Research Article

Functional Approach for Solving Reduced Order of Index-Four Hessenberg Differential-Algebraic Control System

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This research investigates differential-algebraic equations with higher index (index four). Specifically, a functional analytic approach is proposed to find the solution of (index four) Hessenberg differential-algebraic equations (DAEs). The approximate solution of the proposed functional approach for a suitable separable Hilbert space is obtained with the help of the direct optimisation technique and Ritz basis functions method. Illustrations have been ranked from an 8×8 test system of index-four Hessenberg linear DAEs to a 4×4 DAE of rotating masses as well as an 8×8 differential-algebraic generator model, where it reformulated index-four linear Hessenberg DAEs. Their approximate solutions were obtained using the present approach with comparisons. The numerical results demonstrate the simplicity of the proposed approach and express suitable accuracy and efficiency.

1. Introduction

Many life applications can be modelled as differential-algebraic systems, including optimal control problems, constrained robotic systems, and constrained electrical networks [1–3]. In the literature, the index property, of which several definitions exist, is usually utilised to characterise the differential-algebraic system [4, 5]. To determine an equivalent ordinary differential equation, the minimum number of times that the algebraic constraint of DAEs must be differentiated to time is represented with the frequently used differentiation index [5]. The higher index DAEs present a greater challenge in being solved than those in the lower index system because it makes applications more cumbersome [6]. In general DAEs, with a special structure, Hessenberg DAEs are an exceedingly important class implemented in various engineering and scientific applications [7, 8].

The authors in [2, 3, 9] presented numerical methods to solve index two and index three. These methods utilize the index reduction method to reduce the higher DAE index to be lower one [10–12].

To solve higher index (index four) Hessenberg linear DAE systems, this article develops an analytical

approximation method to be efficiently and easily implemented. It is based on the functional analytical theory, which will help to determine a functional with critical points that serve as solutions for the fourth index Hessenberg DAEs and vice versa. To present the solution as a linear combination of the basic elements of the given setting space (i.e., separable Banach space), this work introduces a parameterisation approach inserted into the obtained variational formulation. Subsequently, to determine the solution of the variational formulation for unknown parameters, the method uses a direct method of calculus of variation. Finally, solving the linear algebraic system obtains the unknown parameters.

2. Fourth-Order Hessenberg DAEs Problem Formulation

Index-four linear Hessenberg DAEs with control input are considered as follows:

$$\dot{x}_1 + A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 = B_1u + f_1, \quad (1a)$$

$$\dot{x}_2 + A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = B_2u + f_2, \quad (1b)$$

$$\dot{x}_3 + A_{32}x_2 + A_{33}x_3 = B_3u + f_3, \quad (1c)$$

$$A_{43}x_3 = B_4u + f_4, \quad (1d)$$

where

- (1) $A_{43}A_{32}A_{21}A_{14} \triangleq c$ is a square invertible matrix, and $(x_1, x_2, x_3, x_4)^T$ denotes independent state vectors
- (2) $A_{ij}; i, j = 1, \dots, 4$, are constant matrices of appropriate dimension, and A_{ij} is not required to be invertible or a square matrix
- (3) $B_j, j = 1, \dots, 4$ are suitable constant vectors, and u is the given single control input with $u \in C^3[T; R] \triangleq \mathcal{D}_u$ as the class of admissible control. $T = [t_0, t_f]$, $t_0 < t_f$; and $f_j; j = 1, \dots, 4$ denotes differentiable functions, $f_j \in C^3[T; R]$

Using the implicit function theorem, which can be observed in [13, 14], (1d) can be differentiated with respect to t :

$$A_{43}\dot{x}_3 = B_4\dot{u} + \dot{f}_4. \quad (2)$$

From equation (1c), the following is obtained:

$$-B_4\dot{u} - \dot{f}_4 + A_{43}[B_3u + f_3 - A_{32}x_2 - A_{33}x_3] = 0. \quad (3)$$

Now, in equation (3), a new constraint is conjugated with equations (1a)–(1d) (i.e., differential-algebraic system). The differentiability of equation (3) continues until the state x_4 is estimated to have a class of differential equations with states x_1, x_2, x_3 defined on a manifold of the resulting algebraic constraints.

Furthermore, x_4 (state variable) does not exist in equation (3); accordingly, equation (1c) can be differentiated (with respect to t) again.

$$-B_4\ddot{u} - \ddot{f}_4 + A_{43}[B_3\dot{u} + \dot{f}_3 - A_{32}\dot{x}_2 - A_{33}\dot{x}_3] = 0, \quad (4)$$

$$\begin{aligned} & -B_4\ddot{u} + A_{43}B_3\ddot{u} - (A_{43}A_{32}B_2 + A_{43}A_{33}B_3)u - \ddot{f}_4 + A_{43}\dot{f}_3 - A_{43}A_{32}\dot{f}_2 - A_{43}A_{33}\dot{f}_3 + A_{43}A_{32}A_{21}x_1 \\ & + (A_{43}A_{32}A_{22} + A_{43}A_{33}A_{32})x_2 + (A_{43}A_{32}A_{23} + A_{43}A_{33}A_{32})x_3 = 0. \end{aligned} \quad (5)$$

Hence, another algebraic constraint (inserted into the differential-algebraic system (1a)–(1d)) is represented by equation (5). To determine the state variable x_4 and until the

index-four condition ($A_{43}A_{32}A_{21}A_{14}$) is satisfied, the derivation is continued:

$$\begin{aligned} & -B_4\ddot{u} + A_{43}B_3\ddot{u} - (A_{43}A_{32}B_2 + A_{43}A_{33}B_3)u + (A_{43}A_{32}A_{21}B_1 + A_{43}A_{32}A_{22}B_2 + \\ & \cdot A_{43}A_{33}A_{32}B_2 + A_{43}A_{32}A_{23}B_3 + A_{43}A_{33}A_{32}B_3)u - \ddot{f}_4 + A_{43}\dot{f}_3 - A_{43}A_{32}\dot{f}_2 + \\ & \cdot A_{43}A_{32}A_{21}\dot{f}_1 (A_{43}A_{32}A_{22} + A_{43}A_{33}A_{32})\dot{f}_2 + (A_{43}A_{32}A_{23} + A_{43}A_{33}A_{32})\dot{f}_3 - \\ & \cdot (A_{43}A_{32}A_{21}A_{11} + A_{43}A_{32}A_{22}A_{21} + A_{43}A_{33}A_{32}A_{21})x_1 - \\ & \cdot (A_{43}A_{32}A_{21}A_{12} + A_{43}A_{32}A_{22}A_{22} + A_{43}A_{33}A_{32}A_{22} + A_{43}A_{32}A_{23}A_{32} + A_{43}A_{32}A_{23}A_{32})x_2 - \\ & \cdot (A_{43}A_{32}A_{21}A_{13} + A_{43}A_{32}A_{22}A_{23} + A_{43}A_{33}A_{32}A_{23} + A_{43}A_{32}A_{23}A_{33} + A_{43}A_{33}A_{32}A_{33})x_3 - \\ & \cdot (A_{43}A_{32}A_{21}A_{14})x_4 = 0. \end{aligned} \quad (6)$$

Then, since $A_{43}A_{32}A_{21}A_{14}$ is invertible,

$$x_4 = c^{-1}(\ddot{u} + \ddot{f} + \tilde{A}_1x_1 + \tilde{A}_2x_2 + \tilde{A}_3x_3) = L(x_1) + L(x_2) + L(x_3) + L(\ddot{u}, \ddot{f}), \quad (7)$$

where $c = A_{43}A_{32}A_{21}A_{14}$, $L(x_1) = c^{-1}\tilde{A}_1x_1$, $L(x_2) = c^{-1}\tilde{A}_2x_2$, $L(x_3) = c^{-1}\tilde{A}_3x_3$, $L(\ddot{u}, \ddot{f}) = c^{-1}(\ddot{u} + \ddot{f})$,

$$\begin{aligned} \ddot{u} = & -B_4\ddot{u} + A_{43}B_3\ddot{u} - (A_{43}A_{32}B_2 + A_{43}A_{33}B_3)u \\ & + (A_{43}A_{32}A_{21}B_1 + A_{43}A_{32}A_{22}B_2 + A_{43}A_{33}A_{32}B_2 + A_{43}A_{32}A_{23}B_3 + A_{43}A_{33}A_{32}B_3)u \end{aligned}$$

$$\begin{aligned}
\ddot{f} &= -\dot{f}_4 + A_{43}\ddot{f}_3 - A_{43}A_{32}\dot{f}_2 + A_{43}A_{32}A_{21}\dot{f}_1 + (A_{43}A_{32}A_{22} + A_{43}A_{33}A_{32})\dot{f}_2 + (A_{43}A_{32}A_{23} + A_{43}A_{33}A_{32})\dot{f}_3, \\
\tilde{A}_1 &= -(A_{43}A_{32}A_{21}A_{11} + A_{43}A_{32}A_{22}A_{21} + A_{43}A_{33}A_{32}A_{21}), \\
\tilde{A}_2 &= -(A_{43}A_{32}A_{21}A_{12} + A_{43}A_{32}A_{22}A_{22} + A_{43}A_{33}A_{32}A_{22} + A_{43}A_{32}A_{23}A_{32} + A_{43}A_{32}A_{23}A_{32}), \\
\tilde{A}_3 &= -(A_{43}A_{32}A_{21}A_{13} + A_{43}A_{32}A_{22}A_{23} + A_{43}A_{33}A_{32}A_{23} + A_{43}A_{32}A_{23}A_{33} + A_{43}A_{33}A_{32}A_{33}).
\end{aligned} \tag{8}$$

Since system (1) is the index-four Hessenberg DAEs system, the initial condition must be selected in such a way as to satisfy all algebraic constraints in equations (1d), (3), and (5). From the additional two hidden constraint equations (1c)–(1d) and from equation (7), over the class of $C^3[T; R]$, the following form is the reduced type index-four Hessenberg DAEs (1) with control $u, \dot{u}, \dots, u$.

$$\begin{aligned}
\dot{x}_1 + (A_{11} + A_{14}c^{-1}\tilde{A}_1)x_1 + (A_{12} + A_{14}c^{-1}\tilde{A}_2)x_2 \\
+ (A_{13} + A_{14}c^{-1}\tilde{A}_3)x_3 = B_1u + f_1 - A_{14}L(\tilde{u}, \tilde{f}),
\end{aligned} \tag{9}$$

$$\dot{x}_2 + A_{21}x_1 + \dots + A_{23}x_3 = B_2u + f_2, \tag{10}$$

$$\dot{x}_3 + A_{32}x_2 + A_{33}x_3 = B_3u + f_3, \tag{11}$$

$$\text{with algebraic constraint one : } A_{43}x_3 = B_4u + f_4, \tag{12}$$

$$\begin{aligned}
\text{algebraic constraint two : } A_{43}A_{32}x_2 + A_{43}A_{33}x_3 \\
= -B_4\dot{u} + A_{43}B_3u - \dot{f}_4 + A_{43}f_3,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\text{algebraic constraint three : } A_{43}A_{32}A_{21}x_1 + \tilde{A}_2x_2 + \tilde{A}_3x_3 \\
= B_4\ddot{u} - A_{43}B_3\dot{u} \\
+ (A_{43}A_{32}B_2 + A_{43}A_{33}B_3)u + \ddot{f}_4 \\
- A_{43}\dot{f}_3 + A_{43}A_{32}\dot{f}_2 + A_{43}A_{33}\dot{f}_3.
\end{aligned} \tag{14}$$

Set $\mathfrak{F} = \{(x_1(\cdot), x_2(\cdot), x_3(\cdot)) \in C^3(T; R^3) | \text{where (12)–(14) are satisfied. Ultimately, the following represents the class of consistent initial condition:}$

$$J^0 = \{(x_1(t_0), x_2(t_0), x_3(t_0)) \in R^3 | J \text{ is satisfied at } t = t_0\}. \tag{15}$$

3. Fourth Index Hessenberg DAE Solution via Variational Formulation Technique

The present approach is step by step represented as follows:

Step 1: define the operator $L: D(L) \subset H \rightarrow R(L) \subset H$, with $\overline{D(L)} = H$, where H is an appropriate Hilbert space (e.g., $H \triangleq L_2(T); T = [t_0, t_f], t_0 < t_f$) by $Lx = F(t; \tilde{u}, \tilde{f})$. The operator L refers to a linear differential and algebraic operator can be written as follows:

$$Lx = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} + \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{21} & A_{21} \\ 0 & 0 & A_{21} \\ 0 & \mathcal{A}_4 & \mathcal{A}_5 \\ \mathcal{A}_6 & A_{21} & A_{21} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{16}$$

where $\mathcal{A}_1 = A_{11} + A_{14}c^{-1}\tilde{A}_1$, $\mathcal{A}_2 = A_{12} + A_{14}c^{-1}\tilde{A}_2$, $\mathcal{A}_3 = A_{13} + A_{14}c^{-1}\tilde{A}_3$, $\mathcal{A}_4 = A_{43}A_{32}$, $\mathcal{A}_5 = A_{43}A_{33}$, $\mathcal{A}_6 = A_{43}A_{32}A_{21}$, and $F(t; \tilde{u}, \tilde{f}) \triangleq F(t; u, \dot{u}, \ddot{u}, u, f_j, \dot{f}_j, \ddot{f}_j, f_j^{(3)})$, for all $u \in C^3[T; R] \triangleq \Delta_u$.

By using the variational formulation of obtained functional corresponding to (13) on $D(L) = \{x(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot))^T \in L_2[T; R^3] | x(t) \in \mathfrak{F}, x(t_0) \in \mathfrak{F}^0 \text{ and } x(t) \text{ satisfy (13)}\}$.

In almost everywhere in H , the DAEs are satisfied by the linearly invariant (reduced type) index Hessenberg DAEs system. Furthermore, almost everywhere over the setting class $C^1[T; R^3]$, the solution should be a continuous function that satisfies equations (12)–(14).

Step 2: in equation (16), the operator L is not symmetric to the usual bilinear form $(x, \tilde{x}) = \int x^T(t) \tilde{x}(t) dt$, $x = (x_1(t), x_2(t), x_3(t))$, $\tilde{x} = (\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t))$ because of the existence of the operator $d./dt$. To satisfy the nondegenerate condition on the range and the domain of operator L , the bilinear form needs to be chosen (i.e., for every $x \in D(L)$, $(x, \tilde{x}) = 0$, then $\tilde{x} = 0$, and for every $\tilde{x} \in D(L)$, $(x, \tilde{x}) = 0$ (then $x = 0$)). Therefore, a usual variational formulation exists only if the usual bilinear (inner product) or the linear operator is redefined form using $\langle x, \tilde{x} \rangle \triangleq (x, L\tilde{x})$, $x, \tilde{x} \in D(L)$, where $D(L) \subset H_L \subset H$ with $\|x\|_L = \sqrt{(x, Lx)}$, and $(H_L, d(x, \tilde{x}))$ is the complete Hilbert space (where L refers to the asymmetric linear operator and separable Hilbert space is denoted by H_L). The reader is recommended to [10, 11] for more details. $J(x)$ is defined if L (on $\overline{D(L)}$) is positive definite; otherwise, it is needed to determine $J_{\mathfrak{F}^0}(x)$ (the critical points). Over the class of the consistent initial conditions and

$u \in C^3[T; R]$, the augmented functional of $J_{\mathfrak{F}^0}(x)$ can be defined as follows:

$$J_{\mathfrak{F}^0}(x) = \frac{1}{2} \left(L_1^T x(t), L_1 x(t) \right) - \left(F_1^T, L_1 x(t) \right) + \cdots + \frac{1}{2} \left(L_6^T x(t), L_6 x(t) \right) - \left(F_6^T, L_6 x(t) \right) + \frac{1}{2} \left(L_4^T x(t_0), L_4 x(t_0) \right) \\ - \left(F_4^T(t_0), L_4 x(t_0) \right) + \frac{1}{2} \left(L_5^T x(t_0), L_5 x(t_0) \right) - \left(F_5^T(t_0), L_5 x(t_0) \right) + \frac{1}{2} \left(L_6^T x(t_0), L_6 x(t_0) \right) - \left(F_6^T(t_0), L_6 x(t_0) \right). \quad (17)$$

where $L_1 x(t) = \dot{x}_1 + \mathcal{A}_1 x_1 + \mathcal{A}_2 x_2 + \mathcal{A}_3 x_3$, $L_2 x(t) = \dot{x}_2 + A_{21} x_1 + \cdots + A_{23} x_3$, $L_3 x(t) = \dot{x}_3 + A_{32} x_2 + A_{33} x_3$, $L_4 x(t) = A_{43} x_3$, $L_5 x(t) = \mathcal{A}_4 x_2 + \mathcal{A}_5 x_3$, $L_6 x(t) = \mathcal{A}_6 x_1 + \mathcal{A}_2 x_2 + \mathcal{A}_3 x_3$, and $F_1 = B_1 u + f_1 - A_{14} L(\tilde{u}, \tilde{f})$, $F_2 = B_2 u + f_2$, $F_3 = B_3 u + f_3$, $F_4 = B_4 u + f_4$, $F_5 = -B_4 \dot{u} + A_{43} B_3 u - \dot{f}_4 + A_{43} f_3$, and $F_6 = B_4 \ddot{u} - A_{43} B_3 \dot{u} + (\mathcal{A}_4 B_2 + \mathcal{A}_5 B_3) u + f_4 - A_{43} \dot{f}_3 + \mathcal{A}_4 f_2 + \mathcal{A}_5 f_3$.

Thus, the critical points of $J_{\mathfrak{F}^0}(x)$ is guaranteed by the functional as the solution of equations (12)–(14) and vice versa when $(x, \tilde{x})$ is chosen in the form of nondegenerate bilinear.

Step 3: the critical points of equations (12)–(14) are the solutions of equation (12). Furthermore, the critical points are a solution of equations (12)–(14) by the nondegenerate property on $D(L), R(L)$. A countable linearly independent (bases) system $(\varphi_1, \varphi_2, \dots \text{ etc.})$ (complete set of functions; base) can be used to approximate the solution, where $x_j(t) = \sum_{\zeta=1}^n a_{\zeta} \varphi_{\zeta}$, n is a positive number, $j = 1, 2, 3$ and a_{ζ} is determined so that $J(x_1(t), x_2(t), x_3(t))$ is extremized.

Step 4: with respect to a_{ζ} , $\forall \zeta = 1, \dots, n$, J can be derived. To determine these critical points as a function of a_{ζ} parameters, n is given a suitable positive number. To obtain the system of a solvable linear algebraic, one can set $dJ/da_{\zeta} = 0, \forall \zeta = 1, \dots, n$. Furthermore, $\sum_{\zeta=1}^n M(\zeta, j) a_{\zeta} = N_j, \forall j = 1, \dots, n$, $M = (M(\zeta, j))_{\zeta=1, j=1}^{n,3}$ is a nonsingular matrix, where the coefficients can be determined from $dJ/da_{\zeta} = 0$, if $x(t_0) \in \mathfrak{F}^0$. Accordingly, $a_{\zeta} = M^{-1}(\zeta, j) N_j$, and the unique solution is guaranteed. Else, if $x(t_0)$ is

chosen as arbitrary, there may be infinitely solutions or no solution.

4. Illustration Examples

4.1. *Test Example.* The following test example is designed with a suitable exact solution in mind to confirm the validity of the aforementioned approach, where

- (1) The system is 8×8 DAEs with control
- (2) The system is represented as fourth order Hessenberg linear DAE with control

$$\dot{x}_{11} + 2x_{11} + 3x_{12} + x_{31} + x_{41} = f_{11}, \quad (18)$$

$$\dot{x}_{12} - 2x_{11} - 2x_{21} + x_{22} + x_{31} + x_{41} = u + f_{12}, \quad (19)$$

$$\dot{x}_{21} + 4x_{11} + x_{12} - 2x_{21} + x_{31} = 3u + f_{21}, \quad (20)$$

$$\dot{x}_{22} + x_{12} - 2x_{21} + x_{32} = f_{22}, \quad (21)$$

$$\dot{x}_{31} + 3x_{21} - 2x_{22} + x_{31} = f_{31}, \quad (22)$$

$$\dot{x}_{32} + 3x_{21} + x_{32} = -2u + f_{32}, \quad (23)$$

$$x_{31} = -u + f_{42}, \quad (24)$$

$$x_{32} = f_{42}, \quad (25)$$

with

$$f_{11} = t^4 + t + 3, f_{12} = t^5 + t^3 - t^2 - \sin t - 4,$$

$$f_{21} = 5t + t^4 - 2t^2 - 3 \sin t - 2, f_{22} = t^5 + t^3 - 2t^2 - t + 1,$$

$$f_{31} = t^4 + 4t^3 + 3t^2 - 4t + 4, f_{32} = t^5 + 5t^4 + t^3 + 6t^2 + 2 \sin t + 3,$$

$$f_{41} = t^4 - \sin t - 1, f_{42} = t^5 + t^3,$$

$$A_{11} = \begin{bmatrix} 2 & 3 \\ -2 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}, A_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{14} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{21} = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix}, A_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{32} = \begin{bmatrix} 3 & -2 \\ 3 & 0 \end{bmatrix}, A_{33} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{43} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, B_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}, x_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}, x_3 = \begin{bmatrix} x_{31} \\ x_{32} \end{bmatrix}, x_4 = \begin{bmatrix} x_{41} \\ x_{42} \end{bmatrix}. \quad (26)$$

Note that $(A_{43}A_{32}A_{21}A_{14})$ is an invertible matrix.

Significantly, in the DAE system, variable x_4 does not appear. Therefore, the algebraic constraint equations (18)–(23) can be differentiated with respect to t . Then, to obtain the following, substitute the differential constraints to equations (24) and (25).

$$A_{43}\dot{x}_3 = B_4\dot{u} + \dot{f}_4, \quad (27)$$

$$A_{43}(B_3u + f_3 - A_{32}x_2 - A_{33}x_3) = B_4\dot{u} + \dot{f}_4.$$

Thus, equation (27) is considered a new constraint inserted to equations (18)–(25). In the last constraint, variable x_4 is not exist. Therefore, to obtain the following, equation (27) can be differentiated one more time with respect to t .

$$-B_4\ddot{u} + A_{43}B_3\ddot{u} - (A_4B_2 + A_5B_3)u - \ddot{f}_4 + A_{43}\dot{f}_3 - A_4f_2 - A_5f_3 + A_4A_{21}x_1$$

$$+ (A_4A_{22} + A_5A_{32})x_2 + (A_4A_{23} + A_5A_{33})x_3 = 0, \quad (28)$$

where $A_4 = A_{43}A_{32}$, $A_5 = A_{43}A_{33}$.

Next, one can differentiate equation (28) again for the t set: $c \triangleq (A_{43}A_{32}A_{21}A_{14})^{-1} = \begin{bmatrix} 0.1250 & -0.0417 \\ -0.5 & 0.5 \end{bmatrix}$.

Step 1: x_4 is estimated as $x_4 = L(x_1, x_2, x_3) + L(u, \dot{u}, \ddot{u}, f_1, f_2, f_3, \dot{f}_2, \dot{f}_3, \ddot{f}_3)$, where the derivative of the class of the consistent initial conditions can be

$$\mathfrak{F}^0 = \left\{ \begin{array}{l} (x_1(t_0), x_2(t_0), x_3(t_0)) \\ A_{43}x_3(t_0) = B_4u(t_0) + f_4(t_0) \\ -A_{43}A_{32}x_2(t_0) - A_{43}A_{33}x_3(t_0) = B_4\dot{u}(t_0) + \dot{f}_4(t_0) - A_{43}B_3u(t_0) - A_{43}f_3(t_0) \\ \mathcal{A}_4A_{21}x_1(t_0) + \mathcal{A}_5A_{22}x_2(t_0) + \mathcal{A}_4A_{23}x_3(t_0) + \mathcal{A}_5A_{32}x_2(t_0) + \mathcal{A}_5A_{33}x_3(t_0) \\ = B_4\ddot{u}(t_0) + \ddot{f}_4(t_0) - A_{43}B_3\ddot{u}(t_0) - A_{43}\dot{f}_3(t_0) + \mathcal{A}_4B_2u(t_0) + \mathcal{A}_4f_2(t_0) + \mathcal{A}_5B_3u(t_0) + \mathcal{A}_5f_3(t_0) \end{array} \right\}. \quad (29)$$

Step 2: as can be seen in the following, the definition of the variational function with the class of the consistent initial conditions can be written as follows:

$$J\mathfrak{F}^0[x] = \int_{t_0}^{t_f} \frac{1}{2} [\dot{x}_1 + \mathcal{A}_1x_1 + \mathcal{A}_2x_2 + \mathcal{A}_3x_3]^T [\dot{x}_1 + \mathcal{A}_1x_1 + \mathcal{A}_2x_2 + \mathcal{A}_3x_3]$$

$$+ [\dot{x}_2 + \mathcal{A}_{21}x_1 + \mathcal{A}_{22}x_2 + \mathcal{A}_{23}x_3]^T [\dot{x}_2 + \mathcal{A}_{21}x_1 + \mathcal{A}_{22}x_2 + \mathcal{A}_{23}x_3] + [\dot{x}_3 + \mathcal{A}_{32}x_2 + \mathcal{A}_{33}x_3]^T$$

$$\cdot [\dot{x}_3 + \mathcal{A}_{32}x_2 + \mathcal{A}_{33}x_3] + [A_{43}x_3]^T [A_{43}x_3] + [\mathcal{A}_4x_2 + \mathcal{A}_5x_3]^T [\mathcal{A}_4x_2 + \mathcal{A}_5x_3] +$$

$$\cdot [A_6x_1 + A_2x_2 + \mathcal{A}_3x_3]^T [A_6x_1 + A_2x_2 + A_3x_3] + [A_{43}x_3(t_0)]^T [A_{43}x_3(t_0)]$$

$$\begin{aligned}
& + [A_4 x_2(t_0) + A_5 x_3(t_0)]^T [A_4 x_2(t_0) + A_5 x_3(t_0)] + [\mathcal{A}_6 x_1(t_0) + \mathcal{A}_2 x_2(t_0) + \mathcal{A}_3 x_3(t_0)]^T \\
& \cdot [\mathcal{A}_6 x_1(t_0) + \mathcal{A}_2 x_2(t_0) + \mathcal{A}_3 x_3(t_0)] - [F_1]^T [\dot{x}_1 + \mathcal{A}_1 x_1 + \mathcal{A}_2 x_2 + \mathcal{A}_3 x_3] + \\
& \cdot [F_2]^T [\dot{x}_2 + A_{21} x_1 + A_{22} x_2 + A_{23} x_3] + [F_3]^T [\dot{x}_3 + A_{32} x_2 + A_{33} x_3] + [F_4]^T [A_{43} x_3] + \\
& \cdot [F_5]^T [\mathcal{A}_4 x_2 + \mathcal{A}_5 x_3] + [F_6]^T [\mathcal{A}_6 x_1 + \mathcal{A}_2 x_2 + \mathcal{A}_3 x_3] + [F_4(t_0)]^T [A_{43} x_3(t_0)] \\
& + [F_5(t_0)]^T [\mathcal{A}_4 x_2(t_0) + \mathcal{A}_5 x_3(t_0)] + [F_6(t_0)]^T [\mathcal{A}_6 x_1(t_0) + \mathcal{A}_2 x_2(t_0) + \mathcal{A}_3 x_3(t_0)] dt,
\end{aligned} \tag{30}$$

where $F_1 = B_1 u + f_1 - c A_{43} B_3 \ddot{u} - c A_{43} \ddot{f}_3 + c B_4 \dot{u} + c f_4 + c \mathcal{A}_4 B_2 \dot{u} + c \mathcal{A}_4 f_2 - c \mathcal{A}_4 A_{21} B_1 u - c \mathcal{A}_4 A_{21} f_1 - c \mathcal{A}_4 A_{22} B_2 u - c \mathcal{A}_4 A_{22} f_2 - c \mathcal{A}_4 A_{23} B_3 u - c \mathcal{A}_4 A_{23} f_3 + c \mathcal{A}_5 B_3 \dot{u} + c \mathcal{A}_5 \dot{f}_3 - c \mathcal{A}_5 A_{32} B_2 u - c \mathcal{A}_5 A_{32} f_2 - c \mathcal{A}_5 A_{33} B_3 u - c \mathcal{A}_5 A_{33} f_3$, $F_2 = B_2 u + f_2$, $F_3 = B_3 u + f_3$, $F_4 = B_4 u + f_4$, $F_5 = B_4 \dot{u} + \dot{f}_4 - A_{43} B_3 u - A_{43} f_3$, $F_6 = B_4 \ddot{u} + \ddot{f}_4 - A_{43} B_3 \dot{u} - A_{43} \dot{f}_3 + \mathcal{A}_4 B_2 u + \mathcal{A}_4 f_2 + \mathcal{A}_5 B_3 u + \mathcal{A}_5 f_3$.

Step 3: one can set $x_{11}(t) = \sum_{i=0}^5 a_i \varphi_i(t)$, $x_{12}(t) = \sum_{i=0}^5 b_i \varphi_i(t)$,

$x_{21}(t) = \sum_{i=0}^5 c_i \varphi_i(t)$, $x_{22}(t) = \sum_{i=0}^5 d_i \varphi_i(t)$, $x_{31}(t) = \sum_{i=0}^5 v_i \varphi_i(t)$,

and $x_{32}(t) = \sum_{i=0}^5 w_i \varphi_i(t)$, with $\varphi_i(t) = t^i$ to obtain $x_4(t) = L(x_1, x_2, x_3, t) + L(u, \dot{u}, \ddot{u}, \dot{f}_1, f_2, f_3, \dot{f}_2, \dot{f}_3, \ddot{f}_3)$

Step 4: as mentioned above, the functional J is a function of $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}$, and $\vec{w}$. As $J(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}, \vec{w})$ is extremized, one can set $\partial J / \partial \vec{a} = 0$, $\partial J / \partial \vec{b} = 0$, $\partial J / \partial \vec{c} = 0$, $\partial J / \partial \vec{d} = 0$, $\partial J / \partial \vec{v} = 0$, and $\partial J / \partial \vec{w}$. It leads to the linear algebraic equation $M(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}, \vec{w}) = N$, which is solvable directly for $(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}, \vec{w}) = M^{-1}N$. By deriving J with respect to $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}$, and $\vec{w}$, M (the matrix coefficient) can be determined. Hence, the

approximate solution $(x_1(t), x_2(t), x_3(t))$ is obtained. The numerical values of the unknown coefficients $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}$, and $\vec{w}$ are listed in Table 1.

Figure 1 depicts the numerical solutions with equality states compared with the exact solution $x_1 = \begin{bmatrix} t \\ 1-t \end{bmatrix}$, $x_2 = \begin{bmatrix} t^2 + 1 \\ 2t - 1 \end{bmatrix}$, $x_3(t) = \begin{bmatrix} t^4 - 1 \\ t^3 + t^5 \end{bmatrix}$, and $x_4(t) = \begin{bmatrix} 2t \\ t^2 \end{bmatrix}$, for a given $u(t) \triangleq \sin t$.

4.2. Application of the Generator Model. Consider the model of a generator described with the following parameters. On the right axis, the input is the angle θ connected to mass with inertia K which is rotated about the angle θ with angular velocity v . The torque acting on the right and left sides are T_1 and T_2 , respectively. The mass is then connected to the second axis with a generator. The electrical quantities and the variables described in the second axis are then assumed as in [15]. The rest of the electrical circuit includes one inductor and resistor. The model is illustrated in Figure 2.

The system is now modelled using the equations representing the different parts and their connections.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{\mathcal{V}} \\ \emptyset \\ \dot{I} \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \mathcal{V} \\ \emptyset \\ I \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \theta, \tag{31}$$

with $\theta(t) \triangleq \sin t \in C^2[I, R]$,

TABLE 1: Unknown coefficients of the representation of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

Coef. of x_{11} states	a_5	a_4	a_3	a_2	a_1	a_0
Numerical values	1×10^{-39}	-8×10^{-39}	1×10^{-38}	-6×10^{-39}	1	0
Coef. of x_{12} states	b_5	b_4	b_3	b_2	b_1	b_0
Numerical values	1×10^{-38}	-3×10^{-38}	3×10^{-38}	-1×10^{-38}	-1	1
Coef. of x_{21} states	c_5	c_4	c_3	c_2	c_1	c_0
Numerical values	2×10^{-38}	-7×10^{-38}	6×10^{-38}	1	1×10^{-39}	1
Coef. of x_{22} states	d_5	d_4	d_3	d_2	d_1	d_0
Numerical values	1×10^{-40}	-1×10^{-39}	1×10^{-39}	-1×10^{-39}	2	-1
Coef. of x_{31} states	v_5	v_4	v_3	v_2	v_1	v_0
Numerical values	2×10^{-40}	1	-3×10^{-39}	2×10^{-39}	-3×10^{-40}	-1
Coef. of x_{32} states	w_5	w_4	w_3	w_2	w_1	w_0
Numerical values	1	1×10^{-38}	1	9×10^{-39}	-1×10^{-39}	0

FIGURE 1: The comparisons between the exact solution and approximate solution of the test example.

FIGURE 2: A model of generator.

$$A_{11} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, A_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{14} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{21} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix},$$

$$\begin{aligned}
A_{22} &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{32} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, A_{33} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_{43} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_1 = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, x_2 = \begin{bmatrix} \mathcal{V} \\ \emptyset \end{bmatrix}, x_3 = \begin{bmatrix} I \\ u_1 \end{bmatrix}, x_4 = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}.
\end{aligned} \tag{32}$$

and $(A_{43}A_{32}A_{21}A_{14})$ is an invertible matrix.

In the DAE system, variable x_4 does not appear. Therefore, the algebraic constraint equation (31) can be differentiated with respect to t . Then, three times until x_4 is yielded in the following:

$$A_{43}\dot{x}_3 = B_4\dot{\theta}, \tag{33}$$

$$A_{43}(B_3\theta + f_3 - A_{32}x_2 - A_{33}x_3) = B_4\dot{\theta}, \tag{34}$$

$$\begin{aligned}
&-B_4\ddot{\theta} + A_{43}B_3\dot{\theta} - (\mathcal{A}_4B_2 + \mathcal{A}_5B_3)\theta + \mathcal{A}_4A_{21}x_1 \\
&+ (\mathcal{A}_4A_{22} + \mathcal{A}_5A_{32})x_2 + (\mathcal{A}_4A_{23} + \mathcal{A}_5A_{33})x_3 = 0,
\end{aligned} \tag{35}$$

where $\mathcal{A}_4 = A_{43}A_{32}$, $\mathcal{A}_5 = A_{43}A_{33}$.

Next, one can differentiate equation (27) again for the t and set $c \triangleq (A_{43}A_{32}A_{21}A_{14})^{-1} = \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} \therefore$

Step 1: it is mentioned above that x_4 is obtained as $x_4 = L(x_1, x_2, x_3) + L(\theta, \dot{\theta}, \ddot{\theta})$,

The derivative of the class of the consistent initial conditions is as follows:

$$\mathfrak{F}^0 = \left\{ \begin{array}{l} (x_1(t_0), x_2(t_0), x_3(t_0)) \\ A_{43}x_3(t_0) = B_4\theta(t_0) \\ -A_{43}A_{32}x_2(t_0) - A_{43}A_{33}x_3(t_0) = B_4\dot{\theta}(t_0) - A_{43}B_3\theta(t_0) \\ \mathcal{A}_4A_{21}x_1(t_0) + \mathcal{A}_4A_{22}x_2(t_0) + \mathcal{A}_4A_{23}x_3(t_0) + \mathcal{A}_5A_{32}x_2(t_0) + \mathcal{A}_5A_{33}x_3(t_0) \\ = B_4\ddot{\theta}(t_0) - A_{43}B_3\dot{\theta}(t_0) + \mathcal{A}_4B_2\theta(t_0) + \mathcal{A}_5B_3\theta(t_0) \end{array} \right\}, \tag{36}$$

Step 2: the following is the definition of the variational function with the class of the consistent initial condition.

$$\begin{aligned}
J\mathfrak{F}^0[x] &= \int_{t_0}^{t_f} \frac{1}{2} [[\dot{x}_1 + A_1x_1 + A_2x_2 + A_3x_3]^T [\dot{x}_1 + A_1x_1 + A_2x_2 + A_3x_3] + \\
&\cdot [\dot{x}_2 + A_{21}x_1 + A_{22}x_2 + A_{23}x_3]^T [\dot{x}_2 + A_{21}x_1 + A_{22}x_2 + A_{23}x_3] + [\dot{x}_3 + A_{32}x_2 + A_{33}x_3]^T \\
&\cdot [\dot{x}_3 + A_{32}x_2 + A_{33}x_3] + [A_{43}x_3]^T [A_{43}x_3] + [A_4x_2 + A_5x_3]^T [A_4x_2 + A_5x_3] + \\
&\cdot [A_6x_1 + A_2x_2 + A_3x_3]^T [A_6x_1 + A_2x_2 + A_3x_3] + [A_{43}x_3(t_0)]^T [A_{43}x_3(t_0)] + \\
&\cdot [A_4x_2(t_0) + A_5x_3(t_0)]^T [A_4x_2(t_0) + A_5x_3(t_0)] + [A_6x_1(t_0) + A_2x_2(t_0) + A_3x_3(t_0)]^T \\
&\cdot [A_6x_1(t_0) + A_2x_2(t_0) + A_3x_3(t_0)]] - [[F_1]^T [\dot{x}_1 + A_1x_1 + A_2x_2 + A_3x_3] + \\
&\cdot [F_2]^T [\dot{x}_2 + A_{21}x_1 + A_{22}x_2 + A_{23}x_3] + [F_3]^T [\dot{x}_3 + A_{32}x_2 + A_{33}x_3] + [F_4]^T [A_{43}x_3] + \\
&\cdot [F_5]^T [A_4x_2 + A_5x_3] + [F_6]^T [A_6x_1 + A_2x_2 + A_3x_3] + [F_4(t_0)]^T [A_{43}x_3(t_0)] + [F_5(t_0)]^T \\
&\cdot [A_4x_2(t_0) + A_5x_3(t_0)] + [F_6(t_0)]^T [A_6x_1(t_0) + A_2x_2(t_0) + A_3x_3(t_0)]] dt,
\end{aligned} \tag{37}$$

where $F_1 = B_1\theta - cA_{43}B_3\ddot{\theta} + cB_4\dot{\theta} + cs_1B_2\dot{\theta} - cs_1A_{21}B_1\theta - cs_1A_{22}B_2\theta - cs_1A_{23}B_3\theta + cs_2B_3\theta + cs_2A_{32}B_2\theta - cs_2A_{33}B_3\theta$, $F_2 = B_2\theta$, $F_3 = B_3\theta$, $F_4 = B_4\theta$, $F_5 = B_4\theta - A_{43}B_3\theta$, $F_6 = B_4\theta - A_{43}B_3\theta + s_1B_2\theta + s_2B_3\theta$.

Step 3: one can set $x_{11}(t) = \sum_{i=0}^5 a_i \varphi_i(t)$, $x_{12}(t) = \sum_{i=0}^5 b_i \varphi_i(t)$,

$x_{21}(t) = \sum_{i=0}^5 c_i \varphi_i(t)$, $x_{22}(t) = \sum_{i=0}^5 d_i \varphi_i(t)$, $x_{31}(t) = \sum_{i=0}^5 v_i \varphi_i(t)$, and $x_{32}(t) = \sum_{i=0}^5 w_i \varphi_i(t)$, with $\varphi_i(t) = t^i$ to obtain

$$x_4(t) = L \left(\sum_{i=0}^5 a_i t^i, \sum_{i=0}^5 b_i t^i, \sum_{i=0}^5 c_i t^i, \sum_{i=0}^5 d_i t^i, \sum_{i=0}^5 v_i t^i, \sum_{i=0}^5 w_i t^i, t \right) + L(\theta, \dot{\theta}, \ddot{\theta}, \theta \diamond). \quad (38)$$

Step 4: J is a function of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{v}$, and $\vec{w}$, and as $J(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}, \vec{w})$ is extremized, one can set $\partial J / \partial \vec{a} = 0$, $\partial J / \partial \vec{b} = 0$, $\partial J / \partial \vec{c} = 0$, $\partial J / \partial \vec{d} = 0$, $\partial J / \partial \vec{v} = 0$, and $\partial J / \partial \vec{w} = 0$. It is led to the linear algebraic equation $M(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}, \vec{w}) = N$, which is solvable directly for $(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{v}, \vec{w}) = M^{-1}N$. Hence, the approximate solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ is determined. Lastly, the derivative of J with respect to $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{v}$, and $\vec{w}$ is used to obtain the matrix coefficient M .

Step 5: to evaluate the solution's accuracy without knowing the exact solution, the model utilises L_2 -norm by substituting the values of $x_{11}(t) = \sum_{i=0}^5 a_i t^i$, $x_{12}(t) = \sum_{i=0}^5 b_i t^i$, $x_{21}(t) = \sum_{i=0}^5 c_i t^i$, $x_{22}(t) = \sum_{i=0}^5 d_i t^i$,

$x_{31}(t) = \sum_{i=0}^5 v_i t^i$, and $x_{32}(t) = \sum_{i=0}^5 w_i t^i$ in system (31). The error presented in the test example at each point along time $t \in [0, 1]$ and for different kinds of θ are displayed (Figures 3–6).

4.3. Application of the Rotating Masses System. Consider a system of two rotating masses that are described by the torques $T_1(t)$, $T_2(t)$, $T_3(t)$, and $T_4(t)$ and the angular velocities $\alpha_1(t)$ and $\alpha_2(t)$. The masses have the moments of inertia γ_1 and γ_2 . Figure 7 illustrates this scenario.

The system to matrixly describes this model is provided as

$$\begin{bmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{T}_2 \\ \dot{T}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_4 \end{bmatrix}, \quad (39)$$

with $\gamma_1 = \gamma_2 = 1$ and $T_1 = t^5 - 4t$, $T_4 = 2t^2 + 3$.

Step 1: since $A_{43}A_{32}A_{21}A_{14}$ exists as an invertible matrix, the variable T_3 does not appear and $\dot{T}_2 = 4t$ through differentiation of the algebraic constraint of

system (39), we can continue to differentiate until T_3 obtained as $T_3 = \alpha_1 + t^5 - 4t$ with consistent initial condition:

$$\mathfrak{F}^0 = \{(\alpha_1(t_0), \alpha_2(t_0), T_2(t_0)) | T_2 = 2t_0^2 + 3, \alpha_2 + T_2 = 4t_0, -\alpha_1 + \alpha_2 = 4\}. \quad (40)$$

Step 2: using the variational formulation associated with this consistent initial condition,

$$\begin{aligned} J_{\mathfrak{F}^0}[x] = & \int_{t_0}^{t_f} \frac{1}{2} \left[[\dot{\alpha}_1 + T_2 + T_3]^T [\dot{\alpha}_1 + T_2 + T_3] + [\dot{\alpha}_2 + \alpha_1 + T_2]^T [\dot{\alpha}_2 + \alpha_1 + T_2] \right. \\ & \cdot + [\dot{T}_2 - \alpha_2 - T_2]^T [\dot{T}_2 - \alpha_2 - T_2] + [T_2]^T [T_2] + [\alpha_2 + T_2]^T [\alpha_2 + T_2] + [-\alpha_1 + \alpha_2]^T [-\alpha_1 + \alpha_2] \Big] dt, \\ & \cdot \left[[t^5 - 4t]^T [\dot{\alpha}_1 + T_2 + T_3] + [2t^2 + 3]^T [T_2] + 4t[\alpha_2 + T_2] + 4[-\alpha_1 + \alpha_2] \right] dt. \end{aligned} \quad (41)$$

FIGURE 3: The error of the generator system solution for $\theta = e^{-t}$. FIGURE 4: The error of the generator system solution for $\theta = \sin t$. FIGURE 5: The error of the generator system solution for $\theta = \sin t + \cos t$.

FIGURE 6: The error of the generator system solution for $\theta = t^4 + t^2$.

FIGURE 7: Connected rotating masses.

One can set $\alpha_1(t) = \sum_{i=0}^5 p_i t^i$, $\alpha_2(t) = \sum_{i=0}^5 \eta_i t^i$, $T_2(t) = \sum_{i=0}^5 \mu_i t^i$ to obtain

$$T_3 = L \left(\sum_{i=0}^5 p_i t^i, \sum_{i=0}^5 \eta_i t^i, \sum_{i=0}^5 \mu_i t^i, t \right) + L \left(T_1, T_4, \dot{T}_4, \ddot{T}_4, \ddot{T}_4 \right). \quad (42)$$

Step 3: as mentioned above, the functional J is a function of $\vec{p}$, $\vec{\eta}$, and $\vec{\mu}$. As $J(\vec{p}, \vec{\eta}, \vec{\mu})$ is extremized, one can set $\partial J / \partial \vec{p} = 0$, $\partial J / \partial \vec{\eta} = 0$, and $\partial J / \partial \vec{\mu} = 0$. This yields the linear algebraic equation

$M(\vec{p}, \vec{\eta}, \vec{\mu}) = N$, which is solvable directly for $(\vec{p}, \vec{\eta}, \vec{\mu}) = M^{-1}N$. Hence, the approximate solution $(\alpha_1(t), \alpha_2(t), T_2(t))$ is determined. The derivative of J with respect to $\vec{p}$, $\vec{\eta}$, and $\vec{\mu}$ is used to obtain the matrix coefficient M .

Step 4: to evaluate the solution's accuracy without knowing the exact solution, we use L_2 -norm where the values of $\alpha_1(t) = \sum_{i=0}^5 p_i t^i$, $\alpha_2(t) = \sum_{i=0}^5 \eta_i t^i$, and $T_2(t) = \sum_{i=0}^5 \mu_i t^i$ are substituted in system (39).

For differential equations,

$$\begin{aligned} \|\dot{\alpha}_1 + T_2 + T_3 - t^5 + 4t\|_2 &= \left[\int_0^1 |\dot{\alpha}_1 + T_2 + T_3 - t^5 + 4t|^2 dt \right]^{1/2} = 0.0035, \\ \|\dot{\alpha}_2 + \alpha_1 + T_2\|_2 &= \left[\int_0^1 |\dot{\alpha}_2 + \alpha_1 + T_2|^2 dt \right]^{1/2} = 2 \times 10^{-5}, \\ \|\dot{T}_2 - \alpha_2 - T_2\|_2 &= \left[\int_0^1 |\dot{T}_2 - \alpha_2 - T_2|^2 dt \right]^{1/2} = 7 \times 10^{-3}. \end{aligned} \quad (43)$$

For equality algebraic constraint, $[\int_0^1 |T_2 - 2t^2 - 3|^2 dt]^{1/2} = 2 \times 10^{-3}$.

5. Results and Discussion

Using the test example to discuss and validate the presented method for the system of index-four Hessenberg differential algebraic equations, the results indicate how the exact and the approximate solutions match exceedingly well in a way that demonstrates the accuracy and efficiency of the model. Applications 4.1 and 4.2 show to employ differential algebraic equations as a modelling environment in system applications.

In examples where a state space cannot effectively describe the whole system, the model system will be reduced to a standard state-space system on the manifold of the resulting algebraic constraints which is provide a simplest system to deal with and this reduction reduce accuracy of the results. These examples are taken as a test for the proposed method. A step-by-step implementation, the approximate solution is parameterised via polynomial base functions, which are dense in $C[I, R^3]$. Despite reasonably small numbers of these polynomials with an unknown coefficient, the obtained solutions prove to be rather accurate and efficient.

6. Conclusions

This research presented an efficient and reliable to solve higher index (index four) linear Hessenberg control DAEs. To ensure a controllable and linearly independent set of functions in its base, the setting space of the solvability is assumed to be the separable Banach space. The differentiation index is adopted and used repeatedly in this study until we achieve the required conditions around the implicit function theorem. This outcome depends on the index of the system. Reduced Hessenberg DAEs are obtained with the help of the implicit function theorem. An operator form has been used to write the latest system. For this case, we developed the variational formulation. The representation of the solution has been as a linear combination of elements in the setting space. The cost for utilising these new measures consists of more smoothness conditions on the control signal.

Generally, the treating of the system has been as a perturbation in the nonhomogenous functions. The example, index four, is efficiently illustrated. Despite a rather simple basis function type, the evaluation of the approach revealed excellent accuracy. Due to the decision-maker, these bases can be modified to include spline and orthonormal bases.

Data Availability

The data used in the article are generated by Octave and can be made available upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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