Hindawi
Journal of Mathematics
Volume 2022, Article ID 9722983, 16 pages
https://doi.org/10.1155/2022/9722983

Research Article

Construction of Mutually Orthogonal Graph Squares Using Novel Product Techniques

A. El-Mesady [1] and Omar Bazighifan [2,3]

[1] *Department of Physics and Engineering Mathematics, Faculty of Electronic Engineering, Menoufia University, Menouf 32952, Egypt*
[2] *Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen*
[3] *Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout 50512, Yemen*

Correspondence should be addressed to Omar Bazighifan; o.bazighifan@gmail.com

Received 13 July 2022; Revised 5 August 2022; Accepted 16 August 2022; Published 10 October 2022

Academic Editor: Mehar Ali Malik

Sets of mutually orthogonal Latin squares prescribe the order in which to apply different treatments in designing an experiment to permit effective statistical analysis of results, they encode the incidence structure of finite geometries, they encapsulate the structure of finite groups and more general algebraic objects known as quasigroups, and they produce optimal density error-correcting codes. This paper gives some new results on mutually orthogonal graph squares. Mutually orthogonal graph squares generalize orthogonal Latin squares interestingly. Mutually orthogonal graph squares are an area of combinatorial design theory that has many applications in optical communications, wireless communications, cryptography, storage system design, algorithm design and analysis, and communication protocols, to mention just a few areas. In this paper, novel product techniques of mutually orthogonal graph squares are considered. Proposed product techniques are the half-starters' vectors Cartesian product, half-starters' function product, and tensor product of graphs. It is shown that by taking mutually orthogonal subgraphs of complete bipartite graphs, one can obtain enough mutually orthogonal subgraphs in some larger complete bipartite graphs. Also, we try to find the minimum number of mutually orthogonal subgraphs for certain graphs based on the proposed product techniques. As a direct application to the proposed different product techniques, mutually orthogonal graph squares for disjoint unions of stars are constructed. All the constructed results in this paper can be used to generate new graph-orthogonal arrays and new authentication codes.

1. Introduction

Graphs are discrete structures consisting of vertices and edges that connect these vertices. Several problems in almost every conceivable discipline can be solved using graph models. Certain problems in physics, chemistry, computer technology, psychology, communication science, linguistics, engineering, sociology, and genetics can be formulated as problems in graph theory. For instance, graphs are used to represent the competition of different species in an environment, to represent who influences whom in an organization, and to represent the outcomes of round-robin tournaments. Also, graphs are used to model relationships between people, collaborations between researchers, telephone calls between telephone numbers, and links between websites, to mention just a few areas. Many branches of mathematics, such as probability, topology, matrix theory, and group theory, have strong connections with graph theory. For standard terminology and notations concerning graph theory, see [1]. Decompositions of complete bipartite graphs have several applications in the design of experiments, graph code generation, and authentication codes [2, 3]. Table 1 shows the nomenclature used in the paper.

In this paper, we are concerned with an area of combinatorial theory that deals with mutually orthogonal F squares where F is a subgraph of $K_{n,n}$. Mutually orthogonal Latin squares (MOLS) are a special case of mutually orthogonal graph squares (MOGS). MOGS are interesting but not attainable for general graphs. Combinatorial design theory has many applications in optical communications, wireless communications, cryptography, storage system design, algorithm design and analysis, and communication protocols, to mention just a few areas.

Table 1: The nomenclature used in the paper.

	Nomenclature
$d(x)$	Degree of a vertex x
$\mathbb{Z}_n$	The group $\{0, 1, \ldots, n-1\}$
K_m	Complete graph with m vertices
$K_{m,n}$	Complete bipartite graph having two independent sets of sizes m and n
C_k	Cycle of length k
P_k	Path on k vertices
$G \cup H$	Disjoint union of G and H
mG	m disjoint copies of G
$M(i, j)$	Entry in row i and column j of a square matrix M
$\otimes$	Cartesian product operation
$\mathbb{P}_n(F)$	F−path obtained by replacing each edge in P_n by the graph F

Definition 1 (see [4]). Let F be a subgraph of $K_{n,n}$ with size n. A square matrix M of order n is an F−square if every element in $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is found exactly n times in M, and the graphs F_k with $E(F_k) = \{(a_0, b_1): M(a_0, b_1) = k: k \in \mathbb{Z}_n\}$ are isomorphic to F. The elements of $\mathbb{Z}_n \times \{0\}$ are used for labeling the rows of M, and the elements of $\mathbb{Z}_n \times \{1\}$ are used for labeling the columns of M. An edge decomposition of $K_{n,n}$ by a graph F can be represented by an F−square.

Definition 2 (see [4]). Suppose M_1 is an F−square of order n with entries from a set C, and M_2 is an F−square of order n with entries from a set D. Then, the two squares M_1 and M_2 are orthogonal if, for every $x \in C$ and for every $y \in D$, there exists exactly one cell (a_0, b_1) such that $M_1(a_0, b_1) = x$ and $M_2(a_0, b_1) = y$. A set of λF−squares of order n, say $M_1, \ldots, M_\lambda$, are called pairwise orthogonal (mutually orthogonal) F−squares (MOGS) if M_p and M_q are orthogonal for all $1 \leq p < q \leq \lambda$. Here, we consider $C = D = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.

Theorem 1. *For the bipartite graph F having n edges, $N(n, F)$ denotes the maximum number k in a largest possible set of MOGS of $K_{n,n}$ by F. For every bipartite graph F with $n \geq 2$ edges, we have $N(n, F) \leq n$.*

Great efforts have been made to get the solution to several problems concerned with the MOLS since Euler first asked about MOLS to solve the thirty-six officer's problem. Famous theorems concerning the MOLS were introduced by Bose, Shrikhande, and Parker [5, 6]. Also, Wilson in [7] handled celebrated theorems concerned with the MOLS. Many efforts have been concentrated on refining and finding novel applications for these approaches. The authors of [8] proposed an integrated firefly algorithm based on MOLS, named FA-MOLS, to address the quadratic assignment problem. Liu [9] introduced the packing of Latin squares by BCL algebras. The authors in [10] focused on the existence of orthogonal large sets of partitioned incomplete Latin squares. A large set of disjoint incomplete Latin squares was introduced in [11]. A strategy for producing group-based Sudoku-pair Latin squares was investigated in [12]. The Latin squares were constructed based on the circulant matrix by the authors of [13]. Authentication codes based on orthogonal arrays and Latin squares were proposed in [14]. For a good survey of MOLS, see [15] and the references therein. El-Shanawany [16] proposed the conjecture, $N(p, P_{p+1}) = p$, where P_{p+1} is a path with $p+1$ vertices and p is a prime number. Sampathkumar et al. [17] solved this conjecture. El-Shanawany [18] found $N(p, \mathbb{P}_{p+1}(F))$. El-Shanawany [19] computed $N(n, F) = r \geq 3$ where F is disjoint copies of some subgraphs of $K_{n,n}$. El-Shanawany and El-Mesady [4] introduced the Kronecker product of MOGS and applied this technique to get a new mutually orthogonal disjoint union of some complete bipartite graph squares. MOGS for disjoint unions of paths were developed in [20]. MOGS for certain graphs were handled by [21]. El-Mesady et al. [22] generalized the MacNeish's Kronecker product theorem of MOLS. MOGS were used to construct graph-transversal designs and graph-authentication codes in [3, 23]. MOGS are used to construct orthogonal arrays that have many applications [24].

The main purpose of this paper is to construct several new results on MOGS. All of the previously mentioned MOGS results motivated us to introduce novel different product techniques to MOGS that yield new MOGS results. The proposed product techniques are the half-starters' vectors Cartesian product, half-starters' function product, and graph tensor product. The novelty of the current paper is demonstrated by the fact that it is the first to introduce the MOGS by the aforementioned product techniques. It is shown that by taking mutually orthogonal subgraphs of complete bipartite graphs, one can obtain enough mutually orthogonal subgraphs in some larger complete bipartite graphs. Also, we try to find the minimum number of mutually orthogonal subgraphs for certain graphs based on the proposed product techniques. As a direct application to the proposed different product techniques, mutually orthogonal graph squares for disjoint unions of stars are constructed. The main difference between this paper and almost all the related study works that we surveyed in this section is that the proposed product techniques are recursive construction techniques that can use all the results in the literature to construct novel results concerned with MOGS. Also, the Kronecker product [4] was applied to the squares, but the half-starters' vectors Cartesian product is applied to the vectors, the half-starters' function product is applied to the functions, and the graph tensor product is applied to graphs.

The remaining part of the present paper is divided as follows: Section 2 is devoted to MOGS from mutually orthogonal half-starters' vectors. Section 3 constructs MOGS based on the Cartesian product of half-starters' vectors. MOGS from mutually orthogonal half-starters' functions are presented in Section 4. Section 5 introduces the tensor products of MOGS. MOGS for complete bipartite graphs by stars based on the tensor product are proved in Section 6. Discussion is presented in Section 7. Section 8 is devoted to the conclusion and future work.

2. MOGS from Mutually Orthogonal Half Starters' Vectors

If we have a graph F which is considered a subgraph of $K_{n,n}$ with n edges, then the graph $F + x$ is called the x−translate of F and $E(F + x) = \{(a_0 + x, b_1 + x): (a_0, b_1) \in E(F)\}$. If the edge $(a_0, b_1) \in E(F)$, then its length is defined by $b - a$, where arithmetic operations are calculated modulo n. A graph F is called a half-starter w.r.t. $\mathbb{Z}_n$ if $|E(F)| = n$ and the edges in F have different lengths that are equivalent to the group $\mathbb{Z}_n$.

Theorem 2 *(see [9]). If F is a half-starter, then an edge decomposition of $K_{n,n}$ can be constructed by finding all the translates of F and taking their union; that is, $\cup_{x\in\mathbb{Z}_n} E(F+x) = E(K_{n,n})$.*

The vector $u(F) = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Z}_n^n = \overbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}^{n\text{-times}}$ can be used to represent the half-starter F where $u_k, k \in \mathbb{Z}_n$, and $(u_k)_0$ is the unique vertex $((u_k, 0) \in \mathbb{Z}_n \times \{0\})$ that belongs to the unique edge of length k in F with $E(F) = \{(u_k, u_k + k)\colon u_k, k \in \mathbb{Z}_n\}$. Two half-starters' vectors $u(H)$ and $u(F)$ are said to be orthogonal if $\{u_k(H) - u_k(F)\colon k \in \mathbb{Z}_n\} = \mathbb{Z}_n$. A set of half-starters' vectors $u(F_0), u(F_1), \ldots, u(F_{\lambda-1})$ is mutually orthogonal if $u(F_k)$ and $u(F_l)$ are orthogonal for every $0 \le k < l \le \lambda - 1$. It is worth noting that each half-starter and its translates of a subgraph F of $K_{n,n}$ are equivalent to F−square. Hence, the set of k mutually orthogonal half-starters and their translates are equivalent to a set of k mutually orthogonal F−squares.

3. MOGS Based on the Cartesian Product of Half-Starters' Vectors

The Cartesian product of half-starters' vectors has been defined in literature for constructing orthogonal double covers of $K_{n,n}$. This method has been applied to construct orthogonal double covers of $K_{n,n}$ by new graph classes. The Cartesian product of two vectors corresponding to two half-starter graphs is considered a very special case of the tensor product of these two half-starter graphs.

Definition 3. The tensor product of two graphs G_1 and $G_2, G_1 \times G_2$ is defined as follows. If a vertex u_1 is adjacent to a vertex v_1 in G_1 and a vertex u_2 is adjacent to a vertex v_2 in G_2, then the vertex (u_1, u_2) is adjacent to the vertex (v_1, v_2) in $G_1 \times G_2$.

Example 1. Figure 1 exhibits an example of the graphs G_1, G_2, and $G_1 \times G_2$.

Definition 4 (see [25]). Let G be a graph, and v belongs to the vertex set of G. The number of edges incident at v in G is called the degree (or valency) of the vertex v in G and is denoted by $d(v)$. From the degrees of vertices of G, we can construct a sequence which is called a degree sequence of G, when the vertices are taken in the same order. It is customary to put this sequence in nondecreasing or nonincreasing order. This gives a unique sequence.

Example 2 (see [25]). In the graph G of Figure 2, the number within the parentheses indicates the degree of the corresponding vertex. The degree sequence of G is $(0, 1, 2, 2, 4, 4, 5)$.

Definition 5. If we have the vector $u(G) = (u_0, u_1, \ldots, u_{(m-1)}) \in \mathbb{Z}_m^m$, then, by determining the repetition number of each element in the vector $u(G)$, we get the vector $N = ((n(0), n(1), \ldots, n((m-1)))$, where $n(i)$ is the repetition number of the element i, $i \in \mathbb{Z}_m$. By the ascending order for the vector N, we get the degree sequence of the vector $u(G)$ defined by $L = (l_0, l_1, \ldots, l_{(m-1)})$, where $l_0 \le l_1 \le \ldots \le l_{(m-1)}$.

Definition 6. If we have the two vectors $u^p(G^p) = (u_0^p, u_1^p, \ldots, u_{(m-1)}^p), u^q(G^q) = (u_0^q, u_1^q, \ldots, u_{(m-1)}^q)$ and the two vectors $u'^P = (u'^P_0, u'^P_1, \ldots, u'^P_{(m-1)}), u'^q = (u'^q_0, u'^q_1, \ldots, u'^q_{(m-1)})$, where $u'^P_i = u_i^p + i$, $u'^q_i = u_i^q + i$, $i \in \mathbb{Z}_m$, then the two half-starters G^p and G^q are isomorphic if $L_p = L_q$ and $L'_P = L'_q$ or $L_p = L'_q$ and $L'_p = L_q$, where L_p is the degree sequence of the vector u^p, L'_p is the degree sequence of the vector $u\prime^p$,L_q is the degree sequence of the vector u^q, and L'_q is the degree sequence of the vector u'^q. In our paper, we consider the case of $L_p = L_q$ and $L'_P = L'_q$ for the isomorphism of the two half-starters G^p and G^q.

For Proposition 1, if we have the two half-starters G and H which are represented by the vectors $v(G) \in \mathbb{Z}_m^m$ and $u(H) \in \mathbb{Z}_n^n$, respectively, then the graph $\mathbb{T} \cong G \otimes H$ is defined by the edge set $E(\mathbb{T}) = E(G \otimes H) = \{(v_i u_j, (v_i + i)(u_j + j))\colon i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$.

Proposition 1. *If there are k mutually orthogonal half-starters' vectors of length m for the graph G, and k mutually orthogonal half-starter' vectors of length n for the graph H, then there are k mutually orthogonal half-starters' vectors of length mn for the graph $\mathbb{T} \cong G \otimes H$.*

Proof 1. For $p \ne q \in \mathbb{Z}_k$, let $v(G^s \cong G)$ be k mutually orthogonal half-starters' vectors of length m and $v(H^s \cong H)$ be k mutually orthogonal half-starters' vectors of length n, where $v^s(G^s) = (v_0^s, v_1^s, \ldots, v_{(m-1)}^s) \in \mathbb{Z}_m^m$ and $u^s(H^s) = (u_0^s, u_1^s, \ldots, u_{n-1}^s) \in \mathbb{Z}_n^n$. Hence,

$$\left\{v_i^p(G^p) - v_i^q(G^q)\colon \quad i \in \mathbb{Z}_m\right\} = \mathbb{Z}_m, \tag{1}$$

$$\left\{u_i^p(H^p) - u_i^q(H^q)\colon \quad i \in \mathbb{Z}_n\right\} = \mathbb{Z}_n. \tag{2}$$

Then, $v^s(G^s) \otimes u^s(H^s) = (v_0^s u_0^s, v_0^s u_1^s, v_0^s u_2^s, \ldots, v_i^s u_j^s, \ldots, v_{(m-1)}^s u_{n-1}^s)$ where $i \in \mathbb{Z}_m, j \in \mathbb{Z}_n$. For $\mathbb{T}^p \cong G^p \otimes H^p, \mathbb{T}^q \cong G^q \otimes H^q$, we conclude from (1) and (2),

$$\left\{v_i^p u_j^p - v_i^q u_j^q\colon \quad ij \in \mathbb{Z}_m \otimes \mathbb{Z}_n\right\} = \left\{\left(v_i^p(G^p) - v_i^q(G^q)\right)\left(u_j^p(H^p) - u_j^q(H^q)\right)\colon i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\right\} = \mathbb{Z}_m \otimes \mathbb{Z}_n. \tag{3}$$

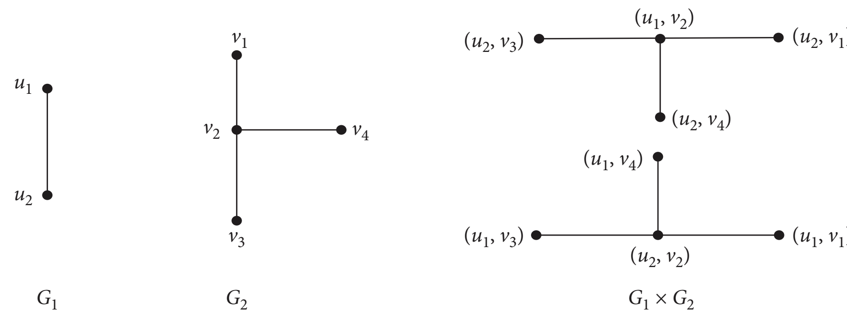

Figure 1: An example for the graphs G_1, G_2, and $G_1 \times G_2$.

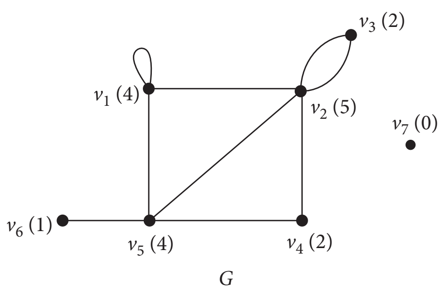

Figure 2: Degrees of vertices for the graph G.

Hence, from (3), the two half-starters $\mathbb{T}^p$ and $\mathbb{T}^q$ are orthogonal. Since $E(G^s) = \{(v_i^s, v_i^s + i): \quad i \in \mathbb{Z}_m\}$ and $E(H^s) = \{(u_j^s, u_j^s + j): \quad j \in \mathbb{Z}_n\}$, then

$$\begin{aligned} E(G^s \otimes H^s) &= E(\mathbb{T}^s), \\ &= \{(v_i^s u_j^s, (v_i^s + i)(u_j^s + j)): \quad i \in \mathbb{Z}_m, \quad j \in \mathbb{Z}_n\}. \end{aligned} \tag{4}$$

Now, we will try to prove the isomorphism of the graphs $\mathbb{T}^s$, $s \in \mathbb{Z}_k$. We have $E(\mathbb{T}^p) = \{(v_i^p u_j^p, (v_i^p + i)\ (u_j^p + j)): \quad i \in \mathbb{Z}_m, \quad j \in \mathbb{Z}_n\} = \{((v_i^p u_j^p, v'^p_i u'^p_j): \quad i \in \mathbb{Z}_m, \quad j \in \mathbb{Z}_n\}$ and $E(\mathbb{T}^q) = \{(v_i^q u_j^q, (v_i^q + i)(u_j^q + j)): \quad i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\} = \{(v_i^q u_j^q, v'^q_i u'^q_j): \quad i \in \mathbb{Z}_m, \quad j \in \mathbb{Z}_n\}$. Since the degree sequence of the vector v^p equals the degree sequence of the vector v^q, the degree sequence of the vector v'^p equals the degree sequence of the vector v'^q, the degree sequence of the vector u^p equals the degree sequence of the vector u^q, and the degree sequence of the vector u'^p equals the degree sequence of the vector u'^q, then the degree sequence of the vector $v^p u^p$ equals the degree sequence of the vector $v^q u^q$, and the degree sequence of the vector $v'^p u'^p$ equals the degree sequence of the vector $v'^q u'^q$. Hence, the two half-starters $\mathbb{T}^p$ and $\mathbb{T}^q$ are isomorphic. □

Example 3. Let q be a prime > 2. Then, we have q mutually orthogonal half-starters' vectors defined by $u_i^s(G^s) = i(s - i)$, where $i, \quad s \in \mathbb{Z}_q$ and $G^s \cong P_{q+1}$, see [11]. Hence, we have 3 mutually orthogonal half-starters' vectors for both P_6 and P_4, which are $v^s(P_6^s)$ and $u^s(P_4^s)$, respectively. Then, we obtain 3 mutually orthogonal half-starters' vectors $w^s(P_6^s \otimes P_4^s)$ as shown in Table 2. See Figure 3.

Also, the three MOGS corresponding to the vectors $v^s(P_6^s)$ are A_0, A_1, and A_2, the three MOGS corresponding to the vectors $u^s(P_4^s)$ are B_0, B_1, and B_2, and the three MOGS corresponding to the vectors $w^s(P_6^s \otimes P_4^s)$ are L_0, L_1, and L_2.

Of course in L_0, L_1, and L_2, one can easily replace the ordered pairs 00, 01, 02, 10, 11, 12, 20, 21, 22, 30, 31, 32, 40, 41, 42 by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 to obtain three mutually orthogonal $(P_6 \otimes P_4)$-squares of order 15, where their elements are the usual symbols.

4. MOGS from Mutually Orthogonal Half-Starters' Functions

In what follows, if $V_1 = \mathbb{Z}_n \otimes \{0\}$ and $V_2 = \mathbb{Z}_n \otimes \{1\}$, then the graphs G_{f_i} will be represented by the functions $f_i : V_1 \longrightarrow V_2$. The edge set of the graphs G_{f_i} is $E(G_{f_i}) = \{(x, f_i(x)): x \in V_1\}$. Every graph from the graphs G_{f_i} represents unions of stars which have the same direction where every vertex x belongs to V_1 has a degree of one, that is, $d(x) = 1$. The graph G_f is called $f-$ half-starter if $E(G_f) = \cup_{x \in V_1}\{(x, f(x))\}$, where G_f is a subgraph of $K_{n,n}$ with n edges and $f : V_1 \longrightarrow V_2$.

Definition 7 (see [9]). Let $z \in \mathbb{Z}_n$. Then, the graph $G_f + z = \{(x, f(x) + z): (x, f(x)) \in E(G_f)\}$ is called the (z, f)–translate of G_f.

Remark 1 (see [9]). The union of all translates of G_f forms an edge decomposition of $K_{n,n}$, that is, $\cup_{z \in \mathbb{Z}_n} E(G_f + z) = E(K_{n,n})$.

Definition 7 and Remark 1 show that every $f-$ half-starter graph G_f and the translates are equivalent to G_f-square.

Let $G_{f_i} \cong G, \quad i \in \mathbb{Z}_k$ be subgraphs of $K_{n,n}$ with $|E(G)| = n$. Then, the set $\{G_{f_i}\}$ is called a set of k mutually orthogonal subgraphs, if $\{f_i(x) - f_j(x): \quad x \in \mathbb{Z}_n\} = \mathbb{Z}_n$ for all $i, \quad j \in \mathbb{Z}_k$ and $i \neq j$. If the two half-starters G_f and G_g are orthogonal, then the two sets of translates of G_f and G_g are orthogonal. A set of edge decompositions $\{\mathscr{H}_{f_i} = \cup_{z \in \mathbb{Z}_n} E(G_{f_i} + z)\}$ is a set of k MOGS if $\mathscr{H}_{f_i}$ and $\mathscr{H}_{f_j}$ are orthogonal for all $i, \quad j \in \mathbb{Z}_k$ and $i \neq j$.

Example 4. Let $f_i(x) = x^2 + ix$ be $f-$ half-starter graphs $G_{f_i} \cong K_2 \cup K_{1,2}$ of $K_{3,3}$ for all $i, \quad x \in \mathbb{Z}_3$. The edge set of the graphs G_{f_i} and their translates are shown in Tables 3, 4, and 5.

Table 2: The used vectors in Example 3.

	$v^s(P_6^s), v'^s(P_6^s)$	$u^s(P_4^s), u'^s(P_4^s)$	$w^s(P_6^s \otimes P_4^s), w'^s(P_6^s \otimes P_4^s)$
$s = 0$	$(0, 4, 1, 1, 4), (0, 0, 3, 4, 3)$	$(0, 2, 2), (0, 0, 1)$	$(00, 02, 02, 40, 42, 42, 10, 12, 12, 10, 12, 12, 40, 42, 42), (00, 00, 01, 00, 00, 01, 30, 30, 31, 40, 40, 41, 30, 30, 31)$
$s = 1$	$(0, 0, 3, 4, 3), (0, 1, 0, 2, 2)$	$(0, 0, 1), (0, 1, 0)$	$(00, 00, 01, 00, 00, 01, 30, 30, 31, 40, 40, 41, 30, 30, 31), (00, 01, 00, 10, 11, 10, 00, 01, 00, 20, 21, 20, 20, 21, 20)$
$s = 2$	$(0, 1, 0, 2, 2), (0, 2, 2, 0, 1)$	$(0, 1, 0), (0, 2, 2)$	$(00, 01, 00, 10, 11, 10, 00, 01, 00, 20, 21, 20, 20, 21, 20), (00, 02, 02, 20, 22, 22, 20, 22, 22, 00, 02, 02, 10, 12, 12)$

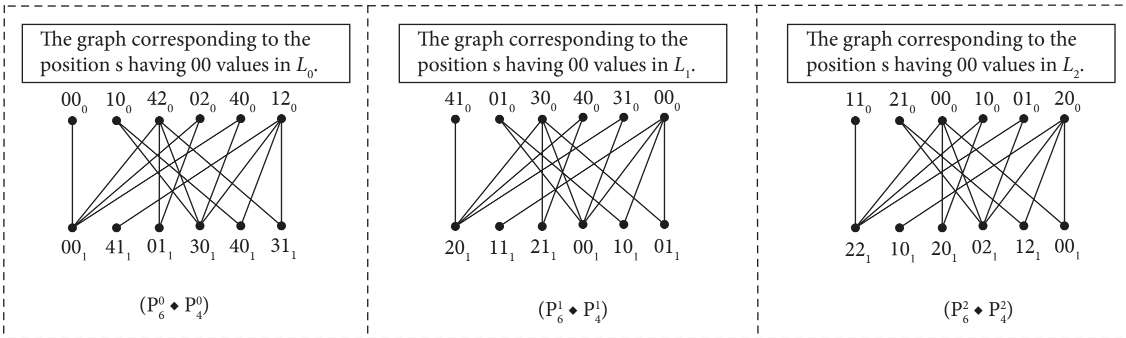

FIGURE 3: Three mutually orthogonal half-starters corresponding to the vectors $w^s(P_6^s \otimes P_4^s)$.

Hence, we deduce the following three mutually orthogonal $(K_2 \cup K_{1,2})$-squares:

$$
\begin{aligned}
L^0 &= \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \\
L^1 &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \\
L^2 &= \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}.
\end{aligned} \tag{5}
$$

Definition 8. For the $f-$ half-starter graph G_f represented by the function $f: V_1 \longrightarrow V_2$, where $V_1 = \mathbb{Z}_n \otimes \{0\}, V_2 = \mathbb{Z}_n \otimes \{1\}$, and $E(G_f) = \{(x, f(x)): \quad x \in V_1\}$, we have $d(x) = 1$ for all $x \in V_1$. By determining the degree of each vertex belonging to V_2, from G_f, we get the vector $N = (d(0_1), d(1_1), \ldots, d((n-1)_1))$, where $d(i_1)$ is the degree of the vertex i_1, $\quad i_1 \in \mathbb{Z}_n \otimes \{1\}$. By the ascending order for the vector N, we get the degree vector of G_f denoted by $L_f = (l_0, l_1, \ldots, l_{n-1})$ where $l_0 \leq l_1 \leq \ldots \leq l_{n-1}$.

Definition 9. If we have the f-half-starter graph G_f represented by the function $f: V_1 \longrightarrow V_2$ and the g-half-starter graph G_g represented by the function $g: V_1 \longrightarrow V_2$, then G_f and G_g are isomorphic if $L_f = L_g$, where L_f is the degree vector of G_f and L_g is the degree vector of G_g.

We shall denote by $N(n, G_{f_i})$ the maximal number of f_i–half-starter graphs G_{f_i} in the largest possible set of mutually orthogonal subgraphs G_{f_i} in $K_{n,n}$, $\quad i \in \mathbb{Z}_k$. Hereafter, if we have $N(m, G_{f_i}) \geq \lambda$, where $i \in \mathbb{Z}_\lambda$ and $G_{f_i} \cong G$ having m edges and $N(n, G_{g_i}) \geq \mu$, where $i \in \mathbb{Z}_\mu$ and $G_{g_i} \cong H$ having n edges and $\min\{\lambda, \mu\} = k$, then we obtain $N(mn, G_{h_i}) \geq k$, where $i \in \mathbb{Z}_k$ and $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$ by Proposition 2. Also, we present some results as direct applications to Proposition 2. In the following, if there is no danger of ambiguity, if $(x, y) \in \mathbb{Z}_m \otimes \mathbb{Z}_n$, we can write (x, y) as xy.

Proposition 2. *If $N(m, G_{f_i}) \geq \lambda$, where $i \in \mathbb{Z}_\lambda$ and $G_{f_i} \cong G$ have m edges, $N(n, G_{g_i}) \geq \mu$, where $i \in \mathbb{Z}_\mu$ and $G_{g_i} \cong H$ have n edges, and $\min\{\lambda, \mu\} = k$, then $N(mn, G_{h_i}) \geq k$, where $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$ are isomorphic graphs having mn edges.*

TABLE 3: The graph G_{f_0} and it's translates for Example 4.

$E(G_{f_0}) = \{(x, f_0(x))\}$	$E(G_{f_0} + 1)$	$E(G_{f_0} + 2)$
$(0, 0)$	$(0, 1)$	$(0, 2)$
$(1, 1)$	$(1, 2)$	$(1, 0)$
$(2, 1)$	$(2, 2)$	$(2, 0)$

TABLE 4: The graph G_{f_1} and it's translates for Example 4.

$E(G_{f_1}) = \{(x, f_1(x))\}$	$E(G_{f_1} + 1)$	$E(G_{f_1} + 2)$
$(0, 0)$	$(0, 1)$	$(0, 2)$
$(1, 2)$	$(1, 0)$	$(1, 1)$
$(2, 0)$	$(2, 1)$	$(2, 2)$

TABLE 5: The graph G_{f_2} and it's translates for Example 4.

$E(G_{f_2}) = \{(x, f_2(x))\}$	$E(G_{f_2} + 1)$	$E(G_{f_2} + 2)$
$(0, 0)$	$(0, 1)$	$(0, 2)$
$(1, 0)$	$(1, 1)$	$(1, 2)$
$(2, 2)$	$(2, 0)$	$(2, 1)$

Proof 2. Let f_i–half-starter graphs G_{f_i} be represented by the functions $f_i(x)$, $\quad i \in \mathbb{Z}_k$, $\quad x \in \mathbb{Z}_m$ and g_i–half-starter graphs G_{g_i} be represented by the functions $g_i(y)$, $\quad i \in \mathbb{Z}_k$, $\quad y \in \mathbb{Z}_n$. Then, we obtain the h_i–half-starter graphs $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$, which are represented by the functions

$$
h_i(xy) = f_i(x) g_i(y). \tag{6}
$$

Since $\{h_p(xy) - h_q(xy) = f_p(x) \quad g_p(y) - f_q(x) g_q(y) = (f_p(x) - f_q(x))(g_p(y) - g_q(y)): p, \quad q \in \mathbb{Z}_k, \quad p \neq q\} = \mathbb{Z}_m \otimes \mathbb{Z}_n$, then G_{hp} and G_{hq} are orthogonal. The edge set of the graphs G_{h_i} can be obtained as follows, since $E(G_{f_i}) = \{(x, f_i(x))\}$ and $E(G_{g_i}) = \{(y, g_i(y))\}$, then $E(G_{h_i}) = \{(xy, f_i(x) g_i(y))\}$. Now, we want to prove the isomorphism of the two graphs G_{hp} and G_{hq}. For $p, \quad q \in \mathbb{Z}_k, \quad p \neq q$, we have $L_{f_p} = L_{f_q}$ and $L_{g_p} = L_{g_q}$, then for $G_{hp} \cong G_{f_p} \otimes G_{g_p}$ and

$G_{hq} \cong G_{f_q} \otimes G_{g_q}$, the degree vector of G_{hp} is equal to the degree vector of G_{hq}. This means that $L_{h_p} = L_{h_q}$. Hence, G_{hp} and G_{hq} are isomorphic.

All the following results are based on (i) Proposition 2 and (ii) the following ingredients (see [9]).

(i) Let $n > 2$ be a prime number. Then, $N(n, K_{1,1} \cup ((n-1)/2)K_{1,2}) = n$, and the f_i–half-starter graphs $G_{f_i} \cong K_{1,1} \cup ((n-1)/2)K_{1,2}$ are represented by the functions $f_i(x) = x^2 + ix$, where $i, \quad x \in \mathbb{Z}_n$.

(ii) Let n be a prime number. Then, $N(n, nK_{1,1}) = n - 1$ and the f_i– half-starter graphs $G_{f_i} \cong nK_{1,1}$ are represented by the functions $f_i(x) = ix$, where $x \in \mathbb{Z}_n, i \in \mathbb{Z}_n/\{0\}$.

(iii) Let $n > 2$ be a prime number. Then, $N(n, (n-2)K_{1,1} \cup K_{1,2}) \geq n - 1$, and the f_i– half-starter graphs $G_{f_i} \cong (n-2)K_{1,1} \cup K_{1,2}$ are represented by the functions $f_i(x) = \begin{cases} 0; & x = 0 \\ 1 + ix; & i, x \in \mathbb{Z}_n. \end{cases}$

(iv) Let $n = 9$. Then, $N(9, K_{1,3} \cup 3K_{1,2}) \geq 3$, and the f_i– half-starter graphs $G_{f_i} \cong K_{1,3} \cup 3K_{1,2}$ are represented by the functions $f_i(x) = x^2 + ix$, where $x \in \mathbb{Z}_9, \quad i \in \mathbb{Z}_3$.

(v) Let $n = 7$. Then $N(7, 3K_{1,1} \cup 2K_{1,2}) \geq 4$, for $i \in \mathbb{Z}_4$, the f_i– half-starter graphs $G_{f_i} \cong 3K_{1,1} \cup 2K_{1,2}$ are represented by the functions

$$f_i(x) = \begin{cases} 0; & x = 0, \\ 1 + 4i; & x = 4, \\ 2 + i; & x = 1, \\ 4 + ix; & x = 2, 5, \\ 6 + ix; & x = 3, 6. \end{cases} \quad (7)$$

These known f_i– half-starter graphs G_{f_i} are the ingredients for the following results. These ingredients are some of the literature results and are not all of the literature results. □

Theorem 3. *Let n, m be odd primes. Then, $N(mn, G_1) \geq k$, where $k = min\{m, n-1\}$ and $G_1 \cong nK_{1,1} \cup (((m-1))n)/2)K_{1,2}$.*

Proof 3. We have k mutually orthogonal f_i– half-starter graphs $G_{f_i} \cong K_{1,1} \cup (((m-1))/2)K_{1,2}$, which are represented by the functions $f_i(x) = x^2 + ix$, where $x \in \mathbb{Z}_m, \quad i \in \mathbb{Z}_k$ (ingredient (i)) and k mutually orthogonal g_i– half-starter graphs $G_{g_i} \cong nK_{1,1}$, which are represented by the functions $g_i(y) = (i+1)y$, where $y \in \mathbb{Z}_n, i \in \mathbb{Z}_k$ (ingredient (ii)). Then, we obtain k mutually orthogonal h_i– half-starter graphs $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$, which are represented by the functions $h_i(xy) = (x^2 + ix)((i+1)y)$. Since $\{h_p(xy)- \quad h_p(xy) = (x^2 + px)((p+1)y) - (x^2 + qx)((q+1)y) = ((x^2 + px) - (x^2 + qx))((p+1)y - (q+1)y)$: $p, \quad q \in \mathbb{Z}_k, \quad p \neq q\} = \mathbb{Z}_m \otimes \mathbb{Z}_n$, then G_{hp} and G_{hq} are orthogonal. The edge set of the graphs G_{h_i} can be obtained as follows, since $E(G_{f_i}) = \{(x, x^2 + ix)\}$ and $E(G_{g_i}) = \{(y, (i+1)y)\}$, then $E(G_{h_i}) = \{(xy, (x^2 + ix)((i+1)y))\}$. Now, we want to prove the isomorphism of the two graphs G_{hp} and G_{hq}. For $p, \quad q \in \mathbb{Z}_k, \quad p \neq q$, we have $L_{f_p} = L_{f_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2}$ and $L_{g_p} = L_{g_q} = \overbrace{(1, 1, \ldots, 1)}^{n}$, then for $G_{hp} \cong G_{f_p} \otimes G_{g_p}$ and $G_{hq} \cong G_{f_q} \otimes G_{g_q}$, the degree vector of G_{hp} is equal to the degree vector of G_{hq}. This means that $L_{h_p} = L_{h_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2} \otimes \overbrace{1, 1, \ldots, 1)}^{n}$. Hence, G_{hp} and G_{hq} are isomorphic. □

Theorem 4. *Let $n, \ m$ be primes. Then, $N(mn, G_2) \geq k$, where $k = min\{m, n-1\}$ and $G_2 \cong (n-2)K_{1,1} \cup (((m-1))(n-2)+2)/2)K_{1,2} \cup (((m-1))/2)K_{1,4}$.*

Proof 4. We have k mutually orthogonal f_i– half-starter graphs $G_{f_i} \cong K_{1,1} \cup (((m-1))/2)K_{1,2}$, which are represented by the functions $f_i(x) = x^2 + ix$, where $x \in \mathbb{Z}_m, \quad i \in \mathbb{Z}_k$ (ingredient (i)) and k mutually orthogonal g_i– half-starter graphs $G_{g_i} \cong (n-2)K_{1,1} \cup K_{1,2}$, which are represented by the functions $g_i(y) = \begin{cases} 0; & y = 0 \\ 1 + (i+1)y; & y \in \mathbb{Z}_n/\{0\}, \quad i \in \mathbb{Z}_k \end{cases}$ (ingredient (iii)). Then, we obtain k mutually orthogonal h_i– half-starter graphs $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$, which are represented by the functions $h_i(xy) = (x^2 + ix)g_i(y)$. Since $\{h \ p(xy) - h_p(xy) = (x^2 + px)g_p(y) - (x^2 + qx)g_q(y) = ((x^2 + px) - (x^2 + qx))(g_p(y) - g_q(y))$: $p, \quad q \in \mathbb{Z}_k, \quad p \neq q\} = \mathbb{Z}_m \otimes \mathbb{Z}_n$, then G_{hp} and G_{hq} are orthogonal. The edge set of the graphs G_{h_i} can be obtained as follows, since $E(G_{f_i}) = \{(x, x^2 + ix)\}$ and $E(G_{g_i}) = \{(y, g_i(y))\}$, then $E(G_{h_i}) = \{(xy, (x^2 + ix)(g_i(y)))\}$. Now, we want to prove the isomorphism of the two graphs G_{hp} and G_{hq}. For $p, \quad q \in \mathbb{Z}_k, \quad p \neq q$, we have $L_{f_p} = L_{f_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2}$ and $L_{g_p} = L_{g_q} = (0, \overbrace{1, 1, \ldots, 1}^{n-2}, 2)$, then for $G_{hp} \cong G_{f_p} \otimes G_{g_p}$ and $G_{hq} \cong G_{f_q} \otimes G_{g_q}$, the degree vector of G_{hp} is equal to the degree vector of G_{hq}. This means that $L_{f_p} = L_{f_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2} \otimes (0, \overbrace{1, 1, \ldots, 1}^{n-2}, 2)$. Hence, G_{hp} and G_{hq} are isomorphic. □

Theorem 5. *Let m be odd prime. Then, $N(9m, G_3) \geq 3$, where $G_3 \cong K_{1,3} \cup 3K_{1,2} \cup (((m-1))/2)K_{1,6} \cup (3((m-1))/2)K_{1,4}$.*

Proof 5. We have m mutually orthogonal f_i– half-starter graphs $G_{f_i} \cong K_{1,1} \cup (((m-1))/2)K_{1,2}$, which are represented by the functions $f_i(x) = x^2 + ix$, where $x \in \mathbb{Z}_m, \quad i \in \mathbb{Z}_k$

(ingredient (i)) and 3 mutually orthogonal g_i- half-starter graphs $G_{g_i} \cong K_{1,3} \cup 3K_{1,2}$, which are represented by the functions $g_i(y) = y^2 + iy$, $y \in \mathbb{Z}_9$, $i \in \mathbb{Z}_3$ (ingredient (iv)). Since $\min\{m, 3\} = 3$, then we obtain 3 mutually orthogonal h_i- half-starter graphs $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$, which are represented by the functions $h_i(xy) = (x^2 + ix)(y^2 + iy)$, $i \in \mathbb{Z}_3$. Since $\{h_p(xy) - h_p(xy) = (x^2 + px)(y^2 + py) - (x^2 + qx)(y^2 + qy) = ((x^2 + px) - (x^2 + qx))((y^2 + py) - (y^2 + qy))$: $p, \quad q \in \mathbb{Z}_3, \quad p \neq q\} = \mathbb{Z}_m \otimes \mathbb{Z}_9$, then G_{hp} and G_{hq} are orthogonal. The edge set of the graphs G_{h_i} can be obtained as follows, since $E(G_{f_i}) = \{(x, x^2 + ix)\}$ and $E(G_{g_i}) = \{(y, y^2 + iy)\}$, then $E(G_{h_i}) = \{(xy, (x^2 + ix)(y^2 + iy))\}$. Now, we want to prove the isomorphism of the two graphs G_{hp} and G_{hq}. For $p, \quad q \in \mathbb{Z}_3, \quad p \neq q$, we have $L_{f_p} = L_{f_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2}$ and $L_{g_p} = L_{g_q} = (0, 0, 0, 0, 0, 2, 2, 2, 3)$, then for $G_{hp} \cong G_{f_p} \otimes G_{g_p}$ and $G_{hq} \cong G_{f_q} \otimes G_{g_q}$, the degree vector of G_{hp} is equal to the degree vector of G_{hq}. This means that $L_{h_p} = L_{h_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2} \otimes (0, 0, 0, 0, 0, 2, 2, 2, 3)$ Hence, G_{hp} and G_{hq} are isomorphic. □

Theorem 6. *Let $m > 3$ be prime. Then, $N(7m, G_4) \geq 4$, where $G_4 \cong 3K_{1,1} \cup ((3((m-1)) + 4)/2)K_{1,2} \cup ((m-1))K_{1,4}$.*

Proof 6. We have m mutually orthogonal f_i- half-starter graphs $G_{f_i} \cong K_{1,1} \cup (((m-1))/2)K_{1,2}$, which are represented by the functions $f_i(x) = x^2 + ix$, where $x \in \mathbb{Z}_m, i \in \mathbb{Z}_k$ (ingredient (i)) and 4 mutually orthogonal g_i- half-starter graphs $G_{g_i} \cong 3K_{1,1} \cup 2K_{1,2}$, which are represented by the functions

$$g_i(y) = \begin{cases} 0; & y = 0, \\ 1 + 4i; & y = 4, \\ 2 + i; & y = 1, \\ 4 + iy; & y = 2, 5, \\ 6 + iy; & y = 3, 6, \end{cases} \tag{8}$$

where $i \in \mathbb{Z}_4$ (ingredient (v)). Since $\min\{m, 4\} = 4$, then we obtain 4 mutually orthogonal h_i- half-starter graphs $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$, which are represented by the functions $h_i(xy) = (x^2 + ix)g_i(y)$. Since $\{h_p(xy) - h \quad p(xy) = (x^2 + px) g_p(y) - (x^2 + qx)g_q(y) = ((x^2 + px) - (x^2 + qx))(g_p(y) - g_q(y))$: $p, \quad q \in \mathbb{Z}_4, \quad p \neq q\} = \mathbb{Z}_m \otimes \mathbb{Z}_7$, then G_{hp} and G_{hq} are orthogonal. The edge set of the graphs G_{h_i} can be obtained as follows, since $E(G_{f_i}) = \{(x, x^2 + ix)\}$ and $E(G_{g_i}) = \{(y, g_i(y))\}$, then $E(G_{h_i}) = \{(xy, (x^2 + ix)(g_i(y)))\}$. Now, we want to prove the isomorphism of the two graphs G_{hp} and G_{hq}. For $p, \quad q \in \mathbb{Z}_4, \quad p \neq q$, we have $L_{f_p} = L_{f_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2}$ and $L_{g_p} = L_{g_q} = (0, 0, 1, 1, 1, 2, 2)$, then for $G_{hp} \cong G_{f_p} \otimes G_{g_p}$ and $G_{hq} \cong G_{f_q} \otimes G_{g_q}$, the degree vector of G_{hp} is equal to the degree vector of G_{hq}. This means that $L_{h_p} = L_{h_q} = \overbrace{(0, 0, \ldots, 0}^{(m-1)/2}, 1, \overbrace{2, 2, \ldots, 2)}^{(m-1)/2} \otimes (0, 0, 1, 1, 1, 2, 2)$. Hence, G_{hp} and G_{hq} are isomorphic. □

Theorem 7. *Let n, m be primes. Then $N(mn, G_5) \geq k$, where $k = min\{m-1, n-1\}$ and $G_5 \cong m(n-2)K_{1,1} \cup m_{1,2}$.*

Proof 7. We have $((m-1))$ mutually orthogonal f_i- half-starter graphs $G_{f_i} \cong mK_{1,1}$, which are represented by the functions $f_i(x) = (i+1)x$, where $x \in \mathbb{Z}_m$, $i \in \mathbb{Z}_k$ (ingredient (ii)) and $(n-1)$ mutually orthogonal g_i- half-starter graphs $G_{g_i} \cong (n-2)K_{1,1} \cup K_{1,2}$, which are represented by the functions $g_i(y) = \begin{cases} 0; & y = 0 \\ 1 + (i+1)y; & y \in \mathbb{Z}_n/\{0\}, \quad i \in \mathbb{Z}_k \end{cases}$ (ingredient (iii)). Then, we obtain k mutually orthogonal h_i- half-starter graphs $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$, which are represented by the functions $h_i(xy) = ((i+1)x)g_i(y)$. Since $\{h \quad p(xy) - h_p(xy) = ((p+1)x)g_p(y) - ((q+1)x)g_q(y) = ((p+1)x - (q+1)x)(g_p(y) - g_q(y))$: $p, q \in \mathbb{Z}_k, p \neq q\} = \mathbb{Z}_m \otimes \mathbb{Z}_n$, then G_{hp} and G_{hq} are orthogonal. The edge set of the graphs G_{h_i} can be obtained as follows, since $E(G_{f_i}) = \{(x, (i+1)x)\}$ and $E(G_{g_i}) = \{(y, g_i(y))\}$, then $E(G_{h_i}) = \{(xy, ((i+1)x)(g_i(y)))\}$. Now, we want to prove the isomorphism of the two graphs G_{hp} and G_{hq}. For $p, \quad q \in \mathbb{Z}_k, \quad p \neq q$, we have $L_{f_p} = L_{f_q} = \overbrace{(1, 1, \ldots, 1)}^{m}$ and $L_{g_p} = L_{g_q} = (0, \overbrace{1, 1, \ldots, 1}^{n-2}, 2)$, then for $G_{hp} \cong G_{f_p} \otimes G_{g_p}$ and $G_{hq} \cong G_{f_q} \otimes G_{g_q}$, the degree vector of G_{hp} is equal to the degree vector of G_{hq}. This means that $L_{h_p} = L_{h_q} = \overbrace{(1, 1, \ldots, 1)}^{m} \otimes (0, \overbrace{1, 1, \ldots, 1}^{n-2}, 2)$. Hence, G_{hp} and G_{hq} are isomorphic. □

Theorem 8. *Let $m > 3$ be prime. Then $N(9m, G_6) \geq 3$, where $G_6 \cong (m-2)K_{1,3} \cup 3(m-2)K_{1,2} \cup K_{1,6} \cup 3K_{1,4}$.*

Proof 8. We have 3 mutually orthogonal f_i- half-starter graphs $G_{f_i} \cong (m-2)K_{1,1} \cup K_{1,2}$, which are represented by the functions $f_i(x) = \begin{cases} 0; & x = 0 \\ 1 + (i+1)x; & x \in \mathbb{Z}_m/\{0\}, i \in \mathbb{Z}_3 \end{cases}$ (ingredient (iii)) and 3 mutually orthogonal g_i- half-starter graphs $G_{g_i} \cong K_{1,3} \cup 3K_{1,2}$, which are represented by the functions $g_i(y) = y^2 + iy, y \in \mathbb{Z}_9, i \in \mathbb{Z}_3$ (ingredient (iv)). Then, we obtain 3 mutually orthogonal h_i- half-starter graphs $G_{h_i} \cong G_{f_i} \otimes G_{g_i}$, which are represented by the functions $h_i(xy) = f_i(x)(y^2 + iy)$. Since $\{h_p(xy) - \quad h_p(xy) = f_p(x)(y^2 + py) - f_q(x)(y^2 + qy) = (f_p(x) - f_q(x)(y^2 + py) - (y^2 + qy))\}$: $\{p, q \in \mathbb{Z}_3, p \neq q\} = \mathbb{Z}_m \otimes \mathbb{Z}_9$, then G_{hp} and G_{hq} are orthogonal. The edge set of the graphs G_{h_i} can be obtained as follows, since $E(G_{f_i}) = \{(x, f_i(x))\}$ and $E(G_{g_i}) = \{(y, y^2 + iy)\}$, then $E(G_{h_i}) = \{(xy, f_i(x)(y^2 + iy))\}$ Now, we want to prove the isomorphism of the two graphs G_{hp}

Table 6: The edge set of the graphs G_{f_0}, G_{f_1}, and G_{f_2} for Example 5.

$E(G_{f_0}) = \{(x, f_0(x))\}$	$E(G_{f_1}) = \{(x, f_1(x))\}$	$E(G_{f_2}) = \{(x, f_2(x))\}$
(0, 0)	(0, 0)	(0, 0)
(1, 1)	(1, 2)	(1, 0)
(2, 1)	(2, 0)	(2, 2)

Table 7: The edge set of the graphs G_{g_0}, G_{g_1}, and G_{g_2} for Example 5.

$E(G_{g_0}) = \{((y, g_0(y)))\}$	$E(G_{g_1}) = \{(y, g_1(y))\}$	$E(G_{g_2}) = \{(y, g_2(y))\}$
(0, 0)	(0, 0)	(0, 0)
(1, 1)	(1, 2)	(1, 3)
(2, 4)	(2, 1)	(2, 3)
(3, 4)	(3, 2)	(3, 0)
(4, 1)	(4, 0)	(4, 4)

Table 8: The edge set of the graphs G_{h_0}, G_{h_1}, and G_{h_2} for Example 5.

$E(G_{h_0})$	$E(G_{h_1})$	$E(G_{h_2})$
(00, 00)	(00, 00)	(00, 00)
(01, 01)	(01, 02)	(01, 03)
(02, 04)	(02, 01)	(02, 03)
(03, 04)	(03, 02)	(03, 00)
(04, 01)	(04, 00)	(04, 04)
(10, 10)	(10, 20)	(10, 00)
(11, 11)	(11, 22)	(11, 03)
(12, 14)	(12, 21)	(12, 03)
(13, 14)	(13, 22)	(13, 00)
(14, 11)	(14, 20)	(14, 04)
(20, 20)	(20, 20)	(20, 20)
(21, 21)	(21, 02)	(21, 23)
(22, 24)	(22, 01)	(22, 23)
(23, 24)	(23, 02)	(23, 20)
(24, 21)	(24, 00)	(24, 24)

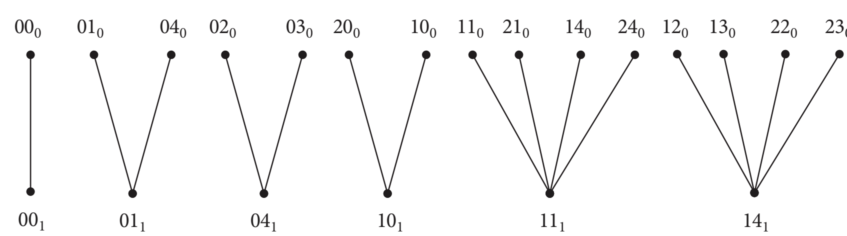

Figure 4: $G_{h_0} \cong K_2 \cup 3K_{1,2} \cup 2K_{1,4}$ corresponding to the 00 values in C_0.

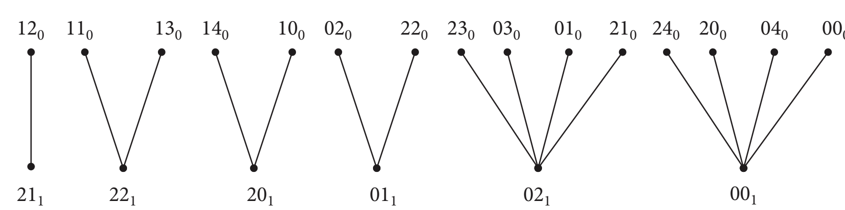

Figure 5: $G_{h_1} \cong K_2 \cup 3K_{1,2} \cup 2K_{1,4}$ corresponding to the 00 values in C_1.

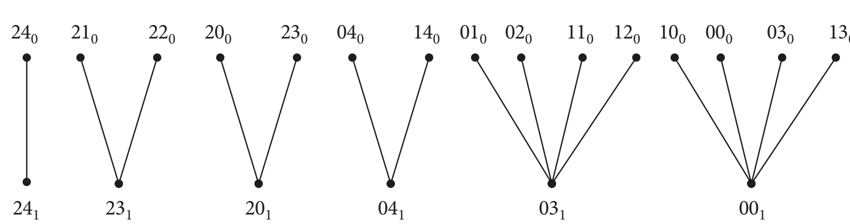

Figure 6: $G_{h_2} \cong K_2 \cup 3K_{1,2} \cup 2K_{1,4}$ corresponding to the 00 values in C_2.

and G_{hq}. For $p, q \in \mathbb{Z}_3, p \neq q$, we have $L_{f_p} = L_{f_q} = (0, \overbrace{1, 1, \ldots, 1}^{m-2}, 2)$ and $L_{g_p} = L_{g_q} = (0, 0, 0, 0, 0, 2, 2, 2, 3)$, then for $G_{hp} \cong G_{f_p} \otimes G_{g_p}$ and $G_{hq} \cong G_{f_q} \otimes G_{g_q}$, the degree vector of G_{hp} is equal to the degree vector of G_{hq}. This means that $L_{h_p} = L_{h_q} = (0, \overbrace{1, 1, \ldots, 1}^{m-2}, 2) \otimes (0, 0, 0, 0, 0, 2, 2, 2, 3)$ Hence, G_{hp} and G_{hq} are isomorphic. □

Example 5. Let $f_i(x) = x^2 + ix$ be f_i- half-starter graphs G_{f_i} of $K_{3,3}$ for all $i, x \in \mathbb{Z}_3$ and $g_i(y) = y^2 + iy$ be g_i- half-starter graphs G_{g_i} of $K_{5,5}$ for all $i, x \in \mathbb{Z}_5$. Since $\min\{3, 5\} = 3$, then we obtain 3 h_i- half-starter graphs G_{h_i} of $K_{15,15}$, which are represented by the functions $h_i(xy) = f_i(x) g_i(y)$ for all $i \in \mathbb{Z}_3$. The edge sets of G_{f_i}, G_{g_i}, and G_{h_i} are shown in Tables 6, 7, and 8, where $G_{f_i} \cong K_2 \cup K_{1,2}, G_{g_i} \cong K_2 \cup 2K_{1,2}$, and $G_{h_i} \cong K_2 \cup 3K_{1,2} \cup 2K_{1,4}$. Also, the three MOGS corresponding to the functions $f_i(x)$ are A_0, A_1, and A_2, the three MOGS corresponding to the functions $g_i(y)$ are B_0, B_1, and B_2, and the three MOGS corresponding to the functions $h_i(xy)$ are C_0, C_1, and C_2. See Figures 4, 5, and 6.

$$A_0 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 2 & 3 & 4 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 12 & 13 & 24 & 20 & 21 & 22 & 23 \\ 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 & 21 & 22 & 23 & 24 & 20 \\ 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 & 21 & 22 & 23 & 24 & 20 \\ 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 14 & 13 & 24 & 20 & 21 & 22 & 23 \\ 20 & 21 & 22 & 23 & 24 & 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 \\ 24 & 20 & 21 & 22 & 23 & 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 12 & 13 \\ 21 & 22 & 23 & 24 & 20 & 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 \\ 21 & 22 & 23 & 24 & 20 & 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 \\ 24 & 20 & 21 & 22 & 23 & 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 12 & 13 \\ 20 & 21 & 22 & 23 & 24 & 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 \\ 24 & 20 & 21 & 22 & 23 & 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 12 & 13 \\ 21 & 22 & 23 & 24 & 20 & 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 \\ 21 & 22 & 23 & 24 & 20 & 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 \\ 24 & 20 & 21 & 22 & 23 & 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 12 & 13 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 03 & 04 & 00 & 01 & 02 & 13 & 14 & 10 & 11 & 12 & 23 & 24 & 20 & 21 & 22 \\ 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 12 & 13 & 24 & 20 & 21 & 22 & 23 \\ 03 & 04 & 00 & 01 & 02 & 13 & 14 & 10 & 11 & 12 & 23 & 24 & 20 & 21 & 22 \\ 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 & 00 & 01 & 02 & 03 & 04 \\ 13 & 14 & 10 & 11 & 12 & 23 & 24 & 20 & 21 & 22 & 03 & 04 & 00 & 01 & 02 \\ 14 & 10 & 11 & 12 & 13 & 24 & 20 & 21 & 22 & 23 & 04 & 00 & 01 & 02 & 03 \\ 13 & 14 & 10 & 11 & 12 & 23 & 24 & 20 & 21 & 22 & 03 & 04 & 00 & 01 & 02 \\ 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 & 00 & 01 & 02 & 03 & 04 \\ 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 03 & 04 & 00 & 01 & 02 & 13 & 14 & 10 & 11 & 12 & 23 & 24 & 20 & 21 & 22 \\ 04 & 00 & 01 & 02 & 03 & 14 & 10 & 11 & 12 & 13 & 24 & 20 & 21 & 22 & 23 \\ 03 & 04 & 00 & 01 & 02 & 13 & 14 & 10 & 11 & 12 & 23 & 24 & 20 & 21 & 22 \\ 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \end{bmatrix}, \tag{9}$$

$$C_2 = \begin{bmatrix} 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 02 & 03 & 04 & 00 & 01 & 12 & 13 & 14 & 10 & 11 & 22 & 23 & 24 & 20 & 21 \\ 02 & 03 & 04 & 00 & 01 & 12 & 13 & 14 & 10 & 11 & 22 & 23 & 24 & 20 & 21 \\ 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 & 21 & 22 & 23 & 24 & 20 \\ 00 & 02 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 02 & 03 & 04 & 00 & 01 & 12 & 13 & 14 & 10 & 11 & 22 & 23 & 24 & 20 & 21 \\ 02 & 03 & 04 & 00 & 01 & 12 & 13 & 14 & 10 & 11 & 22 & 23 & 24 & 20 & 21 \\ 00 & 01 & 02 & 03 & 04 & 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 \\ 01 & 02 & 03 & 04 & 00 & 11 & 12 & 13 & 14 & 10 & 21 & 22 & 23 & 24 & 20 \\ 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 & 00 & 01 & 02 & 03 & 04 \\ 12 & 13 & 14 & 10 & 11 & 22 & 23 & 24 & 20 & 21 & 02 & 03 & 04 & 00 & 01 \\ 12 & 13 & 14 & 10 & 11 & 22 & 23 & 24 & 20 & 21 & 02 & 03 & 04 & 00 & 01 \\ 10 & 11 & 12 & 13 & 14 & 20 & 21 & 22 & 23 & 24 & 00 & 01 & 02 & 03 & 04 \\ 11 & 12 & 13 & 14 & 10 & 21 & 22 & 23 & 24 & 20 & 01 & 02 & 03 & 04 & 00 \end{bmatrix}.$$

In the following section, we present the general tensor product technique for constructing the MOGS. As stated above, the MOGS represent mutually orthogonal covers (MOCs) of complete bipartite graphs. A k mutually orthogonal covers (k MOCs) of the complete bipartite graph $K_{n,n}$ by F is a family $\mathscr{G}$ of isomorphic copies of a given subgraph F such that they cover every edge of $K_{n,n}k$ times and the intersection of any two of them contains at most one edge.

5. Tensor Products of MOCs

Let A and B be simple graphs, then the tensor product, $A \times B$, of A and B, is the graph with the vertex set $V(A) \times V(B)$ and the edge set $E(A \times B) = \{(a, b)(c, d)\colon ac \in E(A)\}$ and $b\ d \in E(B)$. If the simple graphs A and B are bipartite with bipartitions (E, F) and (Y, Z), respectively, then the induced subgraphs $(A \times B)[(E \times Y) \cup (F \times Z)]$ and $(A \times B)[(E \times Z) \cup (F \times Y)]$ are called the weak-tensor products of A and B. We denote the weak-tensor product $(A\times B)[(E\times Y)\cup(F\times Z)]$ by $A\circledast B$.

Proposition 3. *If there are k MOCs of $K_{m,m}$ by A and k MOCs of $K_{n,n}$ by B, then there are k MOCs of $K_{mn,mn}$ by $A\circledast B$.*

Proof 9. Let $\cap_{i=1}^{k}\mathscr{A}_i$, where $\mathscr{A}_i=\{A_1^i,A_2^i,\ldots,A_m^i\}$ be k MOCs of $K_{m,m}$ by A on $V(K_{m,m})=(E,F)$ where $E=\{e_1,\ldots,e_m\}$ and $F=\{f_1,\ldots,f_m\}$ is the bipartition of $K_{m,m}$, and let $\cap_{i=1}^{k}\mathscr{B}_i$, where $\mathscr{B}_i=\{B_1^i,B_2^i,\ldots,B_n^i\}$ be k MOCs of $K_{n,n}$ by B on $V(K_{n,n})=(Y,Z)$ where $Y=\{y_1,\ldots,y_n\}$ and $Z=\{z_1,\ldots,z_n\}$ is the bipartition of $K_{n,n}$. Let $W=V(K_{mn,mn})$ and the partite sets of $K_{mn,mn}$ be $\{(e_p,y_q)\colon 1\le p\le m, 1\le q\le n\}$ and $\{(f_p,z_q)\colon 1\le p\le m,1\le q\le n\}$. Consider the set $\mathscr{C}^i=\{(A_p^i\times B_q^i)[W]\colon 1\le p\le m,1\le q\le n\}, 1\le i\le k,$ of subgraphs of $K_{mn,mn}$. Clearly, $(A_p^i\times B_q^i)[W]\cong A\circledast B$, $1\le p\le m, 1\le q\le n, 1\le i\le k$, since $A_p^i\cong A$ and $B_q^i\cong B$. □

Claim 1. Every edge of $K_{mn,mn}$ occurs in exactly k graphs of $\cup_{i=1}^{k}\mathscr{C}^i$.

Consider an arbitrary edge $(e_s,y_t)(f_u,z_v)$ of $K_{mn,mn}$. Since $\cup_{i=1}^{k}A_i$ and $\cup_{i=1}^{k}B_i$ are k MOCs of $K_{m,m}$ by A and k MOCs of $K_{n,n}$ by B, respectively, the edges e_sf_u and y_tz_v are, respectively, in exactly k graphs of $\cup_{i=1}^{k}A_i$ and $\cup_{i=1}^{k}B_i$. Let the k graphs containing e_sf_u be $A_{t_1}^1,A_{t_2}^2,\ldots,A_{t_k}^k$ and that of y_tz_v be $B_{r_1}^1,B_{r_2}^2,\ldots,B_{r_k}^k$. Then, the k graphs containing the edge $(e_s,y_t)(f_u,z_v)$ are $(A_{t_1}^1\times B_{r_1}^1)[W], (A_{t_2}^2\times B_{r_2}^2)[W],\ldots,(A_{t_k}^k\times B_{r_k}^k)[W]$.

Claim 2. Let $i\in\{1,2,\ldots,k\}$. Any two graphs in $\mathscr{C}^i$ have no edges in common.

The two graphs $(A_{t_1}^i\times B_{r_1}^i)[W]$ and $(A_{t_2}^i\times B_{r_2}^i)[W]$ have no edges in common, because $E(A_{t_1}^i)\cap E(A_{t_2}^i)=\varnothing$ and $E(B_{r_1}^i)\cap E(B_{r_2}^i)=\varnothing$.

Claim 3. Any graph in $\mathscr{C}^x$ and any graph in $\mathscr{C}^y$ have exactly one edge in common, $1\le x<y\le k$.

The two graphs $(A_{t_1}^x\times B_{r_1}^x)[W]$ and $(A_{t_2}^y\times B_{r_2}^y)[W]$ have exactly one edge in common, since $|E(A_{t_1}^x)\cap E(A_{t_2}^y)|=1$ and $|E(B_{r_1}^x)\cap E(B_{r_2}^y)|=1$.

By Claims 1, 2, and 3, $\cup_{i=1}^{k}\mathscr{C}^i$ is k MOCs of $K_{mn,mn}$ by $A\circledast B$.

6. MOCs of Complete Bipartite Graphs Based on Tensor Product

All the following results are based on (i) the tensor product in Proposition 3 and (ii) the existence of MOCs for some classes of graphs. These graphs can be used as ingredients for the tensor product to obtain new MOCs. See [9] for the ingredients from (i) to (iv). Addition and subtraction are calculated modulo n for the following ingredients.

(i) Let $\cup_{i=1}^{n}\mathscr{A}_i$ where $\mathscr{A}_i=\{A_1^i,A_2^i,\ldots,A_n^i\}$ be n MOCs of $K_{n,n}$ by $(K_{1,1}\cup n-1/2K_{1,2})$, $E(A_{j+1}^i)=\{(\beta,j+(i-1)\beta+\beta^2)\}$, $j,\beta\in\mathbb{Z}_n$, n. be a prime >2.

(ii) Let $\cup_{i=1}^{n-1}\mathscr{A}_i$, where $\mathscr{A}_i=\{A_1^i,A_2^i,\ldots,A_n^i\}$ be $(n-1)$ MOCs of $K_{n,n}$ by $((n-2)K_{1,1}\cup K_{1,2})$, $E(A_{j+1}^i)=\{(0,0),(\beta,i\beta+j+1)\}$, $j\in\mathbb{Z}_n$, $\beta\in\{1,2,\ldots,n-1\}$, n be a prime >2.

(iii) If $n=9$, then the 3 MOCs of $K_{9,9}$ by $K_{1,3}\cup 3K_{1,2}$ are $\cup_{i=1}^{3}\mathscr{A}_i$, where $\mathscr{A}_i=A_1^i,A_2^i,\ldots,A_9^i$, $E(A_{j+1}^i)=\{(\alpha,\alpha^2+(i-1)\alpha+j)\}$, $j,\alpha\in\mathbb{Z}_9$.

(iv) If $n=7$, then the 4 MOCs of $K_{7,7}$ by $3K_{1,1}\cup 2K_{1,2}$ are $\cup_{i=1}^{4}\mathscr{A}_i$, where $\mathscr{A}_i=\{A_1^i,A_2^i,\ldots,A_7^i\}$, for $j\in\mathbb{Z}_7$,

$$\begin{aligned}
E\left(A_{j+1}^1\right)&=\{(0,0+j),(1,2+j),(2,4+j),(3,6+j),(4,1+j),(5,4+j),(6,6+j)\},\\
E\left(A_{j+1}^2\right)&=\{(0,0+j),(1,3+j),(2,6+j),(3,2+j),(4,5+j),(5,2+j),(6,5+j)\},\\
E\left(A_{j+1}^3\right)&=\{(0,0+j),(1,4+j),(2,1+j),(3,5+j),(4,2+j),(5,0+j),(6,4+j)\},\\
E\left(A_{j+1}^4\right)&=\{(0,0+j),(1,5+j),(2,3+j),(3,1+j),(4,6+j),(5,5+j),(6,3+j)\}.
\end{aligned}\tag{10}$$

(v) If $n=4$, then the 3 MOCs of $K_{4,4}$ by $2K_{1,2}$ are $\cup_{i=1}^{3}\mathscr{A}_i$ (follows from Theorem 1 in [4], by setting $\alpha=0,\beta=1,\gamma=2,$ and $\delta=3$), where $\mathscr{A}_i=\{A_1^i,A_2^i,A_3^i,A_4^i\}$,

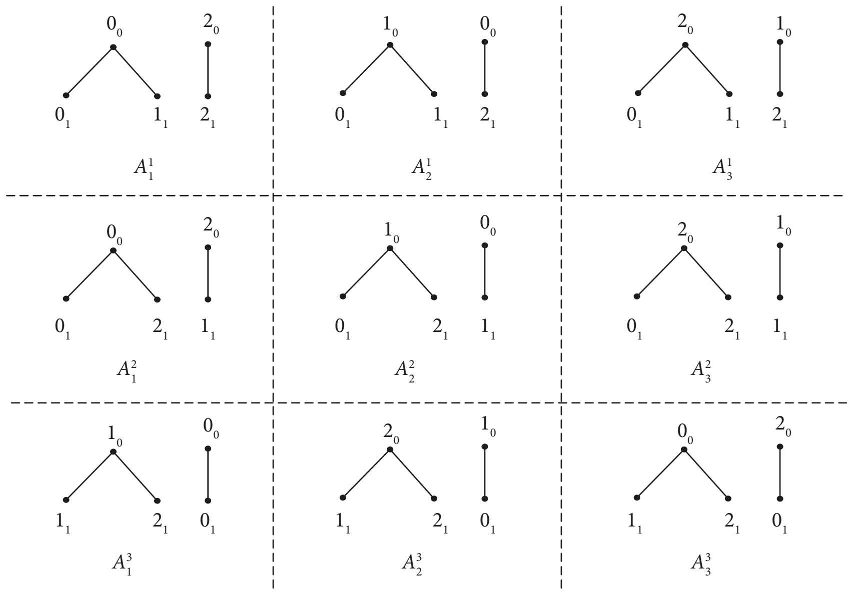

FIGURE 7: 3 mutually orthogonal covers of $K_{3,3}$ by $P_3 \cup K_2$.

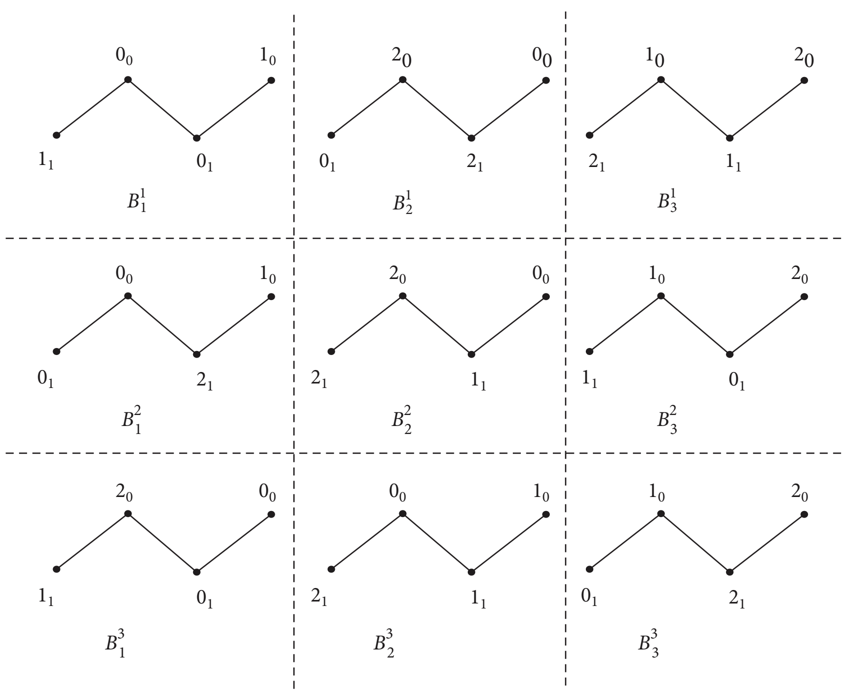

FIGURE 8: 3 mutually orthogonal covers of $K_{3,3}$ by P_4.

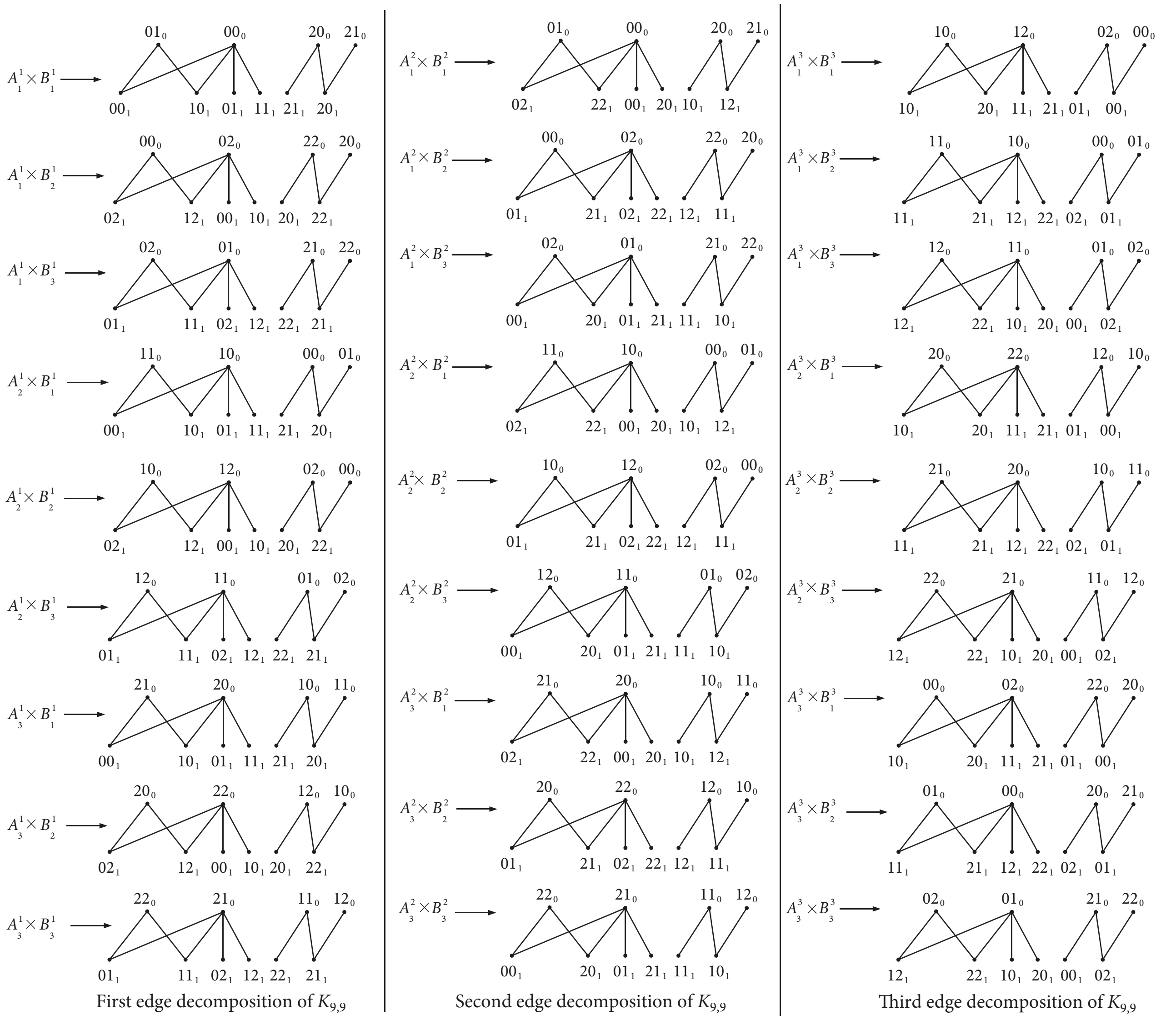

FIGURE 9: 3 mutually orthogonal covers of $K_{9,9}$ by $(P_3 \cup K_2) \circledast P_4$.

$$\begin{aligned}
E\left(A_1^1\right) &= \{(0,0), (1,2), (2,0), (3,2)\},\\
E\left(A_2^1\right) &= \{(0,3), (1,1), (2,3), (3,1)\},\\
E\left(A_3^1\right) &= \{(0,2), (1,0), (2,2), (3,0)\},\\
E\left(A_4^1\right) &= \{(0,1), (1,3), (2,1), (3,3)\},\\
E\left(A_3^2\right) &= \{(0,0), (1,1), (2,1), (3,0)\},\\
E\left(A_4^2\right) &= \{(0,1), (1,0), (2,0), (3,1)\},\\
E\left(A_1^3\right) &= \{(0,1), (1,1), (2,2), (3,2)\},\\
E\left(A_2^3\right) &= \{(0,3), (1,3), (2,1), (3,1)\},\\
E\left(A_3^3\right) &= \{(0,3), (1,3), (2,0), (3,0)\},\\
E\left(A_4^3\right) &= \{(0,0), (1,0), (2,3), (3,3)\}.
\end{aligned} \tag{11}$$

These known MOCs are the ingredients for the tensor product to obtain the following results. Note that we used some of the ingredients from the literature.

Theorem 9. *Let m, n be odd primes and $k = \min\{n, m-1\}$. Then, there are k MOCs of $K_{mn,nm}$ by $(m-2)K_{1,1} \cup (((n-1)(m+2)+2)/2)K_{1,2} \cup ((n-1)/2)K_{1,4}$.*

Proof 10. We have n MOCs of $K_{n,n}$ by $(K_{1,1} \cup n-1/2K_{1,2})$ (ingredient (i)) and $((m-1))$ MOCs of $K_{m,m}$ by $((m-2)K_{1,1} \cup K_{1,2})$ (ingredient (ii)). If $\min\{n, m-1\} = k$, then we construct k MOCs of $K_{mn,nm}$ by $(m-2)K_{1,1} \cup (((n-1)(m+2)+2)/2)K_{1,2} \cup ((n-1)/2)K_{1,4}$ (Proposition 3). □

Theorem 10. *Let n be odd prime. Then, there are 3 MOCs of $K_{9n,9n}$ by $K_{1,3} \cup 3K_{1,2} \cup (n-1)/2K_{1,6} \cup 3(n-1)/2K_{1,4}$.*

Proof 11. We have n MOCs of $K_{n,n}$ by $(K_{1,1} \cup n - 1/2K_{1,2})$ (ingredient (i)) and 3 MOCs of $K_{9,9}$ by $(K_{1,3} \cup 3K_{1,2})$ (ingredient (iii)). Then, we construct 3 MOCs of $K_{9n,9n}$ by $K_{1,3} \cup 3K_{1,2} \cup (n-1)/2K_{1,6} \cup 3(n-1)/2K_{1,4}$ (Proposition 3). □

Theorem 11. *Let n be odd prime. Then, there are* 3 *MOCs of* $K_{4n,4n}$ *by* $2K_{1,2} \cup (n-1)K_{1,4}$.

Proof 12. We have n MOCs of $K_{n,n}$ by $(K_{1,1} \cup (n-1)/2K_{1,2})$ (ingredient (i)) and 3 MOCs of $K_{4,4}$ by $2K_{1,2}$ (ingredient (v)). Then, we construct 3 MOCs of $K_{4n,4n}$ by $2K_{1,2} \cup (n-1)K_{1,4}$ (Proposition 3). □

Theorem 12. *Let n be odd prime. Then, there are* 3 *MOCs of* $K_{9n,9n}$ *by* $(n-2)K_{1,3} \cup 3(n-2)K_{1,2} \cup K_{1,6} \cup 3K_{1,4}$.

Proof 13. We have $(n-1)$ MOCs of $K_{n,n}$ by $((n-2)K_{1,1} \cup K_{1,2})$ (ingredient (ii)) and 3 MOCs of $K_{9,9}$ by $(K_{1,3} \cup 3K_{1,2})$ (ingredient (iii)). Then, we construct 3 MOCs of $K_{9n,9n}$ by $(n-2)K_{1,3} \cup 3(n-2)\ K_{1,2} \cup K_{1,6} \cup 3K_{1,4}$ (Proposition 3). □

Theorem 13. *Let n be odd prime. Then, there are* 4 *MOCs of* $K_{7n,7n}$ *by* $3K_{1,1} \cup (3n-1)/2K_{1,2} \cup (n-1)K_{1,4}$.

Proof 14. We have n MOCs of $K_{n,n}$by$(K_{1,1} \cup (n - 1/2K_{1,2}))$ (ingredient (i)) and 4 MOCs of $K_{7,7}$by $((3K_{1,1} \cup 2K_{1,2})$ (ingredient (iv)). Then, we construct 4 MOCs of $K_{7n,7n}$ by $3K_{1,1} \cup (3n+1)/2K_{1,2} \cup (n-1)K_{1,4}$ (Proposition 3). □

Example 6. We have 3 mutually orthogonal covers of $K_{3,3}$ by $P_3 \cup K_2$, shown in Figure 7, and 3 mutually orthogonal covers of $K_{3,3}$ by P_4, shown in Figure 8. Hence, we construct 3 mutually orthogonal covers of $K_{9,9}$ by $(P_3 \cup K_2) \circledast P_4$, shown in Figure 9.

7. Discussion

All the results in this paper are based on recursive construction techniques as stated above. In the literature, the Kronecker product of graph squares has been used to construct some results for MOGS. Herein, we defined three novel product techniques, which are the Cartesian product of half-starters' vectors, the half-starters' function product, and the graph tensor product. Some graphs can be represented by vectors, so the Cartesian product can be used with this class of graphs. Other graphs cannot be represented by vectors but can be represented by functions; hence, the function product can be used with this class of graphs. In addition, there is a third class of graphs that cannot be represented by vectors and functions; in this case, the tensor product of graphs is applied to construct the MOGS. All the results from the literature of MOGS with small orders can be used to get MOGS with higher orders by applying the new product techniques defined in this paper. The main results are Propositions 1, 2, and 3, which introduce the construction techniques based on the defined novel product techniques. All the remaining results in the paper are direct applications to these propositions. These results are MOGS for disjoint unions of stars such as $nK_{1,1} \cup (((((m-1))n))/2) K_{1,2}, (n-2)K_{1,1} \cup ((((m-1))(n-2)+2)/2)K_{1,2} \cup ((m-1) /2)K_{1,4}, K_{1,3} \cup 3\ K_{1,2} \cup (((m-1))/2)K_{1,6} \cup (3(((m-1))\ /2)) K_{1,4}, m(n-2)K_{1,1} \cup mK_{1,2}$, and $3K_{1,1} \cup 3n + 1/2K_{1,2} \cup\ (n-1)\ K_{1,4}$. All the constructed results in this paper can be used to generate new graph-orthogonal arrays, new graph-authentication codes, and new graph-transversal designs [3, 23]. They can also be used in the design of experiments [24].

8. Conclusion

In conclusion, we can say that the proposed novel product techniques are helping tools for constructing several new results concerned with the MOGS that have not been constructed before. It is clear that the proposed product techniques cannot be used to construct MOGS with prime order. In future work, we will try to find new recursive construction techniques for the MOGS.

Data Availability

The data used to support the findings of this study are available from the corresponding author on request.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] J. Gross and J. Yellen, *Graph Theory and its Applications*, CRC Press, Boca Raton, FL, USA, 1999.

[2] A. El-Mesady, Y. S. Hamed, and H. Shabana, "On the decomposition of circulant graphs using algorithmic approaches," *Alexandria Engineering Journal*, vol. 61, no. 10, pp. 8263–8275, 2022.

[3] A. El-Mesady, Y. S. Hamed, and K. M. Abualnaja, "A novel application on mutually orthogonal graph squares and graph-orthogonal arrays," *AIMS Mathematics*, vol. 7, no. 5, pp. 7349–7373, 2022.

[4] R. El-Shanawany and A. El-Mesady, "Mutually orthogonal graph squares for disjoint union of stars," *Ars Combinatoria*, vol. 149, pp. 83–91, 2020.

[5] R. C. Bose and S. S. Shrikhande, "On the construction of sets of mutually orthogonal Latin squares and the falsity of a conjecture of Euler," *Transactions of the American Mathematical Society*, vol. 95, no. 2, pp. 191–209, 1960.

[6] R. C. Bose, S. S. Shrikhande, and E. T. Parker, "Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture," *Canadian Journal of Mathematics*, vol. 12, pp. 189–203, 1960.

[7] R. M. Wilson, "Concerning the number of mutually orthogonal Latin squares," *Discrete Mathematics*, vol. 9, no. 2, pp. 181–198, 1974.

[8] R. M. Rizk-Allah, A. Slowik, A. Darwish, and A. E. Hassanien, "Orthogonal Latin squares-based firefly optimization algorithm for industrial quadratic assignment tasks," *Neural Computing & Applications*, vol. 33, no. 23, pp. 16675–16696, 2021.

[9] Y. Liu, "Packing Latin squares by BCL algebras," *Journal of applied mathematics & informatics*, vol. 40, pp. 133–139, 2022.

[10] D. Li, C. Shen, L. Wang, and H. Cao, “On a pair of orthogonal large sets of partitioned incomplete Latin squares,” *Journal of Combinatorial Designs*, vol. 30, no. 5, pp. 281–297, 2022.

[11] C. Shen and H. Cao, “On large set plus of disjoint incomplete Latin squares,” *Discrete Mathematics*, vol. 345, no. 9, Article ID 112970, 2022.

[12] W. J. Decapua and J. Lorch, “Sudoku pair Latin squares based on groups,” *Discrete Mathematics*, vol. 345, no. 6, Article ID 112852, 2022.

[13] S. Saurabh and M. K. Singh, “A note on the construction of Latin square type designs,” *Communications in Statistics-Theory and Methods*, vol. 51, no. 10, pp. 3434–3437, 2022.

[14] X. Wang, N. Jin, M. Cao, and Y. Wen, “Construction of authentication codes based on orthogonal array and Latin square,” *Computational and Applied Mathematics*, vol. 41, no. 4, p. 161, 2022.

[15] C. Colbourn and J. Dinitz, “Mutually orthogonal Latin squares: a brief survey of constructions,” *Journal of Statistical Planning and Inference*, vol. 95, no. 1-2, pp. 9–48, 2001.

[16] R. El-Shanawany, *Orthogonal double covers of complete bipartite graphs*, Ph.D. Dissertation, University of Rostock, Rostock, Germany, 2001.

[17] R. Sampathkumar and S. Srinivasan, “Mutually orthogonal graph squares,” *Journal of Combinatorial Designs*, vol. 17, no. 5, pp. 369–373, 2009.

[18] R. El-Shanawany, “On mutually orthogonal graph-path squares,” *Open Journal of Discrete Mathematics*, vol. 06, no. 01, pp. 7–12, 2016.

[19] R. El-Shanawany, “On mutually orthogonal disjoint copies of graph squares,” *Note di Matematica*, vol. 36, pp. 89–98, 2016.

[20] R. El-Shanawany, A. El-Mesady, and S. M. Shaaban, “Mutually orthogonal graph squares for disjoint union of paths,” *Applied Mathematical Sciences*, vol. 12, no. 7, pp. 303–310, 2018.

[21] R. El-Shanawany and A. El-Mesady, “ON mutually orthogonal certain graph squares,” *Online Journal of Analytic Combinatorics*, vol. 14, 2020.

[22] A. El-Mesady and S. M. Shaaban, “Generalization of MacNeish’s Kronecker product theorem of mutually orthogonal Latin squares,” *AKCE International Journal of Graphs and Combinatorics*, vol. 18, no. 2, pp. 117–122, 2021.

[23] A. El-Mesady, O. Bazighifan, and H. M. Shabana, “On graph-transversal designs and graph authentication codes based on mutually orthogonal graph squares,” *Journal of Mathematics*, vol. 2022, Article ID 8992934, 10 pages, 2022.

[24] M. Higazy, A. El-Mesady, and M. S. Mohamed, “On graph-orthogonal arrays by mutually orthogonal graph squares,” *Symmetry*, vol. 12, no. 11, p. 1895, 2020.

[25] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Springer, New York, NY, USA, 2012.