On Geraghty Contractive Mappings and an Application


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1. Introduction

Kramosil and Michalek [1] defined the notion of a fuzzy metric space (FMS) by using the concept of fuzzy sets introduced by Zadeh [2]. Grabiec [3] gave the concept of weak Cauchy sequences, which is called a G-Cauchy sequence and proved the Banach contraction principle (BCP) [4] in the setting of a FMS. George and Veermani [5] modified the definition of a FMS given by Kramosil and Michalek [1] and established some fixed point results. For more works in FMSs, see [6–10].

In 1989, Bakhtin [11] introduced the notion of a b-metric space (BMS). Later on, the concept of a BMS was further used by Czerwick [12] to establish different fixed point results on this platform. The study of b-metric space endows an imperative place in fixed point theory with multiple aspects. Many mathematicians (Abdeljawad et al. [13, 14], Akkouchi [15], Chifu and Karapinar [16], Kadelburg and Radenović [17], Chauhan and Gupta [18], Kamran et al. [19] and Gupta [20], etc) led the foundation to improve fixed point theory in BMSs. Another innovative task has been achieved by Kamran et al. [21] in 2017 by introducing the notion of an extended b-metric space (EBMS), which generalizes the notion of a BMS. Some fixed point results are proved in this new setting. See for instance, the works shown in [22, 23].

By considering an auxiliary function, Geraghty [24] established a generalisation of the Banach contraction principle in the complete metric spaces. Later on, Gupta et al. [25] proved the fixed point theorems for \((\Psi,\beta)\)-Geraghty contraction type maps in ordered metric spaces. For more results using Geraghty contraction type maps in metric spaces can be seen in [26–30].

Nădăban [31] generalized the notion of b-metric space (BMS) by introducing the concept of fuzzy b-metric space. The idea of an extended fuzzy b-metric space (EFBMS) was introduced by Mahmood et al. in [32]. In the present article, some fixed point results for Geraghty-type contractions in \(G\)-complete EFBMS are established. Our results are generalizations of many existing results on FMS. See, for example, [32–35]. At the end, by applying our results, we give a real application.
2. Preliminaries

Recently, the concept of an EFBMS in [32] has been introduced as follows:

**Definition 1** (see [32]). Let $\Delta$ be a non empty set. Given $\theta: \Delta \times \Delta \rightarrow [1, \infty)$ and let $\ast$ be a continuous $t$-norm. A fuzzy set $\tau_\theta$ in $\Delta \times \Delta \times [0, \infty)$ is called an extended fuzzy $b$-metric on $\Delta$ if for all $\omega_1, \omega_2, \omega_3 \in \Delta$, the following conditions hold:

- $[FbM_01]: \tau_\theta(\omega_1, \omega_2, 0) = 0$;
- $[FbM_02]: \tau_\theta(\omega_1, \omega_2, y) = 1, \forall y > 0$ if and only if $\omega_1 = \omega_2$;
- $[FbM_03]: \tau_\theta(\omega_1, \omega_2, y) = \tau_\theta(\omega_2, \omega_1, y)$;
- $[FbM_04]: \tau_\theta(\omega_1, \omega_3, \theta(\omega_1, \omega_3)(y + \alpha)) \geq \theta(\omega_1, \omega_2, y) \ast \tau_\theta(\omega_2, \omega_3, \alpha) \forall y, \alpha \geq 0$;
- $[FbM_05]: \tau_\theta(\omega_1, \omega_2, \cdot): (0, \infty) \rightarrow [0, 1]$ is left continuous, and $\lim_{y \rightarrow \infty} \tau_\theta(\omega_1, \omega_2, y) = 1$.

Here, $(\Delta, \tau_\theta, \ast, \theta(\omega_1, \omega_2))$ is called an EFBMS.

**Remark 1.** Taking $\theta(\omega_1, \omega_2, 1) > 1$, the notion of a FMS defined in [31] is obtained and by taking $\theta(\omega_1, \omega_2) = 1$, the notion of a FMS defined in [1] is obtained.

**Example 1** (see [32]). Let $\Delta = \{1, 2, 3\}$ and define $d_b: \Delta \times \Delta \rightarrow \mathbb{R}$ by $d_b(\omega_1, \omega_2) = (\omega_1 - \omega_2)^2$. Clearly, $(\Delta, d_b)$ is a BMS. Define the mapping $\theta: \Delta \times \Delta \rightarrow [1, \infty)$ by

$$\theta(\omega_1, \omega_2) = 1 + \omega_1 + \omega_2,$$

(1)

Let $\tau_\theta: \Delta \times \Delta \times [0, \infty) \rightarrow [0, 1)$ be defined by

$$\tau_\theta(\omega_1, \omega_2, y) = \frac{y}{\gamma + d_b(\omega_1, \omega_2)},$$

if $y > 0$,

$$0, \quad \text{if } y = 0,$$

and take the continuous $t$-norm $\ast = \wedge$, that is, $t_1 \ast t_2 = t_1 \wedge t_2 = \min\{t_1, t_2\}$. Then $(\Delta, \tau_\theta, \wedge, \theta(\omega_1, \omega_2))$ is an EFBMS.

The notions of convergence, Cauchyness and completeness in an EFBMS can be generalized naturally as follows:

**Definition 2** (see [32]). Let $(\Delta, \tau_\theta, \ast)$ be an EFBMS.

(i) A sequence $[\omega_n]$ in $\Delta$ is said to be convergent if there exists $\omega \in \Delta$ such that

$$\lim_{n \rightarrow \infty} \tau_\theta(\omega_n, \omega, 1) = 1, \forall y > 0.$$

(ii) A sequence $[\omega_n]$ in $\Delta$ is said to be a $G_r$-Cauchy sequence if $\lim_{n \rightarrow \infty} \tau_\theta(\omega_n, \omega_{m+q}, 1) = 1$ for all $y > 0$ and $q > 0$.

(iii) An EFBMS in which every $G_r$-Cauchy sequence is convergent is called a $G_r$-complete EFBMS.

**Lemma 1** (see [34]). Let $(\Delta, F_{b}, \ast)$ be a complete FBMS and $F(\omega_1, \omega_2, k \gamma) \geq F(\omega_1, \omega_2, \gamma)$ for all $\omega_1, \omega_2 \in \Delta$, $k \in (0, 1)$ and $y > 0$, then $\omega_1 = \omega_2$.

Let $(\Delta, \tau_\theta, \ast)$ be a $G_r$-complete FBMS. Throughout this article, let

$$F_{\beta} = \left\{ \beta: [0, \infty) \rightarrow \left[\left[0, \frac{1}{b}\right], \limsup_{n \rightarrow \infty} \beta(t_n) = \frac{1}{b} \text{ implies } \lim_{n \rightarrow \infty} t_n = 0 \right] \right\},$$

(4)

where $b \geq 1$.

3. Main Results

The BCP in the setting of $G_r$-complete EFBMSs, is established as follows:

**Theorem 1.** Let $(\Delta, \tau_\theta, \ast)$ be a $G_r$-complete EFBMS with $\theta(\omega_1, \omega_2) \geq 1$. Let $T: \Delta \rightarrow \Delta$ be a mapping satisfying

$$\tau_\theta(\omega_1, \omega_{i+1}, y) = \tau_\theta(T(\omega_{i-1}, T(\omega_i, y) \geq \tau_\theta(\omega_{i-1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))})

\geq \tau_\theta(\omega_{i-2}, T(\omega_i, y) \beta(\omega_{i-1}, \omega_i, y)) \beta(\tau_\theta(\omega_{i-2}, \omega_{i-1}, y))$$

for all $\omega_1, \omega_2 \in \Delta$, where $\beta \in F_{b}$ and $\beta(\tau_\theta(\omega_i, \omega_2, y)) \beta(\theta(\omega_1, \omega_2)) < 1$. Then $T$ has a unique fixed point.

**Proof.** Let $\omega_0 \in \Delta$. Generate a sequence $[\omega_i]$ by the iterative process $\omega_i = T^i\omega_0 (i \in \mathbb{N})$. For all $I, y > 0$, by (5), we have
So, we have

\[
\tau_\theta(\omega_0, \omega_1, \gamma) \geq \tau_\theta(\omega_0, \omega_1, \beta(\tau_\theta(\omega_0, \omega_1, \gamma)) \cdot \beta(\tau_\theta(\omega_0, \omega_1, \gamma)) \cdot \beta(\tau_\theta(\omega_0, \omega_1, \gamma)) \cdot \beta(\tau_\theta(\omega_0, \omega_1, \gamma)))
\]  

(6)

For any \( q \in \mathbb{N} \), taking \( \gamma = \gamma/q + \gamma/q + \ldots + \gamma/q \) and using \([FbM_\theta]^4\) repeatedly, one can write

\[
\tau_\theta(\omega_0, \omega_{m+q}, \gamma) \geq \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})}) \cdot \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})}) \cdot \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})}) \cdot \ldots \cdot \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})})
\]  

(8)

Using (7) and \([FbM_\theta]^5\), we get

\[
\tau_\theta(\omega_0, \omega_{m+q}, \gamma) \geq \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})}) \cdot \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})}) \cdot \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})}) \cdot \ldots \cdot \tau_\theta(\omega_0, \omega_{m+q}, \frac{\gamma}{q\theta(\omega_0, \omega_{m+q})})
\]  

(9)

Then
\( \tau_\theta(\bar{\omega}_i, \bar{\omega}_{m+q}, \gamma) \)
\[ \geq \tau_\theta \left( \bar{\omega}_0, \bar{\omega}_1, \frac{b^{l-1}\gamma}{q\theta(\bar{\omega}_1, \bar{\omega}_{m+q})\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_1, \gamma))} \right) \]
\* \( \tau_\theta \left( \bar{\omega}_0, \bar{\omega}_1, \frac{b^{l-1}\gamma}{q\theta(\bar{\omega}_1, \bar{\omega}_{m+q})\beta(\tau_\theta(\bar{\omega}_{l+1}, \bar{\omega}_1, \gamma))} \right) \* \ldots \* \)
\[ \tau_\theta \left( \bar{\omega}_0, \bar{\omega}_1, \frac{b^{l-1}\gamma}{q\theta(\bar{\omega}_1, \bar{\omega}_{m+q})\ldots\theta(\bar{\omega}_{i+q-1}, \bar{\omega}_{m+q})\beta(\tau_\theta(\bar{\omega}_{m+q}, \bar{\omega}_{i+q-1}, \gamma)) \ldots \beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_1, \gamma))} \right) \]

Since for all \( l, q \in \mathbb{N} \), we have
\[ \theta(\bar{\omega}_1, \bar{\omega}_{m+q})\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_1, \gamma)) < 1 \], taking limit as \( l \to \infty \), we get
\[ \lim_{l \to \infty} \tau_\theta(\bar{\omega}_1, \bar{\omega}_{m+q}, \gamma) = 1 * 1 * \ldots * 1 = 1 \].

\[ \tau_\theta(Tp_1, p_1, \gamma) \geq \tau_\theta \left( Tp_1, T\bar{\omega}_1, \frac{\gamma}{2\theta(Tp_1, p_1)} \right) \* \tau_\theta \left( T\bar{\omega}_1, p_1, \frac{\gamma}{2\theta(Tp_1, p_1)} \right) \]
\[ \geq \tau_\theta \left( p_1, \bar{\omega}_1, \frac{\gamma}{2\beta(p_1, \bar{\omega}_1, 1)\theta(Tp_1, p_1)} \right) \* \tau_\theta \left( \bar{\omega}_{i+1}, \bar{\omega}_1, \frac{\gamma}{2\theta(\bar{\omega}_{i+1}, \bar{\omega}_1)} \right) \]
\[ \longrightarrow 1 * 1 = 1. \]

That is, \( Tp_1 = p_1 \) is a fixed point.

**3.1. Uniqueness.** Assume \( Tp_2 = p_2 \) for some \( p_2 \in \Delta \). Then

\[ \tau_\theta(p_2, p_1, \gamma) = \tau_\theta(Tp_2, Tp_1, \gamma) \]
\[ \geq \tau_\theta \left( p_2, \bar{\omega}_1, \frac{\gamma}{\beta(\tau_\theta(p_2, p_1, \gamma))} \right) = \tau_\theta \left( Tp_2, Tp_1, \frac{\gamma}{\beta(\tau_\theta(p_2, p_1, \gamma))} \right) \]
\[ \geq \tau_\theta \left( p_2, p_1, \frac{\gamma}{\beta(\tau_\theta(p_2, p_1, \gamma))} \right) \geq \ldots \geq \tau_\theta \left( p_2, p_1, \frac{\gamma}{\beta(\tau_\theta(p_2, p_1, \gamma))} \right) = \tau_\theta(p_2, p_1, b^n\gamma) \]
\[ \longrightarrow 1 \text{ as } l \to \infty. \]

Hence, the fixed point is unique.

**Example 2.** Let \( \Delta = \{0, 1, 2\} \) and \( \tau_\theta(\bar{\omega}_1, \bar{\omega}_2, \gamma) = \gamma/\gamma + (\bar{\omega}_1 - \bar{\omega}_2)^2 \)
Recall that
\[
\frac{(\omega_1 - \omega_2)^2}{(1 + \omega_1)^2(1 + \omega_1)} \leq (\omega_1 - \omega_2)^2
\]
\[
y + \frac{(\omega_1 - \omega_2)^2}{(1 + \omega_1)^2(1 + \omega_1)} \leq y + (\omega_1 - \omega_2)^2
\]
\[
\frac{1}{y + \left( \frac{(\omega_1 - \omega_2)^2}{(1 + \omega_1)^2(1 + \omega_1)} \right)^2} \geq \frac{1}{y + (\omega_1 - \omega_2)^2}
\]
\[
\frac{y}{y + \left( \frac{(\omega_1 - \omega_2)^2}{(1 + \omega_1)^2(1 + \omega_1)} \right)^2} \geq \frac{y}{y + (\omega_1 - \omega_2)^2}.
\]  
(16)

This implies that
\[
\tau_\theta(T_\omega_1, T_\omega_2, \beta(\tau_\theta(\omega_1, \omega_2, \gamma))) \gamma \geq \min\{ \tau_\theta(T_\omega_1, T_\omega_2, \gamma), \tau_\theta(\omega_1, T_\omega_1, \gamma), \tau_\theta(\omega_2, T_\omega_2, \gamma), \tau_\theta(\omega_1, \omega_2, \gamma) \},
\]  
(18)

for all \( \omega_1, \omega_2 \in \Delta \), where \( \beta \in F_\beta \) and \( \theta(\omega_1, \omega_2) \beta(\tau_\theta(\omega_1, \omega_2, \gamma)) < 1 \). Then \( T \) has a unique fixed point.

Proof. Starting by the same way as in Theorem 1, we have

\[
\tau_\theta(\omega_{l-1}, \omega_l, \gamma) = \tau_\theta(T_\omega_{l-1}, T_\omega_l, \gamma)
\]
\[
\geq \min\left\{ \tau_\theta(\omega_{l-1}, T_\omega_l, \frac{y}{\beta(\tau_\theta(\omega_{l-1}, \omega_l, \gamma))}), \tau_\theta(\omega_{l-1}, T_\omega_l, \frac{y}{\beta(\tau_\theta(\omega_{l-1}, \omega_l, \gamma))}, \tau_\theta(\omega_{l-1}, \omega_l, \frac{y}{\beta(\tau_\theta(\omega_{l-1}, \omega_l, \gamma))}) \right\},
\]
\[
\geq \min\left\{ \tau_\theta(\omega_{l-1}, \omega_{l+1}, \frac{y}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))}), \tau_\theta(\omega_{l-1}, \omega_{l+1}, \frac{y}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))}), \tau_\theta(\omega_{l-1}, \omega_{l+1}, \frac{y}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))}) \right\},
\]
\[
\tau_\theta(\omega_{l-1}, \omega_{l+1}, \frac{y}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))}) \right\},
\]  
(19)

Hence
\[ \tau_\theta(\omega_1, \omega_{i+1}, y) \geq \min \left\{ \tau_\theta \left( \omega_1, \omega_{i+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right), \tau_\theta(\omega_{i-1}, \omega_i, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))}) \right\}, \quad (20) \]

If

\[ \min \left\{ \tau_\theta \left( \omega_1, \omega_{i+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right), \tau_\theta(\omega_{i-1}, \omega_i, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))}) \right\} = \tau_\theta \left( \omega_1, \omega_{i+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right), \quad (21) \]

then (20) implies

\[ \tau_\theta(\omega_1, \omega_{i+1}, y) \geq \tau_\theta \left( \omega_1, \omega_{i+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right). \quad (22) \]

By Lemma 1, \( T \) has a fixed point. If

\[ \min \left\{ \tau_\theta \left( \omega_1, \omega_{i+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right), \tau_\theta(\omega_{i-1}, \omega_i, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))}) \right\} = \tau_\theta \left( \omega_1, \omega_{i+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right), \quad (23) \]

then from (20), we have

\[ \tau_\theta(\omega_1, \omega_{i+1}, y) \geq \tau_\theta \left( \omega_1, \omega_i, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right) \geq \ldots \geq \tau_\theta \left( \omega_0, \omega_i, \frac{\gamma}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \cdot \beta(\tau_\theta(\omega_{i-2}, \omega_{i-1}, y)) \ldots \beta(\tau_\theta(\omega_0, \omega_1, y)) \right). \quad (24) \]

The rest of the proof follows from Theorem 1. \( \square \)

**Remark 4.** Taking \( \theta(\omega_1, \omega_2) = b \) we get Theorem 3.6 of [36].

The following result is an extension of the main result of Gupta at el. [34].

\[ \tau_\theta(T\omega_1, T\omega_2, \beta(\tau_\theta(\omega_1, \omega_2, y))y) \geq \min \left\{ \frac{\tau_\theta(\omega_2, T\omega_2, y)}{1 + \tau_\theta(\omega_1, T\omega_2, y)} \left[ 1 + \frac{\tau_\theta(\omega_1, T\omega_2, y)}{1 + \tau_\theta(\omega_1, \omega_2, y)} \right], \tau_\theta(\omega_1, \omega_2, y) \right\}, \quad (25) \]
for all $\omega_1, \omega_2 \in \Delta$, where $\beta \in F_\beta$ and $\beta(\tau_\theta(\omega_1, \omega_2, \gamma)) \theta(\omega_1, \omega_2) < 1$, then $T$ has a unique fixed point.

Proof. Consider

$$
\tau_\theta(\omega_1, \omega_{l+1}, \gamma) = \tau_\theta(T\omega_l, \omega_l, \gamma)
$$

$$
\geq \min \left\{ \tau_\theta\left(\omega_1, \omega_{l+1}, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l+1}, \omega_1, \gamma)\right)}\right), \tau_\theta\left(\omega_{l-1}, \omega_l, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l-1}, \omega_1, \gamma)\right)}\right) \right\} = \tau_\theta\left(\omega_1, \omega_{l+1}, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l+1}, \omega_1, \gamma)\right)}\right),
$$

(26)

If

$$
\min \left\{ \tau_\theta\left(\omega_1, \omega_{l+1}, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l+1}, \omega_1, \gamma)\right)}\right), \tau_\theta\left(\omega_{l-1}, \omega_l, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l-1}, \omega_1, \gamma)\right)}\right) \right\} = \tau_\theta\left(\omega_1, \omega_{l+1}, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l+1}, \omega_1, \gamma)\right)}\right),
$$

(27)

then (26) implies

$$
\tau_\theta(\omega_1, \omega_{l+1}, \gamma) \geq \tau_\theta\left(\omega_1, \omega_{l+1}, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l+1}, \omega_1, \gamma)\right)}\right).
$$

(28)

Continuing in this way, one writes

$$
\tau_\theta(\omega_1, \omega_{l+1}, \gamma) \geq \tau_\theta\left(\omega_1, \omega_{l+1}, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l+1}, \omega_1, \gamma)\right)}\right).
$$

(29)

then from (26) we have

$$
\tau_\theta(\omega_1, \omega_{l+1}, \gamma) \geq \tau_\theta\left(\omega_1, \omega_{l+1}, \frac{\gamma}{\beta\left(\tau_\theta(\omega_{l+1}, \omega_1, \gamma)\right)}\right).
$$

(30)

Therefore, proceeding as in Theorem 1 after inequality (7), the desired result is obtained.

Theorem 4. Let $(X, \tau_\theta, \ast)$ be a $G_r$-complete EFBMS with $\theta(\omega_1, \omega_2) \geq 1$. Let $T: \Delta \rightarrow \Delta$ be a mapping satisfying

$$
\tau_\theta(T\omega_1, T\omega_2, \beta(\tau_\theta(\omega_1, \omega_2, \gamma))) \geq \min \left\{ \tau_\theta(\omega_2, T\omega_1, \gamma) + \tau_\theta(\omega_2, T\omega_1, \gamma), \tau_\theta(\omega_1, \omega_2, \gamma) \right\},
$$

(32)
for all $\omega_1, \omega_2 \in \Delta$, where $\beta \in F_\beta$ and $\beta((\tau_0(\omega_1, \omega_2, \gamma)) \theta(\omega_1, \omega_2) < 1$, then $T$ has a unique fixed point.

Proof. For $\omega_0 \in \Delta$, we choose a sequence $\{\omega_t\}$ in $\Delta$ and start by $\omega_{t+1} = T\omega_t$. For all $t, \gamma > 0$, we have

\[
\tau_\theta(\omega_t, \omega_{t+1}, \gamma) = \tau_\theta(T\omega_t, T\omega_t, \gamma)
\]

\[
\geq \min \left\{ \tau_\theta(\omega_t, T\omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) \left[ 1 + \tau_\theta(\omega_t, T\omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) + \tau_\theta(\omega_t, T\omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) \right] \right\}
\]

\[
\tau_\theta(\omega_{t+1}, \omega_t, \beta(\tau_\theta(\omega_t, \omega_t, \gamma))) \left[ 1 + \tau_\theta(\omega_t, T\omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) \right] \right\}
\]

\[
= \min \left\{ \tau_\theta(\omega_t, \omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) \left[ 1 + \tau_\theta(\omega_t, \omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) \right] + \tau_\theta(\omega_t, \omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) \right\}
\]

\[
\tau_\theta(\omega_{t+1}, \omega_t, \beta(\tau_\theta(\omega_t, \omega_t, \gamma))) \left[ 1 + \tau_\theta(\omega_t, \omega_t, (y/\beta(\tau_\theta(\omega_t, \omega_t, \gamma)))) \right] \right\}
\]

So, we have

\[
\tau_\theta(\omega_t, \omega_{t+1}, \gamma) \geq \min \left\{ \tau_\theta(\omega_t, \omega_{t+1}, \beta(\tau_\theta(\omega_t, \omega_t, \gamma))) \right\},
\]

If

\[
\min \left\{ \tau_\theta(\omega_t, \omega_{t+1}, \beta(\tau_\theta(\omega_t, \omega_t, \gamma))) \right\}, \tau_\theta(\omega_{t+1}, \omega_t, \beta(\tau_\theta(\omega_t, \omega_t, \gamma))) \right\} = \tau_\theta(\omega_t, \omega_{t+1}, \beta(\tau_\theta(\omega_t, \omega_t, \gamma)))
\]

then from (34)

\[
\tau_\theta(\omega_t, \omega_{t+1}, \gamma) \geq \tau_\theta(\omega_t, \omega_{t+1}, \beta(\tau_\theta(\omega_t, \omega_t, \gamma)))
\]

Then result is trivial by Lemma 1. If

\[
\min \left\{ \tau_\theta(\omega_t, \omega_{t+1}, \beta(\tau_\theta(\omega_t, \omega_t, \gamma))) \right\}, \tau_\theta(\omega_{t+1}, \omega_t, \beta(\tau_\theta(\omega_t, \omega_t, \gamma))) \right\} = \tau_\theta(\omega_t, \omega_{t+1}, \beta(\tau_\theta(\omega_t, \omega_t, \gamma)))
\]

then from (34)
\[ \tau_\theta((\omega, \phi, \gamma)) = \tau_\theta((\omega, \phi, \gamma)) \]

Continuing in this direction, we get

\[ \tau_\theta((\omega, \phi, \gamma)) \geq \tau_\theta((\omega, \phi, \gamma)) \]

The desired result is then established by the same procedure as in Theorem 1. The following result is the extension of the main result of Roshan et al. [35].

\[ \text{Theorem 5. Consider a } G \text{-complete EBFMS } (X, \tau_\theta, \ast) \text{ with } \theta((\omega_1, \omega_2)) \geq 1. \text{ Let } T: \Delta \rightarrow \Delta \text{ be a mapping satisfying} \]

\[ \tau_\theta(T\omega_1, T\omega_2, \beta((\tau_\theta(\omega_1, \omega_2, \gamma)))) \geq \delta(\omega_1, \omega_2, \gamma). \]

for all \( \omega_1, \omega_2 \in \Delta \), where \( \beta \in F_\beta \) and \( \beta((\tau_\theta(\omega_1, \omega_2, \gamma))\theta(\omega_1, \omega_2) < 1 \), where

\[ \delta(\omega_1, \omega_2, \gamma) = \min \left\{ \frac{\tau_\theta(\omega_1, T\omega_1, \gamma)[1 + \tau_\theta(\omega_2, T\omega_2, \gamma)]}{1 + \tau_\theta(T\omega_1, \omega_2, \gamma)}, \frac{\tau_\theta(\omega_2, T\omega_2, \gamma)[1 + \tau_\theta(\omega_1, T\omega_1, \gamma)]}{1 + \tau_\theta(\omega_1, \omega_2, \gamma)} \right\} \]

then \( T \) has a unique fixed point.

\[ \text{Proof. Let } \omega_0 \in \Delta. \text{ Choose a sequence } \{\omega_n\} \text{ in } \Delta \text{ by starting an iterative process } \omega_{n+1} = T\omega_n. \text{ For all } l, \gamma > 0, \text{ we have} \]

\[ \tau_\theta(\omega, \omega_{n+1}, \gamma) = \tau_\theta(T\omega_{n+1}, T\omega_n, \gamma) \geq \delta(\omega_{n+1}, \omega_n, \gamma) \]

where

\[ \delta(\omega_{n+1}, \omega_n, \gamma) = \min \left\{ \frac{\tau_\theta(\omega_{n+1}, T\omega_{n+1}, \gamma)[1 + \tau_\theta(\omega_n, T\omega_n, \gamma)]}{1 + \tau_\theta(T\omega_{n+1}, \omega_n, \gamma)} \right\} \]

Now

\[ \delta(\omega_{n+1}, \omega_n, \gamma) \]

\[ = \min \left\{ \frac{\tau_\theta(\omega_{n+1}, T\omega_{n+1}, \gamma)[1 + \tau_\theta(\omega_n, T\omega_n, \gamma)]}{1 + \tau_\theta(T\omega_{n+1}, \omega_n, \gamma)} \right\} \]

\[ \tau_\theta(\omega_{n+1}, T\omega_{n+1}, \gamma)[1 + \tau_\theta(\omega_n, T\omega_n, \gamma)] \]

\[ \tau_\theta(\omega_{n+1}, T\omega_{n+1}, \gamma)[1 + \tau_\theta(\omega_n, T\omega_n, \gamma)] \]

\[ \tau_\theta(\omega_{n+1}, T\omega_{n+1}, \gamma)[1 + \tau_\theta(\omega_n, T\omega_n, \gamma)] \]

\[ \tau_\theta(\omega_{n+1}, T\omega_{n+1}, \gamma)[1 + \tau_\theta(\omega_n, T\omega_n, \gamma)] \]

That is
One writes

\[ \delta \left( \omega_{l-1}, \omega_l, \frac{y}{\beta(\tau_0(\omega_{l-1}, \omega_l, y))} \right) \]

\[ = \min \left\{ \tau_0(\omega_{l-1}, T\omega_{l-1}, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) + \tau_0(\omega_l, \omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \right\} \]

\[ \tau_0(\omega_{l-1}, T\omega_{l-1}, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))))[1 + \tau_0(\omega_l, T\omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y)))] \]

\[ 1 + \tau_0(T\omega_{l-1}, T\omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \]

(44)

We obtain

\[ \delta \left( \omega_{l-1}, \omega_l, \frac{y}{\beta(\tau_0(\omega_{l-1}, \omega_l, y))} \right) \]

\[ = \min \left\{ \tau_0(\omega_{l-1}, T\omega_{l-1}, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) + \tau_0(\omega_l, \omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \right\} \]

\[ \tau_0(\omega_{l-1}, T\omega_{l-1}, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))))[2 + \tau_0(\omega_l, T\omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) + \tau_0(\omega_l, T\omega_{l-1}, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y)))] \]

\[ 1 + \tau_0(T\omega_{l-1}, T\omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \]

(45)

We obtain

\[ \delta \left( \omega_{l-1}, \omega_l, \frac{y}{\beta(\tau_0(\omega_{l-1}, \omega_l, y))} \right) \]

\[ = \min \left\{ \tau_0(\omega_{l-1}, \omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) + \tau_0(\omega_l, \omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \right\} \]

(46)

If \( \min \{ \tau_0(\omega_{l-1}, \omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \} \) then from (42)

\[ \tau_0(\omega_l, \omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) = \tau_0(\omega_l, \omega_{l+1}, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \]

(47)

Since the range of \( \beta \) is \([0, (1/\beta)]\), the result is obvious by Lemma 1. If \( \min \{ \tau_0(\omega_{l-1}, \omega_l, y/\beta(\tau_0(\omega_{l-1}, \omega_l, y))) \} \)

Then the result is trivial by Lemma 1.

\[ \Box \]
Theorem 6. Let \((\Delta, \tau_\theta, *)\) be a \(G_r\)-complete EFBMS with \(\theta(\omega_1, \omega_2) \geq 1\). Let \(T: \Delta \rightarrow \Delta\) be a mapping satisfying the condition

\[
\tau_\theta(T\omega_1, T\omega_2, \beta(\tau_\theta(\omega_1, \omega_2, \gamma))) \geq \frac{\alpha(\omega_1, \omega_2, \gamma)}{\max\{\tau_\theta(\omega_1, T\omega_1, \gamma), \tau_\theta(\omega_2, T\omega_2, \gamma)\}},
\]

where

\[
\alpha(\omega_1, \omega_2, \gamma) = \min\{\tau_\theta(T\omega_1, T\omega_2, \gamma) : \tau_\theta(\omega_1, T\omega_1, \gamma), \tau_\theta(\omega_2, T\omega_2, \gamma)\},
\]

for all \(\omega_1, \omega_2 \in \Delta\), where \(\beta \in F_\beta\) and \(\theta(\omega_1, \omega_2)\beta(\tau_\theta(\omega_1, \omega_2, \gamma)) < 1\), then \(T\) has a unique fixed point.

Proof. Consider

\[
\tau_\theta(\omega_1, \omega_{l+1}, \gamma) = \tau_\theta(T\omega_{l-1}, T\omega_l, \gamma)
\]

\[
\geq \frac{\alpha(\omega_1, \omega_{l+1}, \gamma/\beta(\tau_\theta(\omega_1, \omega_{l+1}, \gamma)))}{\max\{\tau_\theta(\omega_{l-1}, T\omega_{l-1}, \gamma/\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))), \tau_\theta(\omega_{l}, T\omega_l, \gamma/\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma)))\}}.
\]

Now

\[
\alpha\left(\frac{\gamma}{\beta(\tau_\theta(\omega_1, \omega_{l+1}, \gamma))}\right)
\]

\[
= \min\left\{\tau_\theta\left(\frac{T\omega_{l-1}, T\omega_l, \gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))}\right) : \tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma/\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma)))\right\}
\]

\[
\cdot \tau_\theta\left(\frac{\omega_{l+1}, T\omega_{l+1}, \gamma}{\beta(\tau_\theta(\omega_{l+1}, \omega_{l+1}, \gamma))}\right)
\]

\[
\ {}
\]

\[
\alpha\left(\frac{\gamma}{\beta(\tau_\theta(\omega_1, \omega_{l+1}, \gamma))}\right)
\]

\[
= \min\left\{\tau_\theta\left(\frac{\omega_{l-1}, \omega_{l+1}, \gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))}\right) : \tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma/\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma)))\right\}
\]

\[
\cdot \tau_\theta\left(\frac{\omega_{l+1}, \omega_{l+1}, \gamma}{\beta(\tau_\theta(\omega_{l+1}, \omega_{l+1}, \gamma))}\right)
\]

\[
\]

Using (14) in (52) we get

\[
\tau_\theta(\omega_1, \omega_{l+1}, \gamma) \geq \frac{\tau_\theta(\omega_1, \omega_{l+1}, \gamma/\beta(\tau_\theta(\omega_1, \omega_{l+1}, \gamma))) \cdot \tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma/\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma)))}{\max\{\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma/\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma))), \tau_\theta(\omega_{l}, \omega_{l+1}, \gamma/\beta(\tau_\theta(\omega_{l-1}, \omega_{l+1}, \gamma)))\}}.
\]
If

\[ \max \left\{ \tau_\theta \left( \omega_{l-1}, \omega_{l+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right), \tau_\theta \left( \omega_{l-1}, \omega_{l}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right) \right\} = \tau_\theta \left( \omega_{l-1}, \omega_{l}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right), \quad (55) \]

then (54) implies

\[ \tau_\theta \left( \omega_{l}, \omega_{l+1}, \beta \right) \geq \tau_\theta \left( \omega_{l}, \omega_{l+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right), \quad (56) \]

\[ \max \left\{ \tau_\theta \left( \omega_{l}, \omega_{l+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right), \tau_\theta \left( \omega_{l-1}, \omega_{l}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right) \right\} = \tau_\theta \left( \omega_{l}, \omega_{l+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right), \quad (57) \]

then from (54), we have

\[ \tau_\theta \left( \omega_{l}, \omega_{l+1}, \gamma \right) \geq \tau_\theta \left( \omega_{l-1}, \omega_{l+1}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma))} \right) \geq \ldots \]

\[ \geq \tau_\theta \left( \omega_{l}, \omega_{l+1}, \beta(\tau_\theta(\omega_{l-1}, \omega_{l_1}, \gamma)) \cdot \beta(\tau_\theta(\omega_{l-2}, \omega_{l-1}, \gamma)) \ldots \beta(\tau_\theta(\omega_{l}, \omega_{l_1}, \gamma)) \right), \quad (58) \]

By using the same method as in Theorem 1, one can complete the proof. \(\square\)

**Remark 5.** Taking \(\theta(\omega_{l}, \omega_{l+1}) = b\) we get Theorem 3.9 of [36].

\[ \tau_\theta (T \omega_{l}, T \omega_{l+1}, \beta(\tau_\theta(\omega_{l-1}, \omega_{l}, \gamma))) \geq \lambda(\omega_{l}, \omega_{l+1}, \gamma) \ast \mu(\omega_{l}, \omega_{l+1}, \gamma), \quad (59) \]

where

\[ \begin{align*}
\lambda(\omega_{l}, \omega_{l+1}, \gamma) &= \min \{ \tau_\theta (T \omega_{l}, T \omega_{l+1}, \gamma), \tau_\theta (\omega_{l}, T \omega_{l+1}, \gamma), \tau_\theta (\omega_{l+1}, T \omega_{l}, \gamma), \tau_\theta (\omega_{l+1}, T \omega_{l+1}, \gamma) \}, \\
\mu(\omega_{l}, \omega_{l+1}, \gamma) &= \max \{ \tau_\theta (\omega_{l}, \omega_{l+1}, \gamma), \tau_\theta (T \omega_{l}, \omega_{l+1}, \gamma) \}. 
\end{align*} \quad (60) \]

**Theorem 7.** Let \((\Delta, \tau_\theta, \ast)\) be a \(G_r\)-complete EFBMS with \(\theta(\omega_{l}, \omega_{l+1}) \geq 1\) and \(T: \Delta \rightarrow \Delta\) be a mapping satisfying the condition

\[ \tau_\theta (T \omega_{l}, T \omega_{l+1}, \gamma) \geq \lambda(\omega_{l}, \omega_{l+1}, \gamma) \ast \mu(\omega_{l}, \omega_{l+1}, \gamma), \quad (59) \]

for all \(\omega_{l}, \omega_{l+1} \in \Delta\), where \(\beta \in F_\beta\) and \(\theta(\omega_{l}, \omega_{l+1}) \beta(\tau_\theta(\omega_{l}, \omega_{l+1}, \gamma)) < 1\), then \(T\) has a unique fixed point, where \(a \ast b = \min(a, b)\).

**Proof.** Start by

\[ \tau_\theta (\omega_{l}, \omega_{l+1}, \gamma) = \tau_\theta (T \omega_{l-1}, T \omega_{l}, \gamma) \geq \lambda \left( \omega_{l-1}, \omega_{l}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l}, \gamma))} \right) \ast \mu \left( \omega_{l-1}, \omega_{l}, \frac{\gamma}{\beta(\tau_\theta(\omega_{l-1}, \omega_{l}, \gamma))} \right), \quad (61) \]

Now
Using (62) and (63) in (61), we have

\[
\tau_\theta(\omega_{i+1}, \omega_i, y) \geq \min \left\{ \tau_\theta\left( \omega_{i+1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i+1}, \omega_i, y))} \right), \tau_\theta\left( \omega_{i-1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right) \right\} \ast 1
\]

If

\[
\min \left\{ \tau_\theta\left( \omega_{i+1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i+1}, \omega_i, y))} \right), \tau_\theta\left( \omega_{i-1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right) \right\} = \tau_\theta\left( \omega_{i+1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i+1}, \omega_i, y))} \right),
\]

then (64) implies

\[
\tau_\theta(\omega_{i+1}, \omega_i, y) \geq \tau_\theta\left( \omega_{i+1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i+1}, \omega_i, y))} \right),
\]

then \( T \) has a fixed point by Lemma 1. If

\[
\tau_\theta(\omega_{i+1}, \omega_i, y) \geq \tau_\theta\left( \omega_{i+1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i+1}, \omega_i, y))} \right), \quad \tau_\theta\left( \omega_{i-1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right)
\]

\[
\min \left\{ \tau_\theta\left( \omega_{i+1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i+1}, \omega_i, y))} \right), \tau_\theta\left( \omega_{i-1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i-1}, \omega_i, y))} \right) \right\} = \tau_\theta\left( \omega_{i+1}, \omega_i, \frac{y}{\beta(\tau_\theta(\omega_{i+1}, \omega_i, y))} \right),
\]
then from (64), we have
\[ \tau_\theta(\bar{\omega}_l, \bar{\omega}_{l+1}, y) \geq \tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, \beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y))). \] (68)

Continuing in this way, we will get
\[ \tau_\theta(\bar{\omega}_l, \bar{\omega}_{l+1}, y) \geq \tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, \beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y))) \]
\[ \geq \ldots \geq \tau_\theta(\bar{\omega}_{0}, \bar{\omega}_1, \beta(\tau_\theta(\bar{\omega}_{0}, \bar{\omega}_1, y))) \cdot \beta(\tau_\theta(\bar{\omega}_{0}, \bar{\omega}_1, y)) \ldots \beta(\tau_\theta(\bar{\omega}_0, \bar{\omega}_1, y))). \] (69)

Using the method of Theorem 1 after inequality (7), the desired result is established.

Remark 6. Taking \( \theta(\bar{\omega}_1, \bar{\omega}_2) = b \) we get Theorem 3.11 of [36].

\[ \tau_\theta(T\bar{\omega}_1, T\bar{\omega}_2, \beta(\tau_\theta(\bar{\omega}_1, \bar{\omega}_2, y))) \geq \frac{\lambda(\bar{\omega}_1, \bar{\omega}_2, y) \ast \mu(\bar{\omega}_1, \bar{\omega}_2, y)}{\alpha(\bar{\omega}_1, \bar{\omega}_2, y)}. \] (70)

where
\[ \left\{ \begin{array}{l}
\lambda(\bar{\omega}_1, \bar{\omega}_2, y) = \min\{\tau_\theta(T\bar{\omega}_1, T\bar{\omega}_2, y) \cdot \tau_\theta(\bar{\omega}_1, \bar{\omega}_2, y), \tau_\theta(\bar{\omega}_1, T\bar{\omega}_1, y) \cdot \tau_\theta(\bar{\omega}_2, T\bar{\omega}_2, y)\}
\mu(\bar{\omega}_1, \bar{\omega}_2, y) = \max\{\tau_\theta(\bar{\omega}_1, T\bar{\omega}_1, y) \cdot \tau_\theta(\bar{\omega}_2, T\bar{\omega}_2, y), (\tau_\theta(\bar{\omega}_1, \bar{\omega}_2, y))^2\}
\alpha(\bar{\omega}_1, \bar{\omega}_2, y) = \max\{\tau_\theta(\bar{\omega}_1, T\bar{\omega}_1, y), \tau_\theta(\bar{\omega}_2, T\bar{\omega}_2, y)\}
\end{array} \right\}, \] (71)

for all \( \bar{\omega}_1, \bar{\omega}_2 \in \Delta \), where \( \beta \in F_\beta \) and \( \theta(\bar{\omega}_1, \bar{\omega}_2) \beta(\tau_\theta(\bar{\omega}_1, \bar{\omega}_2, y)) < 1 \). Then \( T \) has a unique fixed point.

Proof. By the same way as in Theorem 1, we have
\[ \tau_\theta(\bar{\omega}_l, \bar{\omega}_{l+1}, y) = \tau_\theta(T\bar{\omega}_l, T\bar{\omega}_{l+1}, y) \geq \frac{\lambda(\bar{\omega}_{l-1}, \bar{\omega}_l, y/\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y))) \ast \mu(\bar{\omega}_{l-1}, \bar{\omega}_l, y/\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y)))}{\alpha(\bar{\omega}_{l-1}, \bar{\omega}_l, y/\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y)))}, \] (72)

\[ \lambda(\bar{\omega}_{l-1}, \bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y))}) \]
\[ = \min\left\{ \tau_\theta(\bar{\omega}_{l-1}, T\bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y))}) \cdot \tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y))}) \right\} \cdot \tau_\theta(\bar{\omega}_{l-1}, T\bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_{l-1}, \bar{\omega}_l, y))}), \]
\[ \tau_\theta(\bar{\omega}_l, T\bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_l, \bar{\omega}_l, y))}) \}
\[ = \min\left\{ \tau_\theta(\bar{\omega}_l, T\bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_l, \bar{\omega}_l, y))}) \cdot \tau_\theta(\bar{\omega}_l, \bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_l, \bar{\omega}_l, y))}) \right\} \cdot \tau_\theta(\bar{\omega}_l, T\bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_l, \bar{\omega}_l, y))}), \]
\[ \tau_\theta(\bar{\omega}_l, T\bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_l, \bar{\omega}_l, y))}) \}
\[ = \tau_\theta\left(\bar{\omega}_l, T\bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_l, \bar{\omega}_l, y))}\right) \cdot \tau_\theta\left(\bar{\omega}_l, \bar{\omega}_l, \frac{y}{\beta(\tau_\theta(\bar{\omega}_l, \bar{\omega}_l, y))}\right). \] (73)
Using (73), (74) and (75) in (72), we have

\[
\mu \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right) = \max \left\{ \tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right), \frac{\tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right)}{\mu(\theta, \omega, \gamma)} \right\}
\]

(74)

\[
\alpha \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right) = \max \left\{ \tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right), \frac{\tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right)}{\alpha(\theta, \omega, \gamma)} \right\}
\]

(75)

\[
\tau_\theta(\omega, \omega, \gamma) \geq \frac{\tau_\theta(\omega, \omega, \gamma)}{\max \left\{ \tau_\theta(\omega, \omega, \gamma), \frac{\tau_\theta(\omega, \omega, \gamma)}{\alpha(\theta, \omega, \gamma)} \right\}}
\]

(76)

If

\[
\max \left\{ \tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right), \frac{\tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right)}{\alpha(\theta, \omega, \gamma)} \right\} = \tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right)
\]

(77)

then (76) implies

\[
\tau_\theta(\omega, \omega, \gamma) \geq \frac{\tau_\theta(\omega, \omega, \gamma)}{\alpha(\theta, \omega, \gamma)}
\]

(78)

then \( T \) has a fixed point by Lemma 1. If

\[
\max \left\{ \tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right), \frac{\tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right)}{\alpha(\theta, \omega, \gamma)} \right\} = \tau_\theta \left( \frac{\gamma}{\beta(\theta, \omega, \gamma)} \right)
\]

(79)

then from (76), we have
\[
\tau_\theta(\omega_1, \omega_{i+1}, y) \geq \tau_\theta(\omega_{i-1}, \omega_i, \beta(\tau_\theta(\omega_{i+1}, \omega_i, y)))
\]  
(80)

Continuing in this way, we will get

\[
\tau_\theta(\omega_1, \omega_{i+1}, y) \geq \tau_\theta(\omega_{i-1}, \omega_i, \beta(\tau_\theta(\omega_{i+1}, \omega_i, y))) \geq \ldots \geq \tau_\theta(\omega_0, \omega_1, \beta(\tau_\theta(\omega_1, \omega_0, y))) \cdot \beta(\tau_\theta(\omega_{i-2}, \omega_{i-1}, y)) \ldots \beta(\tau_\theta(\omega_0, \omega_1, y)).
\]  
(81)

4. Application

The application of our main result stated in Theorem 1 is established here. Let \( \Delta = \mathbb{C}[0, I] \) be the set of real valued continuous functions on \([0, I]\) and define a \( G_r \)-complete EFBMS \( F_\alpha: \Delta \times \Delta \times [0, \infty) \rightarrow [0, 1) \) by

\[
\theta(\omega_1, \omega_2) = 1 + \omega_1 + \omega_2.
\]  
(83)

Consider the integral equation

\[
\omega_1(s) = f(s) + \int_0^I h(s, r)H(s, r, \omega_1(r))\, dr,
\]  
(84)

with

\[
\tau_\theta(\omega_1, \omega_2, y) = \begin{cases} 
\sup_{r \in [0, I]} |\omega_1(r) - \omega_2(r)|^2, & \text{if } y > 0, \\
0, & \text{if } y \leq 0,
\end{cases}
\]  
(82)

where \( I > 0 \), and \( f: [0, I] \rightarrow \mathbb{R}, h: [0, I] \times [0, I] \rightarrow \mathbb{R}, \) and \( H: [0, I] \times [0, I] \times \mathbb{R} \rightarrow \mathbb{R} \) all are continuous functions.

**Theorem 9.** Suppose that the following conditions hold:

(i) for all \( s, r \in [0, I], \omega_1, \omega_2 \in \Delta \) and \( \beta \in F_\alpha \), we have

\[
|H(s, r, \omega_1(r)) - H(s, r, \omega_2(r))| < \beta(\tau_\theta(\omega_1, \omega_2, y))|\omega_1(r) - \omega_2(r)|.
\]  
(85)

(ii) for all \( s, r \in [0, I] \),

\[
\sup_{r \in [0, I]} \int_0^I (h(s, r))^2\, dr \leq \frac{1}{I}
\]  
(86)

Then the integral equation (84) has a solution in \( \Delta \).

**Proof.** Let \( T: \Delta \rightarrow \Delta \) be the integral operator defined by

\[
T\omega_1(s) = f(s) + \int_0^I h(s, r)H(s, r, \omega_1(r))\, dr,
\]  
(87)

for all \( \omega_1 \in \Delta \), and \( s, r \in [0, I] \). Now, for all \( \omega_1, \omega_2 \in \Delta \) and by using Conditions (i) and (ii), we have
\[ \tau_\theta(T\omega_1, T\omega_2, \beta(\tau_\theta(\omega_1, \omega_2, \gamma))) \gamma = e^{\sup_{\nu \in [0,1]}|T\omega_1(\nu) - T\omega_2(\nu)|^\gamma \beta(t_\theta(\omega, \omega, \gamma))} \]

\[ = e^{\left( \sup_{\nu \in [0,1]} \int_0^1 h \int_0^1 (s, r) H(s, r, \omega_1(\nu)) dr - \int_0^1 h(s, r) H(s, r, \omega_2(\nu)) dr \right) \beta(\tau_\theta(\omega_1, \omega_2, \gamma)) \gamma} \]

\[ = e^{\left( \sup_{\nu \in [0,1]} \int_0^1 h(s, r) \left( H(s, r, \omega_1(\nu)) - H(s, r, \omega_2(\nu)) \right) dr \right)^\gamma \beta(\tau_\theta(\omega_1, \omega_2, \gamma)) \gamma} \]

\[ \geq e^{\left( \sup_{\nu \in [0,1]} \int_0^1 (h(s, r))^2 dr \right) \int_0^1 \left| F(s, r, \omega_1(\nu)) - F(s, r, \omega_2(\nu)) \right|^2 \beta(\tau_\theta(\omega_1, \omega_2, \gamma)) \gamma} \]

\[ \geq e^{\left( \sup_{\nu \in [0,1]} \frac{1}{(1/\gamma)} \int_0^1 \left\{ \left[ \beta(t_\theta(\omega_1, \omega_2, \gamma)) \right] \beta(t_\theta(\omega_1, \omega_2, \gamma)) \right\}^2 \beta(\tau_\theta(\omega_1, \omega_2, \gamma)) \gamma} \]

\[ \geq e^{-\left( \sup_{\nu \in [0,1]} |\beta(\omega_1(\nu) - \omega_2(\nu)|^\gamma \beta(\tau_\theta(\omega_1, \omega_2, \gamma)) \gamma} = \tau_\theta(\omega_1, \omega_2, \gamma). \] (89)

That is

\[ \tau_\theta(T\omega_1, T\omega_2, \beta(\tau_\theta(\omega_1, \omega_2, \gamma))) \gamma \geq \tau_\theta(\omega_1, \omega_2, \gamma). \] (90)

Consequently

\[ \tau_\theta(T\omega_1, T\omega_2, \beta(\tau_\theta(\omega_1, \omega_2, \gamma))) \gamma \geq \tau_\theta(\omega_1, \omega_2, \gamma). \]

Since all the conditions of Theorem 1 are satisfied, the mapping \( T \) has a fixed point. That is, the integral equation (84) has a solution. \[ \square \]

Data Availability

No data were used to support this study.

Disclosure

The statements made and views expressed are solely the responsibility of the author.

Conflicts of Interest

The authors declare that they have no Conflicts of Interest.

Authors’ Contributions

The authors contributed equally to this work.

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