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Corrigendum

Corrigendum to "A Shrinking Projection Algorithm with Errors for Costerro Bounded Mappings"

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In the original publication, the title was "A Shrinking Projection Algorithm with Errors for Costerro Bounded Linear Mappings"; however, the author wishes to remove the word "linear" from the title and throughout the manuscript.

This is to make clear that convex sets are for nonlinear maps, and as a result, it is not appropriate to define linear maps on convex sets. Linear maps are mostly defined on subspaces and not on convex sets.

The title is therefore corrected as above. And, the revised article is as follows.

A Shrinking Projection Algorithm with Errors for Costerro Bounded Mappings

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The purpose of this paper is to introduce and analyze the shrinking projection algorithm with errors for a finite set of Costerro bounded mappings in the setting of uniformly convex smooth Banach spaces. Here, under finite-dimensional or compactness restriction or the error term being zero, we study the strong limit point of the sequence stated in our iterative scheme for these mappings in uniformly convex smooth Banach spaces. This paper extends Ezearn and Prempeh's result for nonexpansive mappings in real Hilbert spaces.

1. Introduction

Fixed point theory is a fascinating subject, with a lot of applications in various fields of mathematics and engineering. In a number of situations, one may need to find a common fixed point of a family of mappings. In practice, a modification may be needed to turn the problem into a fixed point problem (see for instance Picard (1) and Lindelöf (2)). For more information on fixed point problem and its applications to certain types of linear and nonlinear problems, interested readers should refer to Tang and Chang (3) (equilibrium problems), Solodov and Svaiter (4) (proximal point algorithm), Takahashi (5, 6) (convex optimization and minimization problems), and Blum and Oettli (7) (variational inequalities).

In practice, finding an exact closed form of a solution to a fixed point problem is almost a difficult task. For this reason, it has been of particular importance in the development of feasible iterative schemes or methods for approximating fixed points of certain maps, most notably, nonexpansive type of mappings. For instance, Halpern (8), Mann (9), and Ishikawa (10) studied and developed an iterative scheme to approximate the fixed points of nonexpansive mappings in Hilbert spaces under certain conditions. In their scheme, strong convergence is always guaranteed for all closed convex subsets of a Hilbert space. Haugazeau (11) initially proposed the projection method which was later developed by Solodov and Svaiter (4). A type of projection method which is of relevance and central to this paper is called the *Shrinking Projection Method with Errors*, which was developed by Takahashi et al. (12) and used by Yasunori (13). The strong convergence result is always guaranteed for all closed convex subsets of a Hilbert space under certain conditions.

In (14), Ezearn and Prempeh improved the boundedness requirement of Yasunori's result (13) regarding a shrinking projection algorithm for common fixed points of nonexpansive mappings in a real Hilbert space. In their results, they showed that the boundedness requirement in Yasunori's results could be removed. That is to say, the convergence of the iterative sequence in the scheme presented in Yasunori's paper, that is, when the error term $\varepsilon_0 = 0$, is independent of the boundedness of the closed convex subset in a real Hilbert space. With the boundedness being removed, Ezearn and Prempeh further provided a better estimate for the convergence result of the iterative sequence in their algorithm especially in finite dimension and further showed that when the closed convex set is compact, their estimates does not involve the diameter of the subset.

In this paper, we show that the strong limit point of the iterative sequence $\{x_n\}_{n\geq 1}$ presented in Section 1.1 always exists in a finite-dimensional space. We also show that, when the space is not finite dimensional, the strong limit point of $\{x_n\}_{n\geq 1}$ is guaranteed when the closed convex subset is compact. Finally, we show that the strong limit point of $\{x_n\}_{n\geq 1}$ also exists when the error term (ε_0) is zero regardless of the compactness of the closed convex subset and the dimension of the space.

Definition 1 (normalised duality mapping, see Lunner (18)). Let $\mathscr{X}$ be a Banach space with the norm $\|\cdot\|$, and let $\mathscr{X}^*$ be the dual space of $\mathscr{X}$. Denote $\langle\cdot,\cdot\rangle$ as the duality product. The normalised duality mapping J from $\mathscr{X}$ to $\mathscr{X}^*$ is defined by

$$Jx \coloneqq \left\{f \in \mathscr{X}^*\colon \|f\|_*^2 = \|x\|^2 = \langle x, f\rangle = fx\right\}, \quad (1)$$

for all $x \in \mathscr{X}$. The Hahn–Banach theorem guarantees that $Jx \neq \varnothing$ for every $x \in \mathscr{X}$. For our purposes in this paper, our interest mostly lies on the case when Jx is single-valued for all $x \in \mathscr{X}$, which is equivalent to the statement that $\mathscr{X}$ is a smooth Banach space.

Throughout this paper, $\mathfrak{R}$ denotes the real part of a complex number. We also use $F(T)$ to denote the set of fixed points of the mapping T (that is, $F(T) = \{x \in \mathscr{C}\colon Tx = x\}$).

The mappings we study in this paper are defined as follows.

Definition 2 (Costerro bounded mappings). Let $\mathscr{X}$ be a strictly convex smooth reflexive space and $\mathscr{C}$ be a closed convex subset of $\mathscr{X}$. A mapping $T\colon \mathscr{C} \longrightarrow \mathscr{X}$ is said to be a Costerro bounded mapping if

$$\|Tx\| \leq \|x\|, \quad (2)$$

such that whenever $z \in F(T)$, then

$$\mathfrak{R}\langle z, JTx - Jx\rangle \geq 0, \quad \forall x, z \in \mathscr{C}. \quad (3)$$

An immediate example of such mappings is the scaling operator given by

$$T(x) = ax, \quad (4)$$

where the scaling factor a lies in the closed unit disk.

In order to state our iterative scheme, let us define the following function.

Definition 3 (generalised projection functional, see (16)). Let $\mathscr{X}$ be a smooth Banach space, and let $\mathscr{X}^*$ be the dual space of $\mathscr{X}$. The generalised projection functional $\phi(\cdot,\cdot)\colon \mathscr{X}\times\mathscr{X} \longrightarrow \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\mathfrak{R}\langle y, Jx\rangle + \|x\|^2, \quad (5)$$

for all $x, y \in \mathscr{X}$, where J is the normalised duality mapping from $\mathscr{X}$ to $\mathscr{X}^*$. It is obvious from the definition that the generalised projection functional $\phi(\cdot,\cdot)$ satisfies the following inequality:

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad (6)$$

for all $x, y \in \mathscr{X}$.

We should note here that the generalised projection functional $\phi(\cdot,\cdot)$ is continuous.

The next function which is stated in our iterative scheme is established via the theorem as follows.

Theorem 1 (generalised projection, see [18]). *Let $\mathscr{X}$ be a uniformly convex smooth Banach space, and let $\mathscr{C} \neq \varnothing$ be a closed convex subset of $\mathscr{X}$. Then, for every $x \in \mathscr{X}$, there exists a unique $y \in \mathscr{C}$ such that*

$$\Lambda(x, C) = \phi(y, x) = \inf_{z\in\mathscr{C}} \phi(z, x). \quad (7)$$

The unique point y satisfying equation (7) is called the generalised projection of x on $\mathscr{C}$. That is, we define the projection operator $\Pi_{\mathscr{C}}\colon \mathscr{X} \longrightarrow \mathscr{C}$ by setting

$$\Pi_{\mathscr{C}} x = y, \quad (8)$$

where y is the only point in $\mathscr{C}$ satisfying equation (7).

Remark 1. In Theorem 1, note that if $\mathscr{X}$ is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$. Hence, the (generalised) projection $\Pi_{\mathscr{C}}$ defined in equation (8) coincides with the metric projection onto $\mathscr{C}$ in the Hilbert space setting. The converse is not necessarily true in a general Banach space.

The iterative scheme is stated as follows.

1.1. Iterative Scheme 1. Let $\mathscr{X}$ be a uniformly convex smooth Banach space, and let $\mathscr{C} \neq \varnothing$ (not necessarily bounded) be a *closed convex subset of $\mathscr{X}$. Let $\{T_k\}_{k=1}^m$ be a finite set of Costerro bounded mappings from $\mathscr{C}$ to $\mathscr{X}$ with $F \coloneqq \cap_{k=1}^m F(T_k) \neq \varnothing$. Let $\{\alpha_{n,k}\}_{n\geq 1}$ and $\{\varepsilon_n\}_{n\geq 1}$ be nonnegative real sequences satisfying the following conditions, for all $1 \leq k \leq m$ and $n \geq 1$:*

(i) $\{\alpha_{n,k}\}_{n\geq 1} \subset [0, 1]$
(ii) $\sum_{k=1}^m \alpha_{n,k} = 1$

(iii) $\alpha_k := \liminf_{n\longrightarrow\infty}\alpha_{n,k} > 0$

(iv) $\varepsilon_0 := \limsup_{n\longrightarrow\infty}\varepsilon_n < \infty$

Then, for any arbitrary $u \in \mathcal{X}$ with the assumptions $x_1 \in \mathcal{C}_1 := \mathcal{C}$ and $\phi(x_1, u) < \varepsilon_1^2$, the sequence $\{x_n\}_{n\geq 1}$ is defined iteratively by the following scheme:

$$\mathcal{B}_n = \left\{ z \in \mathcal{C}\colon \sum_{k=1}^{m} \alpha_{n,k}\phi(z, T_k x_n) \leq \phi(z, x_n) \right\}, \tag{9}$$

$$\mathcal{C}_{n+1} = \mathcal{B}_n \cap \mathcal{C}_n, \tag{10}$$

$$x_{n+1} \in \mathcal{C}_{n+1}, \phi(x_{n+1}, u) \leq \Lambda(u, \mathcal{C}_{n+1}) + \varepsilon_{n+1}^2, \tag{11}$$

for all $n \geq 1$.

2. Preliminaries

The inequality $\mathfrak{R}\langle z, JTx - Jx\rangle \geq 0$ in Definition 2 can be written equivalently in terms of norms. This is achieved via the elementary lemma by Ezearn in (19). We give the proof here for the sake of completeness.

Theorem 2 (see, for instance, (19)). *Let $\mathcal{X}$ be a smooth Banach space, and let $x \in \mathcal{X}\backslash\{0\}$ and any $y \in \mathcal{X}$. Then,*

$$\frac{1}{\|x\|}\mathfrak{R}\langle y, Jx\rangle = \lim_{\alpha\longrightarrow 0}\frac{\|x + \alpha y\| - \|x\|}{\alpha}, \tag{12}$$

for all $\alpha > 0$.

Lemma 1 (19). *Let $\mathcal{X}$ be a smooth Banach space and $x_1, x_2, \ldots, x_m, y \in \mathcal{X}$ where $m \in \mathbb{N}$. Then,*

$$\sum_{k=1}^{m}\|x_k\|^2 \leq \sum_{k=1}^{m}\|x_k\|\|x_k + \alpha y\|, \tag{13}$$

for all $\alpha \in [0, q]$ (where $q \in \mathbb{R}_{>0}$) if and only if

$$\mathfrak{R}\langle y, Jx_1 + Jx_2 + \ldots + Jx_m\rangle \geq 0. \tag{14}$$

Proof. We observe that if $\alpha = 0$, then the lemma is proved trivially and as a result, we assume that $\alpha \neq 0$ (without loss of generality, we can equally assume that $x_k \neq 0$). Now, if $\mathfrak{R}\sum_{k=1}^{m}\langle y, Jx_k\rangle \geq 0$, then

$$\sum_{k=1}^{m}\|x_k\|^2 = \mathfrak{R}\sum_{k=1}^{m}\langle x_k, Jx_k\rangle \leq \mathfrak{R}\langle x_k + \alpha y, Jx_k\rangle \leq \sum_{k=1}^{m}\|x_k + \alpha y\|\|x_k\|. \tag{15}$$

On the other hand, if $\sum_{k=1}^{m}\|x_k\|^2 \leq \sum_{k=1}^{m}\|x_k\|\|x_k + \alpha y\|$ for every $\alpha \in (0, q]$ (where $q \in \mathbb{R}_{>0}$), then

$$0 \leq \frac{1}{\alpha}\sum_{k=1}^{m}\|x_k\|\left(\|x_k + \alpha y\| - \|x_k\|\right) = \sum_{k=1}^{m}\|x_k\|\frac{\|x_k + \alpha y\| - \|x_k\|}{\alpha}. \tag{16}$$

Taking the limit as $\alpha \longrightarrow 0$, then by Theorem 2, equation (16) becomes

$$\frac{1}{\| x_k \|}\mathfrak{R}\sum_{k=1}^{m}\langle y, Jx_k\rangle \geq 0. \tag{17}$$

Since $x_k \neq 0$, then $\mathfrak{R}\sum_{k=1}^{m}\langle y, Jx_k\rangle \geq 0$ and hence proved. □

Corollary 1. *The inequality $\mathfrak{R}\langle z, JTx - Jx\rangle \geq 0$ is equivalent to*

$$\|Tx\|^2 + \|x\|^2 \leq \|Tx\|\|Tx + \alpha z\| + \|x\|\|x - \alpha z\|, \tag{18}$$

for all $\alpha \geq 0$.

Proof. By considering Lemma 1 for the case when $m = 2$, the inequality

$$\mathfrak{R}\langle z, Jx_1 - Jx_2\rangle = \mathfrak{R}\langle z, Jx_1 + J(-x_2)\rangle \geq 0 \tag{19}$$

is equivalent to the following condition:

$$\sum_{k=1}^{2}\|x_k\|^2 \leq \sum_{k=1}^{2}\|x_k\|\|x_k + \alpha y\|, \tag{20}$$

$$\|x_1\|^2 + \|x_2\|^2 \leq \|x_1\|\|x_1 + \alpha y\| + \|x_2\|\|x_2 + \alpha y\|. \tag{21}$$

Now, replacing x_1 with Tx, y with z, and x_2 with $(-x)$, the corollary is proved.

In the following, we give a nontrivial example of Costerro bounded mappings which we refer to as Ezearn nonexpansive mapping. Ezearn, in his thesis (19), had defined certain closely related mappings (named Type III variational nonexpansive mappings). □

Corollary 2 (Ezearn nonexpansive mapping). *Let $\mathcal{C}$ be a closed convex subset of a strictly convex smooth reflexive space $\mathcal{X}$. Then, the following is a nontrivial example of a Costerro bounded mapping:*

$$\left(\|Tx\|^2+\|Ty\|^2\right)+\left(\|x\|^2-\|y\|^2\right)\le\|Tx\|\|Tx+\alpha Ty\|+\|x\|\|x-\alpha y\|, \tag{22}$$

for all $x,y\in\mathscr{C}$ *and all* $\alpha\ge 0$.

Proof. For $\alpha=0$, equation (18) reduces to the following:

$$\|Tx\|^2+\|Ty\|^2+\|x\|^2-\|y\|^2\le\|Tx\|^2+\|x\|^2, \tag{23}$$

$$\|Ty\|^2\le\|y\|^2, \tag{24}$$

$$\|Ty\|\le\|y\|, \tag{25}$$

which satisfies the first part of Definition 2. To show the second part of Definition 2, if $y\in F(T)$, where $F(T)$ refers to the fixed point set of T, then equation (18) reduces to the following evaluation:

% $$\left(\|Tx\|^2+\|y\|^2\right)+\left(\|x\|^2-\|y\|^2\right)\le\|Tx\|\|Tx+\alpha y\|+\|x\|\|x-\alpha y\|, \tag{26}$$

$$\|Tx\|^2+\|x\|^2\le\|Tx\|\|Tx+\alpha y\|+\|x\|\|x-\alpha y\|, \tag{27}$$

which by Corollary 1 is equivalent to $\mathfrak{R}\langle y,JTx-Jx\rangle\ge 0$. Hence proved. □

Lemma 2 (see, for instance, Ezearn, (19)). *Let*$\{\mathscr{C}_n\}_{n\ge1}$*be a sequence of nonempty closed convex subsets of a uniformly convex smooth Banach space*$\mathscr{X}$*such that*$\mathscr{C}_{n+1}\subset\mathscr{C}_n$*. Suppose further that*$\mathscr{C}_\infty=\cap_{n\ge1}\mathscr{C}_n$*is nonempty. Then, the sequence of generalized projections*$\left\{\Pi_{\mathscr{C}_n}x\right\}_{n\ge1}$*converges strongly to*$\Pi_{\mathscr{C}_\infty}x$*for any*$x\in\mathscr{X}$.

Proposition 1 (Alber (20), Alber and Reich (21), Kamimura and Takahashi, (22)). *Let*$\mathscr{X}$*be a real uniformly convex smooth Banach space and*$\mathscr{C}\neq\varnothing$*be a closed convex subset of*$\mathscr{X}$*. Then, the following inequality holds:*

$$\phi(y,\Pi_{\mathscr{C}}x)+\phi(\Pi_{\mathscr{C}}x,x)\le\phi(y,x), \tag{28}$$

for all $y\in\mathscr{C}$ *and* $x\in\mathscr{X}$.

Proposition 2 (continuity in duality pairing). *Let*$\mathscr{X}$*be a Banach space, and let*$\mathscr{X}^*$*be the dual space of*$\mathscr{X}$*. Denote*$\langle\cdot,\cdot\rangle$*as the duality product. Now, for*$\{x_n\}_{n\ge1}\subset\mathscr{X}$*and*$\{f_n\}_{n\ge1}\subset\mathscr{X}^*$*, suppose either of the following conditions hold:*

$x_n x$ *and* $f_n\longrightarrow f$, $x_n\longrightarrow x$ *and* $f_n^* f$

Then, $\lim_{n\longrightarrow\infty}\langle x_n,f_n\rangle=\langle x,f\rangle$.

Lemma 3 (weak star-continuity in smooth spaces). *Let*$\mathscr{X}$*be a real smooth Banach space. Then,*$J\colon\mathscr{X}\longrightarrow\mathscr{X}^*$*is norm-to-weak star continuous, where*J*is the normalized duality mapping.*

Lemma 4 (Kamimura and Takahashi, strong). *Let*$\mathscr{X}$*be a uniformly convex and smooth Banach space, and let*$\{x_n\}$*and*$\{y_n\}$*be two sequences in*$\mathscr{X}$*such that either*$\{x_n\}$*or*$\{y_n\}$*is bounded. If*$\lim_{n\longrightarrow\infty}\phi(x_n,y_n)=0$*, then*$\lim_{n\longrightarrow\infty}\|x_n-y_n\|=0$.

3. Main Results

We give the proof of the main result of this paper, which is accomplished in Theorem 3. The following corollary and lemmas shall aid us in arriving at the conclusion of the main result.

Corollary 3. *Let*$\{T_k\}_{k=1}^m$*be a continuous Costerro bounded mapping. If the sequence*$\{x_n\}_{n\ge1}$*has a strong limit point, say*x*, then*$x\in F=\cap_{k=1}^m Fix(T)_k$.

Proof. Without loss of generality, we assume that the sequence $\{x_n\}_{n\ge1}=x_1,x_2,x_3,\ldots$ is the subsequence converging to x. Now, for $n\ge1$, since the sets $\mathscr{C}_n$ form a decreasing sequence of sets, that is, $\mathscr{C}_{n+1}\subset\mathscr{C}_n$, then from Section 1.1, we have that $x_{n+1}\in\mathscr{C}_{n+1}\subset\mathscr{C}_n$, where $\{x_{n+1}\}_{n\ge1}=x_2,x_3,x_4,\ldots$. Hence, we observe that

$$\sum_{k=1}^m\alpha_{n,k}\phi(x_{n+1},T_kx_n)\le\phi(x_{n+1},x_n). \tag{29}$$

Hence, taking limit as $n\longrightarrow\infty$ of the above inequality, we have

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^m\alpha_{n,k}\phi(x_{n+1},T_kx_n)\le\lim_{n\longrightarrow\infty}\phi(x_{n+1},x_n). \tag{30}$$

By Proposition 2 and Lemma 3, $\lim_{n\longrightarrow\infty}\phi(x_{n+1},x_n)\longrightarrow 0$ and as a result, we obtain

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^m\alpha_{n,k}\phi(x_{n+1},T_kx_n)\le 0. \tag{31}$$

Since the generalised functional $\phi(\cdot,\cdot)$ is nonnegative and the limit infimum of $\{\alpha_{n,k}\}$ is nonzero for all k, we have

$$\lim_{n\longrightarrow\infty}\phi(x_{n+1},T_kx_n)=0, \tag{32}$$

for all. $k\in\{1,\ldots,m\}$.

So by Lemma 4, we have that

$$\lim_{n\longrightarrow\infty}\left\|x_{n+1}-T_kx_n\right\|=0, \tag{33}$$

for all $k\in\{1,\ldots,m\}$ and that proves the corollary due to the continuity of the norm functional and the mappings T_k. □

Lemma 5. *For all*$n\ge1$*, the sets*$\mathscr{B}_n$*and*$\mathscr{C}_n$*in Section1.1are closed convex sets.*

Proof. Because $\mathscr{C}_1\coloneqq\mathscr{C}$ is a closed convex set by assumption, it suffices to show that $\mathscr{B}_n$ is a closed convex set for all n. To prove the closure aspect of the lemma, we observe that if $\left\{z_j\right\}_{j\ge1}\subset\mathscr{B}_n$ converges to $z\in\mathscr{C}$, then via the continuity of the generalised functional $\phi(\cdot,\cdot)$, we have the following:

$$\sum_{k=1}^{m}\alpha_{n,k}\phi(z,T_kx_n)=\lim_{j\longrightarrow\infty}\sum_{k=1}^{m}\alpha_{n,k}\phi\big(z_j,T_kx_n\big)\le\lim_{j\longrightarrow\infty}\phi\big(z_j,x_n\big)=\phi(z,x_n), \tag{34}$$

and as a result, $z\in\mathscr{B}_n$.

Finally, to prove convexity, let $u,v\in\mathscr{B}_n$ and $t\in[0,1]$. First, note that whenever $z\in\mathscr{B}_n$, then we have the following inequality:

$$\sum_{k=1}^{m}\alpha_{n,k}\phi(z,T_kx_n)\le\phi(z,x_n), \tag{35}$$

which can be expanded and observed to be equivalent to

$$\sum_{k=1}^{m}\alpha_{n,k}\Big(\|T_kx_n\|^2-\|x_n\|^2\Big)\le 2\mathfrak{R}\sum_{k=1}^{m}\alpha_{n,k}\langle z,JT_kx_n-Jx_n\rangle. \tag{36}$$

So by making the substitution $z=u$ and multiplying it by t and adding it to $z:=v$ multiplied by $(1-t)$, we obtain

$$\begin{aligned}\sum_{k=1}^{m}\alpha_{n,k}\Big(\|T_kx_n\|^2-\|x_n\|^2\Big)&=t\sum_{k=1}^{m}\alpha_{n,k}\Big(\|T_kx_n\|^2-\|x_n\|^2\Big)+(1\\&-t)\sum_{k=1}^{m}\alpha_{n,k}\Big(\|T_kx_n\|^2-\|x_n\|^2\Big)\le 2t\mathfrak{R}\sum_{k=1}^{m}\alpha_{n,k}\langle u,JT_kx_n-Jx_n\rangle\\&+2(1-t)\mathfrak{R}\sum_{k=1}^{m}\alpha_{n,k}\langle v,JT_kx_n-Jx_n\rangle\\&=2\mathfrak{R}\sum_{k=1}^{m}\alpha_{n,k}\langle tu+(1-t)v,JT_kx_n-Jx_n\rangle.\end{aligned} \tag{37}$$

From this, we conclude that

$$\sum_{k=1}^{m}\alpha_{n,k}\phi(tu+(1-t)v,T_kx_n)\le\phi(tu+(1-t)v,x_n). \tag{38}$$

Hence, $\mathscr{B}_n$ is convex.

Now, let us define

$$\mathscr{C}_\infty := \cap_{n\ge1}\mathscr{C}_n. \tag{39}$$

□

Lemma 6. *The set$\mathscr{C}_\infty$is a closed convex set containing$F$. Hence, the sequence$\{\Pi_{\mathscr{C}_n}x\}_{n\ge1}$of generalised projections converges strongly to$\Pi_{\mathscr{C}_\infty}x$for any arbitrary$x$in a uniformly convex smooth Banach space$\mathscr{X}$.*

Proof. By induction, we observe that the sets $\mathscr{C}_n$ are all closed convex subsets by the help of Lemma 5 and the definition of $\mathscr{C}_{n+1}$ in Section 1.1 Moreover, by inclusion, these sets $\mathscr{C}_n$ form a decreasing sequence of sets. That is, $\mathscr{C}_{n+1}\subset\mathscr{C}_n$ for all $n\ge1$. So $\mathscr{C}_\infty$ is either empty or nonempty. We claim that ${\mathscr{C}_\infty}^F$ by induction. By the assumption in Section 1.1, it is observed that $F\subset\mathscr{C}=\mathscr{C}_1$ and x_1 is given. Now, let us suppose $F\subset\mathscr{C}_m$ for all $m\le n$ and choose arbitrary $z\in F$. Then, we have the following evaluation:

$$\begin{aligned}\sum_{k=1}^{m}\alpha_{n,k}\phi(z,T_kx_n)&=\sum_{k=1}^{m}\alpha_{n,k}\Big(\|z\|^2+\|T_kx_n\|^2-2\mathfrak{R}\langle z,JT_kx_n\rangle\Big)\\&=\sum_{k=1}^{m}\alpha_{n,k}\|z\|^2+\sum_{k=1}^{m}\alpha_{n,k}\|T_kx_n\|^2-2\mathfrak{R}\sum_{k=1}^{m}\alpha_{n,k}\langle z,JT_kx_n\rangle\\&=\sum_{k=1}^{m}\alpha_{n,k}\|z\|^2+\sum_{k=1}^{m}\alpha_{n,k}\|T_kx_n\|^2\\&\quad-2\mathfrak{R}\sum_{k=1}^{m}\alpha_{n,k}\langle z,JT_kx_n\rangle\le\sum_{k=1}^{m}\alpha_{n,k}\|z\|^2+\sum_{k=1}^{m}\alpha_{n,k}\|x_n\|^2\\&\quad-2\mathfrak{R}\sum_{k=1}^{m}\alpha_{n,k}\langle z,Jx_n\rangle\\&=\|z\|^2+\|x_n\|^2-2\mathfrak{R}\langle z,Jx_n\rangle=\phi(z,x_n),\end{aligned} \tag{40}$$

where we have used the fact that the mappings are Costerro bounded mappings in the third step. Hence, we have shown that $z\in\mathscr{C}_{n+1}$. From Lemma 2, we conclude that $\{\Pi_{\mathscr{C}_n}x\}_{n\ge1}$ converges strongly to $\Pi_{\mathscr{C}_\infty}x$. □

Lemma 7. *The sequence$\{x_n\}_{n\ge1}$satisfies the following inequality:*

$$\phi\big(x_n,\Pi_{\mathscr{C}_n}u\big)\le\varepsilon_n^2. \tag{41}$$

Proof. Since $\Lambda(u,\mathscr{C}_n):=\inf_{x\in\mathscr{C}_n}\phi(x,u)=\phi(\Pi_{\mathscr{C}_n}u,u)$, then for every $\varepsilon_n>0$, we can find our $x_n\in\mathscr{C}_n$ such that

$$\phi(x_n,u)\le\Lambda(\mathscr{C}_n,u)+\varepsilon_n^2=\phi\big(\Pi_{\mathscr{C}_n}u,u\big)+\varepsilon_n^2, \tag{42}$$

which implies that

$$\phi(x_n,u)-\phi\big(\Pi_{\mathscr{C}_n}u,u\big)\le\varepsilon_n^2. \tag{43}$$

However, Proposition 1 implies that

$$\phi(x_n,\Pi_{\mathscr{C}}u)\le\phi(x_n,u)-\phi(\Pi_{\mathscr{C}}u,u), \tag{44}$$

and so in addition to equation (43), we have the following inequality:

$$\phi\big(x_n,\Pi_{\mathscr{C}_n}u\big)\le\varepsilon_n^2. \tag{45}$$

This completes the proof.

The main result of this paper is given by the following theorem. □

Theorem 3 (main result). *Let$\mathscr{X}$be a uniformly convex smooth Banach space and suppose any of the following cases hold:*

(1) The space $\mathscr{X}$ is finite dimensional

(2) The convex set $\mathscr{C}$ is compact

(3) $\varepsilon_0=0$

Then, $\omega(\{x_n\}_{n\geq 1}) \neq \varnothing$ *and* $\omega(\{x_n\}_{n\geq 1}) \subseteq \cap_{k=1}^{m} F(T_k)$, *where* ω *denotes the (strong) limit set of the iterative sequence* $\{x_n\}_{n\geq 1}$.

Proof. First we observe that $\{x_n\}_{n\geq 1}$ is a bounded sequence. As a matter of fact, by Lemma 7, we have

$$\phi\left(x_n, \Pi_{\mathscr{C}_n} u\right) = \|x_n\|^2 - 2\Re\langle x_n, J\Pi_{\mathscr{C}_n} u\rangle + \left\|\Pi_{\mathscr{C}_n} u\right\|^2 \leq \varepsilon_n^2, \tag{46}$$

which simplifies to

$$\|x_n\|^2 + \left\|\Pi_{\mathscr{C}_n} u\right\|^2 - 2\|x_n\|\left\|\Pi_{\mathscr{C}_n} u\right\| = \left(\|x_n\| - \left\|\Pi_{\mathscr{C}_n} u\right\|\right)^2 \leq \varepsilon_n^2. \tag{47}$$

Hence, we have $\|x_n\| \leq \|\Pi_{\mathscr{C}_n} u\| + \varepsilon_n$. So by Lemma 6 and the conditions in Section 1.1, we have that $\{x_n\}_{n\geq 1}$ is a bounded sequence.

We now consider the following cases stated in Theorem 3. □

Case 1. Given that $\{x_n\}_{n\geq 1}$ is bounded and $\mathscr{X}$ is finite dimensional, then by the Bolzano–Weierstrass Theorem, $\{x_n\}_{n\geq 1}$ has a limit point, say z, and by Corollary 3, $z \in F$, and as a result, a subsequence of $\{x_n\}_{n\geq 1}$ converges strongly to $z \in F$.

Case 2. Given that $\mathscr{C}$ is compact, since in metric spaces, compactness implies sequential compactness, then $\{x_n\}_{n\geq 1}$ being a bounded sequence has a limit point, say z, and by Corollary 3, $z \in F$, and as a result, a subsequence of $\{x_n\}_{n\geq 1}$ converges strongly to $z \in F$.

Case 3. Given that $\varepsilon_0 = 0$, then by Lemma 7, we have

$$\lim_{n\longrightarrow\infty} \left\|x_n - \Pi_{\mathscr{C}_n} u\right\| = 0. \tag{48}$$

By continuity of the norm function and Lemma 6, we have that $\{x_n\}_{n\geq 1} \longrightarrow \Pi_{\mathscr{C}_\infty} u$ and as result, $\Pi_{\mathscr{C}_\infty} u \in F$ since it is the only limit point of the sequence $\{x_n\}_{n\geq 1}$.

Remark 2. We also note that for infinite dimensions, we can also say that the sequence $\{x_n\}_{n\geq 1}$ *has a weak limit point since a uniformly convex smooth Banach is a reflexive space.*

Conflicts of Interest

The author declares that he has no conflicts of interest.

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