Hindawi
Journal of Mathematics
Volume 2022, Article ID 9773089, 6 pages
https://doi.org/10.1155/2022/9773089

Research Article

Computing Fault-Tolerant Metric Dimension of Connected Graphs

Uzma Ahmad,[1] **Sara Ahmed**,[1] **Muhammad Javaid**,[2] **and Md Nur Alam**[3]

[1]*Department of Mathematics, University of the Punjab, Lahore, Pakistan*
[2]*Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan*
[3]*Department of Mathematics, Pabna University of Science and Technology, Pabna 6600, Bangladesh*

Correspondence should be addressed to Md Nur Alam; nuralam23@pust.ac.bd

Received 15 March 2022; Accepted 26 April 2022; Published 28 May 2022

Academic Editor: Kenan Yildirim

For a connected graph, the concept of metric dimension contributes an important role in computer networking and in the formation of chemical structures. Among the various types of the metric dimensions, the fault-tolerant metric dimension has attained much more attention by the researchers in the last decade. In this study, the mixed fault-tolerant dimension of rooted product of a graph with path graph with reference to a pendant vertex of path graph is determined. In general, the necessary and sufficient conditions for graphs of order at least 3 having mixed fault-tolerant generators are established. Moreover, the mixed fault-tolerant metric generator is determined for graphs having shortest cycle length at least 4.

1. Introduction

The concept of metric dimension is applicable in all those networks where there is a need of localization of particular nodes. It is significantly used in different fields of science such as telecommunication, road networking, chemistry, and image processing to find winning combinations for different games. Slater in 1975 [1] introduced the locating set of graphs, whereas, in 1976, Harary and Melter [1] defined the term resolving set for graphs. Later on, both the terms were emerged and named as metric-based basis or generator. For a graph, $J, W\subseteq V(J)$ is termed as metric generator if for $a, b \in V(J) (a \neq b)$, there exists a vertex $w \in W$, such that $d(a, w) \neq d(b, w)$. Then, the vertex $w \in W$ is said to distinguish (resolve) vertices a and b. If $W = \{w_1, \ldots, w_s\}$, then the distance coordinate vector of $a \in V(J)$ is s-tuple $r(a|W) = (d(a, w_1), \ldots, d(a, w_s))$. This metric generator is extensively studied in literature.

The behavior of metric dimension of graphs relative to different graph products was investigated by different authors, like Cartesian product by Caceres et al. [2], the join product as well as Cartesian product by Hernando et al. [3], and join of different combinations of complete, path, cycle graphs by Sunitha et al. [4]. Even though metric generator was first proposed for problem of robot navigation, now this metric generator along with its various variants can have interesting and significant connections to other fields as well. For example, the determination of local variant named as local metric dimension may be associated to limited function of robot sensors. The mixed metric generator is one of the variants of the metric generator. This mixed version of metric generators was presented in 2017 by Kelenc et al. [5]. A subset $M\subseteq V(J)$ is termed as the mixed metric generator of graph J if coordinate distance vectors relative to M of any two distinct elements of $V(J) \cup E(J)$ are not same. The smallest set which is the mixed generator of graph J is termed as mixed basis, and its cardinality is termed as mixed dimension $(\dim(J))$. Kelenc et al. [5] showed that a necessary and sufficient condition for a graph J of order r to have mixed dimension two is $J \cong P_r$. They also proved that $\dim(C_n) = 3$ and $\dim(P_{r_2} \square P_{r_2}) = 3$ for $r_1 \geq r_2 \geq 2$.

Hernando et al. [6] in 2008 presented the idea of the fault-tolerant metric (FTM) generator. To illustrate this, consider a network where the metric basis represents the censors [7]. In this situation, if some censor is forced to not operate properly, then other censors will not be sufficient to localize all stations or places uniquely and to deliver proper information to confront the problem. This type of frequently occurring situation in networks was resolved by Hernando et al. [6] with the help of applying fault-tolerance in the metric generator. To consider the fault-tolerance in the metric generator, the new generator will be able to transfer the information correctly even when a censor is disabled for any reason. It can be said that this FTM generator is applicable to all those networks where metric dimension has its significance like in optimal flow control problems of interconnecting networks.

The FTM generator and FTM dimension for a tree was determined in [6]. Javaid et al. [8] also explored the FTM dimension and FTM partition dimension of different graphs. They exhibited that for a graph of order $m, m \leq PD^{n-2}\ m$, where P is the fault-tolerant partition dimension and D is the diameter of the graph. The FTM dimension of infinite families of convex polytopes is proved to be constant by Raza et al. [9]. So far, a lot of research works on metric dimension and its different variants have been carried out by different authors, whereas comparatively less investigations have been made in the exploration of mixed generators. The motivation of this research work is to fill this literature gap and to apply the concept of fault-tolerance in mixed generators.

2. Mixed Dimension of Rooted Product

The rooted product $Q_1 \triangleright_o Q_2$ can be constructed by taking connected graphs Q_1 and Q_2 and a root vertex $o \in Q_2$ such that $V(Q_1 \text{ o } Q_2) = \{(a_1, b_1) | a_1 \in V(Q_1), b_1 \in V(Q_2)\}$ and $E(Q_1 \text{ o } Q_2) = \{(a_1, b_1)(a_2, b_2) | a_1 = a_2\ \&\ b_1 \sim b_2 \text{ or } b_1 = b_2 = \text{o}\ \&\ a_1 \sim a_2\}$.

Corollary 1 (see [10]). *A pendant vertex always belongs to every mixed metric generator of the graph.*

Theorem 1. *Consider a graph Q having order $n \geq 2$. If o is the pendant vertex of P_m, then $\dim_m (Q' = Q \triangleright_o P_m) = n$.*

Proof. Let $V(Q) = \{p_1, p_2, \ldots, p_n\}$ and $S = \{w_1, w_2, \ldots, w_k\}$ be the mixed metric basis of Q. Also, suppose that $V(P_m) = \{q_0, q_1, \ldots, q_{m-1}\}$, such that q_0 and q_{m-1} are the pendant vertices and $E(P_m) = \{q_i q_{i-1}\colon 1 \leq i \leq m-1\}$. To construct required $Q' = Q \triangleright_o P_m$, take the vertex $o = q_0$. The vertex set and edge set of $Q' = Q \triangleright_o P_m$ are given as $V(Q') = \{(p_i, q_j\colon 1 \leq i \leq n, 0 \leq j \leq m-1)\}$ and $E(Q') = \{(p_i, q_{j-1})(p_i, q_j), (p_i, o)(p_j, o)\colon p_i p_j \in E(Q)\}$. Clearly, the set of pendant vertices of Q' is $M = \{(p_i, q_{m-1})\colon 1 \leq i \leq n\}$. If S^* is the mixed metric generator of Q', then from Corollary 1, $M = \{(p_i, q_{m-1})\colon 1 \leq i \leq n\} \subseteq S^*$. Hence,

$$\dim_m (Q' = Q \triangleright_o P_m) \geq n. \tag{1}$$

Let $X = \{(p_i, q_j), (p_i, q_{j-1})(p_i, q_j)\colon 1 \leq i \leq n, 1 \leq j \leq m-1\} \subseteq Q'$. Then, $Q = Q' - X$. Now, we prove that M is a mixed metric generator. For this, let $g, h \in Q'$. Then, there arise three cases. □

Case 1: Let $g, h \in X$. If $g = (p_i, q_k)$ (respectively $(p_i, q_{k-1})(p_i, q_k)$) and $h = (p_j, q_k)$ (respectively $(p_j, q_{l-1})(p_j, q_l)$), such that $i \neq j$; then, $d(g, (p_i, q_{m-1})) = t < m, d(h, (p_i, q_{m-1})) \geq m$. If $g = (p_i, q_k)$ and $h = (p_j, q_{l-1})(p_j, q_l)(i \neq j)$, then $d(g, (p_i, q_{m-1})) < m, d(h, (p_i, q_{m-1})) \geq m$. If $g = (p_I, q_k)$ and $h = (p_i, q_{l-1})(p_i, q_l)$ $(k \neq l)$, then clearly, $d(g, (p_i, q_{m-1})) \neq d(h, (p_i, q_{m-1}))$. But if $g = (p_i, q_k)$ and $h = (p_i, q_{k-1})(p_i, q_k)$, then $d(g, (p_i, q_{m-1})) = d(h, (p_i, q_{m-1}))$. Now, as $n \geq 2$, there exist $j \neq i$, such that $(p_j, q_{m-1}) \in X$. But as (p_i, q_{k-1}) is nearer to (p_j, q_{m-1}) as compared to (p_i, q_k), so $d((p_i, q_{k-1})(p_i, q_k), (p_j, q_{m-1})) = d((p_i, q_{k-1}), (p_j, q_{m-1}))$. This shows that

$$\begin{aligned} d(g, (p_j, q_{m-1})) &= d((p_i, q_{k-1}), (p_j, q_{m-1})) + 1, \\ &= d((p_i, q_{k-1})(p_i, q_k), (p_k, q_{m-1})) + 1 \end{aligned} \tag{2}$$

This implies that $d(g, (p_j, q_{m-1})) \neq d(h, (p_j, q_{m-1}))$. Now, if $g = (p_i, q_{k-1})$ and $h = (p_i, q_{k-1})(p_i, q_k)$, then as (p_i, q_k) is nearer to (p_i, q_{m-1}) as compared to (p_i, q_{k-1}) (p_i, q_{k-1}), so $d((p_i, q_{k-1})(p_i, q_k), (p_i, q_{m-1})) = d((p_i, q_k), (p_i, q_{m-1}))$. This shows that

$$\begin{aligned} d(g, (p_i, q_{m-1})) &= d((p_i, q_k), (p_i, q_{m-1})) + 1 \\ &= d((p_i, q_{k-1})(p_i, q_k), (p_i, q_{m-1})) + 1 \end{aligned} \tag{3}$$

This further shows that $d(g, (p_i, q_{m-1})) \neq d(h, (p_i, q_{m-1}))$. Thus, every pair $g, h \in X$ is resolved by M.

Case 2: Let $g, h \in Q$. Since S is the mixed resolving set of Q, there exists $s = (w_i, o) \in S$, such that

$$d(g, s) \neq (h, s). \tag{4}$$

Consider the pendant vertex $(w_i, q_{m-1}) \in S^*$. Then, any path from g (respectively h) to (w_i, q_{m-1}) must contain $s = (w_i, o)$ and

$$\begin{aligned} d(g, (w_i, q_{m-1})) &= d(g, (w_i, o)) + m-1, \\ d(h, (w_i, q_{m-1})) &= d(h, (w_i, o)) + m-1. \end{aligned} \tag{5}$$

As $d(g, s) \neq (h, s)$, so $d(h, (w_i, q_{m-1})) \neq d(g, (w_i, q_{m-1}))$. Thus, all elements of Q are resolved by M.

Case 3: Last, let $g \in Q$, $h \in X$. Then, $h = (p_i, q_j)$ or $h = (p_i, q_{j-1})(p_i, q_j)$. In any case, $d(g, (p_i, q_{m-1})) \geq m-1$ but $d(h, (p_i, q_{m-1})) \leq m-2$, which shows that the pair g, h in this case are also resolved by M. Hence, M is a mixed metric generator and $\dim_m (Q') = n$.

Now, we note that by considering o as the pendant vertex of P_2, the corona graph $Q' = QoK_1$ is actually the rooted product graph of Q by P2, i.e., $QoK_1 = Q \triangleright_O P_2$.

As a consequence, the following corollary can be stated.

Corollary 2. *If Q' is a corona graph with t vertices, then* $\dim_m (Q') = t/2$.

3. Mixed Fault-Tolerant Generators of Graphs

A mixed metric generator M_f of a graph J is termed as the mixed fault-tolerant (MFTM) generator, if for

$u \in M_f, M_f \setminus \{u\}$, u is also a mixed generator. The smallest MFTM generator is termed as MFTM basis, and its cardinality is MFTM dimension $(\dim_{m_f}(J))$.

Remark 1. From definition, it is obvious that the MFTM dimension is always greater than or equal to the mixed metric dimension, i.e., for any graph J,

$$\dim_{m_f}(J) \geq \dim_m(J) + 1. \tag{6}$$

Example 1. Take $P_3 \square P_3$ with labeling as shown in Figure 1 and $M = \{v_1, v_3, v_7, v_9\} \subseteq V(P_3 \square P_3)$. Then, the distance coordinate vectors of all vertices of $P_3 \square P_3$ relative to M are computed as follows:

$$\begin{aligned}
r(v_1|M) &= (0,2,2,4),\\
r(v_2|M) &= (1,1,3,3),\\
r(v_3|M) &= (2,0,4,2),\\
r(v_4|M) &= (1,3,1,3),\\
r(v_5|M) &= (2,2,2,2),\\
r(v_6|M) &= (3,1,3,1),\\
r(v_7|M) &= (2,4,0,2),\\
r(v_8|M) &= (3,3,1,1),\\
r(v_9|M) &= (4,2,2,0).
\end{aligned} \tag{7}$$

Similarly, the distance coordinate vectors of all of its edges relative to M are computed as follows:

$$\begin{aligned}
r(e_1|M) &= (0,\ 1,2,3),\\
r(e_2|M) &= (1,0,3,2),\\
r(e_3|M) &= (1,2,1,2),\\
r(e_4|M) &= (2,1,2,1),\\
r(e_5|M) &= (2,3,0,1),\\
r(e_6|M) &= (3,2,1,0),\\
r(e_7|M) &= (0,2,1,3),\\
r(e_8|M) &= (1,3,0,2),\\
r(e_9|M) &= (1,1,2,2),\\
r(e_{10}|M) &= (2,2,1,1),\\
r(e_{11}|M) &= (2,0,3,1),\\
r(e_{12}|M) &= (3,1,2,0).
\end{aligned} \tag{8}$$

We can see that all distance coordinate vectors are distinguished by at least two points. Thus, M is a MFTM generator, and $\dim_{m_f}(P_3 \square P_3) \leq 4$. But from Proposition 4.4 of [10], $\dim_m(P_3 \square P_3) = 3$. This along with [7] implies that $\dim_{m_f}(P_3 \square P_3) \geq 4$. Thus, we have $\dim_{m_f}(P_3 \square P_3) = 4$.

To recall, for the vertex z in a graph J, the open neighbourhood $N(z)$ is the collection of all vertices adjacent to z and closed neighbourhood is $N[z] = N(z) \cup \{z\}$. For a vertex w, a vertex $q \in N(w)$ is referred to as maximal neighbourhood of w if $N[w] \subseteq N[q]$. A vertex v is referred to as the dominant vertex if it is adjacent to all other vertices of graph.

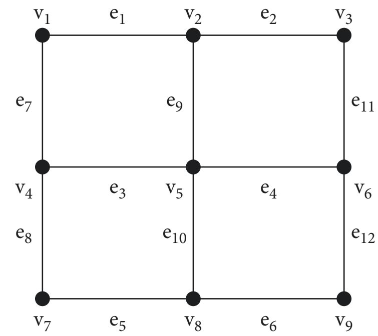

Figure 1: Graph $P_3 \square P_3$.

Lemma 1 (see [10]). *Consider $u \in V(J)$ and $M = V(J) \setminus \{u\}$. Suppose for each $w \in N(u)$, there is $y \in S$, such that $d(vw, x) \neq d(w, y)$. Then, M is a mixed metric generator for J.*

Lemma 2 (See [6]). *For a graph J, let S be a metric generator and $T(y) = \{w \in V(J)\colon N(y) \subseteq N(w)\}$ for $\in \in S$. Then, $S_f = \cup_{y \in S}(N[y] \cup T(y))$ is a FTM generator.*

Theorem 2. *A connected graph of order at least 3 contains a mixed fault-tolerant (MFTM) generator if and only every vertex is neither a pendant vertex nor possesses a maximal neighbourhood.*

Proof. Let $M = \{q_1, q_2, \ldots, q_s\}$ be a MFTM generator of connected graph J. Suppose p_2 is a pendant vertex of J. Then, there is a unique pendant edge with one end vertex as p_1, say $p = p_1 p_2$. Then, $p_2 \in M$ by Corollary 1, i.e., $p_2 = q_j$ for some $j = 1, \ldots, s$. Now, assume that the distance coordinate vector of p_1 relative to M is given as

$$r(p_1|M) = \left(d_1, d_2, \ldots, d_{j-1}, 1, d_{j+1}, \ldots, d_s\right). \tag{9}$$

As p_1 is adjacent to p_2 and $p_2 = q_j$, therefore j^{th} coordinate of $r(p_1|M)$ is equal to 1. Since p_2 is a pendant vertex, every path from any vertex of J to p_2 must contain vertex p_1. This implies that p_1 is nearer to any vertex of J than p_2. This further implies that for $p_2 \neq y \in V(J) \cup E(J)$, we have $d(y, p) = d(y, p_1)$ but $d(p_2, p)$. Thus, the distance coordinate vector of edge p relative to M is given as

$$r(p|M) = \left(d_1, d_2, \ldots, d_{j-1}, 0, d_{j+1}, \ldots, d_s\right). \tag{10}$$

The j^{th} coordinate is 0 because $p = p_1 p_2$ and $p_2 = q_j$. It is cleared from [3, 11] that the coordinate vectors of p_1 and p differ exactly by one coordinate. This further implies that $r(p_1|M \setminus \{p_2\}) = r(p|M \setminus \{p_2\})$. This shows that M cannot be a MFTM generator, which leads to a contradiction. Next, we may assume that the graph J has no pendant vertex, but there exists a vertex t having maximal neighbourhood c, i.e., $N[t] \subseteq N[c]$. This implies that $d(c, t) = 1$ but $d(tc, t) = 0$. The elements tc and c can only be distinguished by the vertex t because if there is a vertex $m \in M$, such that $m \neq t$ and

$d(c, m) \neq d(m, tc)$, then as $d(m, tc) = \min(d(t, m), d(c, m))$, so

$$d(t, m) < d(c, m). \tag{11}$$

This further implies that any shortest path from m to t must have a vertex from $N(t)$ different from c. As $\deg(t) \geq 2$, there is a vertex $t_1 \in N(t) \setminus \{c\}$ on the shortest path from m to t. Since $N[t] \subseteq N[c]$, $c \sim t_1$. Therefore,

$$d(m, c) \leq d(m, t_1) + d(t_1, c) = 1 + d(m, t_1) = d(m, t). \tag{12}$$

This contradicts (6). Thus, tc and c can only be distinguished by t. Now, if $\notin M$, then M cannot be a mixed generator. On the other hand, if $t \in M$, then $M \setminus \{t\}$ cannot not be the mixed generator. In any case, M is not a MFTM generator which again leads to a contradiction. Conversely, suppose that any vertex of J is the nonpendant vertex and does not have any maximal neighbourhood, i.e., for $a \in V(J)$, $\deg(a) \geq 2$ and $N[a] \nsubseteq N[y]$ for any $y \in N(a)$. For any $a \in V(J)$, we exhibit that the set $V(J) \backslash \{a\}$ is a mixed generator. Let $b \in N(a)$. As $\deg(a) \geq 2$ and $N[a] \nsubseteq N[b]$ (by assumption), there is a vertex a_1 (different from a and b), such that $a_1 \sim a$ but $a_1 \nsim b$. Then, $d(a_1, b) > 1$ but $d(a_1, a) = 1$. This shows that $d(a_1, ab) = d(a_1, a) \neq d(a_1, b)$. Using Lemma 1, it is easy to see that $V(J) \backslash \{a\}$ is a mixed generator for any $a \in V(J)$. Thus, the vertex set $V(J)$ itself is the MFTM generator. □

Corollary 3. *A tree does not possess the mixed fault-tolerant generator.*

Proof. Since a tree must have minimum two pendant vertices, the result follows using Theorem 2. □

Corollary 4. *A connected graph containing a dominant vertex d have no mixed fault-tolerant generator.*

Proof. Let d be a dominant vertex of connected graph J. Then, d serves as maximal neighbourhood of every vertex of $V(J) \backslash \{d\}$. The result follows by using Theorem 2. □

Theorem 3. *Suppose J is a graph having girth at least four and contains the MFTM generator. If M is a mixed basis of graph J, then* $M_f = \cup_{\{m \in M\}} (N[m] \cup (\cap_{\{m_i \in N(m)\}} N(m_i)))$ *is a MFTM generator for J.*

Proof. Suppose $a \in V(J)$. If $a \notin M$, then as $M \subseteq M_f \backslash \{a\}$, so $M_f \backslash \{a\}$ is a mixed generator. Thus, we may assume that $a \in M$. Now, we show that $M_f \backslash \{a\}$ still is a mixed generator for J. For this, take two elements $g, h \in V(J) \cup E(J)$. Now, there arise the following cases:

Case (1): suppose $g, h \in V(J)$, such that $g \neq h$. As g and h both are vertices, so using Lemma 1, these can be distinguished by a vertex of $S_f \backslash \{a\}$, where $S_f = \cup_{\{m \in M\}} (N[m] \cup T(m))$ and $T(m) := \{m_1 \in V(J): N(m) \subseteq N(m_1)\}$. To show g and h are distinguished by some element of $M_f \backslash \{a\}$, it is enough to exhibit that $S_f \subseteq M_f$. To show this, take $a' \in T(m) \subseteq S_f$. Then, clearly, $N(m) \subseteq N(a')$, that is for every $b' \in N(m), a' \sim b'$. This implies that $a' \in N(b')$ for every $b' \sim m$ or we can say that $a' \in \cap_{\{b' \in N(m)\}} N(b') \subseteq M_f$.

Case (2): now suppose g and h are the two distinct edges, such that $g = g_1 g_2$ and $h = h_1 h_2$. Clearly, one of g_i must be different from one of h_j. Suppose there does not exist any vertex in $M_f \backslash \{a\}$ that distinguished these edges. Then, as M is mixed metric and $M \subseteq M_f$, so the edges g and h must be distinguished by the vertex a, that is, $d(a, g) \neq d(a, h)$ $d(a, g)$. Assume without any loss of generality (WLG) that

$$d(a, g) < d(a, h). \tag{13}$$

Now, there are two possibilities, either all vertices on these edges are distinct or there is some common vertex between them.

Case 2(a): suppose g_1, g_2, h_1, and h_2 are distinct, i.e., edges are nonadjacent. Using (13), we have

$$\begin{gathered} d(a, g) \leq d(a, h) - 1, \\ \min\{d(a, g_1), d(a, g_2)\} \leq \min\{d(a, h_1), d(a, h_2)\} - 1. \end{gathered} \tag{14}$$

Assume WLG that g_1 and h_2 are nearer to the vertex a relative to g_2 and h_1, respectively. Then, from (14), we have

$$d(a, g_1) \leq d(a, h_2) - 1. \tag{15}$$

Now assume that $g_1 \neq a$ and $a' \in N(a)$ lies on the minimum path between a and g_1. Then, clearly, $a' \in M_f \backslash \{a\} \in F_m \setminus \{a\}$ being member of neighbourhood of a, we can write

$$d(a, g_1) = 1 + d(a', g_1) \tag{16}$$

By using (15) and (16), we have

$$d(a', g_1) + 1 \leq d(a, h_2) - 1 \tag{17}$$

Now, by taking the path $a - a' - \cdots - h_2$, we have $d(a', h_2) + 1 \geq d(a, h_2)$. Then, using (17), we have

$$\begin{aligned} d(a', g_1) + 1 \leq d(a, h_2) - 1 \leq d(a', h_2)&, \\ d(a', g_1) \leq d(a', h_2)& \\ d(a', g_1) < d(a', h_2)&. \end{aligned} \tag{18}$$

Since $a' \in N(a)$ and lies on the minimum path between g_1 and a and $d(a, g_1) < d(a, g_2)$, therefore $d(a', g_1) < d(a, g_1) < d(a, g_2)$. Now, by taking the path $a - a' - \cdots - g_2$, we have $d(a', g_1) < d(a, g_2) \leq d(a', g_2) + 1$. This implies that $d(a', g_1) \leq d(a', g_2)$. Thus, we can write

$$d(a', g) = \min(d(a', g_1), d(a', g_2)) = d(a', g_1). \tag{19}$$

Also, we have

$$d(a', h) = \min(d(a', h_2), d(a', h_1)). \quad (20)$$

We claim that $d(a', g_1) < d(a', h_1)$, for otherwise, $d(a, g_1) = 1 + d(a', g_1) \geq 1 + d(h_1, a') \geq d(a, h_1) \geq d(a, h_2)$ which contradicts [10]. Hence, $d(a', g_1) < d(a', h_2)$, and using [1, 4, 13], we can write

$$d(a', g) = d(a', g_1) < \min(d(a', h_2), d(a', h_1)) = d(a', h), \quad (21)$$

Which implies that edges g and h are distinguished by $a' \in M_f \backslash \{a\}$.

Now, if a coincides with g_1, then $g_2 \in M_f \backslash \{a\}$, such that $d(g, g_2) = 0$ and $d(h, g_2) \geq 1$. Hence, g and h are distinguished by g_2.

Case 2(b): now suppose that g and h are adjacent and g_2 is their common vertex, i.e., $g = g_1 g_2$ and $h = g_2 h_1$. Furthermore, suppose that a and g_1 are distinct. This implies that $a \neq g_1, g_2, h_1$ and $d(g_1, a) < d(g_2, a)$. Then, (13) becomes

$$d(a, g_1) < \min\{d(a, g_2), d(a, h_1)\}. \quad (22)$$

If $a' \in N(a)$ lies on the minimum path between a and g_1, then we have

$$d(a', g) = d(g_1, a') < d(g_2, a'). \quad (23)$$

Also, we have

$$d(a', h) = \min\{d(g_2, a'), d(h_1, a')\}. \quad (24)$$

We claim that $d(a', g_1) < d(a', h_1)$; for otherwise, we have

$$d(a, g_1) = 1 + d(a, g_2) \geq 1 + d(a', h_1) \geq d(a, h_1). \quad (25)$$

This contradicts (22). Thus, $d(a', g_1) < d(a', h_1)$, and using (23) and (24), we can write

$$d(a', g) = d(g_1, a') < \min\{d(g_2, a'), d(h_1, a')\} = d(a', h). \quad (26)$$

Thus, the edges are distinguished by $a' \in M_f \backslash \{a\}$. Now, suppose a lies on one of the edges g and h. Using (13), it is easy to see that only possibility is that the vertex a coincides with vertex g_1. If the vertex $h_1 \in M_f \backslash \{a\}$, then g and h are distinguished by h_1. Thus, we may suppose that $h_1 \notin M_f = N[m] \cup (\cap_{m_i \in N(m)} N(m_i))$. This further implies that there exists a vertex $a' \in N(a)$, such that $h_1 \nsim a'$. If $g_2 \sim a'$, then we have a triangle $g_1 - a' - g_2 - g_1$, which is not possible as the graph J has girth at least four. Thus, $g_2 \nsim a'$ which implies that $d(a', h) \geq 2$, but $d(a', g) = 1$. Hence, this case is completed.

Case (3): finally, suppose that g is a vertex and $h = h_1 h_2$ is an edge of J, such that there does not exist any vertex in $M_f \backslash \{a\}$ that distinguishes them. Then, g and h can only be distinguished by a, i.e., $d(a, g) \neq d(a, h)$. Now, the vertex g may lie on h or not.

Case 3a: suppose $g \neq h_1, h_2$ and a is distinct from g, h_1, and h_2. Furthermore, suppose

$$d(g, a) < d(h, a). \quad (27)$$

If the vertex a is nearer to h_1 as compared to h_2, then from (27), we have

$$d(g, a) \leq d(h_1, a) - 1. \quad (28)$$

Consider $a' \in N(a)$ on the minimum path between a and g. Then, $d(g, a') = d(g, a) - 1$, and from (28), we have

$$d(g, a') + 1 \leq d(h_1, a) - 1. \quad (29)$$

From the path $a - a' - \cdots - h_1$, we can see that

$$d(a, h_1) \leq d(a', h_1) + 1. \quad (30)$$

Now, using (29) and (30),

$$\begin{gathered} d(g, a') + 1 \leq d(a', h_1), \\ d(g, a') < d(a', h_1). \end{gathered} \quad (31)$$

$$d(a', h) = \min(d(a', h_1), d(a', h_2)). \quad (32)$$

We claim that $d(a', g) < d(a', h_2)$; for otherwise, $d(a, g) = 1 + d(a', g) \geq 1 + d(a', h_2) \geq d(a, h_2)$, that is, $d(a, g) \geq d(a, h_2) \geq d(a, h)$, which contradicts (27). Thus,

$$d(a', g) < d(a', h_2). \quad (33)$$

Using (31), (32), and (33), we can write

$$d(a', g) < \min(d(a', g_1), d(a', h_2)) = d(a', h). \quad (34)$$

This implies that the vertex g and edge h are distinguished by $a' \in M_f \backslash \{a\}$. The case $d(a, g) > d(a, h)$ can be dealt using same arguments. Now, suppose that $a = g$. If h_1 or $(h_2) \in M_f$, then g and h are distinguished by h_1 (or h_2). Therefore, we may suppose that $h_1, h_2, \notin M_f$ and $d(h_1, a) \leq d(h_2, a)$. Then, there exists $a' \in N(a)$, such that $h_1 \nsim a'$. By using the paths $a - a' - \cdots - h_2$ and $a - a' - \cdots - h_1$, we can see that

$$d(a', h_2) \geq d(a, h_2) - 1 \geq d(a, h_1) - 1 \geq 1 + d(a', h_1) - 1 = d(h_1, a') \geq 2. \quad (35)$$

Since $a = g$ and $a'' \sim a$, so $d(g, a') = 1$, i.e., $a' \in M_f \backslash \{a\}$ resolves g and h. Now, assume that vertex a coincides with some vertex of h, say h_1, i.e., $h = h_1 h_2 = a h_2$. If $g \in M_f$, then g and h are distinguished by g. But if $g \notin$, then g is not adjacent to some vertex $b' \in N(a)$. This implies that $d(b', h) = 1$ and $d(b', g) \geq 2$. This completes this subcase.

Case 3b: suppose that the vertex g lies on h. Let $h_1 = g$. If $a \neq g, h_2$, then we have

$$d(h,a) = d(h_2,a) < d(g,a) \leq d(h_2,a) + 1. \tag{36}$$

Thus,

$$d(a,g) = d(a,h_2) + 1. \tag{37}$$

If $h_2, \in M_f$, then g and h are distinguished by h_2, but if $h_2, \notin M_f$, then $h_2, \nsim a$. This implies that there exists a vertex $b' \in N(a)$ on the minimum path between a and h_2. Using (37), we have

$$d(b',g) = d(a,h_2) = 1 + d(b',h_2). \tag{38}$$

This implies that $d(b',g) > d(b',h_2)$. Hence, g and h are distinguished by $b' \in M_f \backslash \{a\}$. Finally, suppose that the vertices a and g both are on the edge h, that is, $h = ag$. As the graph J possesses the MFTM generator, so by Theorem 2, the vertex a is not a pendant vertex and therefore $\deg(a) > 1$. This further implies that there exists a vertex $c \in N(a) \subseteq M_f \backslash \{a\}$ other than g, i.e., $c \neq g$. As J has girth at least four, so c is not adjacent to g ,and we have $d(c,h) = 1$, whereas $d(c,g) \geq 2$. Thus, in all cases, any pair of element of J is distinguished by vertices of $M_f \backslash \{a\}$. This completes the proof. □

4. Conclusion

In this study, it is shown that the mixed metric dimension for the rooted product of graph of order $n \geq 2$ n by path graph by taking pendant vertex as root vertex is n. As a consequence, it is presented that the mixed metric dimension of the corona graph with t vertices is $t/2$. The notion of the mixed fault-tolerant metric generator is defined for the mixed generator. As the graphs like path graph, tree, and complete graph do not have any mixed metric generators, so it is important to classify those graphs which possess the mixed generator. This problem is settled in this study, and the graphs having the mixed fault-tolerant metric generator are characterized. Specifically, it is shown that the necessary and sufficient conditions for existence of the mixed generator for a graph Q are that the graph Q does not have pendant vertices and does not contain any vertex having maximal neighbourhood. Moreover, the mixed fault-tolerant metric resolving set for a graph Q with girth at least 4 is presented as $F_m = \cup_{t \in A} (N[t] \cup (\cap_{t_i \in N[t]} N(t_i)))$, where $N[t]$ and $N(t)$ are the closed and open neighbourhoods, respectively, and A is the mixed basis for Q.

Data Availability

The data used to support the findings of this study are included within the article and are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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