Hindawi
Journal of Mathematics
Volume 2023, Article ID 1001901, 7 pages
https://doi.org/10.1155/2023/1001901

Research Article

Core-EP-Nilpotent Decomposition and Its Applications

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Received 3 August 2022; Revised 16 September 2022; Accepted 26 October 2022; Published 23 January 2023

Academic Editor: Xiaogang Liu

We give the illustration of a new decomposition: the core-EP-nilpotent decomposition, which is on the basis of the core-EP decomposition and EP-nilpotent decomposition for some square matrices in this thesis. According to the new decomposition, we show the definitions and characteristics of two new orders: core-E-N partial order and core-E-S partial order. We also illustrate relations of the two orders under some restricted conditions.

1. Introduction

First of all, some mathematical notations are introduced as follows: $\mathbb{C}_{m,n}$ denotes the $m \times n$ matrices in the complex field. A^* is the conjugate transpose, $\mathfrak{R}(A)$ denotes the range space (or column space), and $rk(A)$ denotes the rank of $A \in \mathbb{C}_{m,n}$. I_n is the identity matrix of order n. The index of $A \in \mathbb{C}_{n,n}$, which is denoted by Ind (A), satisfies $rk(A^{k+1}) = rk(A^k)$ where k is the smallest positive integer. The symbol $\mathbb{C}_n^{CM}$ stands for a set of $n \times n$ matrices of index less than or equal to 1.

A unique matrix $X \in \mathbb{C}_{n,m}$, which is called the Moore–Penrose inverse of $A \in \mathbb{C}_{m,n}$, satisfies the equations

$$\begin{aligned} &(1) AXA = A,\\ &(2) XAX = X,\\ &(3) (AX)^* = AX,\\ &(4) (XA)^* = XA, \end{aligned} \tag{1}$$

and then, it is usually denoted as $X = A^\dagger$. Furthermore, we denote $P_A = AA^\dagger$. The unique matrix $X \in \mathbb{C}_{n,n}$ which is the group inverse of $A \in \mathbb{C}_{n,n}$ satisfies the equations

$$\begin{aligned} &(1) AXA = A,\\ &(2) XAX = X,\\ &(5) AX = XA, \end{aligned} \tag{2}$$

and then, it is usually denoted as $X = A^\#$.

The definition of core invertible matrix A is defined as there can be at most one matrix X such that

$$\begin{aligned} AX &= AA^\dagger,\\ \mathfrak{R}(X) &\subseteq \mathfrak{R}(A), \end{aligned} \tag{3}$$

if equation (3) is satisfied. In this case, X is the core inverse of A and we denote . In [1], it had been proved that $A \in \mathbb{C}_{n,n}$ is core invertible if and only if $A \in \mathbb{C}_n^{CM}$.

We denote a set of EP matrices over $\mathbb{C}_{n,n}$ by $\mathbb{C}_n^{\mathrm{EP}}$, where

$$\mathbb{C}_n^{\mathrm{EP}} = \left\{ A \mid \mathfrak{R}(A) = \mathfrak{R}(A^*), \quad A \in \mathbb{C}_{n,n} \right\}. \tag{4}$$

As it is well known that $\mathbb{C}_n^{\mathrm{EP}}$ contains many special types of matrices, such as Hermitian matrices, normal matrices, and nonsingular matrices.

Matrix decomposition and its research have been a hot research direction in recent years, among which some special matrices occupy the core position in matrix decomposition. For example, a decomposition named Toeplitz decomposition or Cartesian decomposition is introduced in [2, 3]. More details about other matrix decompositions can refer to [4–10]. Furthermore, there are some research issues about the systems of matrix equations by using generalized inverses of matrices which refer to [11–13].

In this paper, we will adopt Schur upper triangulation matrix decomposition, construct a new matrix decomposition based on the known core-EP decomposition and EP-nilpotent decomposition, and investigate related properties of this new matrix decomposition.

2. Preliminary Results

In this section, we give some preliminary results, (refer to [14], Theorem 5.4.1; [4], Theorem 2.1; [5], Theorem 2.1) which will be used in the next section.

Lemma 1 (Schur Decomposition). *Let $A \in \mathbb{C}_{n,n}$. Then, there exist a unitary matrix $U \in \mathbb{C}_{n,n}$ and an upper-triangular matrix $B \in \mathbb{C}_{n,n}$ such that*

$$A = UBU^{*}. \tag{5}$$

Lemma 2 (Core-EP decomposition). *Let $A \in \mathbb{C}_{n,n}$ with Ind $(A) = k$. Then, A can be written as the sum of matrices A_1 and A_2, i.e., $A = A_1 + A_2$, where*

(1) $A_1 \in \mathbb{C}_n^{CM}$;

(2) $A_2^k = 0$;

(3) $A_1^{*}A_2 = A_2A_1 = 0$.

Here, one or both of A_1 and A_2 can be null.

Lemma 3 (EP-nilpotent Decomposition). *Let $A \in \mathbb{C}_{n,n}$ with Ind $(A) = k$, $rk(A) = r$, and $rk(A^k) = s$. Then, A can be written as the sum of matrices A_1 and A_2, i.e., $A = A_1 + A_2$, where*

(1) $A_1 \in \mathbb{C}_n^{EP}$

(2) $A_2^{k+1} = 0$

(3) $A_2A_1 = 0$.

Here one or both of A_1 and A_2 can be null.

3. Main Results

First, we give a lemma as follows:

Lemma 4. *Let $A \in \mathbb{C}_{n,n}$. There exists a unitary matrix $U \in \mathbb{C}_{n,n}$ such that*

$$A = U\begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & N \end{bmatrix}U^{*}, \tag{6}$$

and thus, A can be written as $A = A_1 + A_2 + A_3$, where

$$\begin{aligned} A_1 &= U\begin{bmatrix} T_1 & S_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}U^{*}, \\ A_2 &= U\begin{bmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}U^{*}, \\ A_3 &= U\begin{bmatrix} 0 & 0 & S_2 \\ 0 & 0 & S_3 \\ 0 & 0 & N \end{bmatrix}U^{*}, \end{aligned} \tag{7}$$

where T_1 and T_2 are nonsingular and upper-triangular, the main diagonal of T_1 and T_2 are the eigenvalues of A, the last column of the T_1 except the main diagonal exists at least one nonzero element, N is an upper-triangular matrix with zero elements on main diagonal, and S_1, S_2, and S_3 are arbitrary matrices.

Proof. According to Lemma 1 and [4] (Theorem 2.2), for $A \in \mathbb{C}_{n,n}$ and A can be written as follows:

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^{*}, \tag{8}$$

where T is nonsingular and upper-triangular, the main diagonal of T are the eigenvalues of A, N is an upper-triangular matrix with zero elements on main diagonal, and S is arbitrary matrix. Moreover, using the block matrix method to decompose T, we can derive that $A = U\begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & N \end{bmatrix}U^{*}$. Then, the forms of A_1, A_2, and A_3 can easily be obtained.

Based on Lemma 4, we give a theorem of decomposition for some square matrices below. □

Theorem 1. *Let $A \in \mathbb{C}_{n,n}$ Ind $(A) = k$, $rk(A^k) = s$. A unique decomposition of $A = A_1 + A_2 + A_3$, where the forms of A_1, A_2, and A_3 are written as Lemma 4, where*

(i) $A_1 \in \mathbb{C}_n^{CM}$

(ii) $A_2 \in \mathbb{C}_n^{EP}$ or $A_2 \in \mathbb{C}_n^{EP}$, $rk(A_2) = l - 1$

(iii) $A_3^{k+1} = 0$

(iv) $A_1^{*}A_2 = A_2A_1 = 0$, $A_1A_3^{*} = A_3A_1 = 0$, $A_2A_3^{*} = A_3A_2 = 0$

Here A_1, A_2, and A_3 can be null, and we denote $l\,(2 \le l < s)$, which represents the number of columns of T_1 and T_2 that have the nonzero element except the main diagonal.

Proof. First, by calculating, we can obviously find that the matrices A_1, A_2, and A_3 satisfy all four conditions of the theorem. Next, we will illustrate the uniqueness of the decomposition. It follows from equation (8) that

$$A^k\left(A^k\right)^{\dagger} = U\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{9}$$

With Lemma 1, we can convert equation (9) as follows:

$$A^k\left(A^k\right)^{\dagger} = U\begin{bmatrix} I_{s_1} & 0 & 0 \\ 0 & I_{s_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}U^*, \tag{10}$$

where $rk(T_1) = s_1$, $rk(T_2) = s_2$, and $s_1 + s_2 = s$. Therefore, with equation (10) and Lemma 4, we obtain the following equation:

$$\begin{aligned} A_1 + A_2 &= AA^k\left(A^k\right)^{\dagger} = (A_1 + A_2)A^k\left(A^k\right)^{\dagger}, \\ A_3 &= A - AA^k\left(A^k\right)^{\dagger}. \end{aligned} \tag{11}$$

According to equation (11), we know that A can be written as $A_1 + A_2$ and A_3 uniquely. Moreover, with the restricted condition of T_1 and $A = U\begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & N \end{bmatrix}U^*$ in Lemma 4, we can derive that when $l = 1$, the decomposition of $A_1 + A_2$ can be written uniquely. When $l \geq 2$, the condition $rk(A_2) = l - 1$ can always guarantee the decomposition of $A_1 + A_2$ uniquely. In conclusion, A can be uniquely written as A_1, A_2, and A_3.

According to Theorem 1, we give a definition of new decomposition as follows: □

Definition 1. Let $A \in \mathbb{C}_{n,n}$ and Ind $(A) = k$, $rk(A^k) = s$. If the matrix decomposition satisfies Theorem 1, we say it is the core-EP-nilpotent decomposition.

A binary relation on a nonempty set is called partial order if it satisfies reflexivity, transitivity, and antisymmetry. It is significant to establish the partial orders by using matrix decomposition. Here are some well-known partial orders such as minus, sharp, star, core, E-N, and E-S partial orders, which are defined as follows:

(a) $A \overset{-}{\leq} B$: $A, B \in \mathbb{C}_{m,n}$, $rk(B) - rk(A) = rk(B - A)$
(b) $A^{\#} \leq B$: $A, B \in \mathbb{C}_n^{CM}$, $AA^{\#} = BA^{\#}$ and $A^{\#}A = A^{\#}B$
(c) $A_* \leq B$: $A, B \in \mathbb{C}_{m,n}$, $AA^* = BA^*$ and $A^*A = A^*B$
(d) $A \overset{\textcircled{\#}}{\leq} B$: $A, B \in \mathbb{C}_n^{CM}$, $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$ and $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$
(e) $A \leq^{\mathrm{EN}} B$: $A, B \in \mathbb{C}_{n,n}$, $A_1 \overset{-}{\leq} B_1$ and $A_2 \overset{-}{\leq} B_2$, in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the EP-nilpotent decompositions of A and B, respectively
(f) $A \leq^{\mathrm{ES}} B$: $A, B \in \mathbb{C}_{n,n}$, $A_1^{\#} \leq B_1$ and $A_2 \overset{-}{\leq} B_2$, in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the EP-nilpotent decompositions of A and B, respectively

Next, based on the E-N and E-S partial orders, we will introduce two new partial order relations and describe some related properties of these two new partial orders.

Definition 2 (Core-E-N order). Let $A, B \in \mathbb{C}_{n,n}$, $A = A_1 + A_2 + A_3$ and $B = B_1 + B_2 + B_3$ be decomposed as Definition 1, where A_1 and B_1 are core invertible, A_2 and B_2 are EP, and A_3 and B_3 are nilpotent. We consider the binary operation:

$$A \overset{CEN}{\leq} B\colon\ A_1 \overset{\textcircled{\#}}{\leq} B_1,\ A_2 \overset{-}{\leq} B_2 \text{ and } A_3 \overset{-}{\leq} B_3. \tag{12}$$

Theorem 2. *Operation (12) is called the core-E-N partial order.*

Proof. Reflexivity of the relation is obvious. If $A \overset{CEN}{\leq} B$ and $B \overset{CEN}{\leq} A$, with equation (12), and the definitions of core partial order, and minus partial order, we can easily obtain that $A_1 = B_1$, $A_2 = B_2$, and $A_3 = B_3$, i.e., $A = B$. The antisymmetry condition holds. Then, we suppose $A \overset{CEN}{\leq} B$ and $B \overset{CEN}{\leq} C$, applying the decomposition of equation (12) and definition of core partial order, $A_1 \overset{\textcircled{\#}}{\leq} B_1$ and $B_1 \overset{\textcircled{\#}}{\leq} C_1$ can imply that $A_1 \overset{\textcircled{\#}}{\leq} C_1$. Similarly, with the definition of minus partial order, we can derive that $A_2 \overset{-}{\leq} C_2$ and $A_3 \overset{-}{\leq} C_3$. By (12), we have $A \overset{CEN}{\leq} C$. The transitivity condition holds. The proof is complete.

The constructional form in the following theorem is referred to [5] and by calculating with Lemma 4. □

Theorem 3. *Let $A, B \in \mathbb{C}_{n,n}$. Then, $A \overset{CEN}{\leq} B$ if and only if there exists a unitary matrix U such that*

$$\begin{aligned} A &= U\begin{bmatrix} T_1 & T_2 & S_1 & S_2 \\ 0 & T_4 & 0 & S_3 \\ 0 & 0 & 0 & S_4 \\ 0 & 0 & 0 & N_1 \end{bmatrix}U^*, \\ B &= U\begin{bmatrix} T_1 & T_2 & S_1 & \widetilde{S}_2 \\ 0 & T_3 + T_4 + D_1T_5D_2 & D_1T_5 + \widetilde{S}_1 & \widetilde{S}_3 \\ 0 & T_5D_2 & T_5 & \widetilde{S}_4 \\ 0 & 0 & 0 & N_2 \end{bmatrix}U^*, \end{aligned} \tag{13}$$

where T_1, T_3, T_4, and T_5 are nonsingular and upper-triangular, N_1 and N_2 are nilpotent and upper-triangular, D_1 and D_2 are arbitrary matrices, and $[S_2^*S_3^*S_4^*N_1^*] \overset{-}{\leq} [\widetilde{S}_2^*\widetilde{S}_3^*\widetilde{S}_4^*N_2^*]$.

Proof. Let $A \overset{CEN}{\leq} B$, where A and B are decomposed as Definition 1. Then, $A_1 \overset{\textcircled{\#}}{\leq} B_1$, $A_2 \overset{-}{\leq} B_2$, and $A_3 \overset{-}{\leq} B_3$. Because of $A_1, B_1 \in \mathbb{C}_n^{CM}$, $A_1 \overset{\textcircled{\#}}{\leq} B_1$, and $A_2, B_2 \in \mathbb{C}_n^{\mathrm{EP}}$, $A_2 \overset{-}{\leq} B_2$, we can derive the following equation:

$$A_1 = U\begin{bmatrix} T_1 & T_2 & S_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}U^*,$$

$$A_2 = U\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & T_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}U^*,$$

$$B_1 = U \begin{bmatrix} T_1 & T_2 & S_1 & 0 \\ 0 & T_3 & \widetilde{S}_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} U^*,$$

$$B_2 = U \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & T_4 + D_1 T_5 D_2 & D_1 T_5 & 0 \\ 0 & T_5 D_2 & T_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} U^*, \tag{14}$$

where T_1, T_3, T_4, and T_5 are nonsingular and upper-triangular. Applying Definition 2, we can know that

$$A_3 = U \begin{bmatrix} 0 & 0 & 0 & S_2 \\ 0 & 0 & 0 & S_3 \\ 0 & 0 & 0 & S_4 \\ 0 & 0 & 0 & N_1 \end{bmatrix} U^*, \quad B_3 = U \begin{bmatrix} 0 & 0 & 0 & \widetilde{S}_2 \\ 0 & 0 & 0 & \widetilde{S}_3 \\ 0 & 0 & 0 & \widetilde{S}_4 \\ 0 & 0 & 0 & N_2 \end{bmatrix} U^*, \tag{15}$$

where N_1 and N_2 are nilpotent and upper-triangular. Applying Definition 2 again, it follows from $A_3 \overset{-}{\leq} B_3$ that

$$[S_2^* S_3^* S_4^* N_1^*] \overset{-}{\leq} [\widetilde{S}_2^* \widetilde{S}_3^* \widetilde{S}_4^* N_2^*]. \tag{16}$$

The proof is complete.

It is worth noting that a few famous partial orders are not the minus type. Therefore, we need to study whether the core-E-N partial order described in equation (12) is a minus type or not; the following example will illustrate. □

Example 1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

in which A and B are decomposed as Definition 1, where A_1 and B_1 are core invertible, A_2 and B_2 are EP, and A_3 and B_3 are nilpotent. Then,

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

We can easily check that $A_1 \overset{\textcircled{\#}}{\leq} B_1$, $A_2 \overset{-}{\leq} B_2$ and $A_3 \overset{-}{\leq} B_3$, i.e., $A \overset{CEN}{\leq} B$.

Since $rk(B) - rk(A) = 0 \neq rk(B - A) = 1$, by condition (a), we can get that A is not below B under the minus partial order and a corollary as follows:

Corollary 1. *Core-E-N partial order is not the minus type.*

Next, we will give another example to illustrate whether the minus partial order can lead to core-E-N partial order or not.

Example 2. Let

$$A=\begin{bmatrix}1&1&0&1\\0&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}1&1&0&1\\0&1&1&1\\0&0&1&0\\0&0&0&0\end{bmatrix}, \tag{19}$$

in which the decompositions of A and B and the definitions of their components are the same as Example 1. Then,

$$A_1=\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad A_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad A_3=\begin{bmatrix}0&0&0&1\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad B_1=\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\quad B_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad B_3=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}. \tag{20}$$

By calculating, we have $rk(B-A)=1$, $rk(B)=3$, $rk(A)=2$, and $rk(B)-rk(A)=1$. With condition (a), we have $A\overset{-}{\leq}B$. However, as $rk(B_3-A_3)=1\neq rk(B_3)-rk(A_3)=-1$, that is, A_3 is not below B_3 under the minus partial order, we derive that A is not below B under the core-E-N partial order. Therefore, the minus partial order cannot lead to the core-E-N partial order.

Next, we characterize and introduce another new partial order.

Definition 3 (Core-E-S order). Let $A, B\in\mathbb{C}_{n,n}$, and they are decomposed as Definition 1 where A_1 and B_1 are core invertible, A_2 and B_2 are EP, A_3 and B_3 are nilpotent. We consider the binary operation:

$$A\overset{CES}{\leq}B\colon\ A_1\overset{\textcircled{\#}}{\leq}B_1,\ A_2\overset{\#}{<}=B_2\ \text{and}\ A_3\overset{-}{\leq}B_3. \tag{21}$$

Theorem 4. *Operation (21) is called the core-E-S partial order.*

Proof. Similar to the proof of Theorem 2, according to Definition 3, we can easily conclude that the binary relation is partial order.

The following theorem works in a similar way to Theorem 3. □

Theorem 5. *Let $A, B\in\mathbb{C}_{n,n}$. Then, $A\overset{CES}{\leq}B$ if and only if there exists a unitary matrix U satisfying*

$$A=U\begin{bmatrix}T_1&T_2&S_1&S_2\\0&T_4&0&S_3\\0&0&0&S_4\\0&0&0&N_1\end{bmatrix}U^*,\quad B=U\begin{bmatrix}T_1&T_2&S_1&\widetilde{S}_2\\0&T_3+T_4&\widetilde{S}_1&\widetilde{S}_3\\0&0&T_5&\widetilde{S}_4\\0&0&0&N_2\end{bmatrix}U^*, \tag{22}$$

where T_1, T_3, T_4, and T_5 are invertible and upper-triangular, N_1 and N_2 are nilpotent and upper-triangular, and $[S_2^*S_3^*S_4^*N_1^*]\overset{-}{\leq}[\widetilde{S}_2^*\widetilde{S}_3^*\widetilde{S}_4^*N_2^*]$.

Proof. Let $A\overset{CES}{\leq}B$, where A and B are decomposed as Definition 1. Then, $A_1\overset{\textcircled{\#}}{\leq}B_1$, $A_2\overset{\#}{\leq}B_2$, and $A_3\overset{-}{\leq}B_3$. Because of $A_1, B_1\in\mathbb{C}_n^{CM}$, $A_1\overset{\textcircled{\#}}{\leq}B_1$ and $A_2, B_2\in\mathbb{C}_n^{\mathrm{EP}}$, $A_2\overset{\#}{\leq}B_2$, we can imply that

$$A_1=U\begin{bmatrix}T_1&T_2&S_1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}U^*,\quad A_2=U\begin{bmatrix}0&0&0&0\\0&T_4&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}U^*,$$

$$B_1 = U\begin{bmatrix} T_1 & T_2 & S_1 & 0 \\ 0 & T_3 & \widetilde{S}_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}U^*,$$

$$B_2 = U\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & T_4 & 0 & 0 \\ 0 & 0 & T_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}U^*, \quad (23)$$

where $T_1, T_3, T_4,$ and T_5 are invertible and upper-triangular. Because of Definition 3, we can derive that

$$A_3 = U\begin{bmatrix} 0 & 0 & 0 & S_2 \\ 0 & 0 & 0 & S_3 \\ 0 & 0 & 0 & S_4 \\ 0 & 0 & 0 & N_1 \end{bmatrix}U^*,$$

$$B_3 = U\begin{bmatrix} 0 & 0 & 0 & \widetilde{S}_2 \\ 0 & 0 & 0 & \widetilde{S}_3 \\ 0 & 0 & 0 & \widetilde{S}_4 \\ 0 & 0 & 0 & N_2 \end{bmatrix}U^*, \quad (24)$$

where N_1 and N_2 are nilpotent and upper-triangular. Since $A_3 \bar{\leq} B_3$, we have the following results:

$$\left[S_2^* S_3^* S_4^* N_1^*\right] \bar{\leq} \left[\widetilde{S}_2^* \widetilde{S}_3^* \widetilde{S}_4^* N_2^*\right]. \quad (25)$$

The proof is complete.

Next, we will investigate whether core-E-S partial order is minus type or not. □

Example 3. We assume A, B which their forms as shown in the Example 1. We can check that $A_1 \overset{\circledast}{\leq} B_1$, $A_2 \overset{\#}{\leq} B_2$ and $A_3 \bar{\leq} B_3$ by calculating, which implies $A \overset{CEN}{\leq} B$. However, since $rk(B-A) = 1 \neq rk(B) - rk(A) = 0$, it is contradicted with condition (a) and we obtain the following corollary.

Corollary 2. *The core-E-S partial order is not the minus type.*

After the above discussion, we know that both the core-E-N and core-E-S partial orders are not minus type. Therefore, we will study the relationship between core-E-N partial order and core-E-S partial order under some conditions.

It is worth noting that when $A, B \in \mathbb{C}_n^{\mathrm{EP}}$, we assume

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{C}_3^{\mathrm{EP}}. \quad (26)$$

By calculating, we can easily check that $A \bar{\leq} B$ and $A \overset{CEN}{\leq} B$. However,

$$AB = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (27)$$

With the definition of (b), we get that A and B do not hold sharp partial order relationship, i.e., $A \bar{\leq} B \nRightarrow A^{\#} \leq B$. With the Definitions 2 and 3, we derive that $A \overset{CEN}{\leq} B \nRightarrow A \overset{CES}{\leq} B$.

According to ([15], Remark 4.2.2), it has been proved that $A \overset{\#}{\leq} B \Rightarrow A \bar{\leq} B$, so we can draw a corollary as follows:

Corollary 3. *Let $A, B \in \mathbb{C}_n^{EP}$, $A \overset{CES}{\leq} B \Rightarrow A \overset{CEN}{\leq} B$.*

On the basis of [16] (Theorem 2.1), if $A, B \in \mathbb{C}_n^{EP}$, then $A \overset{\#}{\leq} B \Leftrightarrow A^* \leq B$. Moreover, we can get another way of defining core-E-S partial order as follows:

$$A \overset{CES}{\leq} B\colon\ A_1 \overset{\circledast}{\leq} B_1,\ A_2{}^* \leq B_2 \text{ and } A_3 \bar{\leq} B_3, \quad (28)$$

where the decomposition of A and B are the core-EP-nilpotent decompositions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This paper was supported by the Fundamental Research Funds for the Central Universities (No. 2572021BC03).

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