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Research Article

3-Group Divisible Designs with 3 Groups and Block Size 5

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A 3-GDD $(n, 2, k, \lambda_1, \lambda_2)$ was defined by combining the definitions of a group divisible design and a t-design. In this paper, we extend the definitions to 3 groups and block size 5, and we denote such GDD by 3-GDD $(n, 3, 5, \mu_1, \mu_2)$. Some necessary conditions for the existence of these GDDs are developed, and several new constructions and specific instances of nonexistence are given.

1. Introduction

A group divisible design, GDD $(n, m, k, \lambda_1, \lambda_2)$, is an ordered triple $(\mathbb{V}, \mathbb{G}, \mathbb{B})$, where $\mathbb{V}$ is a mn-set of symbols, $\mathbb{G}$ is a partition of $\mathbb{V}$ into m sets called groups of size n each, and $\mathbb{B}$ is a collection of k-subsets called blocks of $\mathbb{V}$, such that each pair of symbols from the same group occurs in exactly λ_1 blocks and each pair of symbols from different groups occurs in exactly λ_2 blocks [1–5]. Any two symbols occurring together in the same group are called first associates, and pairs of symbols occurring in different groups are called second associates.

Group divisible designs (GDDs) have been studied for their usefulness in statistics, coding and for their universal application to constructions of new designs (for instance balanced incomplete design, orthogonal arrays, and transversal designs etc.) [6–8]. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 that began classifying such designs [9]. GDDs and t-designs have been studied by many authors [10–13] and the references therein. Recently, a 3-GDD $(n, 2, k, \lambda_1, \lambda_2)$ was defined by extending the definitions of a group divisible designs and a t-design, and some necessary conditions for its existence were given [14, 15].

In this paper, these recent works are extended to include more than two groups. We mainly continue to focus on the definition of 3-GDDs, and we explicitly consider the case when the required designs have three groups of size n each and block size 5. Throughout this paper, such GDD is denoted by 3-GDD $(n, 3, 5, \mu_1, \mu_2)$. In this work, some necessary conditions for the existence of such GDDs are determined, the existence of some GDDs will be proved and their constructions are produced. Furthermore, several specific instances of nonexistence are proved.

This work is organized as follows. In Section 2, we present some well-known definitions and examples that will be used to proof the main results. In Section 3, some necessary conditions for the existence of such designs together with their proofs are given. In Section 4, the proofs of some constructions especially when $\mu_2 = 0$ are presented. Finally, in Section 5, an infinite families of existences for the 3-GDD when $n = 3$ are given.

2. Preliminaries

In this section, we present some well-known definitions and concepts which will be used in the subsequent sections.

Definition 1 (see [14]). A t-(v, k, λ) design, or a t-design is a pair $(\mathbb{X}, \mathbb{B})$, where $\mathbb{X}$ is a v-set of points and $\mathbb{B}$ is a collection of k-subsets (blocks) of $\mathbb{X}$ with the property that every t-subset of $\mathbb{X}$ is contained in exactly λ blocks. The parameter λ is called the index of the design.

A necessary condition for the existence of a t-(v,k,λ) design is known [13] to be the following equation:

$$\lambda.\frac{\begin{pmatrix} v-h \\ t-h \end{pmatrix}}{\begin{pmatrix} k-h \\ t-h \end{pmatrix}} = \text{integer}, h = 0,1,2,3,\ldots,t-1. \quad (1)$$

The problem of determining the values of k, t, and $\mathbb{X}$ for which this condition is also sufficient is not yet solved completely, and less is known about configurations with $t = 3$.

Example 1. A 3-$(8,4,1)$ design on the set $\mathbb{X} = \{1,2,\ldots,8\}$ is given by the blocks: $\{1,2,5,6\}$, $\{3,4,7,8\}$, $\{1,3,5,7\}$, $\{2,4,6,8\}$, $\{1,4,5,8\}$, $\{2,3,6,7\}$, $\{1,2,3,4\}$, $\{5,6,7,8\}$, $\{1,2,7,8\}$, $\{3,4,5,6\}$, $\{1,3,6,8\}$, $\{2,4,5,7\}$, $\{1,4,6,7\}$ and $\{2,3,5,8\}$.

The concepts of a GDD and a 3-design can be merged to define a 3-GDD as follows.

Definition 2. A $3'$-GDD $(n,m,k,\Lambda_1,\Lambda_2)$, for $m > 2$, is a set $\mathbb{X}$ of mn symbols partitioned into m parts of size n called groups together with a collection of k-subsets of $\mathbb{X}$ called blocks, such that

(i) Every 3-subset of each group occurs in Λ_1 blocks

(ii) Every mixed 3-subset, meaning either two symbols are from one group and one symbol from the other group or all three symbols are from different groups, occurs in Λ_2 blocks

The above mentioned definition was given in [14, 15], in which case a 3-subset of $\mathbb{X}$ has only two choices, all symbols from one of the two groups or one symbol from a group and two symbols from the other group. Such GDD was denoted by a 3-GDD (n, 2, 4, λ_1, λ_2).

(i) The necessary conditions are sufficient for the existence of a 3-GDD (n, 2, 4, λ_1, λ_2) for $n \equiv 1,7,9 \pmod{12}$ [14]

(ii) Except possibly when $n \equiv 1,3 \pmod 6$, $n \neq 3$, 7, 13, the necessary conditions are sufficient for the existence of a 3-GDD (n, 2, 4, λ_1, λ_2) and $\lambda_1 > \lambda_2$ [15]

Lemma 3 (see [14]). *If a 3-(2n, 4, λ_2) and a 3-(2n, 4, $\lambda_1 - \lambda_2$) exists, then a 3-GDD (n, 2, 4, λ_1, λ_2) exists.*

Example 2. Here, $\mathbb{X} = \{1,2,3,a,b,c\}$, $G_1 = \{1,2,3\}$ and $G_2 = \{a,b,c\}$, the blocks of a 3-GDD $(3,2,4,3,1)$ are: $\{1,2,3,a\}$, $\{1,2,3,b\}$, $\{1,2,3,c\}$, $\{a,b,c,1\}$, $\{a,b,c,2\}$ and $\{a,b,c,3\}$.

Following necessary conditions (Table 1, where the values of λ_1 and λ_2 are given modulo 6) and the existence results of a 3-GDD $(n,2,4,\lambda_1,\lambda_2)$ are given in [14, 15].

One may relax condition (ii) in Definition 2 to define a 3-PBIBD as follows.

Definition 4. A 3-PBIBD (partially balanced incomplete block design), 3-PBIBD (n, m, k, θ_1, θ_2, θ_3) is a collection of k-subsets of an mn set $\mathbb{X}$, where $\mathbb{X}$ is partitioned in to m groups of order n such that

(i) Every triple formed from symbols of only a single group occurs in θ_1 blocks

(ii) Every triple formed from symbols of only two groups occurs in θ_2 blocks

(iii) Every triple formed from symbols of all three groups occurs in θ_3 blocks

Definition 5. A 3-GDD (n, m, k, μ_1, μ_2) is a pair ($\mathbb{X}$, $\mathbb{B}$), where $\mathbb{X}$ is a set of mn elements partitioned into mn-subsets (groups) and $\mathbb{B}$ is a collection of k-subsets (blocks) of $\mathbb{X}$ such that

(i) Every triple occurs in exactly μ_1 blocks if it contains elements from at most 2 groups

(ii) It occurs in exactly μ_2 blocks if it has all three elements from different groups

Definition 5 is the subject matter of this paper and we explicitly consider the case in which $m = 3$ and $k = 5$. Such GDD is denoted by 3-GDD (n, 3, 5, μ_1, μ_2).

Remark 6. A 3-GDD (n, 3, 5, μ_1, μ_2) is a 3-PBIBD (n, 3, 5, θ_1, θ_2, θ_3), where $\theta_1 = \theta_2$, denoted by μ_1, while θ_3 is denoted by μ_2.

When $\mu_1 = \mu_2$, a 3-GDD (n, 3, 5, μ_1, μ_2) is actually a 3-($3n$, 5, μ_1). It follows that a 3-GDD (n, 3, 5, μ_1, μ_2) exists if and only if a 3-($3n$, 5, μ_1) exists.

The next three sections of this paper discuss our research findings.

3. Necessary Conditions

In this section, assuming a 3-GDD (n, 3, 5, μ_1, μ_2) exists, we obtain some necessary conditions for the existence of a 3-GDD (n, 3, 5, μ_1, μ_2).

Let λ_1 (first associate pair) denote the number of blocks containing $\{x_1, x_2\}$, where x_1 and x_2 are from the same group, λ_2 (second associate pair) denote the number of blocks containing $\{x, y\}$ where x and y are from different groups, r and b, respectively, denote the replication number and the number of blocks in a 3-GDD.

Theorem 7. *Given a 3-GDD* $(n,3,5,\mu_1,\mu_2)$,

$$r = \frac{(n-1)(7n-2)\mu_1 + 2n^2\mu_2}{12}, \quad (2)$$

$$b = \frac{n}{20}\left((n-1)(7n-2)\mu_1 + 2n^2\mu_2\right), \quad (3)$$

$$\lambda_1 = \frac{(3n-2)}{3}\mu_1, \quad (4)$$

$$\lambda_2 = \frac{2(n-1)\mu_1 + n\mu_2}{3}, \quad (5)$$

Table 1: Congruence restrictions for 3-GDD $(n, 2, 4, \lambda_1, \lambda_2)$.

λ_1/λ_2	0	1	2	3	4	5
0	All n	n even	All n	n even	All n	n even
1	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$
2	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$
3	n even	All n	n even	All n	n even	All n
4	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$
5	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$	$2, 4 \pmod 6$	$1, 2, 4, 5 \pmod 6$

Proof. The abovementioned results can be proved as follows:

(1) We count the number of triples containing a fixed element x in the design in two ways.
First, given an element x, it appears in $\binom{n-1}{2}$ triples of the type $(3, 0)$ and in $2n(n-1)+2\binom{n}{2}$ triples of the type $(2,1)$ (there are $2n(n-1)$ triples of the type $(2,1)$ where x and an element from the same group as x appear with an element of another group, and there are $2\binom{n}{2}$ triples of type $(2,1)$ where x appears with two elements from another group). So, in sum, x is in $\left(\binom{n-1}{2}+3n(n-1)\right)\mu_1$ triples of the form $(3,0)$ or $(2,1)$ in the design.
Similarly, there are n^2 triples of the type $(1,1,1)$ containing x, and are repeated μ_2 times.
All together, there are $\left(\binom{n-1}{2}+3n(n-1)\right)\mu_1 + n^2\mu_2$ triples containing x.
Second, in every block containing x, there are 6 triples containing x and x occurs in r blocks, which means there are $6r$ triples containing x in the design.
Hence, $6r = \left(\binom{n-1}{2}+3n(n-1)\right)\mu_1 + n^2\mu_2$, and $r = ((n-1)(7n-2)\mu_1 + 2n^2\mu_2)/12$

(2) Again, counting in two ways, in a design with block size 5 and b blocks, there are $b\binom{5}{3} = 10b$ triples. On the other hand, there must be exactly $3\binom{n}{3}\mu_1$ triples of type $(3,0)$, $3n^2(n-1)\mu_1$ triples of type $(2,1)$, and $n^3\mu_2$ triples of type $(1,1,1)$ in the design.
Therefore, $10b = 3\binom{n}{3}\mu_1 + 3n^2(n-1)\mu_1 + n^3\mu_2$ gives equation (3).

(3) Let (x_1, x_2) be a first associate pair and occurs in λ_1 blocks. The triples containing (x_1, x_2) can only be of type $(3,0)$ or $(2, 1)$. There are $(n-2)$ triples of the type $(3,0)$ and $2n$ triples of the type $(2,1)$ containing (x_1, x_2). Now, since these triples occur μ_1 times and since the block containing a first associate pair, say (x_1, x_2) contains three triples containing (x_1, x_2), we have: $\lambda_1 = ((3n-2)/3)\mu_1$.

(4) Let (x, y) be a second associate pair. This pair occurs in triples of the type $(2, 1)$ and $(1, 1, 1)$ only. It occurs in $2(n-1)\mu_1$ triples of type $(2,1)$ and $n\mu_2$ triples of type $(1, 1, 1)$.

Each of the λ_2 blocks containing the pair (x, y) has three triples containing (x, y).
Hence, $3\lambda_2 = 2(n-1)\mu_1 + n\mu_2$ and $\lambda_2 = (2(n-1)\mu_1 + n\mu_2)/3$. □

As a consequence of Theorem 7, we have the following corollaries.

Corollary 8. *For $\mu_1 \equiv 0 \pmod 3$, a 3-GDD $(n, 3, 5, \mu_1, \mu_2)$ does not exist.*

Proof. From equation (3) in Theorem 7, λ_1 is not an integer when $\mu_1 \equiv 0 \pmod 3$. □

Corollary 9. *In a 3-GDD $(n, 3, 5, \mu_1, \mu_2)$, $\lambda_1 \neq 0$.*

Proof. As the number of groups is less than the block size, $\mu_1 \neq 0$. Again from (3) in Theorem 7, λ_1 is a multiple of μ_1, and hence $\lambda_1 \neq 0$

From the original necessary conditions in Theorem 7, when $\mu_1 = \mu_2 = \lambda$, we get 3-$(3n, 5, \lambda)$, and from (4) and (5), we must have $\lambda \equiv 0 \pmod 3$. In addition, from (2) and (3) in Theorem 7, the following result follows. □

Corollary 10. *The necessary conditions for the existence of 3-$(3n, 5, \lambda)$ are satisfied only under the following cases:*

(i) If $n \equiv 2, 7, 10, 14, 15 \pmod{20}$, then $\lambda \equiv 0 \pmod 3$
(ii) If $n \equiv 0, 4, 5, 9, 12, 17 \pmod{20}$, then $\lambda \equiv 0 \pmod 6$
(iii) If $n \equiv 3, 6, 11, 18 \pmod{20}$, then $\lambda \equiv 0 \pmod{15}$
(iv) If $n \equiv 1, 8, 13, 16 \pmod{20}$, then $\lambda \equiv 0 \pmod{30}$

Theorem 11. *Given a 3-GDD $(n, 3, 5, \mu_1, \mu_2)$,*

(i) $\mu_2 < 3\mu_1$
(ii) $b \geq (3n^2(n-1)\mu_1 + n^3\mu_2)/15$

Proof

(i) In a 3-GDD $(n, 3, 5, \mu_1, \mu_2)$, triples of the type $(1, 1, 1)$ occur in the blocks of type $(3, 1, 1)$ and $(2, 2, 1)$ only. In each of these blocks which can have a $(1, 1, 1)$ triples, the number of $(2, 1)$ triples are more than the number of $(1, 1, 1)$ triples.

Hence, $3n^2(n-1)\mu_1 > n^3\mu_2$, implies $\mu_2 < 3(n-1)\mu_2/n < 3\mu_1$.

(ii) We calculated the value of $b = (n/20)((n-1)(7n-2)\mu_1 + 2n^2\mu_2)$ in Theorem 7, if we do not include the triples of the form (3, 0), in the calculations, we obtain the following equation:

$$b \geq \frac{3n^2(n-1)\mu_1 + n^3\mu_2}{15}. \tag{6}$$

□

Theorem 12. *For $n \leq 5$, a 3-GDD $(n, 3, 5, \mu_1, 0)$ does not exist.*

Proof. As $\mu_2 = 0$, blocks are of configuration $(5, 0)$, $(4, 1)$ and $(3, 2)$ only. The maximum number of $(2, 1)$ triples occur in $(3, 2)$ configuration blocks and ratio is $9:1$ (in each block of configuration $(3, 2)$, there are nine $(2, 1)$ and one $(3, 0)$ triples). Counting the number of $(2, 1)$ triples $(3n^2(n-1)\mu_1)$ and the number of $(3, 0)$ triples $(n(n-1)(n-2)/2)$, their ratio must be less than or equal to 9.

Thus, $3n^2(n-1)\mu_1/n(n-1)(n-2)/2 \leq 9 \Rightarrow 6n/n-2 \leq 9$, which gives $n \geq 6$.

From (4) and (5) in Theorem 7, the necessary conditions are satisfied under the following conditions. □

Lemma 13. *In regards to*

(i) λ_1, *for every* n, $\mu_1 \equiv 0 \pmod 3$ *and* μ_2 *is free*

(ii) λ_2, *when*

- *(i)* $n \equiv 0 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ *and* μ_2 *is free*
- *(ii)* $n \equiv 1 \pmod 3$, μ_1 *is and* $\mu_2 \equiv 0 \pmod 3$
- *(iii)* $n \equiv 2 \pmod 3$, $\mu_1 + \mu_2 \equiv 0 \pmod 3$

As the values of b and r must also be integers, Table 2 is a table of congruence restrictions for a 3-GDDs with 3 groups and block size 5 (all values are considered to be in terms of $\pmod{60}$ *unless otherwise stated).*

where

(i) $\lambda_1 = 4\mu_1 + \mu_2 \equiv 0 \pmod 5$, $\lambda_2 = \mu_1 + \mu_2 \equiv 0 \pmod{10}$, $\lambda_3 = 9\mu_1 + \mu_2 \equiv 0 \pmod{10}$, $\lambda_4 = 4\mu_1 + \mu_2 \equiv 0 \pmod{10}$, $\lambda_5 = 5\mu_1 + \mu_2 \equiv 0 \pmod{10}$, $\lambda_6 = 8\mu_1 + 3\mu_2 \equiv 0 \pmod{10}$, $\lambda_7 = \mu_1 + \mu_2 \equiv 0 \pmod 2$, *and* $\lambda_8 = \mu_1 + \mu_2 \equiv 0 \pmod 5$

(ii) $*_1 = 5\mu_1 + 3\mu_2 \equiv 0 \pmod 6$, $*_2 = 3\mu_1 + 2\mu_2 \equiv 0 \pmod 6$, $*_3 = 3\mu_1 + \mu_2 \equiv 0 \pmod 6$, *and* $*_4 = 2\mu_1 + 3\mu_2 \equiv 0 \pmod 6$.

Table 2: Congruence restrictions for 3-GDD $(n, 3, 5, \mu_1, \mu_2)$.

	r	r	b	b
n	μ_1	μ_2	μ_1	μ_2
0	0 (mod 6)	All	All	All
1, 41	All	0 (mod 6)	All	0 (mod 10)
2, 14, 22, 34, 42, 54	All	0 (mod 3)	λ_1	λ_1
3	$*_1$	$*_1$	λ_2	λ_2
4, 32, 44, 52	$*_2$	$*_2$	λ_1	λ_1
5, 25	All	0 (mod 6)	All	0 (mod 2)
6	0 (mod 3)	All	All	0 (mod 5)
7, 19, 47, 59	$*_3$	$*_3$	λ_3	λ_3
8	$*_2$	$*_2$	λ_8	λ_8
9, 57	$*_4$	$*_4$	λ_4	λ_4
10, 50	All	0 (mod 3)	All	All
11, 31	$*_3$	$*_3$	λ_5	λ_5
12, 24	0 (mod 6)	All	λ_1	λ_1
13, 53	All	0 (mod 6)	λ_6	λ_6
15	$*_1$	$*_1$	λ_7	λ_7
16, 56	$*_2$	$*_2$	All	0 (mod 5)
17, 29, 37, 49	All	0 (mod 6)	λ_4	λ_4
18	0 (mod 3)	All	λ_8	λ_8
20, 40	$*_2$	$*_2$	All	All
21	$*_4$	$*_4$	All	0 (mod 10)
23, 43	$*_3$	$*_3$	λ_2	λ_2
26, 46	All	0 (mod 3)	All	0 (mod 5)
27, 39	$*_1$	$*_1$	λ_3	λ_3
28	$*_2$	$*_2$	λ_7	λ_7
30	0 (mod 3)	All	All	All
33	$*_4$	$*_4$	λ_6	λ_6
35, 55	$*_3$	$*_3$	λ_7	λ_7
36	0 (mod 6)	All	All	0 (mod 5)
38, 58	All	0 (mod 3)	λ_8	λ_8
45	$*_4$	$*_4$	All	0 (mod 2)
48	0 (mod 6)	All	λ_7	λ_7
51	$*_1$	$*_1$	λ_5	λ_5

b, r, λ_1, and λ_2 must be positive integers for a design to exist. From Lemma 13 and Table 2, we have more compact necessary conditions for different values of n as follows.

Lemma 14. *For*

(i) $n \equiv 0 \pmod{10}$, μ_1 *and* μ_2 *are free in regards to the number of blocks b*

(ii) $n \equiv 0 \pmod 3$ *and* $\mu_1 \equiv 0 \pmod 3$, λ_1*&*λ_2 *are integers for any chosen* μ_2

(iii) $n \equiv 1, 2 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ *and* $\mu_2 \equiv 0 \pmod 3$, λ_1 *&* λ_2 *are integers*

(iv) $n \equiv 1, 5 \pmod{12}$ *and* $\mu_2 \equiv 0 \pmod 6$ *or* $n \equiv 2, 10 \pmod{12}$ *and* $\mu_2 \equiv 0 \pmod 3$, *r is an integer for any chosen* μ_1

(v) $n \equiv 1, 2, 4, 7, 10 \pmod{12}$ *and* $n \equiv 5, 17, 41, 53 \pmod{60}$, μ_2 *is multiple of 3*

(vi) $n \equiv 6 \pmod{10}$ *and* $n \equiv 1 \pmod{20}$, μ_2 *is multiple of 5*

4. Some Constructions When $\mu_2 = 0$

In this section, the proofs of some constructions when $\mu_2 = 0$ are given.

Theorem 15

(a) *All blocks of a design are of type* $(3, 2)$ *constitute a 3-PBIBD (n, 3, 5,* $n(n-1)$, $3(n-1)(n-2)/2$*, 0), where the number of blocks of such design is* $6\binom{n}{3}\binom{n}{2}$

(b) *There exists a 3-PBIBD (n, 3, 5,* $(n-3)(n-4)/2$*, 0, 0)*

Proof

(a) When all blocks are of type $(3, 2)$, the number of blocks of such design is $6\binom{n}{3}\binom{n}{2}$, where $3\binom{n}{3}$ gives the number of 3-subsets of 3 groups and $2\binom{n}{2}$ gives the number of 2-subsets for each choice of a 3-subset. This also tells the number of blocks containing every $(3, 0)$ triple is $2\binom{n}{2} = n(n-1)$.
Similarly, let $\{a, b, x\}$ be $(2, 1)$ triple, where the pair $\{a, b\}$ is from say G_1 and the element x is from G_2. There are $(n-2)(n-1)$ triples containing $\{a, b, x\}$ (one element each from G_1 and G_2) or there are $\binom{n-1}{2}$ triples containing $\{a, b, x\}$ (3-subsets of G_2), and hence in total there are $3(n-1)(n-2)/2$ blocks of size five containing $\{a, b, x\}$.

(b) When all blocks are of type $(5, 0)$, the number of blocks containing every $(3, 0)$ triple is $\binom{n-3}{2} = (n-3)(n-4)/2$. □

Example 3. A 3-GDD (6, 3, 5, 30, 0) exists.

In Theorem 15 (a), if $n = 6$, then, $\theta_1 = \theta_2 = n(n-1) = 3(n-1)(n-2)/2 = 30$.

Blocks are of type (4, 1) if there are four elements from a single group and one element from other group.

Theorem 16. *If all the blocks are of type* $(4, 1)$*, then a 3-GDD (n, 3, 5,* μ_1*, 0) does not exist.*

Proof. when all blocks are of type $(4, 1)$, a fixed $(3, 0)$ triple occurs in $2n(n-3)$ and a fixed $(2, 1)$ triple occurs in $\binom{n-2}{2}$ blocks.

Hence, $2n(n-3) = \binom{n-2}{2}$, which is true only for $3n = -2$, which is impossible.

In Theorem 12, a 3-GDD (n, 3, 5, μ_1, 0) does not exist when $n \leq 5$. But, when $n \geq 6$, a 3-GDD (n, 3, 5, μ_1, 0) exists for some μ_1. □

Theorem 17. *For* $n \geq 6$*, a 3-GDD (n, 3, 5,* $3(n-1)(n-2)(n-3)(n-4)/4$*, 0) exists.*

Proof. When $n \geq 6$, taking $\binom{n-3}{2} = (n-3)(n-4)/2$ copies of the blocks of Theorem 15(a) together with $3(n-1)(n-2)/2 - n(n-1) = (n-1)(n-6)/2$ copies of the blocks of Theorem 15(b) give the blocks of 3-GDD (n, 3, 5, $3(n-1)(n-2)(n-3)(n-4)/4$, 0). □

Example 4. By Theorem 17, the following 3-GDDs exist:

3-GDD (6, 3, 5, 90, 0), 3-GDD (7, 3, 5, 270, 0), 3-GDD (8, 3, 5, 630, 0)

3-GDD (9, 3, 5, 1260, 0), 3-GDD (10, 3, 5, 2268, 0), 3-GDD (11, 3, 5, 3780, 0)

3-GDD (12, 3, 5, 5940, 0), etc.

Theorem 18. *There exists a 3-GDD (n, 3, 5,* $(n-1)(n-2)(n-3)(n-4)/4$*, 0) for* $n \geq 6$ *and* $n \equiv 2 \pmod 3$*.*

Proof. If $n \equiv 2 \pmod 3$ and $n \geq 6$, then both $(n-1)(n-6)/2$ and $(n-3)(n-4)/2$ are divisible by 3. Joining $(n-3)(n-4)/6$ copies of the blocks of Theorem 15(a) together with $(n-1)(n-6)/6$ copies of the blocks of Theorem 15(b) give the blocks of 3-GDD (n, 3, 5, $(n-1)(n-2)(n-3)(n-4)/4$, 0). □

Example 5. By Theorem 18 above, the following 3-GDDs exist:

3-GDD (6, 3, 5, 30, 0), 3-GDD (7, 3, 5, 90, 0), 3-GDD (9, 3, 5, 420, 0)

3-GDD (10, 3, 5, 756, 0), 3-GDD (12, 3, 5, 1980, 0), 3-GDD (13, 3, 5, 2970, 0)

3-GDD (15, 3, 5, 6006, 0), etc.

5. 3-GDD (3, 3, 5, μ_1, μ_2)

In this section, we give some families of existences along with several examples of the 3-GDD when $n = 3$. In addition, some relationship between $3'$-GDD (n, m, k, Λ_1, Λ_2)(Definition 2) and 3-PBIBD (n, m, k, θ_1, θ_2, θ_3) (Definition 4) is designed. We begin with the following three examples (Examples 1–3) of a 3-PBIBD (3, 3, 5, θ_1, θ_2, θ_3).

Example 6. A 3-PBIBD (3, 3, 5, 0, 3, 4) with $G_1 = \{1, 2, 3\}$, $G_2 = \{a, b, c\}$, $G_3 = \{x, y, z\}$ exists and the blocks (written vertically) of this design are as follows.

1	1	1	1	1	1	2	2	2	3	1	2	1	2	3	2	3	1	1	1	2	1	1	2	1	1	2
2	2	2	3	3	3	3	3	3	a	a	a	a	a	a	b	b	b	2	3	3	2	3	3	2	3	3
a	a	b	a	a	b	a	a	b	b	b	b	c	c	c	c	c	c	a	b	c	b	c	a	c	a	b
b	c	c	b	c	c	b	c	c	x	x	y	x	x	y	x	x	y	x	x	x	x	x	x	y	y	y
y	z	x	z	x	y	x	y	z	y	z	z	y	z	z	y	z	z	y	y	y	z	z	z	z	z	z

Example 7. A 3-PBIBD (3, 3, 5, 6, 3, 0) exists and the blocks of this design are obtained by union of every G_i with every two element set of G_j for $i \neq j$, where $i,\ j = 1,\ 2,\ 3$.

Example 8. 3-PBIBD (3, 3, 5, 9, 3, 3) exists. Its blocks are obtained by union of every group with each singleton set of both remaining groups.

Remark 19. When $n = 3$, from the original necessary conditions in Theorem 7; r, b, λ_1 and λ_2, respectively, are integers if $*_1$: μ_1 +3 $\mu_2 \equiv$ (mod 6), $*_2$: $\mu_1 + \mu_2 \equiv$ (mod 10), $\mu_1 \equiv 0 \pmod 3$, and $\mu_1 \equiv 0 \pmod 3$.

Thus, the values of μ_1 and μ_1 satisfying the necessary conditions are as follows: $\mu_1 \equiv 0 (\text{mod } 3) \cap *_1 \cap *_2$ and $\mu_2 \equiv *_1 \cap *_2$.

For $n = 3$, taking all the possible combinations, the original necessary conditions are satisfied when

(1) $\mu_1 \equiv 3 (\text{mod } 30)$ and $\mu_2 \equiv 7 (\text{mod } 10)$
(2) $\mu_1 \equiv 9 (\text{mod } 30)$ and $\mu_2 \equiv 1 (\text{mod } 10)$
(3) $\mu_1 \equiv 15 (\text{mod } 30)$ and $\mu_2 \equiv 5 (\text{mod } 10)$
(4) $\mu_1 \equiv 21 (\text{mod } 30)$ and $\mu_2 \equiv 9 (\text{mod } 10)$
(5) $\mu_1 \equiv 27 (\text{mod } 30)$ and $\mu_2 \equiv 3 (\text{mod } 10)$
(6) $\mu_1 \equiv 0 (\text{mod } 30)$ and $\mu_2 \equiv 0 (\text{mod } 10)$
(7) $\mu_1 \equiv 6 (\text{mod } 30)$ and $\mu_2 \equiv 4 (\text{mod } 10)$
(8) $\mu_1 \equiv 12 (\text{mod } 30)$ and $\mu_2 \equiv 8 (\text{mod } 10)$
(9) $\mu_1 \equiv 18 (\text{mod } 30)$ and $\mu_2 \equiv 2 (\text{mod } 10)$
(10) $\mu_1 \equiv 24 (\text{mod } 30)$ and $\mu_2 \equiv 6 (\text{mod } 10)$

Example 9. 3-GDD (3, 3, 5, 9, 11) exists and its blocks are obtained by combining the blocks of Example 3 with two copies of the blocks of Example 6.

Example 10. 3-GDD (3, 3, 5, 6, 4) exists and its blocks are obtained by joining blocks of Examples 6 and 7.

Hence, we have 3-GDD (3, 3, 5, $6t$, $4t$), for $t \geq 1$.

Theorem 20. *There exists a 3-GDD (3, 3, 5, $9q + 3p$, $11q - 3p$) for $p \leq q$.*

Proof. Blocks of the design are obtained by taking p copies of 3-GDD (3, 3, 5, 12, 8) together with $q - p$ copies of Example 9. □

Corollary 21. *There exists a 3-GDD (3, 3, 5, $9q + 3p + 6$, $11q - 3p + 4$) for $p \leq q$.*

Proof. Taking blocks of 3-GDD (3, 3, 5, $9q + 3p$, $11q - 3p$) together with blocks of Example 10 give 3-GDD (3, 3, 5, $9q + 3p + 6$, $11q - 3p + 4$). □

Corollary 22. *For all $p \geq 1$, 3-(9, 5, $30p$) exists.*

Proof. Blocks of such design are obtained by taking p-copies of 3-GDD (3, 3, 5, 12, 8) together with $2p$-copies of Example 9. □

Corollary 23. *There exists a 3-(9, 5, $30p + 15$) for all $p \geq 0$.*

Proof. For $p \geq 0$, blocks of 3-(9, 5, $30p + 15$) are obtained by joining p-copies of blocks of 3-GDD (3, 3, 5, 12, 8), $2p + 1$-copies of blocks of Example 9 and blocks of Example 10.

The next Theorem discusses the situation in which the existence of a $3'$-GDD (Definition 2) guarantees the existence of 3-PBIBD (Definition 4). □

Theorem 24 (Block Complementation). *Let $k \leq mn - 3$. If a $3'$-GDD (n, m, k, Λ_1, Λ_2) exists, then a 3-PBIBD (n, m, $nm - k$, θ_1, θ_2, θ_3) also exists, where $\theta_1 = b - 3r + 3\lambda_1 - \Lambda_1$, $\theta_2 = b - 3r + \lambda_1 + 2\lambda_2 - \Lambda_2$, $\theta_3 = b - 3r + 3\lambda_2 - \Lambda_2$. r, and λ_1&λ_2, respectively, are the replication number, number of first associate pairs, and number of second associate pairs in the $3'$-GDD.*

Proof. Suppose a $3'$-GDD (n, m, k, Λ_1, Λ_2) exists.

Let $(\mathbb{X}, \mathbb{B})$ is a 3-GDD ($n, m, k, \Lambda_1, \Lambda_2$), where $\mathbb{X}$ is a mn set of points partitioned in to m groups of size n each, and $\mathbb{B}$ is a collection of k-subsets (blocks) of $\mathbb{X}$.

To Show $(\mathbb{X}, \{\mathbb{X}/\beta\colon \beta \in \mathbb{B}\})$ is a 3-PBIBD ($n, m, mn - k, \theta_1$, θ_2, θ_3).

This design has mn points partitioned in to m-parts (groups) of size n, and every block contains $mn - k \geq 3$ symbols.

To show every triple of type,

(i) $(3, 0)$ occurs in $\theta_1 = b - 3r + 3\lambda_1 - \Lambda_1$ blocks

(ii) $(2, 1)$ occurs in $\theta_2 = b - 3r + \lambda_1 + 2\lambda_1 - \Lambda_2$ blocks

(iii) $(1, 1, 1)$ occurs in $\theta_3 = b - 3r + 3\lambda_2 - \Lambda_2$ blocks

(1) To show every triple of type $(3, 0)$ occurs in $\theta_1 = b - 3r + 3\lambda_1 - \Lambda_1$ blocks,

Let (x, y, z) is triple of type $(3, 0)$ in $(\mathbb{X}, \mathbb{B})$. There are $3r - 3\lambda_1 + \Lambda_1$ blocks containing x or y or z (containing at least one of them), which means $b - 3r + 3\lambda_1 - \Lambda_1$ blocks in $(\mathbb{X}, \mathbb{B})$ do not contain any one of the three elements.

Therefore, in $(\mathbb{X}, \{\mathbb{X}/\beta\colon \beta \in \mathbb{B}\})$, and (x, y, z) occurs in $\theta_1 = b - 3r + 3\lambda_1 - \Lambda_1$ blocks.

(2) To show every triple of type $(2, 1)$ occurs in $\theta_2 = b - 3r + \lambda_1 + 2\lambda_2 - \Lambda_2$ blocks. Let (x, y, z) is triple of type $(2, 1)$ in $(\mathbb{X}, \mathbb{B})$. There are $3r - \lambda_1 - 2\lambda_2 + \Lambda_2$ blocks containing x or y or z (each block containing (x, y, z) contains one first and two second associate pairs).

Thus, in $(\mathbb{X}, \{\mathbb{X}/\beta\colon \beta \in \mathbb{B}\})$, there are $\theta_2 = b - 3r + \lambda_1 + 2\lambda_1 - \Lambda_2$ blocks containing x, y, and z.

(3) Similarly, if (x, y, z) is triple of type $(1, 1, 1)$ in $(\mathbb{X}, \mathbb{B})$, there are $3r - 3\lambda_2 + \Lambda_2$ blocks containing x or y or z (and each block containing (x, y, z) contains three second associate pairs), and as a result there are $\theta_3 = b - 3r + 3\lambda_2 - \Lambda_2$ blocks not containing the triple.

Thus, in $(\mathbb{X}, \{\mathbb{X}/\beta\colon \beta \in \mathbb{B}\})$, there are $\theta_3 = b - 3r + 3\lambda_2 - \Lambda_2$ blocks containing x, y and z. □

Corollary 25

(1) If a $3'$-GDD $(3, 3, 4, \Lambda_1, \Lambda_2)$ exists then 3-GDD $(3, 3, 5, (9\Lambda_2 + \Lambda_1)/4, (11\Lambda_2 - \Lambda_1)/4)$ exists

(2) If a 3-GDD $(3, 3, 5, \mu_1, \mu_2)$ exists, then $3'$-GDD $(3, 3, 4, (11\mu_1 - 9\mu_2)/5, (\mu_1 + \mu_2)/5)$ exists

Proof

(a) Suppose $3'$-GDD $(3, 3, 4, \Lambda_1, \Lambda_2)$ exists

By Theorem 24, when $m = 3$ and $k = 4$, a 3-PBIBD $(3, 3, 5, \theta_1, \theta_2, \theta_3)$ exists, and $n = 3$ yields $\theta_1 = \theta_2 = (9\Lambda_2 + \Lambda_1)/4$ and $\theta_3 = (11\Lambda_2 - \Lambda_1)/4$.

(b) If a 3-GDD $(3, 3, 5, \mu_1$, and $\mu_2)$ exists, then taking the complement of its blocks gives a 3-PBIBD $(3, 3, 4, \beta_1, \beta_2, \beta_3)$, where $\beta_1 = (11\mu_1 - 9\mu_2)/5$, $\beta_2 = \beta_3 = (\mu_1 + \mu_2)/5$.

As a result of Corollary 25(b), we have the following examples of a $3'$-GDDs when $k = 4$. □

Example 11. A $3'$-GDD $(3, 3, 4, 6, 2)$ exists and its blocks are obtained by taking the complement of the blocks of Example 10.

Example 12. A 3-(9, 4, 6) exists. Its blocks are obtained by combining the complements of the blocks of Examples 9 and 10.

In general, taking the complements of the blocks of Corollary 23, the following result follows.

Corollary 26. *For all $p \geq 0$, a 3–(9, 4, $12p + 6$) exists.*

6. Conclusions

Both GDDs and t-designs have been studied and have important applications in the field of Combinatorics. These two concepts have been combined to define a 3-GDD. This new definition has the potential to raise many more generalizations and challenging existence problems.

In this paper, we mainly used Definition 5 to study a special type of combinatorial design called a 3-GDD with three groups and block size five, denoted by 3-GDD $(n, 3, 5, \mu_1, \mu_2)$. In this work, we have established some necessary conditions for the existence of such designs (Theorems 7 and 11), proved the nonexistence of a 3-GDD $(n, 3, 5, \mu_1, 0)$ when $n \leq 5$ (Theorem 12), and when $n \geq 6$, we have shown several existence cases for 3-GDD $(n, 3, 5, \mu_1, 0)$ (Theorems 17 and 18). In addition, the relationship between $3'$-GDD $(n, m, k, \Lambda_1, \Lambda_2)$ (Definition 2) and 3-PBIBD is explained in Theorem 24, and when $n = 3$, Corollary 25 shows the existence of a $3'$-GDD $(n, 3, 4, \Lambda_1, \Lambda_2)$ guarantees the existence of 3-GDD$(3, 3, 5, (9\Lambda_2 + \Lambda_1)/4, (11\Lambda_2 - \Lambda_1)/4)$, while the existence of 3-GDD $(3, 3, 5, \mu_1, \mu_2)$ shows the existence of $3'$-GDD $(3, 3, 4, (11\mu_1 - 9\mu_2)/5, (\mu_1 + \mu_2)/5)$. Lastly, we have presented many families of existences for the 3-GDD when $n = 3$.

6.1. Future Work

(1) This research work covers more possibilities for further studies and applications. For example, we can consider a t-GDD instead of only $t = 3$.

We are also thankful to a referee for suggesting future direction for this study as follows.

(2) To increase the importance of such designs, overall efficiency for a 3-GDD should be defined.

(3) A GDD with parameters: $v = mn, b, r, k, \mu_1 = 0, \mu_2 = 1$ is applicable in the construction of earlier regular low-density parity-check (LDPC) codes free of 4–cycles [16]. Is a 3-GDD under certain conditions also applicable in LDPC codes?

(4) Discuss the resolvability of 3-GDDs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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