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Research Article

Results on Implicit Fractional Pantograph Equations with Mittag-Leffler Kernel and Nonlocal Condition

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In this study, the main focus is on an investigation of the sufficient conditions of existence and uniqueness of solution for two-classess of nonlinear implicit fractional pantograph equations with nonlocal conditions via Atangana–Baleanu–Riemann–Liouville (ABR) and Atangana–Baleanu–Caputo (ABC) fractional derivative with order $\sigma \in (1, 2]$. We introduce the properties of solutions as well as stability results for the proposed problem without using the semigroup property. In the beginning, we convert the given problems into equivalent fractional integral equations. Then, by employing some fixed-point theorems such as Krasnoselskii and Banach's techniques, we examine the existence and uniqueness of solutions to proposed problems. Moreover, by using techniques of nonlinear functional analysis, we analyze Ulam–Hyers (UH) and generalized Ulam–Hyers (GUH) stability results. As an application, we provide some examples to illustrate the validity of our results.

1. Introduction and Motivation

Fractional calculus and its applications have increased in popularity because of its utility in modeling a wide range of intricate processes in science and engineering [1–5]. In order to meet the need to model more real-world problems, new approaches and techniques have been created in various fields of science and engineering to characterize the dynamics of real-world events. Until 2015, all fractional derivatives had single kernels. So, simulating physical events based on these singularities is difficult. In 2015, Caputo and Fabrizio (C-F) studied a novel type of fractional derivative (FD) in the exponential kernel [6]. In [7], Atangana and Baleanu (AB) investigated a novel form of FD using Mittag-Leffler kernels. In [8], Abdeljawad expanded the Atangana and Baleanu FD to higher arbitrary orders and established the integral operators associated with them. In [9, 10], Abdeljawad and Baleanu discussed the discrete forms of the new operators. For some theoretical work on Atangana–Baleanu FD, we refer the reader to a series of papers [11–14]. Traditional fractional operators cannot adequately describe some models of dissipative events, which is why fractional derivatives with nonsingular kernels are useful. For further details on the modeling and applications of the AB fractional operator (see [15–17]). The ABC fractional derivative is often used to simulate physical dynamical systems because it accurately represents the processes of heterogeneity and diffusion at various scales (see [18–21]). For the existence and uniqueness, as well as stability results regarding ABC and ABR operators, we refer the readers to a series of papers [22–25]. The challenge arises from the fact that the semigroup property in the ABC fractional derivative is not satisfied. In this paper, we introduce some properties of solutions to the implicit pantograph fractional differential equation without using the semigroup property.

The topic of stability arose from Ulam's question regarding the stability of group homomorphisms in 1940 (see

[26]). In the next year, Hyers [27] offered a positive interpretation of the Ulam issue in Banach spaces, which was the first significant development and step toward additional answers in this area. Since then, some researchers have published different generalizations of the Ulam result and Hyers theory. In 1978, Rassias [28] presented a generalized Hyers concept of mappings over Banach spaces. The Rassias result grabbed the attention of a large number of mathematicians from across the world, who began investigating the problems of functional equation stability. In stochastic analysis, financial mathematics, and actuarial science, these stability results are often employed. Calculating the Lyapunov stability for various nonlinear fractional differential equations is difficult and time-consuming, as everyone knows, and constructing the correct Lyapunov function is also a difficulty. Stability means that the solution of the differential equation will not leave the ϵ-ball. But asymptotic stability means that the solution does not leave the ϵ-ball and goes to the origin. Asymptotic stability implies stability, but the converse is not true in general (see [29]). For nonlinear fractional differential equations that deal with the nonlocal conditions, Ulam–Hyers's stability is ideal. Not only Ulam-Hyers's stability but also the existence and uniqueness of fractional differential equation solutions have attracted a large number of scholars.

The pantograph is a vital component of electric trains that collects electric current from overload lines. The pantograph equations have been modeled by Ockendon and Tayler [30]. Many researchers who are convinced of the relevance of these equations have extended them into numerous types and shown the solvability of such problems both theoretically and quantitatively (for additional details, see [31–35] and the references therein). Many researchers have investigated the existence and UH stability results of fractional pantograph differential equations using various forms of FD. For example, Almalahi et al. [36] studied the existence and uniqueness results of the following Hilfer–Katugampola boundary value problems.

$$\begin{cases} {}^{\rho}\mathbb{D}_{a^+}^{\sigma,\beta}\nu(\iota) = f\left(\iota,\nu(\iota),\nu(\lambda\iota),{}^{\rho}\mathbb{D}_{a^+}^{\varrho,\beta}\nu(\iota)\right), & \lambda\in(0,1), \iota\in J \coloneqq (a,b], \\ \sum_{i=1}^{m}\theta_i {}^{\rho}\mathbb{I}_{a^+}^{\psi_i}\nu(\varpi_i) + \sum_{j=1}^{n}\tau_j {}^{\rho}\mathbb{D}_{a^+}^{\lambda_j,\beta}\nu\left(\kappa_j\right) = B\in\mathbb{R}, \end{cases} \tag{1}$$

where ${}^{\rho}\mathbb{D}_{a^+}^{\sigma,\beta}(\cdot), {}^{\rho}\mathbb{D}_{a^+}^{\lambda_j,\beta}(\cdot)$ are the Hilfer–Katugampola (FD) of order σ and λ_j, respectively, $\sigma\in(0,1)<1$ and type $\beta\in[0,1]$, $\sigma\geq\lambda_j+\beta(1-\lambda_j)$, $(j=0,1,2,\ldots,n)$, ${}^{\rho}\mathbb{I}_{a^+}^{\psi_i}$, ${}^{\rho}\mathcal{J}_{a^+}^{\sigma}$ are the generalized fractional integral of order ψ_i,σ, $(i=0,1,2,\ldots,m)$, respectively, $\theta_i,\tau_j\in\mathbb{R}$, and $\varpi_i,\kappa_j\in J$ are prefixed points.

Ahmed et al. [37] studied some properties of the solutions of the boundary impulsive fractional pantograph differential equation. In [38], the authors considered the pantograph problem as follows:

$$\begin{cases} {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = f(\iota,\nu(\iota),\nu(\lambda\iota)), \\ \nu(a) = \sum_{j=1}^{n}\tau_j\nu\left(\kappa_j\right), \quad \kappa_j\in(a,T), \end{cases} \tag{2}$$

the existence and uniqueness results were investigated using Banach's contraction principle and Krasnoselskii fixed point theorem, and the Ulam–Hyers stabilities were addressed using Gronwall's inequality in the context of ABC. Almalahi et al. [39] via Banach's contraction principle and Krasnoselskii fixed point theorem studied the existence, uniqueness, and UH stability results of the following problems:

$$\begin{cases} {}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = f(\iota,\nu(\iota)), & \iota\in[a,b], \\ \nu(a)=0, \nu(b) = {}^{\mathrm{AB}}\mathbb{I}_{a^+}^{\delta}\nu(\zeta), & \zeta\in(a,b), \end{cases}$$
$$\begin{cases} {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = f(\iota,\nu(\iota)), & \iota\in[a,b], \\ \nu(a)=0, \nu(b) = {}^{\mathrm{AB}}\mathbb{I}_{a^+}^{\delta}\nu(\zeta), & \zeta\in(a,b), \end{cases} \tag{3}$$

where ${}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}$ and ${}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}$ are the ABR and ABC fractional derivatives of order $\sigma\in(2,3]$ and $\sigma\in(1,2]$, respectively, ${}^{\mathrm{AB}}\mathbb{I}_{a^+}^{\delta}$ is the AB-integral operator such that $\delta\in(0,1]$, $\zeta\in(a,b)$, and $f\colon [a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function.

Motivated by the argumentations above and due to the fact that the nonlocal condition is a suitable tool to describe memory phenomena like nonlocal elasticity, propagation in complex media, polymers, biological, porous media, viscoelasticity, electromagnetics, electrochemistry, etc. We intend to analyze and investigate the sufficient conditions of solution for the following two-class of nonlinear implicit fractional pantograph equations with ABR and ABC fractional derivatives in order $1<\sigma\leq 2$ with nonlocal conditions as follows:

$$\begin{cases} {}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = f(\iota,\nu(\iota),\nu(\lambda\iota), & {}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota), \lambda\in(0,1), \\ \nu(b) = \sum_{i=1}^{m}\theta_i\nu(\varpi_i), & \varpi_i\in(a,b), \end{cases} \tag{4}$$

$$\begin{cases} {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = f(\iota,\nu(\iota),\nu(\lambda\iota), & {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota), \lambda\in(0,1), \\ \nu(a)=0, \nu(b) = \sum_{j=1}^{n}\tau_j\nu\left(\kappa_j\right), & \kappa_j\in(a,b), \end{cases} \tag{5}$$

where ${}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}, {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}$ are respectively the ABR and ABC-FD of order $\sigma\in(1,2]$, $\theta_i,\tau_j\in\mathbb{R}$ and $\varpi_i,\kappa_j\in(a,b)$ are prefixed points such that $a<\varpi_1\leq\varpi_2\leq\cdots\leq\varpi_i<b$, $a<\kappa_1\leq\kappa_2\leq\cdots\leq\kappa_j<b$ $(i=1,2,\ldots,m$ and $j=1,2,\ldots,n)$, and $f\colon [a,b]\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is continuous function satisfies some condition described later.

It is notable that nonlocal Cauchy type problems may be employed to explain differential rules in the growth of a system. These equations are frequently used to explain nonnegative values such as a species' concentration or the distribution of mass or temperature. Before studying any model of real-world phenomena, the first question to address is whether the problem genuinely exists or not. The fixed-point theory provides the answer to this question.

The contribution of the current works is as follows:

(i) In this paper, we will study two types of fractional problems involving new higher-order fractional operators via ABC and ABR operators, which have recently been expanded by Abdeljawad.

(ii) To our knowledge, this is the first study that deals with high-order ABC and ABR fractional derivatives. As a result, our findings will be a valuable addition to the current literature on these fascinating operators.

(iii) We use a novel method to establish the existence and uniqueness of solutions for problems (4) and (5), as well as different types of stability results, without relying on the semigroup property and with a minimal number of hypotheses.

(iv) If $\lambda = 1$, then problems (4) and (5), respectively, reduces to the following implicit fractional differential equations:

$$\begin{cases} {}^{ABR}\mathbb{D}^{\sigma}_{a^+}\nu(\iota) = f\left(\iota, \nu(\iota), {}^{ABR}\mathbb{D}^{\sigma}_{a^+}\nu(\iota)\right), \\ \nu(b) = \sum_{i=1}^{m}\theta_i \nu(\varpi_i), \quad \varpi_i \in (a,b), \end{cases}$$
$$\begin{cases} {}^{ABC}\mathbb{D}^{\sigma}_{a^+}\nu(\iota) = f\left(\iota, \nu(\iota), {}^{ABC}\mathbb{D}^{\sigma}_{a^+}\nu(\iota)\right), \\ \nu(a) = 0, \nu(b) = \sum_{j=1}^{n}\tau_j \nu(\kappa_j), \quad \kappa_j \in (a,b). \end{cases} \tag{6}$$

The rest of this paper is organized as follows: in Section 2, we review several notations, definitions, and lemmas that are necessary for our analysis. In Section 3, we examine the existence and uniqueness results for problems (4) and (5) with ABC and ABR derivatives with the nonlocal condition. In section 4, we address the stability results of problems (4) and (5). We present two examples to demonstrate the validity of our results in section 5. In the concluding part, we will provide some last observations regarding our findings.

2. Preliminaries and Auxiliary Results

Let $\mathcal{J} = [a,b]$, $\mathcal{J}' = (a,b) \subset \mathbb{R}$, and $C(\mathcal{J},\mathbb{R})$ be the space of continuous functions $\nu\colon \mathcal{J} \longrightarrow \mathbb{R}$ with the norm $\|\nu\| = \max\{|\nu(\iota)|\colon \iota \in \mathcal{J}\}$. Then $(C(\mathcal{J},\mathbb{R}), \|\cdot\|)$ is a Banach space.

Definition 1 (see [7]). Let $0 < \sigma \leq 1$. Then, the following expressions,

$$\begin{aligned} {}^{ABR}\mathbb{D}^{\sigma}_{a^+}\nu(\iota) &= \frac{\mathfrak{B}(\sigma)}{1-\sigma}\frac{d}{d\iota}\int_0^{\iota} E_{\sigma}\left(\frac{\sigma}{\sigma-1}(\iota-\theta)^{\sigma}\right)\nu'(\theta)d\theta, \quad \iota > a, \\ {}^{ABC}\mathbb{D}^{\sigma}_{a^+}\nu(\iota) &= \frac{\mathfrak{B}(\sigma)}{1-\sigma}\int_0^{\iota} E_{\sigma}\left(\frac{\sigma}{\sigma-1}(\iota-\theta)^{\sigma}\right)\nu'(\theta)d\theta, \quad \iota > a, \end{aligned} \tag{7}$$

are called ABR and ABC fractional derivatives of order σ for a function ν, respectively. $\mathfrak{B}(\sigma)$ is the normalization function that satisfies $\mathfrak{B}(\sigma) = (\sigma/(2-\sigma)) > 0$ and $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$, and E_{σ} is the Mittag-Leffler function defined by

$$\mathbf{E}_{\sigma}(\nu) = \sum_{i=0}^{\infty}\frac{\nu^i}{\Gamma(i\sigma+1)}, \quad \operatorname{Re}(\sigma) > 0, \nu \in \mathbb{C}. \tag{8}$$

The AB fractional integral is given by

$${}^{AB}\mathbb{I}^{\sigma}_{a^+}\nu(\iota) = \frac{1-\sigma}{\mathfrak{B}(\sigma)}\nu(\iota) + \frac{\sigma}{\mathfrak{B}(\sigma)\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}\nu(s)ds. \tag{9}$$

Definition 2 (see [8] Definition 3.1). Let us assume that $\sigma \in (n, n+1]$ and $\nu^{(n)} \in H^1(\mathcal{J})$. We set $\beta = \sigma - n$. Then, $0 < \beta \leq 1$ and the following expressions

$$\begin{aligned} \left({}^{ABR}\mathbb{D}^{\sigma}_{a^+}\nu\right)(\iota) &= \left({}^{ABR}\mathbb{D}^{\beta}_{a^+}\nu^{(n)}\right)(\iota), \\ \left({}^{ABC}\mathbb{D}^{\sigma}_{a^+}\nu\right)(\iota) &= \left({}^{ABC}\mathbb{D}^{\beta}_{a^+}\nu^{(n)}\right)(\iota), \end{aligned} \tag{10}$$

are called the left-sided ABR and ABC fractional derivatives of order σ for a function ν. The correspondent (FI) is given by

$$\left({}^{AB}\mathbb{I}^{\sigma}_{a^+}\nu\right)(\iota) = \left(\mathbb{I}^{n}_{a^+}\,{}^{AB}\mathbb{I}^{\beta}_{a^+}\nu\right)(\iota). \tag{11}$$

Lemma 1 (see [8] Proposition 3.1). *If $\nu(\iota)$ is a function defined on $[0,b]$ and $\sigma \in (n, n+1]$, then, for some $n \in \mathbb{N}_0$, we have*

(i) $\left({}^{ABR}\mathbb{D}^{\sigma}_{a^+}\,{}^{AB}\mathbb{I}^{\sigma}_{a^+}\nu\right)(\iota) = \nu(\iota)$.

(ii) $\left({}^{AB}\mathbb{I}^{\sigma}_{a^+}\,{}^{ABR}\mathbb{D}^{\sigma}_{a^+}\nu\right)(\iota) = \nu(\iota) - \sum_{i=0}^{n-1}\left(\nu^{(i)}(a)/i!\right)(\iota-a)^i$.

(iii) $\left({}^{AB}\mathbb{I}_{a^+}^{\sigma}\,{}^{ABC}\mathbb{D}_{a^+}^{\sigma}\nu\right)(\iota) = \nu(\iota) - \sum_{i=0}^{n}\left(\nu^{(i)}(a)/\,i!\right)(\iota - a)^i$.

Theorem 1 (see [40]). *Let $\mathcal{S} \neq \varnothing$ be a closed subset from a Banach space $\mathcal{K}$, and let $\Pi: \mathcal{S} \longrightarrow \mathcal{S}$ be a strict contraction such that $\|\Pi(\nu) - \Pi(y)\| \leq \rho\|\nu - y\|$ for some $0 < \rho < 1$ for all $\nu, y \in \mathcal{S}$. Then Π has a fixed point in $\mathcal{S}$.*

Theorem 2 (see [41]). *Let Δ be a Banach space, let a set $\digamma \subset \Delta$ be a nonempty, closed, convex, and bounded set. If there are two operators Φ^1, Φ^2 such that (i) $\Phi^1 x + \Phi^2 \nu \in \Delta$, for all $x, \nu \in \Delta$, (ii) Φ^1 is compact and continuous, and (iii) Φ^2 is a contraction mapping, then there exists a function $z \in \digamma$ such that $z = \Phi^1 z + \Phi^2 z$.*

Lemma 2 (see [8] example 3.3). *Let $\sigma \in (1,2]$ and $\hbar \in C(\mathcal{J}, \mathbb{R})$. Then, the solution to the following linear problem*

$$\begin{cases} {}^{ABC}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = \hbar(\iota), \\ \nu(a) = c_1, \nu'(a) = c_2, \end{cases} \tag{12}$$

is given by

$$\nu(\iota) = c_1 + c_2(\iota - a) + {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\iota), \tag{13}$$

where

$$\begin{aligned} {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\iota) &= \frac{2-\sigma}{\mathfrak{B}(\sigma-1)}\int_a^{\iota}\hbar(s)\mathrm{d}s \\ &+ \frac{\sigma-1}{\mathfrak{B}(\sigma-1)\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}\hbar(s)\mathrm{d}s. \end{aligned} \tag{14}$$

3. Equivalent Integral Equations

In this section, we will derive the formula of the equivalent integral equations for problems (4) and (5).

3.1. Equivalent Integral Equations for the Problem (4)

Lemma 3. *Let $\sigma \in (1,2]$ and $\hbar \in C(\mathcal{J}, \mathbb{R})$. A function $\nu \in C(\mathcal{J}, \mathbb{R})$ is a solution to the following ABR-problem*

$$\begin{cases} {}^{ABR}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = \hbar(\iota), \quad \iota \in (a,b], \\ \nu(b) = \sum_{i=1}^{m}\theta_i\nu(\varpi_i), \varpi_i \in (a,b), \end{cases} \tag{15}$$

then, ν satisfies the following fractional integral equation:

$$\begin{aligned} \nu(\iota) &= \frac{1}{1-\sum_{i=1}^{m}\theta_i}\left(\sum_{i=1}^{m}\theta_i\,{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i) - {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(b)\right) \\ &+ {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\iota). \end{aligned} \tag{16}$$

Proof. By (see [8] Theorem 4.2), the solution of ${}^{ABR}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) = \hbar(\iota)$ is given as

$$\nu(\iota) = c + {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\iota). \tag{17}$$

where c is an arbitrary constant and

$$\begin{aligned} {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\iota) &= \frac{2-\sigma}{\mathfrak{B}(\sigma-1)}\int_a^{\iota}\hbar(s)\mathrm{d}s \\ &+ \frac{\sigma-1}{\mathfrak{B}(\sigma-1)\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}\hbar(s)\mathrm{d}s. \end{aligned} \tag{18}$$

Now, we replace ι with ϖ_i into (17) and multiply by θ_i, we get

$$\sum_{i=1}^{m}\theta_i\nu(\varpi_i) = \sum_{i=1}^{m}\theta_i c + \sum_{i=1}^{m}\theta_i\,{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i). \tag{19}$$

Making use of the condition $(\nu(b) = \sum_{i=1}^{m}\theta_i\nu(\varpi_i))$, we have

$$c = \frac{1}{1-\sum_{i=1}^{m}\theta_i}\left(\sum_{i=1}^{m}\theta_i\,{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i) - {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(b)\right). \tag{20}$$

Substituting c in (17), we get (16). Conversely, let us assume that ν satisfies (16). Then, by applying the operator ${}^{ABC}\mathbb{D}_{a^+}^{\sigma}$ on both sides of (16) and using Lemmas 1, we obtain

$$\begin{aligned} {}^{ABR}\mathbb{D}_{a^+}^{\sigma}\nu(\iota) &= {}^{ABR}\mathbb{D}_{a^+}^{\sigma}\frac{1}{1-\sum_{i=1}^{m}\theta_i}\left(\sum_{i=1}^{m}\theta_i\,{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i) - {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(b)\right) \\ &+ {}^{ABR}\mathbb{D}_{a^+}^{\sigma}\,{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\iota) \\ &= \varpi(\iota). \end{aligned} \tag{21}$$

Next, we replace ι by ϖ_i in (16) and multiply by θ_i, we get

$$\begin{aligned}
\sum_{i=1}^{m}\theta_i\nu(\varpi_i) &= \frac{\sum_{i=1}^{m}\theta_i}{1-\sum_{i=1}^{m}\theta_i}\left(\sum_{i=1}^{m}\theta_i{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i) - {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(b)\right)\\
&\quad+\sum_{i=1}^{m}\theta_i{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i)\\
&= \frac{1-\left(1-\sum_{i=1}^{m}\theta_i\right)}{1-\sum_{i=1}^{m}\theta_i}\left(\sum_{i=1}^{m}\theta_i{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i) - {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(b)\right)\\
&\quad+\sum_{i=1}^{m}\theta_i{}^{AB}\mathbb{I}_{a^+}^{\sigma}\varpi(\varpi_i)\\
&= \frac{1}{1-\sum_{i=1}^{m}\theta_i}\left(\sum_{i=1}^{m}\theta_i{}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(\varpi_i) - {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(b)\right) + {}^{AB}\mathbb{I}_{a^+}^{\sigma}\hbar(b)\\
&= \nu(b).
\end{aligned} \tag{22}$$

Thus, the nonlocal condition is satisfied. □

Theorem 3. *Let $\sigma\in(1,2]$, F_ν: $\mathcal{J}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ be a continuous function such that $F_\nu(\iota)=f(\iota,\nu(\iota),\nu(\lambda(\iota)),{}^{ABR}\mathbb{D}_{a^+}^{\sigma}\nu(\iota))$ and $\sum_{i=1}^{m}\theta_i\neq 1$. A function $\nu\in C(\mathcal{J},\mathbb{R})$ is a solution to the problem (4) if and only if ν satisfies the following fractional integral equation:*

$$\begin{aligned}
\nu(\iota) &= \frac{1}{1-\sum_{i=1}^{m}\theta_i}\\
&\quad\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right.\\
&\quad\left.-\left(\mathfrak{p}_1\int_a^{b}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right]\\
&\quad+\left(\mathfrak{p}_1\int_a^{\iota}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\mathfrak{p}_1 &= \frac{2-\sigma}{\mathfrak{B}(\sigma-1)},\\
\mathfrak{p}_2 &= \frac{\sigma-1}{\mathfrak{B}(\sigma-1)}.
\end{aligned} \tag{24}$$

Proof. According to Lemma 3, the solution to problem (4) is given by

$$\begin{aligned}
\nu(\iota) &= \frac{1}{1-\sum_{i=1}^{m}\theta_i}\left[\sum_{i=1}^{m}\theta_i{}^{AB}\mathbb{I}_{a^+}^{\sigma}F_\nu(\varpi_i) - {}^{AB}\mathbb{I}_{a^+}^{\sigma}F_\nu(b)\right]\\
&\quad+{}^{AB}\mathbb{I}_{a^+}^{\sigma}F_\nu(\iota).
\end{aligned} \tag{25}$$

By definition ${}^{AB}\mathbb{I}_{a^+}^{\sigma}$ in the case $\sigma\in(1,2]$, we have

$${}^{AB}\mathbb{I}_{a^+}^{\sigma}F_\nu(\iota) = \frac{2-\sigma}{\mathfrak{B}(\sigma-1)}\int_a^{\iota}F_\nu(s)\mathrm{d}s + \frac{\sigma-1}{\mathfrak{B}(\sigma-1)\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_\nu(s)\mathrm{d}s. \tag{26}$$

By (26), we can rewrite (25) as follows:

$$
\begin{aligned}
\nu(\iota) = {} & \frac{1}{1-\sum_{i=1}^{m}\theta_i} \\
& \left[\sum_{i=1}^{m}\theta_i\left(\frac{2-\sigma}{\mathfrak{B}(\sigma-1)}\int_a^{\varpi_i}F_\nu(s)\mathrm{d}s+\frac{\sigma-1}{\mathfrak{B}(\sigma-1)\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right. \\
& \left.-\left(\frac{2-\sigma}{\mathfrak{B}(\sigma-1)}\int_a^{b}F_\nu(s)\mathrm{d}s+\frac{\sigma-1}{\mathfrak{B}(\sigma-1)\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right] \\
& +\left(\frac{2-\sigma}{\mathfrak{B}(\sigma-1)}\int_a^{\iota}F_\nu(s)\mathrm{d}s+\frac{\sigma-1}{\mathfrak{B}(\sigma-1)\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right).
\end{aligned}
\tag{27}
$$

By (24), we get

$$
\begin{aligned}
\nu(\iota) = {} & \frac{1}{1-\sum_{i=1}^{m}\theta_i} \\
& \left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right. \\
& \left.-\left(\mathfrak{p}_1\int_a^{b}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right] \\
& +\left(\mathfrak{p}_1\int_a^{\iota}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right).
\end{aligned}
\tag{28}
$$

□

3.2. Equivalent Integral Equations for the Problem (5)

Theorem 4. *Let $\sigma \in (1,2], F_\nu\colon \mathcal{J}\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ be a continuous function such that $F_\nu(\iota) = f(\iota,\nu(\iota),\nu(\lambda(\iota)), {}^{ABC}\mathbb{D}_{a^+}^{\sigma}\nu(\iota))$ and $\sum_{j=1}^{n}\tau_j \neq 1$. A function $\nu \in C(\mathcal{J},\mathbb{R})$ is a solution to the problem (5) if and only if ν satisfies the following fractional integral equation:*

$$
\begin{aligned}
\nu(\iota) = {} & \frac{(\iota-a)}{1-\sum_{j=1}^{n}\tau_j}\left[\sum_{j=1}^{n}\tau_j\left(\mathfrak{p}_1\int_a^{\kappa_j}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\kappa_j}\left(\kappa_j-s\right)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right. \\
& \left.-\left(\mathfrak{p}_1\int_a^{b}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right] \\
& +\mathfrak{p}_1\int_a^{\iota}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_\nu(s)\mathrm{d}s,
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
\mathfrak{p}_1 &= \frac{2-\sigma}{\mathfrak{B}(\sigma-1)}, \\
\mathfrak{p}_2 &= \frac{\sigma-1}{\mathfrak{B}(\sigma-1)}.
\end{aligned}
\tag{30}
$$

Proof. Let us assume that ν is a solution of the first equation of (5). Then, by Lemma 2, we get

$$\nu(\iota) = c_1 + c_2(\iota-a) + {}^{AB}\mathbb{I}_{a^+}^{\sigma}F_\nu(\iota). \tag{31}$$

By conditions $(\nu(a)=0, \nu(b)=\sum_{j=1}^{n}\tau_j\nu(\kappa_j))$ and by the same technique of Theorem 3, we can easily get (29). □

4. Main Results

4.1. Existence and Uniqueness of Solutions for Problem (4). In this subsection, we will discuss the existence and uniqueness results for the ABR problem (4). For simplicity, we set

$$
\begin{aligned}
\Theta_i &= \left(\mathfrak{p}_1(\varpi_i-a)+\frac{\mathfrak{p}_2(\varpi_i-a)^{\sigma}}{\Gamma(\sigma+1)}\right), \\
\mathcal{R}_{B,\sigma} &= \left(\mathfrak{p}_1(b-a)+\frac{\mathfrak{p}_2(b-a)^{\sigma}}{\Gamma(\sigma+1)}\right), \\
\mathfrak{A} &= \frac{2\mathfrak{N}_f}{\left(1-\mathfrak{N}_f\right)}\left(\frac{\sum_{i=1}^{m}\theta_i\Theta_i+\mathcal{R}_{B,\sigma}}{1-\sum_{i=1}^{m}\theta_i}+\mathcal{R}_{B,\sigma}\right).
\end{aligned}
\tag{32}
$$

Theorem 5. *Suppose that $F_\nu\colon \mathcal{J}\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function such that $F_\nu(\iota) = f(\iota,\nu(\iota),\nu(\lambda(\iota)),{}^{\mathrm{ABR}}\mathbb{D}^{\sigma}_{a^+}\nu(\iota))$ and $\sum_{i=1}^{m}\theta_i \neq 1$. Moreover, we assume that there is a constant number $\mathfrak{N}_f > 0$ such that*

$$\begin{aligned}(H_1)\colon &|f(\iota,x,\nu,z) - f(\iota,\overline{x},\overline{\nu},\overline{z})|\\ &\leq \mathfrak{N}_f(|x-\overline{x}| + |\nu-\overline{\nu}| + |z-\overline{z}|).\end{aligned} \tag{33}$$

Then the ABR problem (4) has a unique solution provided that $\mathfrak{A} < 1$.

Proof. On the light of Theorem 3, we define the operator $\mathbf{K}\colon C(\mathcal{J},\mathbb{R}) \longrightarrow C(\mathcal{J},\mathbb{R})$

$$\begin{aligned}(\mathbf{K}\nu)(\iota) = &\frac{1}{1-\sum_{i=1}^{m}\theta_i}\\ &\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}F_\nu(s)\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i - s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right.\\ &\left.-\left(\mathfrak{p}_1\int_a^{b}F_\nu(s)\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b - s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right]\\ &+\left(\mathfrak{p}_1\int_a^{\iota}F_\nu(s)\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota - s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right).\end{aligned} \tag{34}$$

Let us consider a closed ball Π_δ defined as

$$\Pi_\delta = \{\vartheta \in C(\mathcal{J},\mathbb{R})\colon \quad \|\vartheta\| \leq \delta\}, \tag{35}$$

with radius $\delta \geq (\mathfrak{A}_1/(1-\mathfrak{A}))$, where

$$\begin{aligned}\mathfrak{A}_1 &= \left(\frac{\sum_{i=1}^{m}\theta_i\Theta_i + \mathcal{R}_{B,\sigma}}{1-\sum_{i=1}^{m}\theta_i} + \mathcal{R}_{B,\sigma}\right)\omega_f,\\ \omega_f &= \max_{\iota\in\mathcal{J}}|f(\iota,0,0,0)|.\end{aligned} \tag{36}$$

Now, we show that $\Xi\Pi_\delta \subset \Pi_\delta$. For all $\vartheta \in \Pi_\delta$ and $\iota \in \mathcal{J}$, we have

$$\begin{aligned}|(\mathbf{K}\nu)(\iota)| \leq &\frac{1}{1-\sum_{i=1}^{m}\theta_i}\\ &\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}|F_\nu(s)|\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i - s)^{\sigma-1}|F_\nu(s)|\mathrm{d}s\right)\right.\\ &\left.+\left(\mathfrak{p}_1\int_a^{b}|F_\nu(s)|\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b - s)^{\sigma-1}|F_\nu(s)|\mathrm{d}s\right)\right]\\ &+\left(\mathfrak{p}_1\int_a^{\iota}|F_\nu(s)|\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota - s)^{\sigma-1}|F_\nu(s)|\mathrm{d}s\right).\end{aligned} \tag{37}$$

By (H_1), we have

$$\begin{aligned}\left|F_\nu(\iota)\right| &= \left|f\left(\iota, \nu(\iota), \nu(\lambda(\iota)), {}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota)\right)\right| \\ &= \left|f\left(\iota, \nu(\iota), \nu(\lambda(\iota)), {}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota)\right) - f(\iota,0,0,0)\right| + |f(\iota,0,0,0)| \\ &\leq \mathfrak{N}_f\left(|\vartheta(\iota)| + |\nu(\lambda(\iota))| + \left|{}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota)\right|\right) + |f(\iota,0,0,0)| \\ &\leq \frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}|\vartheta(\iota)| + \omega_f.\end{aligned} \tag{38}$$

Hence

$$\begin{aligned}\|\mathbf{K}\nu\| &\leq \frac{2\mathfrak{N}_f}{\left(1-\mathfrak{N}_f\right)}\left(\frac{\sum_{i=1}^m \theta_i\Theta_i + \mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^m\theta_i} + \mathscr{R}_{B,\sigma}\right)\delta \\ &\quad + \left(\frac{\sum_{i=1}^m \theta_i\Theta_i + \mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^m\theta_i} + \mathscr{R}_{B,\sigma}\right)\omega_f \\ &= \mathfrak{A}\delta + \mathfrak{A}_1 \leq \delta.\end{aligned} \tag{39}$$

Thus, $\mathbf{K}\nu \in \Pi_\delta$. Now, we will prove that $\mathbf{K}$ is a contraction map. Let $\nu, \widehat{\nu} \in \Pi_\delta$ and $\iota \in \mathscr{J}$. Then

$$\begin{aligned}&|(\mathbf{K}\nu)(\iota) - (\mathbf{K}\widehat{\nu})(\iota)| \\ &\leq \frac{1}{1-\sum_{i=1}^m\theta_i}\left[\sum_{i=1}^m \theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}\left|F_\nu(s) - F_{\widehat{\nu}}(s)\right|\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i - s)^{\sigma-1}\left|F_\nu(s) - F_{\widehat{\nu}}(s)\right|\mathrm{d}s\right)\right. \\ &\quad \left. + \left(\mathfrak{p}_1\int_a^b\left|F_\nu(s) - F_{\widehat{\nu}}(s)\right|\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^b(b-s)^{\sigma-1}\left|F_\nu(s) - F_{\widehat{\nu}}(s)\right|\mathrm{d}s\right)\right] \\ &\quad + \left(\mathfrak{p}_1\int_a^\iota\left|F_\nu(s) - F_{\widehat{\nu}}(s)\right|\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^\iota(\iota-s)^{\sigma-1}\left|F_\nu(s) - F_{\widehat{\nu}}(s)\right|\mathrm{d}s\right).\end{aligned} \tag{40}$$

From our assumption, we obtain

$$\begin{aligned}\left|F_\nu(s) - F_{\widehat{\nu}}(s)\right| &\leq \mathfrak{N}_f\left(|\nu(s) - \widehat{\nu}(s)| + |\nu(\lambda(s)) - \widehat{\nu}(\lambda(s))| + \left|F_\nu(s) - F_{\widehat{\nu}}(s)\right|\right) \\ &\leq \frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\|\nu - \widehat{\nu}\|.\end{aligned} \tag{41}$$

Hence

$$\|\mathbf{K}\nu-\mathbf{K}\widehat{\nu}\|\leq\frac{1}{1-\sum_{i=1}^{m}\theta_i}\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1(\varpi_i-a)+\frac{\mathfrak{p}_2(\varpi_i-a)^{\sigma}}{\Gamma(\sigma+1)}\right)+\left(\mathfrak{p}_1(b-a)+\frac{\mathfrak{p}_2(b-a)^{\sigma}}{\Gamma(\sigma+1)}\right)\right]\frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\|\nu-\widehat{\nu}\|+\left(\mathfrak{p}_1(\iota-a)+\frac{\mathfrak{p}_2(\iota-a)^{\sigma}}{\Gamma(\sigma+1)}\right)\frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\|\nu-\widehat{\nu}\|=\mathfrak{P}\|\nu-\widehat{\nu}\|. \tag{42}$$

Since $\mathfrak{A}<1$, we deduce that $\mathbf{K}$ is a contraction. Hence, Theorem 1 implies that $\mathbf{K}$ has a unique fixed point. Consequently, the ABR problem (4) has a unique solution. □

Theorem 6. *Let us assume that the hypothesis in Theorem 5 is satisfied. Then, the ABR problem (4) has at least one solution.*

Proof. Let us consider the operator $\mathbf{K}$, which is defined in Theorem 5 such that $(\mathbf{K}\nu)(\iota)=(\mathbf{K}_1\nu)(\iota)+(\mathbf{K}_2\nu)(\iota)$, where

$$\begin{aligned}(\mathbf{K}_1\nu)(\iota)&=\frac{1}{1-\sum_{i=1}^{m}\theta_i}\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)-\left(\mathfrak{p}_1\int_a^{b}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_\nu(s)\mathrm{d}s\right)\right],\\(K_2\nu)(\iota)&=\mathfrak{p}_1\int_a^{\iota}F_\nu(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_\nu(s)\mathrm{d}s.\end{aligned} \tag{43}$$

Let Π_δ be a closed ball defined as

$$\Pi_\delta=\{\vartheta\in C(\mathcal{J},\mathbb{R}):\quad \|\vartheta\|\leq\delta\}, \tag{44}$$

with radius $\delta\geq(\mathfrak{A}_1/(1-\mathfrak{A}))$, where

$$\mathfrak{A}_1=\left(\frac{\sum_{i=1}^{m}\theta_i\Theta_i+\mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^{m}\theta_i}+\mathscr{R}_{B,\sigma}\right)\omega_f,\qquad \omega_f=\max_{\iota\in\mathcal{J}}|f(\iota,0,0,0)|. \tag{45}$$

In order to apply Krasnoselskii fixed point theorem, we split the proof into the following steps: □

Step 1. We show that $\mathbf{K}_1\nu+\mathbf{K}_2\widetilde{\nu}\in\Pi_\delta$ for all $\nu,\widetilde{\nu}\in\Pi_\delta$. First, for the operator $\mathbf{K}_1$. For $\nu\in\Pi_\delta$ and $\iota\in\mathcal{J}$, we have

$$(\mathbf{K}_1\nu)(\iota)\leq\frac{1}{1-\sum_{i=1}^{m}\theta_i}\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}|F_\nu(s)|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}|F_\nu(s)|\mathrm{d}s\right)+\left(\mathfrak{p}_1\int_a^{b}|F_\nu(s)|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}|F_\nu(s)|\mathrm{d}s\right)\right]. \tag{46}$$

By (38), we have

$$\|\mathbf{K}_1\nu\| \leq \frac{1}{1-\sum_{i=1}^m \theta_i}\left(\frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\delta+\omega_f\right)\left(\sum_{i=1}^m \theta_i\Theta_i+\mathscr{R}_{B,\sigma}\right). \tag{47}$$

Next, for the operator $\mathbf{K}_2$, we have

$$\|\mathbf{K}_2\nu\| \leq \left(\frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\delta+\omega_f\right)\mathscr{R}_{B,\sigma}. \tag{48}$$

By inequalities (47) and (48), we have

$$\begin{aligned}
&\|\mathbf{K}_1\nu+\mathbf{K}_2\nu\| \\
&\leq \|\mathbf{K}_1\nu\|+\|\mathbf{K}_2\nu\| \\
&\leq \frac{2\mathfrak{N}_f}{\left(1-\mathfrak{N}_f\right)}\left(\frac{\sum_{i=1}^m \theta_i\Theta_i+\mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^m \theta_i}+\mathscr{R}_{B,\sigma}\right)\delta \\
&\quad+\left(\frac{\sum_{i=1}^m \theta_i\Theta_i+\mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^m \theta_i}+\mathscr{R}_{B,\sigma}\right)\omega_f \\
&= \mathfrak{A}\delta+\mathfrak{A}_1<\delta.
\end{aligned} \tag{49}$$

Thus $\mathbf{K}_1\nu+\mathbf{K}_2\widetilde{\nu}\in\Pi_\delta$.

Step 2. $\mathbf{K}_1$ is a contraction map. Due to the operator $\mathbf{K}$ being a contraction map, we conclude that $\mathbf{K}_1$ is a contraction too.

Step 3. $\mathbf{K}_2$ is continuous and compact. Since f is continuous, $\mathbf{K}_2$ is continuous too. Also, by (48), $\mathbf{K}_2$ is uniformly bounded on Π_δ. Now, we show that $\mathbf{K}_2(\Pi_\delta)$ is equicontinuous. For this purpose, let $\nu\in\Pi_\delta$, $a\leq\iota_1<\iota_2\leq b$. Then, we have

$$\begin{aligned}
&\left|(\mathbf{K}_2\nu)(\iota_2)-(\mathbf{K}_2\nu)(\iota_1)\right| \\
&\leq \mathfrak{p}_1\int_{\iota_1}^{\iota_2}\left|F_\nu(s)\right|\mathrm{d}s \\
&\quad+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota_1}\left[(\iota_2-s)^{\sigma-1}-(\iota_1-s)^{\sigma-1}\right]\left|F_\nu(s)\right|\mathrm{d}s \\
&\quad+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_{\iota_1}^{\iota_2}(\iota_2-s)^{\sigma-1}\left|F_\nu(s)\right|\mathrm{d}s \\
&\leq \left(\frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\delta+\omega_f\right)\mathfrak{p}_1(\iota_2-\iota_1) \\
&\quad+\left(\frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\delta+\omega_f\right)\frac{\mathfrak{p}_2\left[(\iota_2-\iota_1)^\sigma-(\iota_2-a)^\sigma+(\iota_1-a)^\sigma\right]}{\Gamma(\sigma+1)}.
\end{aligned} \tag{50}$$

Thus

$$\left\|(\mathbf{K}_2\nu)(\iota_2)-(\mathbf{K}_2\nu)(\iota_1)\right\|\longrightarrow 0,\quad \text{as } \iota_2\longrightarrow\iota_1. \tag{51}$$

In view of the previous steps with the theorem of Arzela–Ascoli, we deduce that $(\mathbf{K}_2\Pi_\delta)$ is relatively compact. Consequently, $\mathbf{K}_2$ is completely continuous. Hence, Theorem 2 shows that ABR problem (4) has at least one solution.

4.2. Existence of Unique Solutions for Problem (5)

Theorem 7. *Suppose that* $F_\nu\colon \mathscr{J}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ *is a continuous function such that* $F_\nu(\iota)=f(\iota,\nu(\iota),\nu(\lambda(\iota)),{}^{\mathrm{ABC}}\mathbb{D}_{a^+}^\sigma\nu(\iota))$ *and* $\sum_{j=1}^m\tau_j\neq 1$. *Moreover, we assume that there is a constant number* $\mathfrak{N}_f>0$ *such that*

$$|f(\iota,x,\nu,z)-f(\iota,\overline{x},\overline{\nu},\overline{z})|\leq\mathfrak{N}_f(|x-\overline{x}|+|\nu-\overline{\nu}|+|z-\overline{z}|). \tag{52}$$

Then the ABC problem (5) has a unique solution, provided that

$$\Upsilon=\frac{2\mathfrak{N}_f}{1-\mathfrak{N}_f}\left(\frac{(b-a)\left[\sum_{j=1}^n\tau_j\Theta_j+\mathscr{R}_{B,\sigma}\right]}{1-\sum_{j=1}^n\tau_j}+\mathscr{R}_{B,\sigma}\right)<1, \tag{53}$$

where

$$\Theta_j = \left(\mathfrak{p}_1 \left(\kappa_j - a \right) + \frac{\mathfrak{p}_2 \left(\kappa_j - a \right)^{\sigma}}{\Gamma(\sigma+1)} \right). \tag{54}$$

Proof. In view of Theorem 4, we define the operator $\Omega: C(\mathcal{J}, \mathbb{R}) \longrightarrow C(\mathcal{J}, \mathbb{R})$ by

$$\begin{aligned}(\Omega \nu)(\iota) &= \frac{(\iota - a)}{1 - \sum_{j=1}^{n} \tau_j} \left[\sum_{j=1}^{n} \tau_j \left(\mathfrak{p}_1 \int_a^{\kappa_j} F_\nu(s) \mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\kappa_j} \left(\kappa_j - s \right)^{\sigma-1} F_\nu(s) \mathrm{d}s \right) \right. \\ &\quad \left. - \left(\mathfrak{p}_1 \int_a^b F_\nu(s) \mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^b (b-s)^{\sigma-1} F_\nu(s) \mathrm{d}s \right) \right] \\ &\quad + \mathfrak{p}_1 \int_a^{\iota} F_\nu(s) \mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\iota} (\iota - s)^{\sigma-1} F_\nu(s) \mathrm{d}s.\end{aligned} \tag{55}$$

Let us consider a closed ball Π_φ^* as

$$\Pi_\varphi^* = \{ \nu \in C(\mathcal{J}, \mathbb{R}): \quad \|\vartheta\| \leq \varphi \}, \tag{56}$$

with radius $\varphi \geq (\Upsilon_1 / (1 - \Upsilon))$, where

$$\Upsilon_1 = \left(\frac{(b-a) \left[\sum_{j=1}^{n} \tau_j \Theta_j + \mathcal{R}_{B,\sigma} \right]}{1 - \sum_{j=1}^{n} \tau_j} + \mathcal{R}_{B,\sigma} \right) \omega_f. \tag{57}$$

Now, we show that $\Omega \Pi_\varphi^* \subset \Pi_\varphi^*$. For all $\nu \in \Pi_\varphi^*$ and $\iota \in \mathcal{J}$, we have

$$\begin{aligned}|(\Omega \nu)(\iota)| &\leq \frac{(\iota - a)}{1 - \sum_{j=1}^{n} \tau_j} \left[\sum_{j=1}^{n} \tau_j \left(\mathfrak{p}_1 \int_a^{\kappa_j} \left| F_\nu(s) \right| \mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\kappa_j} \left(\kappa_j - s \right)^{\sigma-1} \left| F_\nu(s) \right| \mathrm{d}s \right) \right. \\ &\quad \left. + \left(\mathfrak{p}_1 \int_a^b \left| F_\nu(s) \right| \mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^b (b-s)^{\sigma-1} \left| F_\nu(s) \right| \mathrm{d}s \right) \right] \\ &\quad + \mathfrak{p}_1 \int_a^{\iota} \left| F_\nu(s) \right| \mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\iota} (\iota - s)^{\sigma-1} \left| F_\nu(s) \right| \mathrm{d}s.\end{aligned} \tag{58}$$

By (H_2), we have

$$\begin{aligned}\left| F_\nu(\iota) \right| &= \left| f\left(\iota, \nu(\iota), \nu(\lambda(\iota)), {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma} \nu(\iota) \right) \right| \\ &= \left| f\left(\iota, \nu(\iota), \nu(\lambda(\iota)), {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma} \nu(\iota) \right) - f(\iota, 0, 0, 0) \right| + |f(\iota, 0, 0, 0)| \\ &\leq \mathfrak{N}_f \left(|\vartheta(\iota)| + |\nu(\lambda(\iota))| + \left| {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma} \nu(\iota) \right| \right) + |f(\iota, 0, 0, 0)| \\ &\leq \frac{2\mathfrak{N}_f}{1 - \mathfrak{N}_f} |\vartheta(\iota)| + \omega_f.\end{aligned} \tag{59}$$

Hence

$$\begin{aligned}\|\Omega\nu\| &\leq \frac{2\mathfrak{N}_f}{\left(1-\mathfrak{N}_f\right)}\left(\frac{(b-a)\left[\sum_{j=1}^{n}\tau_j\Theta_j+\mathscr{R}_{B,\sigma}\right]}{1-\sum_{j=1}^{n}\tau_j}+\mathscr{R}_{B,\sigma}\right)\varphi \\ &\quad+\left(\frac{(b-a)\left[\sum_{j=1}^{n}\tau_j\Theta_j+\mathscr{R}_{B,\sigma}\right]}{1-\sum_{j=1}^{n}\tau_j}+\mathscr{R}_{B,\sigma}\right)\omega_f \\ &= \Upsilon\varphi+\Upsilon_1\leq\varphi.\end{aligned} \tag{60}$$

Thus, $\Omega\nu\in\Pi_{\varphi}^{*}$. Now, we prove that Ω is a contraction. Let $\nu,\widehat{\nu}\in\Pi_{\varphi}^{*}$ and $\iota\in\mathscr{J}$. Then

$$\begin{aligned}&|(\Omega\nu)(\iota)-(\Omega\widehat{\nu})(\iota)| \\ &\leq\frac{(\iota-a)}{1-\sum_{j=1}^{n}\tau_j} \\ &\left[\sum_{j=1}^{n}\tau_j\left(\mathfrak{p}_1\int_a^{\kappa_j}\left|F_\nu(s)-F_{\widehat{\nu}}(s)\right|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\kappa_j}\left(\kappa_j-s\right)^{\sigma-1}\left|F_\nu(s)-F_{\widehat{\nu}}(s)\right|\mathrm{d}s\right)\right. \\ &\left.+\left(\mathfrak{p}_1\int_a^{b}\left|F_\nu(s)-F_{\widehat{\nu}}(s)\right|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}\left|F_\nu(s)-F_{\widehat{\nu}}(s)\right|\mathrm{d}s\right)\right] \\ &+\mathfrak{p}_1\int_a^{\iota}\left|F_\nu(s)-F_{\widehat{\nu}}(s)\right|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}\left|F_\nu(s)-F_{\widehat{\nu}}(s)\right|\mathrm{d}s.\end{aligned} \tag{61}$$

From (41), we obtain

$$\begin{aligned}&\|\Omega\nu-\Omega\widehat{\nu}\| \\ &\leq\frac{2\mathfrak{N}_f}{\left(1-\mathfrak{N}_f\right)}\left(\frac{(b-a)\left[\sum_{j=1}^{n}\tau_j\Theta_j+\mathscr{R}_{B,\sigma}\right]}{1-\sum_{j=1}^{n}\tau_j}+\mathscr{R}_{B,\sigma}\right)\|\nu-\widehat{\nu}\| \\ &=\Upsilon\|\nu-\widehat{\nu}\|.\end{aligned} \tag{62}$$

Due to condition (53), we conclude that Ω is a contraction. Hence, via Theorem 1, we conclude that Ω has a unique fixed point. Consequently, the ABC problem (5) has a unique solution. □

4.3. Ulam–Hyers Stability for the Problem (4). The UH and GUH stabilities for problem (4) are discussed in this subsection. For $\varepsilon>0$, the following inequality is taken into account:

$$\left|{}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\widehat{\nu}(\iota)-F_{\widehat{\nu}}(\iota)\right|\leq\varepsilon,\quad \iota\in\mathscr{J}. \tag{63}$$

Definition 3 (see [42]). The ABR problem (4) is UH stable if there exists a real number $C_f>0$ such that, for each $\varepsilon>0$ and each solution $\widehat{\nu}\in C(\mathscr{J},\mathbb{R})$ of inequality (63), there is a unique solution $\nu\in C(\mathscr{J},\mathbb{R})$ of (4) with

$$|\widehat{\nu}(\iota)-\nu(\iota)|\leq C_f\varepsilon. \tag{64}$$

Furthermore, the ABR problem (4) is GUH stable if we can identify $\varphi_f\colon\mathbb{R}_+\longrightarrow\mathbb{R}_+$ with $\varphi_f(0)=0$ such that

$$|\widehat{\nu}(\iota)-\nu(\iota)|\leq\varphi_f\varepsilon. \tag{65}$$

Remark 1. Let $\widehat{\nu}\in C(\mathscr{J},\mathbb{R})$ be the solution to inequality (63) if and only if we have a function $\Bbbk\in C(\mathscr{J},\mathbb{R})$ that depends on ν such that

(i) $|\Bbbk(\iota)|\leq\varepsilon$ for all $\iota\in\mathscr{J}$,

(ii) ${}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\widehat{\nu}(\iota)=F_{\widehat{\nu}}(\iota)+\Bbbk(\iota)$, $\iota\in\mathscr{J}$.

Lemma 4. *If $\nu\in C(\mathscr{J},\mathbb{R})$ is a solution to inequality (63), then ν satisfies the following inequality:*

$$\begin{aligned}&\left|\widehat{\nu}(\iota)-\Psi_{\widehat{\nu}}-\mathfrak{p}_1\int_a^{\iota}F_{\widehat{\nu}}(s)\mathrm{d}s-\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right| \\ &\leq\varepsilon\left(\frac{\sum_{i=1}^{m}\theta_i\Theta_i+\mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^{m}\theta_i}+\mathscr{R}_{B,\sigma}\right),\end{aligned} \tag{66}$$

where

$$\Psi_{\widehat{\nu}} = \frac{1}{1-\sum_{i=1}^{m}\theta_i}\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}F_{\widehat{\nu}}(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right)\right.$$
$$\left.-\left(\mathfrak{p}_1\int_a^{b}F_{\widehat{\nu}}(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right)\right]. \tag{67}$$

Proof. In view of Remark 1, we have

$$\begin{aligned}&{}^{ABR}\mathbb{D}_{a^+}^{\sigma}\widehat{\nu}(\iota) = F_{\widehat{\nu}}(\iota)+\Bbbk(\iota),\\ &\widehat{\nu}(a)=0,\widehat{\nu}(b)=\sum_{i=1}^{m}\theta_i\widehat{\nu}(\varpi_i).\end{aligned} \tag{68}$$

Then, by Lemma 3, we get

$$\widehat{\nu}(\iota) = \frac{1}{1-\sum_{i=1}^{m}\theta_i}\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}\left(F_{\widehat{\nu}}(s)+\Bbbk(s)\right)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}\left(F_{\widehat{\nu}}(s)+\Bbbk(s)\right)\mathrm{d}s\right)\right.$$
$$\left.-\left(\mathfrak{p}_1\int_a^{b}\left(F_{\widehat{\nu}}(s)+\Bbbk(s)\right)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}\left(F_{\widehat{\nu}}(s)+\Bbbk(s)\right)\mathrm{d}s\right)\right] \tag{69}$$
$$+\left(\mathfrak{p}_1\int_a^{\iota}\left(F_{\widehat{\nu}}(s)+\Bbbk(s)\right)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}\left(F_{\widehat{\nu}}(s)+\Bbbk(s)\right)\mathrm{d}s\right),$$

which implies

$$\begin{aligned}&\left|\widehat{\nu}(\iota)-\Psi_{\widehat{\nu}}-\mathfrak{p}_1\int_a^{\iota}F_{\widehat{\nu}}(s)\mathrm{d}s-\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right|\\
&\leq\frac{1}{1-\sum_{i=1}^{m}\theta_i}\\
&\left[\sum_{i=1}^{m}\theta_i\left(\mathfrak{p}_1\int_a^{\varpi_i}|\Bbbk(s)|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\varpi_i}(\varpi_i-s)^{\sigma-1}|\Bbbk(s)|\mathrm{d}s\right)\right.\\
&\left.+\mathfrak{p}_1\int_a^{b}|\Bbbk(s)|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}|\Bbbk(s)|\mathrm{d}s\right]\\
&+\mathfrak{p}_1\int_a^{\iota}|\Bbbk(s)|\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}|\Bbbk(s)|\mathrm{d}s\\
&\leq\varepsilon\left(\frac{\sum_{i=1}^{m}\theta_i\Theta_i+\mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^{m}\theta_i}+\mathscr{R}_{B,\sigma}\right).\end{aligned} \tag{70}$$

□

Theorem 8. *Suppose that $F_\nu\colon \mathscr{J}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is a continuous function such that $F_\nu(\iota)=f(\iota,\nu(\iota),\nu(\lambda(\iota)),{}^{ABR}\mathbb{D}_{a^+}^{\sigma}\nu(\iota))$ and $\sum_{i=1}^{m}\theta_i\neq 1$. Moreover, we assume that there is a constant number $\mathfrak{N}_f>0$ such that*

$$(H_1): |f(\iota, x, v, z) - f(\iota, \overline{x}, \overline{v}, \overline{z})| \leq \mathfrak{N}_f(|x-\overline{x}| + |v-\overline{v}| + |z-\overline{z}|). \tag{71}$$

If

$$\frac{2\mathfrak{N}_f \mathscr{R}_{B,\sigma}}{1-\mathfrak{N}_f} < 1, \tag{72}$$

then, the ABR problem (4) is UH stable.

Proof. Let $\varepsilon > 0$ and $\widehat{\nu} \in C(\mathscr{J}, \mathbb{R})$ satisfies the inequality (63), and let $\nu \in C(\mathscr{J}, \mathbb{R})$ be a unique solution to the following problem:

$$\begin{cases} {}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\widehat{\nu}(\iota) = F_{\widehat{\nu}}(\iota), \\ \nu(a) = \widehat{\nu}(a) = 0, \\ \nu(b) = \widehat{\nu}(b) = \sum_{i=1}^{m} \theta_i \widehat{\nu}(\varpi_i). \end{cases} \tag{73}$$

Then, by Lemma 3, we get

$$\widehat{\nu}(\iota) = \Psi_{\widehat{\nu}} + \mathfrak{p}_1 \int_a^{\iota} F_{\widehat{\nu}}(s)\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\iota} (\iota-s)^{\sigma-1} F_{\widehat{\nu}}(s)\mathrm{d}s, \tag{74}$$

where

$$\begin{aligned} \Psi_{\widehat{\nu}} = \frac{1}{1-\sum_{i=1}^{m}\theta_i} \Bigg[\sum_{i=1}^{m} \theta_i \Bigg(\mathfrak{p}_1 \int_a^{\varpi_i} F_{\widehat{\nu}}(s)\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\varpi_i} (\varpi_i - s)^{\sigma-1} F_{\widehat{\nu}}(s)\mathrm{d}s \Bigg) \\ - \Bigg(\mathfrak{p}_1 \int_a^{b} F_{\widehat{\nu}}(s)\mathrm{d}s + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{b} (b-s)^{\sigma-1} F_{\widehat{\nu}}(s)\mathrm{d}s \Bigg) \Bigg]. \end{aligned} \tag{75}$$

Since $\nu(a) = \widehat{\nu}(a) = 0$ and $\nu(b) = \widehat{\nu}(b) = \sum_{i=1}^{m}\theta_i\widehat{\nu}(\varpi_i)$. Then $\Psi_{\widehat{\nu}} = \Psi_{\nu}$ and hence by Lemma 4, we have

$$\begin{aligned} &|\widehat{\nu}(\iota) - \nu(\iota)| \\ &\leq \left| \widehat{\nu}(\iota) - \Psi_{\widehat{\nu}} - \mathfrak{p}_1 \int_a^{\iota} F_{\widehat{\nu}}(s)\mathrm{d}s - \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\iota} (\iota-s)^{\sigma-1} F_{\widehat{\nu}}(s)\mathrm{d}s \right| \\ &\quad + \mathfrak{p}_1 \int_a^{\iota} \left| F_{\widehat{\nu}}(s) - F_{\nu}(s) \right| \mathrm{d}s \\ &\quad + \frac{\mathfrak{p}_2}{\Gamma(\sigma)} \int_a^{\iota} (\iota-s)^{\sigma-1} \left| F_{\widehat{\nu}}(s) - F_{\nu}(s) \right| \mathrm{d}s \\ &\leq \varepsilon \left(\frac{\sum_{i=1}^{m}\theta_i\Theta_i + \mathscr{R}_{B,\sigma}}{1-\sum_{i=1}^{m}\theta_i} + \mathscr{R}_{B,\sigma} \right) + \frac{2\mathfrak{N}_f \mathscr{R}_{B,\sigma}}{1-\mathfrak{N}_f} \|\nu - \widehat{\nu}\|. \end{aligned} \tag{76}$$

Thus

$$\|\nu - \widehat{\nu}\| \leq C_f \varepsilon, \tag{77}$$

where

$$C_f = \frac{\left(\left(\left(\sum_{i=1}^{m}\theta_i\Theta_i + \mathscr{R}_{B,\sigma}\right)/\left(1-\sum_{i=1}^{m}\theta_i\right)\right) + \mathscr{R}_{B,\sigma}\right)}{1-\left(2\mathfrak{N}_f\mathscr{R}_{B,\sigma}/\left(1-\mathfrak{N}_f\right)\right)}. \tag{78}$$

Now, by choosing $\varphi_f(\varepsilon) = C_f\varepsilon$ such that $\varphi_f(0) = 0$, then the ABR problem (4) has GUH stability. □

4.4. Ulam–Hyers Stability for the Problem (5). The UH and generalized UH stabilities for problem (5) are discussed in this subsection.

Lemma 5. *If $\nu \in C(\mathscr{J}, \mathbb{R})$ is a solution of the inequality*

$$\left| {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\widehat{\nu}(\iota) - F_{\widehat{\nu}}(\iota) \right| \leq \varepsilon, \tag{79}$$

then ν satisfies the following inequality:

$$\left|\widehat{\nu}(\iota)-\Psi_{\widehat{\nu}}-\mathfrak{p}_1\int_a^{\iota}F_{\widehat{\nu}}(s)\mathrm{d}s-\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right| \le \varepsilon\left(\frac{(b-a)\left(\sum_{j=1}^{m}\tau_j\Theta_j+\mathscr{R}_{B,\sigma}\right)}{1-\sum_{j=1}^{m}\tau_j}+\mathscr{R}_{B,\sigma}\right), \tag{80}$$

where

$$\Psi_{\widehat{\nu}}=\frac{(\iota-a)}{1-\sum_{j=1}^{m}\tau_j}\left[\sum_{j=1}^{m}\tau_j\left(\mathfrak{p}_1\int_a^{\varpi_i}F_{\widehat{\nu}}(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\kappa_j}\left(\kappa_j-s\right)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right) -\left(\mathfrak{p}_1\int_a^{b}F_{\widehat{\nu}}(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right)\right],$$
$$\Theta_j=\left(\mathfrak{p}_1\left(\kappa_j-a\right)+\frac{\mathfrak{p}_2\left(\kappa_j-a\right)^{\sigma}}{\Gamma(\sigma+1)}\right). \tag{81}$$

Proof. By the same technique of Lemma 4, one can prove it. So, we omit the proof here. □

Theorem 9. *Suppose that $F_\nu\colon \mathscr{J}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is a continuous function such that $F_\nu(\iota)=f(\iota,\nu(\iota),\nu(\lambda(\iota)),{}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota))$ and $\sum_{j=1}^{n}\tau_j\neq 1$. Moreover, we assume that there is a constant number $\mathfrak{N}_f>0$ such that*

$$|f(\iota,x,\nu,z)-f(\iota,\overline{x},\overline{\nu},\overline{z})| \le \mathfrak{N}_f\left(|x-\overline{x}|+|\nu-\overline{\nu}|+|z-\overline{z}|\right). \tag{82}$$

If

$$\frac{2\mathfrak{N}_f\mathscr{R}_{B,\sigma}}{1-\mathfrak{N}_f}<1, \tag{83}$$

then the ABC problem (5) is UH stable.

Proof. Let $\varepsilon>0$ and $\widehat{\nu}\in C(\mathscr{J},\mathbb{R})$ satisfies inequality (79), and let $\nu\in C(\mathscr{J},\mathbb{R})$ be the unique solution to the following problem:

$$\begin{cases} {}^{\mathrm{ABC}}\mathbb{D}_{a^+}^{\sigma}\widehat{\nu}(\iota)=F_{\widehat{\nu}}(\iota),\\ \nu(a)=\widehat{\nu}(a)=0,\\ \nu(b)=\widehat{\nu}(b)=\sum_{j=1}^{n}\tau_j\nu\left(\kappa_j\right). \end{cases} \tag{84}$$

Then, by Theorem 4, we get

$$\widehat{\nu}(\iota)=\Psi_{\widehat{\nu}}+\mathfrak{p}_1\int_a^{\iota}F_{\widehat{\nu}}(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s, \tag{85}$$

where

$$\Psi_{\widehat{\nu}}=\frac{(\iota-a)}{1-\sum_{j=1}^{n}\tau_j}\left[\sum_{j=1}^{m}\tau_j\left(\mathfrak{p}_1\int_a^{\varpi_i}F_{\widehat{\nu}}(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\tau_j}\left(\tau_j-s\right)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right) -\left(\mathfrak{p}_1\int_a^{b}F_{\widehat{\nu}}(s)\mathrm{d}s+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{b}(b-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right)\right]. \tag{86}$$

Since $\nu(a)=\widehat{\nu}(a)=0$ and $\nu(b)=\widehat{\nu}(b)=\sum_{j=1}^{n}\tau_j\widehat{\nu}(\kappa_j)$. Then $\Psi_{\widehat{\nu}}=\Psi_\nu$ and hence by Lemma 5, we have

$$\begin{aligned}
&|\widehat{\nu}(\iota)-\nu(\iota)| \\
&\leq \left|\widehat{\nu}(\iota)-\Psi_{\widehat{\nu}}-\mathfrak{p}_1\int_a^{\iota}F_{\widehat{\nu}}(s)\mathrm{d}s-\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}F_{\widehat{\nu}}(s)\mathrm{d}s\right| \\
&\quad+\mathfrak{p}_1\int_a^{\iota}\left|F_{\widehat{\nu}}(s)-F_{\nu}(s)\right|\mathrm{d}s \\
&\quad+\frac{\mathfrak{p}_2}{\Gamma(\sigma)}\int_a^{\iota}(\iota-s)^{\sigma-1}\left|F_{\widehat{\nu}}(s)-F_{\nu}(s)\right|\mathrm{d}s \\
&\leq \varepsilon\left(\frac{(b-a)\left(\sum_{j=1}^{m}\tau_j\Theta_j+\mathscr{R}_{B,\sigma}\right)}{1-\sum_{j=1}^{m}\tau_j}+\mathscr{R}_{B,\sigma}\right)+\frac{2\mathfrak{N}_f\mathscr{R}_{B,\sigma}}{1-\mathfrak{N}_f}\|\nu-\widehat{\nu}\|.
\end{aligned} \tag{87}$$

Thus

$$\|\nu-\widehat{\nu}\|\leq C_f^{*}\varepsilon, \tag{88}$$

where

$$C_f^{*}=\frac{\varepsilon\left(\left(\left((b-a)\left(\sum_{j=1}^{m}\tau_j\Theta_j+\mathscr{R}_{B,\sigma}\right)\right)/\left(1-\sum_{j=1}^{m}\tau_j\right)\right)+\mathscr{R}_{B,\sigma}\right)}{1-\left(\left(2\mathfrak{N}_f\mathscr{R}_{B,\sigma}\right)/\left(1-\mathfrak{N}_f\right)\right)}. \tag{89}$$

Now, by choosing $\varphi_f(\varepsilon)=C_f^{*}\varepsilon$ such that $\varphi_f(0)=0$, then the ABC problem (5) has GUH stability. □

4.5. Examples

Example 1. Consider the following ABR fractional problem:

$$\begin{cases}
{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)=\dfrac{\iota^2}{20e^{\iota}}\left(e^{-\iota}+\dfrac{|\nu(\iota)|}{1+|\nu(\iota)|}+\dfrac{|\nu(\iota/3)|}{1+|\nu(\iota/3)|}+\dfrac{{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}{1+{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}\right), & \iota\in(0,1) \\
\nu(0)=0,\ \nu(1)=\dfrac{1}{4}\nu\left(\dfrac{1}{2}\right). &
\end{cases} \tag{90}$$

Here $\sigma=(3/2)\in(1,2], a=0, b=1, \theta_1=(1/4),\ m=1, \varpi_1=(1/2)$ and

$$f\left(\iota,\nu(\iota),\nu(\lambda\iota),{}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\nu(\iota)\right)=\frac{\iota^2}{10e^{\iota}}\left(e^{-\iota}+\frac{|\nu(\iota)|}{1+|\nu(\iota)|}+\frac{|\nu(\iota/3)|}{1+|\nu(\iota/3)|}+\frac{{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}{1+{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}\right). \tag{91}$$

Let $\iota\in[0,1], \nu,\overline{\nu}\in\mathbb{R}$. Then

$$\begin{aligned}
&\left|f\left(\iota,\nu(\iota),\nu\left(\frac{\iota}{3}\right),{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\nu(\iota)\right)-f\left(\iota,\overline{\nu}(\iota),\overline{\nu}\left(\frac{\iota}{3}\right),{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)\right)\right| \\
&\leq\frac{\iota^2}{20e^{\iota}}\left(e^{-\iota}+\frac{|\nu(\iota)|}{1+|\nu(\iota)|}+\frac{|\nu(\iota/3)|}{1+|\nu(\iota/3)|}+\frac{{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}{1+{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}\right) \\
&\quad+\frac{\iota^2}{20e^{\iota-1}}\left(e^{-\iota}+\frac{|\overline{\nu}(\iota)|}{1+|\overline{\nu}(\iota)|}+\frac{|\overline{\nu}(\iota/3)|}{1+|\overline{\nu}(\iota/3)|}+\frac{{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}{1+{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)}\right) \\
&\leq\frac{1}{20}\left(|\nu(\iota)-\overline{\nu}(\iota)|+\left|\nu\left(\frac{\iota}{3}\right)-\overline{\nu}\left(\frac{\iota}{3}\right)\right|+\left|{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\nu(\iota)-{}^{\mathrm{ABR}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)\right|\right).
\end{aligned} \tag{92}$$

Therefore, hypothesis (H_1) holds with $\mathfrak{N}_f = 1/20$. Also $\Theta_i = 1.14, \mathscr{R}_{B,\sigma} = 2.62$, and $\mathfrak{A} = 0.68 < 1$. Then all conditions in Theorem 5 are satisfied and hence the ABR-problem (4) has a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\hat{\nu} \in C(\mathscr{J}, \mathbb{R})$ satisfies

$$\left|{}^{\mathrm{ABR}}\mathbb{D}_{a^+}^{\sigma}\hat{\nu}(\iota) - F_{\hat{\nu}}(\iota)\right| \leq \varepsilon. \tag{93}$$

There exists a solution $\nu \in C(\mathscr{J}, \mathbb{R})$ to the ABC problem (5) with

$$\|\hat{\nu} - \nu_1\| \leq C_f \varepsilon, \tag{94}$$

where

$$C_f = \frac{\left(\left(\left(\sum_{i=1}^{m}\theta_i\Theta_i + \mathscr{R}_{B,\sigma}\right)/\left(1 - \sum_{i=1}^{m}\theta_i\right)\right) + \mathscr{R}_{B,\sigma}\right)}{1 - \left(2\mathfrak{N}_f\mathscr{R}_{B,\sigma}/\left(1 - \mathfrak{N}_f\right)\right)} = 8.9 > 0. \tag{95}$$

Therefore, all conditions in Theorem 8 are satisfied and hence the ABR problem (4) is UH stable.

Example 2. Consider the following ABC fractional problem

$$\begin{cases} {}^{\mathrm{ABC}}\mathbb{D}_{0^+}^{3/2}\nu(\iota) = \dfrac{\iota^2}{20e^{\iota}}\left(e^{-\iota} + \dfrac{|\nu(\iota)|}{1 + |\nu(\iota)|} + \dfrac{|\nu(\iota/3)|}{1 + |\nu(\iota/3)|} + \dfrac{{}^{\mathrm{ABC}}\mathbb{D}_{0^+}^{3/2}\nu(\iota)}{1 + {}^{\mathrm{ABC}}\mathbb{D}_{0^+}^{3/2}\nu(\iota)}\right), & \iota \in (0,1), \\ \nu(1) = \dfrac{1}{4}\nu\left(\dfrac{1}{2}\right). \end{cases} \tag{96}$$

Here $\sigma = (3/2) \in (1,2], a = 0, b = 1, \tau_1 = (1/4), n = 1, \kappa_1 = (1/2)$. Let $\iota \in [0,1], \nu, \overline{\nu} \in \mathbb{R}$. Then

$$\begin{aligned} &\left|f\left(\iota, \nu(\iota), \nu\left(\frac{\iota}{3}\right), {}^{\mathrm{ABC}}\mathbb{D}_{0^+}^{3/2}\nu(\iota)\right) - f\left(\iota, \overline{\nu}(\iota), \overline{\nu}\left(\frac{\iota}{3}\right), {}^{\mathrm{ABC}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)\right)\right| \\ &\leq \frac{1}{20}\left(|\nu(\iota) - \overline{\nu}(\iota)| + \left|\nu\left(\frac{\iota}{3}\right) - \overline{\nu}\left(\frac{\iota}{3}\right)\right| + \left|{}^{\mathrm{ABC}}\mathbb{D}_{0^+}^{3/2}\nu(\iota) - {}^{\mathrm{ABC}}\mathbb{D}_{0^+}^{3/2}\overline{\nu}(\iota)\right|\right). \end{aligned} \tag{97}$$

Therefore, the hypothesis (H_1) holds with $\mathfrak{N}_f = (1/20)$. Also $\Theta_j = 1.14, \mathscr{R}_{B,\sigma} = 2.62$ and $\Upsilon = 0.68 < 1$. Then all conditions in Theorem 7 are satisfied and hence the ABC problem (5) has a unique solution.

5. Conclusion remarks

The theory of fractional operators in the Atangana–Baleanu framework has recently sparked interest, prompting some scholars to investigate and create certain qualitative features of solutions to FDEs employing such operators. We developed and investigated adequate guarantee conditions for the existence and uniqueness of solutions for two classes of nonlinear implicit fractional pantograph equations with the interval ABC and ABR fractional derivatives, subjected to nonlocal condition.

The reduction of ABC-type pantograph FDEs to FIEs, as well as various Banach and Krasnoselskii's fixed point theorems, are the foundations of our technique. In addition, we used Gronwall's inequality in the context of the AB fractional integral operator to derive suitable conclusions for various forms of UH stability. The results are supported by relevant instances. The problems under consideration are also true in some particular circumstances, i.e., they may be reduced to problems containing the Caputo–Fabrizio fractional derivative operator. Furthermore, the examination of the generated findings was kept to a bare minimum.

Data Availability

The data available upon requested.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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