,\sigma)}{\Gamma(p-1)}\mathrm{d}\sigma=\frac{\Psi_0^{p-1}(\tau)}{\Gamma(p)},\\
&\int_0^{s}\frac{\Psi_0^{p-1}(\tau)}{\Gamma(p)}\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\\
&\le\frac{\Psi_0^{p-1}(s)}{\lambda\Gamma(p)}\int_0^{s}\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\\
&=\frac{\Psi_0^{p-1}(s)}{\lambda\Gamma(p)}\left(1-e^{-\lambda\Psi_0(s)}\right).
\end{aligned}\tag{31}$$

From the above integrals and left side of *(1)*, we obtain

$$\begin{aligned}
&\left|\int_0^{\theta}\frac{\Psi^{q-1}(\theta,s)}{\Gamma(q)}H(s)\int_0^{s}\int_0^{\tau}\frac{\Psi^{p-2}(\theta,\sigma)}{\Gamma(p-1)}F(\sigma)\mathrm{d}\sigma\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\mathrm{d}s\right|\\
&\le\|H\|\|F\|\int_0^{\theta}\frac{\Psi^{q-1}(\theta,s)}{\Gamma(q)}\frac{\Psi_0^{p-1}(s)}{\lambda\Gamma(p)}\left(1-e^{-\lambda\Psi_0(s)}\right)\mathrm{d}s
\end{aligned}$$

$$\leq \|H\|\|F\|\frac{\left(1-e^{-\lambda\Psi_0(\varpi)}\right)}{\lambda\Gamma(p)}\int_0^\theta \frac{\Psi^{q-1}(\theta,s)}{\Gamma(q)}\Psi_0^{p-1}(s)\mathrm{d}s$$
$$\leq \frac{\Psi_0^{p+q-1}(\varpi)}{\lambda\Gamma(p+q)}\left(1-e^{-\lambda\Psi_0(\varpi)}\right)\|H\|\|F\|. \tag{32}$$

The proofs of properties *(2)* and *(3)* are similar to proof of *(1)*. □

Now, we consider the following assumptions:

(H1) $h\in C(\mathfrak{J}\times\mathbb{R},\mathbb{R}\setminus 0)$ and $f\in C(\mathfrak{J}\times\mathbb{R}^2,\mathbb{R})$, and there exist positive functions $L(\theta)$ and $M(\theta)$, such that

$$\begin{aligned}\left|h(\theta,\varkappa)-h(\theta,\varkappa^*)\right|&\leq L(\theta)\left|\varkappa-\varkappa^*\right|,\\ \left|f(\theta,\varkappa_1,\varkappa_2)-f(\theta,\varkappa_1^*,\varkappa_2^*)\right|&\leq M(\theta)\left(\left|\varkappa_1-\varkappa_1^*\right|+\left|\varkappa_2-\varkappa_2^*\right|\right),\end{aligned} \tag{33}$$

for $\theta\in\mathfrak{J}$ and $\varkappa_1,\varkappa_2,\varkappa_1^*,\varkappa_2^*\in\mathbb{R}$.

(H2) There exist functions $\varphi_i,\chi,\vartheta\in\mathbf{E}$ such that

$$\begin{aligned}\left|g_i(\theta,\varkappa)\right|&\leq\varphi_i(\theta), &&\text{for each }\theta,\varkappa\in\mathfrak{J}\times\mathbb{R},\\ |h(\theta,\varkappa)|&\leq\chi(\theta), &&\text{for each }\theta,\varkappa\in\mathfrak{J}\times\mathbb{R},\\ |f(\theta,\varkappa,\overline{\varkappa})|&\leq\vartheta(\theta), &&\text{for each }\theta,\varkappa,\overline{\varkappa}\in\mathfrak{J}\times\mathbb{R}\times\mathbb{R}.\end{aligned} \tag{34}$$

(H3) There exist $0<\Lambda$ and $\Upsilon<1$, such that

$$\frac{1-e^{-\lambda\Psi_0(\varpi)}}{\lambda}\left(\Lambda\|\chi\|\|M\|+\Upsilon\|L\|\|\vartheta\|\right)<1, \tag{35}$$

where

$$\begin{aligned}\Lambda&=\frac{\Psi_0^{p+q-1}(\varpi)}{\Gamma(p+q)}+\frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p+q-2}(\varpi)}{\Gamma(\gamma+p+q-1)}+\frac{\Psi_0^{q}(\varpi)}{\Delta\Gamma(q+1)}\\ &\quad\times\left(\left(1-e^{-\lambda\Psi_0(\varpi)}\right)\left(\frac{\Psi_0^{p-1}(\varpi)}{\Gamma(p)}+\frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p-2}(\varpi)}{\Gamma(\gamma+p-1)}\right)\right)\\ &\quad+\rho\left(1-e^{-\lambda\Psi_0(\xi)}\right)\left(\frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)}+\frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p-2}(\xi)}{\Gamma(\gamma+p-1)}\right)\Bigg),\\ \Upsilon&=\frac{\Psi_0^{p+q-1}(\varpi)}{\Gamma(p+q)}+\frac{\Psi_0^{q}(\varpi)}{\Delta\Gamma(q+1)\Gamma(p-1)}\left(\left(1-e^{-\lambda\Psi_0(\varpi)}\right)\Psi_0^{p-2}(\varpi)\right.\\ &\quad\left.+\rho\left(1-e^{-\lambda\Psi_0(\xi)}\right)\Psi_0^{p-2}(\xi)\right).\end{aligned} \tag{36}$$

3.1. Existence of Solutions. In this section, we prove the existence of a solution for problems (6) and (7) by applying Dhage FPT [27].

Theorem 1. *Suppose (H1)–(H3) hold; then, problems (6) and (7) have at least one solution in* $\mathbf{E}$.

proof. First, we set $\sup_{\theta\in\mathfrak{J}}|L(\theta)|=\|L\|$, $\sup_{\theta\in\mathfrak{J}}|M(\theta)|=\|M\|$, $\sup_{\theta\in\mathfrak{J}}|\chi(\theta)|=\|\chi\|$, $\sup_{\theta\in\mathfrak{J}}|\vartheta(\theta)|=\|\vartheta\|$, and $\sup_{\theta\in\mathfrak{J}}|\varphi_i(\theta)|=\|\varphi\|, i=1,2,\ldots,m$.

Now, we define $B_r\subset\mathbf{E}$ as

$$B_r=\{\varkappa\in E:\ \|\varkappa\|\leq r\}. \tag{37}$$

Define two operators $C:\mathbf{E}\longrightarrow\mathbf{E}$ and $D:\mathbf{E}\longrightarrow\mathbf{E}$ as

$$\begin{aligned}C\varkappa(\theta)&=\frac{1}{\Gamma(p-1)}\int_0^\theta\psi'(s)(\psi(\theta)-\psi(s))^{p-2}\\ &\qquad\cdot f\left(s,\varkappa(s),\delta I^{\gamma;\psi}\varkappa(s)\right)\mathrm{d}s,\quad\theta\in\mathfrak{J},\\ D\varkappa(\theta)&=h(\theta,\varkappa(\theta)),\quad\theta\in\mathfrak{J}.\end{aligned} \tag{38}$$

Then, using assumptions (H1)-(H2), we have, for $\varkappa,\overline{\varkappa}\in B_r$ and each $\theta\in\mathfrak{J}$,

$$\begin{aligned}
&|C\varkappa(\theta)-C\overline{\varkappa}(\theta)| \\
&\leq \frac{1}{\Gamma S(p-1)}\int_0^\theta \Psi^{p-2}(\theta,s)\left|f\left(s,\varkappa(s),\delta I^{\gamma;\psi}\varkappa(s)\right)-f\left(s,\overline{\varkappa}(s),\delta I^{\gamma;\psi}\overline{\varkappa}(s)\right)\right|\mathrm{d}s \\
&\leq \frac{1}{\Gamma(p-1)}\int_0^\theta \Psi^{p-2}(\theta,s)M(s)\left(|\varkappa(s)-\overline{\varkappa}(s)|+\frac{\delta}{\Gamma(\gamma)}\int_0^s \Psi^{\gamma-1}(\tau,s)|\varkappa(\tau)-\overline{\varkappa}(\tau)|\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \|M\|\|\varkappa(\cdot)-\overline{\varkappa}(\cdot)\|\frac{1}{\Gamma(p-1)}\int_0^\theta \Psi^{p-2}(\theta,s)\left(1+\frac{\delta}{\Gamma(\gamma+1)}\Psi_0^{\gamma}(s)\right)\mathrm{d}s \\
&\leq \left(\frac{1}{\Gamma(p)}\Psi_0^{p-1}(\theta)+\frac{\delta\Gamma(\gamma-1)}{\Gamma(\gamma+p-1)}\Psi_0^{\gamma+p-2}(\theta)\right)\|M\|\|\varkappa(\cdot)-\overline{\varkappa}(\cdot)\|, \\
&|C\varkappa(\theta)|\leq \frac{\Psi_0^{p-2}(\theta)}{\Gamma(p-1)}\|\vartheta\|.
\end{aligned} \tag{39}$$

Also,

$$|D\varkappa(\theta)-D\overline{\varkappa}(\theta)|\leq \|L\|\|\varkappa(\cdot)-\overline{\varkappa}(\cdot)\|,\quad |D\varkappa(\theta)|\leq \|\chi\|. \tag{40}$$

Now, we also consider two operators A: $\mathbf{E}\longrightarrow\mathbf{E}$ and B: $B_r\longrightarrow\mathbf{E}$ defined by

$$\begin{aligned}
A\varkappa(\theta) &= I^{q;\psi}\left(D\varkappa(s)\int_0^s C\varkappa(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right)(\theta) \\
&\quad +\frac{1}{\Delta}I^{q;\psi}\left(D\varkappa(s)\left(1-e^{-\lambda\Psi_0(s)}\right)\right)(\theta) \\
&\quad \times\left(\int_0^{\varpi} C\varkappa(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau-\rho\int_0^{\xi} C\varkappa(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right), \\
B\varkappa(\theta) &= \sum_{i=1}^m I^{\eta_i+q;\psi}g_i(s,\varkappa(s))(\theta).
\end{aligned} \tag{41}$$

We need to prove that A and B satisfy all assumptions of Dhage's theorem [27]. This can be proven in the forthcoming steps:

Step 1: A is a contraction map. Indeed, let $\varkappa(\theta),\overline{\varkappa}(\theta)\in B_r$. Then,

$$\begin{aligned}
&|A\varkappa(\theta)-A\overline{\varkappa}(\theta)| \\
&\leq I^{q;\psi}\left(|D\varkappa(s)|\int_0^s |C\varkappa(\tau)-C\overline{\varkappa}(\tau)|\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right)(\theta) \\
&\quad + I^{q;\psi}\left(|D\varkappa(s)-D\overline{\varkappa}(s)|\%\int_0^s |C\overline{\varkappa}(\tau)|\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right)(\theta) \\
&\quad +\frac{1}{\Delta}I^{q;\psi}\left(\|D\varkappa(s)\|\left(\%1-e^{-\lambda\Psi_0(s)}\right)\right)(\theta)\left(\int_0^{\varpi}|\%C\varkappa(\tau)-C\overline{\varkappa}(\tau)|\psi'(\tau)\right. \\
&\quad \left.\times e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau+\rho\int_0^{\xi}|C\varkappa(\tau)-C\overline{\varkappa}(\tau)|\%\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\%\right)
\end{aligned}$$

$$+\frac{1}{\Delta}I^{q;\psi}\Big(|D\varkappa(s)-D\%\bar{\varkappa}(s)|\Big(1-e^{-\lambda\Psi_0(s)}\Big)\Big)(\theta)$$
$$\left(\int_0^{\varpi}|C\varkappa(\tau)|\psi'(\tau)\times e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau-\rho\int_0^{\xi}|C\varkappa(\tau)|\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right). \tag{42}$$

Using Lemma 3 and hypotheses (H1)-(H2), we obtain

$$\begin{aligned}
&|A\varkappa(\theta)-A\bar{\varkappa}(\theta)|\\
&\leq\frac{1-e^{-\lambda\Psi_0(\varpi)}}{\lambda}\Bigg\{\|\chi\|\|M\|\Bigg(\frac{\Psi_0^{p+q-1}(\varpi)}{\Gamma(p+q)}+\frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p+q-2}(\varpi)}{\Gamma(\gamma+p+q-1)}\\
&\quad+\frac{\Psi_0^{q}(\varpi)}{\Delta\Gamma(q+1)}\Bigg(\Big(1-e^{-\lambda\Psi_0(\varpi)}\Big)\Bigg(\frac{\Psi_0^{p-1}(\varpi)}{\Gamma(p)}+\frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p-2}(\varpi)}{\Gamma(\gamma+p-1)}\Bigg)\\
&\quad+\rho\Big(1-e^{-\lambda\Psi_0(\xi)}\Big)\Bigg(\frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)}+\frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p-2}(\xi)}{\Gamma(\gamma+p-1)}\Bigg)\Bigg)\Bigg)\\
&\quad+\|L\|\|\vartheta\|\Bigg(\frac{\Psi_0^{p+q-1}(\varpi)}{\Gamma(p+q)}+\frac{\Psi_0^{q}(\varpi)}{\Delta\Gamma(q+1)\Gamma(p-1)}\\
&\quad\times\Big(\Big(1-e^{-\lambda\Psi_0(\varpi)}\Big)\Psi_0^{p-2}(\varpi)+\rho\Big(1-e^{-\lambda\Psi_0(\xi)}\Big)\\
&\quad\times\Psi_0^{p-2}(\xi)\Big)\Bigg)\Bigg\}\|\varkappa(\cdot)-\bar{\varkappa}(\cdot)\|.
\end{aligned}\tag{43}$$

Moreover,

$$\|A\varkappa(\theta)-A\bar{\varkappa}(\theta)\|\leq\frac{1-e^{-\lambda\Psi_0(\varpi)}}{\lambda}(\Lambda\|\chi\|\|M\|+\Upsilon\|L\|\|\vartheta\|)\|\varkappa-\bar{\varkappa}\|. \tag{44}$$

Hence, by (50), A is a contraction map.

Step 2 : B is compact and continuous on B_r. Firstly, we prove that B is continuous on B_r.

Let $\varkappa_n(\theta)$ be a sequence such that $\varkappa_n(\theta)\longrightarrow B_r$ in B_r. It follows from Lebesgue dominant convergence theorem that, for all $\theta\in\mathfrak{I}$,

$$\begin{aligned}
B\varkappa_n(\theta)&=\lim_{n\longrightarrow\infty}\sum_{i=1}^{m}\frac{1\%}{\Gamma(\eta_i+q)}\int_a^{\theta}\psi'(s)(\psi(\theta)-\psi(s))^{\eta_i+q-1}g_i(s,\varkappa_n(s))\mathrm{d}s\\
&=\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_a^{\theta}\Psi^{\eta_i+q-1}(\theta,s)\lim_{n\longrightarrow\infty}g\%_i(s,\varkappa_n(s))\mathrm{d}s\\
&=\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_a^{\theta}\Psi^{\eta_i+q-1}(\theta,s)g_i(s,\varkappa(s))\mathrm{d}s.
\end{aligned}\tag{45}$$

Hence, $\lim_{n\longrightarrow\infty} B\varkappa_n(\theta) = B\varkappa(\theta)$. Thus, B is a continuous on B_r. Besides, we prove that B is uniformly bounded on B_r. Indeed, for any $\varkappa \in B_r$, we have

$$\begin{aligned}\|B\varkappa(\theta)\| &\leq \sup_{\theta\in\mathfrak{I}}\left\{\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_a^\theta \Psi^{\eta_i+q-1}(\theta,s)\left|g_i(s,\varkappa(s))\right|\mathrm{d}s\right\}\\ &\leq \sum_{i=1}^{m}\frac{\Psi_0^{\eta_i+q}(\varpi)}{\Gamma(\eta_i+q+1)}\|\varphi_i\| \coloneqq \Phi.\end{aligned} \tag{46}$$

Therefore, $B\varkappa(\theta) \leq \Phi$, for all $\theta \in \mathfrak{I}$, which implies that B is uniformly bounded on B_r. Now, we show that B is equicontinuous. Let $\theta_1, \theta_2 \in \mathfrak{I}$ with $\theta_1 > \theta_2$. Then, for any $\varkappa(\theta) \in B_r$, we have

$$\begin{aligned}&\left\|B\varkappa(\theta_1) - B\varkappa(\theta_2)\right\|\\ &\leq \sup_{\theta\in\mathfrak{I}}\left\{\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\left|\int_0^{\theta_2}\left(\Psi^{\eta_i+q-1}(\theta_1,s) - \Psi^{\eta_i+q-1}(\theta_2,s)\right)g_i(s,\varkappa(s))\mathrm{d}s\right|\right.\\ &\left.+\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\left|\int_{\theta_2}^{\theta_1}\Psi^{\eta_i+q-1}(\theta_1,s)g_i(s,\varkappa(s))\mathrm{d}s\right|\right\}\\ &\leq \sum_{i=1}^{m}\frac{\|\varphi_i\|}{\Gamma(\eta_i+q+1)}\left(2(\psi(\theta_1)-\psi(\theta_2))^{\eta_i+q} + \left|\Psi_0^{\eta_i+q}(\theta_2) - \Psi_0^{\eta_i+q}(\theta_1)\right|\right)\\ &\leq \sum_{i=1}^{m}\frac{2\|\varphi_i\|}{\Gamma(\eta_i+q+1)}(\psi(\theta_1)-\psi(\theta_2))^{\eta_i+q}.\end{aligned} \tag{47}$$

As $\theta_2 \longrightarrow \theta_1$, $\|B\varkappa(\theta_1) - B\varkappa(\theta_2)\| \longrightarrow 0$. This means that B is equicontinuous. Thus, Arzelá–Ascoli theorem shows that B is a compact operator on B_r.

Step 3 : we prove that $\varkappa = A\varkappa + B\overline{\varkappa}$, for all $\overline{\varkappa} \in B_r \Rightarrow \varkappa \in B_r$. For any $\overline{\varkappa} \in B_r$, we have

$$\begin{aligned}\|\varkappa(\theta)\| &= \|A\varkappa(\theta) + B\overline{\varkappa}(\theta)\|\\ &\leq \|A\varkappa(\theta)\| + \|B\overline{\varkappa}(\theta)\|\\ &\leq \sup_{\theta\in\mathfrak{I}}\left\{\left|I^{q;\psi}\left(D\varkappa(s)\int_0^s C\varkappa(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right)(\theta)\right.\right.\\ &\quad+\left(\int_0^{\varpi}C\varkappa(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau - \rho\int_0^{\xi}C\varkappa(\tau)\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\right)\\ &\quad\left.\left.+\frac{1}{\Delta}I^{q;\psi}\left(D\varkappa(s)\left(1-e^{-\lambda\Psi_0(s)}\right)\right)(\theta)\right| + \left|\sum_{i=1}^{m}I^{\eta_i+q;\psi}g_i(s,\overline{\varkappa}(s))(\theta)\right|\right\}\\ &\leq \frac{\left(1-e^{-\lambda\Psi_0(\varpi)}\right)}{\lambda}\left(\frac{\Psi_0^{p+q-1}(\varpi)}{\Gamma(q)\Gamma(p+q)} + \frac{\Psi_0^{q}(\varpi)}{\Gamma(q+1)\Delta}\right.\\ &\quad\left.\times\left(\frac{\Psi_0^{p-1}(\varpi)}{\Gamma(p)}\left(1-e^{-\lambda\Psi_0(\varpi)}\right) - \rho\frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)}\left(1-e^{-\lambda\Psi_0(\xi)}\right)\right)\right)\\ &\quad\times\|\vartheta\|\|\chi\| + \sum_{i=1}^{m}\frac{\Psi_0^{\eta_i+q}(\varpi)}{\Gamma(\eta_i+q+1)}\|\varphi_i\|\\ &\leq r,\end{aligned} \tag{48}$$

which implies $\|\varkappa\| \leq r$, so $\varkappa \in B_r$. Hence, all assumptions of Dhage's theorem [27] are satisfied. So, the equation $\varkappa(\theta) = A\varkappa(\theta) + B\varkappa(\theta)$ has at least one solution in B_r. So, there exists a solution of problems (6) and (7) in $\mathfrak{J} = [0, \varpi]$. □

3.2. Uniqueness of Solutions. Here, we prove the uniqueness theorem of (6) and (7) relying on Banach's FPT [28].

Theorem 2. *Suppose that (H1)-(H2) and the following hypothesis hold:*

(H4) $g_i \in C(\mathfrak{J} \times \mathbb{R}, \mathbb{R})$, and there exist positive function $K_i(\theta)$, such that

$$\left|g_i(\theta, \varkappa) - g_i(\theta, \overline{\varkappa})\right| \leq K_i(\theta)|\varkappa - \overline{\varkappa}|. \tag{49}$$

If

$$\Xi := \frac{1 - e^{-\lambda \Psi_0(\varpi)}}{\lambda}(\Lambda\|\chi\|\|M\| + \Upsilon\|L\|\|\vartheta\|) + \sum_{i=1}^{m} \frac{\|K_i\|}{\Gamma(\eta_i + q + 1)} \Psi_0^{\eta_i + q}(\varpi) < 1, \tag{50}$$

then problems (6) and (7) have a unique solution.

proof. We set the operator G: $\mathbf{E} \longrightarrow \mathbf{E}$ as

$$G\varkappa(\theta) = A\varkappa(\theta) + B\varkappa(\theta). \tag{51}$$

Consider $B_R = \{\varkappa \in \mathbf{E}: \|\varkappa\| \leq R\}$, and we set $\sup_{\theta\in\mathfrak{J}}|K_i(\theta)| = \|K\|, i = 1, 2, \ldots, m$. First, we show that $G(B_R) \in B_R$. As in the previous proof (Step 3) of Theorem 1, we can obtain the following.

For $\varkappa \in B_R$ and $\theta \in \mathfrak{J}$,

$$\begin{aligned}\|G\varkappa\| &\leq \frac{\left(1 - e^{-\lambda\Psi_0(\varpi)}\right)}{\lambda}\left(\frac{\Psi_0^{p+q-1}(\varpi)}{\Gamma(q)\Gamma(p+q)} + \frac{\Psi_0^{q}(\varpi)}{\Gamma(q+1)\Delta}\left(\frac{\Psi_0^{p-1}(\varpi)}{\Gamma(p)}\left(1 - e^{-\lambda\Psi_0(\varpi)}\right)\right.\right.\\ &\quad\left.\left.- \rho\frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)}\left(1 - e^{-\lambda\Psi_0(\xi)}\right)\right)\right)\|\vartheta\|\|\chi\| + \sum_{i=1}^{m}\frac{\Psi_0^{\eta_i+q}(\varpi)}{\Gamma(\eta_i+q+1)}\|\varphi_i\|\\ &\leq R.\end{aligned} \tag{52}$$

This shows that $G(B_R) \subseteq B_R$. Next, we prove that G is a contraction. For $\varkappa, \overline{\varkappa} \in B_R$,

$$\|G\varkappa(\theta) - G\overline{\varkappa}(\theta)\| \leq \|A\varkappa(\theta) - A\overline{\varkappa}(\theta)\| + \|B\varkappa(\theta) - B\overline{\varkappa}(\theta)\|, \tag{53}$$

$$\begin{aligned}&\|B\varkappa(\theta) - B\overline{\varkappa}(\theta)\|\\ &\leq \sup_{\theta\in\mathfrak{J}}\left\{\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i + q)}\int_0^\theta \Psi^{\eta_i+q-1}(\theta, s)\left|g_i(s, \varkappa(s)) - g_i(s, \overline{\varkappa}(s))\right|ds\right\}\\ &\leq \sum_{i=1}^{m}\frac{\|K_i\|}{\Gamma(\eta_i + q + 1)}\Psi_0^{\eta_i+q}(\varpi)\|\varkappa - \overline{\varkappa}\|.\end{aligned} \tag{54}$$

From (44), (50), and (54), we obtain

$$\|G\varkappa(\theta) - G\overline{\varkappa}(\theta)\| \leq \left(\frac{1 - e^{-\lambda\Psi_0(\varpi)}}{\lambda}(\Lambda\|\chi\|\|M\| + \Upsilon\|L\|\|\vartheta\|) + \sum_{i=1}^{m}\frac{\|K_i\|}{\Gamma(\eta_i + q + 1)}\Psi_0^{\eta_i+q}(\varpi)\right)\|\varkappa - \overline{\varkappa}\|. \tag{55}$$

As $\Xi < 1$, G is contractive map. Consequently, by Banach's FPT [28], we conclude that G has a unique fixed point, which is a solution of (6) and (7). □

3.3. Stability Analysis. In this portion, we discuss the U-H and generalized U-H stabilities of the solution of the proposed problem. We adopt the following definitions from [29].

Let $\varepsilon > 0$. Consider the subsequent inequality:

$$\left|L_\psi^p\left[\frac{{}^cD^{q;\psi}\varkappa(\theta) - \sum_{i=1}^m I^{\eta_i;\psi}g_i(\theta, \varkappa(\theta))}{h(\theta, \varkappa(\theta))}\right] - f\left(\theta, \varkappa(\theta), \delta I^{\gamma;\psi}\varkappa(\theta)\right)\right| \leq \varepsilon, \quad \theta \in \mathfrak{J}. \tag{56}$$

Definition 3. *Problems* (6) and (7) *is said to be U-H stable if there exists*$C_f > 0$*s.t for each*$\varepsilon > 0$, *and for each solution*$w \in \mathbf{E}$*of* (56), *there exists a solution*$\varkappa \in \mathbf{E}$*of* (6) and (7) *with*

$$|w(\theta) - \varkappa(\theta)| \le C_f \varepsilon, \quad \theta \in \mathfrak{J}. \tag{57}$$

Definition 4. *Problems* (6) and (7) *are said to be generalized U-H stable if there exists*$\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$*with*$\Theta(0) = 0$*s.t; for each solution*$w \in \mathbf{E}$*of* (56), *there exists a solution*$\varkappa \in \mathbf{E}$*of* (6), (7) *with*

$$|w(\theta) - \varkappa(\theta)| \le \Theta(\varepsilon), \quad \theta \in \mathfrak{J}. \tag{58}$$

Remark 1. *A function* $w \in \mathbf{E}$ *satisfies problems* (6) and (7) *if there exists a*$\phi \in \mathbf{E}$*(which depends on*w*) such that*

(i) $|\phi(\theta)| \le \varepsilon$, $\theta \in \mathfrak{J}$.

(ii) *For*$\theta \in \mathfrak{J}$,

$$\begin{aligned} &L_{\psi}^{p}\left[\frac{{}^{c}D^{q;\psi}w(\theta) - \sum_{i=1}^{m} I^{\eta_i;\psi} g_i(\theta, w(\theta))}{h(\theta, w(\theta))}\right] \\ &= f\left(\theta, w(\theta), \delta I^{\gamma;\psi} w(\theta)\right) + \phi(\theta). \end{aligned} \tag{59}$$

Theorem 3. *Let (H1) and (H3) are fulfilled. Then, problems (6) and (7) are U-H and generalized U-H stable*

proof. Let $w \in \mathbf{E}$ be a solution of inequality (56), for each $\varepsilon > 0$. Then, from Remark 1 and Lemma 2, we have

$$\begin{aligned} w(\theta) = {} & I^{q;\psi}\left(h(s, w(s)) \int_0^s I^{p-1;\psi}\left(f\left(\theta, w(\theta), \delta I^{\gamma;\psi} w(\theta)\right) + \phi(\theta)\right)(\tau)\psi'(\tau) e^{-\lambda\left(\Psi_0(s) - \Psi_0(\tau)\right)} \mathrm{d}\tau\right)(\theta) \\ & + \left(\int_0^{\varpi} I^{p-1;\psi}\left(f\left(\theta, w(\theta), \delta I^{\gamma;\psi} w(\theta)\right) + \phi(\theta)\right)(\tau)\psi'(\tau) e^{-\lambda\left(\Psi_0(\varpi) - \Psi_0(\tau)\right)} \mathrm{d}\tau \right. \\ & \left. - \rho \int_0^{\xi} I^{p-1;\psi}\left(f\left(\theta, w(\theta), \delta I^{\gamma;\psi} w(\theta)\right) + \phi(\theta)\right)(\tau)\psi'(\tau) e^{-\lambda\left(\Psi_0(\xi) - \Psi_0(\tau)\right)} \mathrm{d}\tau\right) \\ & \times \frac{1}{\Delta} I^{q;\psi}\left(h(s, w(s))\left(1 - e^{-\lambda \Psi_0(s)}\right)\right)(\theta) + \sum_{i=1}^{m} I^{\eta_i + q;\psi} g_i(s, w(s))(\theta). \end{aligned} \tag{60}$$

Then, by Remark 1, Lemma 2, and (H1)-(H3), we obtain

$$\begin{aligned} |w(\theta) - Gw(\theta)| = {} & \left|I^{q;\psi}\left(h(s, w(s)) \int_0^s I^{p-1;\psi} \phi(\tau)\psi'(\tau) e^{-\lambda\left(\Psi_0(s) - \Psi_0(\tau)\right)} \mathrm{d}\tau\right)(\theta) \right. \\ & + \left(\int_0^{\varpi} I^{p-1;\psi} \phi(\tau)\psi'(\tau) e^{-\lambda\left(\Psi_0(\varpi) - \Psi_0(\tau)\right)} \mathrm{d}\tau \right. \\ & \left. - \rho \int_0^{\xi} I^{p-1;\psi} \phi(\tau)\psi'(\tau) e^{-\lambda\left(\Psi_0(\xi) - \Psi_0(\tau)\right)} \mathrm{d}\tau\right) \\ & \left. \times \frac{1}{\Delta} I^{q;\psi}\left(h(s, w(s))\left(1 - e^{-\lambda \Psi_0(s)}\right)\right)(\theta)\right| \\ & \le \varepsilon \|\chi\| \Omega, \end{aligned} \tag{61}$$

where

$$\begin{aligned} \Omega = {} & \left[\left(1 - e^{-\lambda \Psi_0(\xi)}\right)\left(\frac{\Psi_0^{p+q-1}(\varpi)}{\lambda \Gamma(p+q)\Gamma(q)} + \left(\frac{\Psi_0^{p-1}(\varpi)}{\lambda \Gamma(p)}\left(1 - e^{-\lambda \Psi_0(\varpi)}\right) \right.\right.\right. \\ & \left.\left.\left. + \rho \frac{\Psi_0^{p-1}(\xi)}{\lambda \Gamma(p)}\left(1 - e^{-\lambda \Psi_0(\xi)}\right)\right) \frac{\Psi_0^{q}(\varpi)}{\Delta \Gamma(q+1)}\right)\right], \end{aligned} \tag{62}$$

which is satisfied inequality (56); then, for each $\theta \in \mathfrak{J}$, we have

$$
\begin{aligned}
&|w(\theta) - \varkappa(\theta)| \\
&= |w(\theta) - I^{q;\psi}\Big(h(s,\varkappa(s))\int_0^s I^{p-1;\psi} f\left(\tau,\varkappa(\tau),\delta I^{\gamma;\psi}\varkappa(\tau)\right)\psi'(\tau)e^{-\lambda\left(\Psi_0(s)-\Psi_0(\tau)\right)}\mathrm{d}\tau\Big)(\theta) \\
&\quad + \Big(\int_0^{\varpi} I^{p-1;\psi} f\left(\tau,\varkappa(\tau),\delta I^{\gamma;\psi}\varkappa(\tau)\right)\psi'(\tau)e^{-\lambda\left(\Psi_0(\varpi)-\Psi_0(\tau)\right)}\mathrm{d}\tau \\
&\quad - \rho\int_0^{\xi} I^{p-1;\psi} f\left(\tau,\varkappa(\tau),\delta I^{\gamma;\psi}\varkappa(\tau)\right)\psi'(\tau)e^{-\lambda\left(\Psi_0(\xi)-\Psi_0(\tau)\right)}\mathrm{d}\tau\Big) \\
&\quad \times \frac{1}{\Delta} I^{q;\psi}\Big(h(s,\varkappa(s))\left(1-e^{-\lambda\Psi_0(s)}\right)\Big)(\theta) + \sum_{i=1}^{m} I^{\eta_i+q;\psi} g_i(s,\varkappa(s))(\theta)| \\
&\le |w(\theta) - Gw(\theta)| + |Gw(\theta) - G\varkappa(\theta)| \\
&\le \Omega\|\chi\|\varepsilon + \frac{1-e^{-\lambda\Psi_0(\varpi)}}{\lambda}(\Lambda\|\chi\|\|M\| + \Upsilon\|L\|\|\vartheta\|) \\
&\quad + \sum_{i=1}^{m}\frac{\|K_i\|}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(\varpi)\|w(\theta)-\varkappa(\theta)\|.
\end{aligned}
\tag{63}
$$

Then,

$$
\|w(\theta) - \varkappa(\theta)\| = \frac{\Omega\|\chi\|}{1-\Xi_1}\varepsilon, \tag{64}
$$

where

$$
\Xi_1 = (\Lambda\|\chi\|\|M\| + \Upsilon\|L\|\|\vartheta\|) + \sum_{i=1}^{m}\frac{\|K_i\|}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(\varpi). \tag{65}
$$

By setting $C_f = (\Omega\|\chi\|/1 - \Xi_1)$, we obtain

$$
|w(\theta) - \varkappa(\theta)| \le C_f\varepsilon. \tag{66}
$$

Therefore, BVP (6) and (7) is U-H stable.

Similarly, for $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that $\Theta(\varepsilon) = C_f\varepsilon$ along with $\Theta(0) = 0$, the solution of problems (6) and (7) is generalized U-H stable. □

4. Example

Consider the following ψ-Caputo HFSIDE:

$$
\begin{cases}
\left[{}^{c}D^{(3/2);\theta} + 3\,{}^{c}D^{(1/2);\theta}\right]\left[\dfrac{{}^{c}D^{(4/5);\theta}\varkappa(\theta) - \sum_{i=1}^{4} I^{\eta_i;\psi} g_i(\theta,\varkappa(\theta))}{h(\theta,\varkappa(\theta))}\right] = \mathrm{f}\Big(\theta,\varkappa(\theta),\dfrac{1}{2}I^{(5/2);\theta}\varkappa(\theta)\Big), \quad \theta \in [0,1], \\
\varkappa(0) = 0, \\
{}^{C}D^{(4/5);\theta}\varkappa(0) = 0, \\
\left[\dfrac{{}^{c}D^{(4/5);\theta}\varkappa(\theta) - \sum_{i=1}^{4} I^{\eta_i;\psi} g_i(\theta,\varkappa(\theta))}{h(\theta,\varkappa(\theta))}\right]_{\theta=1} = \dfrac{7}{13}\left[\dfrac{{}^{c}D^{(4/5);\theta}\varkappa(\theta) - \sum_{i=1}^{4} I^{\eta_i;\psi} g_i(\theta,\varkappa(\theta))}{h(\theta,\varkappa(\theta))}\right]_{\theta=(5/6)},
\end{cases}
\tag{67}
$$

where

$$\sum_{i=1}^{4} I^{\eta_i;\theta} g_i(\theta, \varkappa(\theta))(s)$$
$$= I^{(1/4);\theta}\left(\frac{1}{18}\left(s\sqrt{s^2+1} + \sin s + \cos\varkappa(s)\right)\right)(\theta) + I^{(9/5);\theta}\left(\frac{\sin\varkappa(s)}{4\pi\sqrt{36+s^2}}\right)(\theta)$$
$$+ I^{(4/3);\theta}\left(\frac{\sin^2\varkappa(s)}{12(s+1)^2}\right)(\theta) + I^{(5/2);\theta}\left(\frac{1}{3\pi\sqrt{64+s^3}}\frac{|\varkappa(s)|}{2+|\varkappa(s)|}\right)(\theta), \quad (68)$$
$$h(\theta, \varkappa(\theta)) = \frac{e^{-\theta}\sin\varkappa(\theta)}{\theta+30} + \frac{1}{60}\left(\theta^2+1\right),$$
$$f\left(\theta, \varkappa(\theta), \frac{1}{2}I^{(5/2);\theta}\varkappa(\theta)\right) = \frac{1}{\sqrt{\theta+81}}\left(\frac{|\varkappa(\theta)|}{1+|\varkappa(\theta)|} + \arctan\left(\frac{1}{2}I^{(5/2);\theta}\varkappa(\theta)\right)\right).$$

Here $p = (3/2), q = (4/5), m = 4, \eta_1 = (7/4), \eta_2 = (4/3), \eta_3 = (2/3), \eta_4 = (5/6), \delta = (1/2), \gamma = (5/2), \rho = (7/13), \xi = (5/6)$, and $g_1 = (1/18)(\theta\sqrt{\theta^2+1} + \sin\theta + \cos\varkappa(\theta))$, $g_2 = (\sin x(\theta)/4\pi\sqrt{36+\theta^2})$, $g_3 = (\sin^2 x(\theta)/12(\theta+1)^2)$, and $g_4 = (1/3\pi\sqrt{64+\theta^3})(|x(\theta)|/2+|x(\theta)|)$.

The hypothesis (H1), (H2), and (H4) are satisfied with the following positives functions: $L(\theta) = (e^{-\theta}/\theta+3), \chi(\theta) = (e^{-\theta}/\theta+3) + (\theta^2+1/6), M(\theta) = \vartheta(\theta) = (3/2\sqrt{\theta+81})$, $K_1(\theta) = (1/18), \varphi_1(\theta) = (1/18)(\theta\sqrt{\theta^2+1}+2)$, $K_2(\theta) = \varphi_2(\theta) = (1/4\pi\sqrt{36+\theta^2})$, $K_3(\theta) = \varphi_3(\theta) = (1/12(\theta+1)^2)$, and $K_4(\theta) = \varphi_4(\theta) = (1/3\pi\sqrt{64+\theta^3})$, which gives the norms, $\|L\| = (1/3), \|\chi\| = (1/2), \|M\| = \|\vartheta\| = (1/6), \|K_1\| = (1/18), \|\varphi_1\| = (\sqrt{2}+2/18), \|K_2\| = \|\varphi_2\| = (1/24\pi), \|K_3\| = \|\varphi_3\| = (1/12)$, and $\|K_4\| = \|\varphi_4\| = (1/24\pi)$.

With the given data, we find that

$$\Delta \simeq 0.45595101,$$
$$\Lambda \simeq 5.3500159, \quad (69)$$
$$\Upsilon \simeq 2.8388423,$$

and hypothesis (H3) is satisfied by

$$\frac{1-e^{-\lambda}}{\lambda}(\Lambda\|\chi\|\|M\| + \Upsilon\|L\|\|\vartheta\|) + \sum_{i=1}^{m}\frac{\|K_i\|}{\Gamma(\eta_i+q+1)} \simeq 0.3153266651 < 1. \quad (70)$$

In the view of Theorem 2, problem (67) has a unique solution. In addition, Theorem 3 ensures that (6) and (7) are U-H and generalized U-H stable. As shown in Theorem 3, for every $\epsilon > 0$, if $w \in \mathbb{R}$ satisfies

$$\left|L_{\psi}^{p}\left[\frac{{}^{c}D^{q;\psi}w(\theta) - \sum_{i=1}^{m}I^{\eta_i;\psi}g_i(\theta, w(\theta))}{h(\theta, w(\theta))}\right] - f\left(\theta, w(\theta), \delta I^{\gamma;\psi}w(\theta)\right)\right| \leq \varepsilon, \quad \theta \in [0,1], \quad (71)$$

then there exists a unique solution $\varkappa \in \mathbb{R}$ such that

$$|w(\theta) - \varkappa(\theta)| \leq C_f \varepsilon, \quad \theta \in [0,1], \quad (72)$$

where

$$C_f \simeq 0.81 < 1. \quad (73)$$

Hence, problem (67) is U-H and generalized U-H stable.

5. Conclusion

In this study, we have successfully investigated the existence, uniqueness, and two kinds of stability in the sense of Ulam of the solutions for a new class ψ-Caputo-type hybrid FDEs with hybrid conditions. The existence of solutions is provided by using a generalization of Krasnoselskii's FPT due to Dhage [27], whereas the uniqueness result is achieved by Banach's FPT. After that, we have studied the concept of U-L and generalized U-L stabilities of (6) and (7). Also, we have presented an illustrative example to support our main results from a numerical point of view.

In future works, many results can be established when one takes a more generalized operator. Precisely, it will be of interest to study the current problem in this work for the fractional operator with variable order [30] and ψ-Hilfer fractional operator [31].

Data Availability

The data of this study were used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have equally contributed to this manuscript.

Acknowledgments

This work was supported by Taif University Researchers Supporting Project number (TURSP-2020/252), Taif University, Taif, Saudi Arabia.

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