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Research Article

On the Global Well-Posedness for a Hyperbolic Model Arising from Chemotaxis Model with Fractional Laplacian Operator

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Several cells and microorganisms, such as bacteria and somatic, have many essential features, one of which can be modeled by the chemotaxis system, which we consider to be our main interest in this article. More precisely, we studied the hyperbolic system derived from the chemotaxis model with fractional dissipation, which is a generalization for the hyperbolic system with classical dissipation. The results of this article are divided into two parts. In the first part, we used energy methods to obtain the existence of small solutions in the Besov spaces. The second one deals with the optimal decay of perturbed solutions using a refined time-weighted energy combined with the Littlewood-Paley decomposition technique. To the authors' best knowledge, this type of system (with fractional dissipation) has not been studied in the literature.

1. Introduction

This present article aims to study the hyperbolic chemotaxis system, which is governed by the following Cauchy problem:

$$\begin{cases} \partial_t \tilde{p} + \varpi \Lambda^\sigma \tilde{p} = \operatorname{div}(\tilde{p} q) & \text{if } (\mathrm{t}, \mathrm{x}) \in \mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}}, \\ \partial_t q + \lambda \Lambda^\sigma q = \nabla\left(\tilde{p} + \lambda |q|^2\right) & \text{if } (\mathrm{t}, \mathrm{x}) \in \mathbb{R}_+ \times \mathbb{R}^{\mathrm{d}}, \\ (\tilde{p}, q)_{|t=0} = (\tilde{p}_0, q_0), \end{cases} \tag{1}$$

where $\tilde{p}(t,x)$ represents the cell density, $q(t,x) = -(\nabla v / v)$ with v is the chemical concentration, $\varpi > 0$ and $\lambda \geq 0$ describe the cell and chemical diffusion coefficients, respectively, and the fractional operator $\Lambda^\sigma = (-\Delta)^{\sigma/2}$ denotes the fractional Laplacian operator. In the whole space $\mathbb{R}^d$, the operator Λ^σ is defined via the following Fourier transform:

$$\left(\widehat{\Lambda^\sigma f}\right)(\xi) \triangleq |\xi|^\sigma \left(\widehat{f}\right)(\xi). \tag{2}$$

Chemotaxis model (1) describes the ability of free-moving organisms to react to chemical substances or their concentration differences with specific, directed movements. On the other hand, the fractional dissipation has several applications in the molecular biology, we mention as an example the anomalous diffusion and chemical attraction to organisms in semiconductor growth, see for instance [1]. It is well known that the fractional chemotaxis system (1) is derived from the following canonical formulation of the famous "Keller-Segel" model [2, 3]:

$$\begin{cases} \partial_t u = -\varpi \Lambda^\sigma u - \operatorname{div}(\chi u \nabla \Psi(v)) & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t v = -\lambda \Lambda^\sigma v + f(u, v) & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (p, q)_{|t=0} = (p_0, q_0), \end{cases} \tag{3}$$

where χ is a constant that represents the chemosensitivity; when the coefficient $\chi > 0$, we say that the system is attractive and in the case when the coefficient $\chi < 0$, the system is repulsive. The function Ψ describes the mechanism of signal detection. In the situation, when the function $\Psi(u) = \nabla(\Lambda^{-\sigma/2}u)$ with $\sigma \in (1,2]$ Biler and Wu [4] investigated system (3) in the Besov spaces. In particular, they have established the local existence and uniqueness of solutions. Later, Zhai [5] studied the system in the case when $f(u,v) = u$; more precisely, Zhai [5] showed that system (3) admits a mild solutions. There is a large literature on the analysis of system (3), we refer the reader to [6–10] and the references therein. For the case, when $\lambda = 0$, then system (1) turns out to be the hyperbolic-parabolic chemotaxis; in this context, when the fractional Laplacian operator is substituted by the classical Laplacian operator, Zhang and Zhu [11] investigated the Cauchy problem for system (1) $\sigma = 2, \lambda = 0$ with small initial data. Lately, Jun et al. [12] explored the global existence of system (1) $\sigma = 2, \lambda = 0$ with large initial data. Recently, Hao [13] showed that system (1) $\sigma = 2, \lambda = 0$ admits a unique solution near to some constant equilibrium state in the critical hybrid Besov spaces $B^{d/2-2}(\mathbb{R}^d) \times B^{d/2-2,d/2-1}(\mathbb{R}^d)$. In their very recent work Nie and Yuan [14] investigated the well-posedness and the ill-posednes in the critical Besov spaces $\dot{B}^{d/2-2}_{p,q}(\mathbb{R}^d) \times \dot{B}^{d/2-1}_{p,q}(\mathbb{R}^d)$. For this hyperbolic-parabolic system, there have been a large number of results concerning the long-time dynamics to the solutions, blow-up phenomenon, and the existence of global solutions (see for instance, [15–19]). Here, we focus on the chemically diffusible model corresponding to the case of $\lambda > 0$. As far as we know, the results obtained for the Hyperbolic model (1) $\lambda > 0$ are less compared to Hyperbolic-Parabolic model (1) $\lambda = 0$. In the situation, when the fractional Laplacian is changed by the full Laplacian operator, Tao et al. [20] proved that system (1) $\lambda > 0$ is globally well posed; moreover, they have obtained the long-time behavior, and diffusion limit of one-dimensional large-amplitude classical solutions on finite intervals subject to the Neumann-Dirichlet boundary conditions. Thereafter, in [21], Li and Zhao explored the same issues of [20] but for the Dirichlet-Dirichlet boundary conditions. Wang et al. [22] investigated the global existence, asymptotic decay rates, and diffusion convergence rate of small solutions in the Sobolev framework. Recently, Martinez et al. [23] proved a set of results, such as, the global asymptotic stability of constant ground states and the explicit decay rate of solutions. More recently, Wu and Su [24] showed that system (1) $\sigma = 2$ admits global solution and also they have obtained a decay rate of solutions in the Besov spaces. For the traveling wave solution of problem (1) with $\sigma = 2$ and its nonlinear stability, see the series of papers [25–28].

Stimulated by the above works, especially with [24], the main objective of this article is to investigate system (1) with the fractional Laplacian operator Λ^σ, with $\sigma \in (1,2)$. To be more precise, we have the following two outcomes:

Theorem 1. *Assume that $1 < \sigma < 2, d \geq 2$ and $v \in [1,\infty]$. Let $(\pi_0, q_0) \in \dot{B}^{\alpha}_{2,v}$. We assume that there exists a constant $\epsilon > 0$ such that*

$$\|\pi_0\|_{\dot{B}^{\alpha}_{2,v}} + \|q_0\|_{\dot{B}^{\alpha}_{2,v}} \leq \epsilon, \tag{4}$$

where $\alpha = d/2 - \sigma + 1$ and $\pi_0 = \tilde{p}_0 - \overline{p}$, for some equilibrium state $\overline{p} > 0$. Then, system (1) admits a global unique solution $(\tilde{p}, q)$ satisfying for any $T > 0$:

$$\tilde{p} - \overline{p}, q \in L^\infty\left(0, T; \dot{B}^{(d/2)-\sigma+1}_{2,v}\right) \cap L^1\left(0, T; \dot{B}^{(d/2)+1}_{2,v}\right). \tag{5}$$

Remark 1. From the identity $\dot{B}^s_{2,2} \approx \dot{H}^s$, then for $v = 2$, Theorem 1 implies the global well-posedness in the scale of (homogeneous) Sobolev spaces.

Our existence theorem is based on the energy estimates. Due to the presence of the terms $\mathrm{div}(\tilde{p}q)$ and $\nabla(|q|^2)$, we established with the help of Bony's decomposition (see the next section) a product estimates, thus we can get the a priori estimates of our system, which leads us to get the global small solutions in $\dot{B}^{(d/2)-\sigma+1}_{2,v}$.

The second result of this article deals with the time decay rates of strong solutions for system (1), which is given as follows:

Theorem 2. *Let $1 < \sigma < 2, d \geq 2$ and $v \in [1,\infty]$. Let $(\tilde{p}_0 - \overline{p}, q_0) \in B^{\alpha}_{2,1} \cap \dot{B}^0_{1,\infty}$. There exists a constants $\epsilon > 0$ such that if*

$$\|\pi_0\|_{B^{\alpha}_{2,1} \cap \dot{B}^0_{1,\infty}} + \|q_0\|_{B^{\alpha}_{2,1} \cap \dot{B}^0_{1,\infty}} \leq \epsilon, \tag{6}$$

where $\alpha = (d/2) - \sigma + 1$ and $\pi_0 = \tilde{p}_0 - \overline{p}$, then system (1) has a unique global solution as follows:

$$\pi, q \in L^\infty\left(0, T; B^{(d/2)-\sigma+1}_{2,1}\right), \tag{7}$$

where $\pi = \tilde{p} - \overline{p}$. Furthermore, there exists a positive constant M_0 such that

$$\|(\pi, q)\|_{B^{(d/2)-\sigma+1}_{2,1}} \leq M_0 (1+t)^{-(d/2\sigma)}. \tag{8}$$

Remark 2. Theorem 2 gives the optimal decay of solutions for system (1) due to embedding $B^{(d/2)-\sigma+1}_{2,1} \longrightarrow L^2$.

Remark 3. In the sequel and for the simplicity we assume (without loss of generality) $\lambda = \varpi = \overline{p} = 1$.

The content of this article is arranged as follows. We introduced the notations and some useful definitions and results in Section 2, then we reformulated system (1) and we established some useful a priori estimates in section 3.

Section 4 is devoted to the proof of the existence of solutions. Finally, we proved Theorem 2 in Section 5.

1.1. Notation. C or M stands for some positive constant and may represent different values in different lines, the notation $X \lesssim Y$ means that there exist a constant $M_0 > 0$ such that $X \leq M_0 Y$, where M_0 is a constant depending on the initial data.

2. Preparatory

In this section, we recall some ingredients, such as the famous Littlewood–Paley operators and Bony decomposition, also some function spaces will be introduced.

2.1. Littlewood–Paley Theory. We start this section by giving the definition of the dyadic partition of unity.

Definition 1. There exists $(\chi, \psi) \in \mathfrak{D}(\mathbb{R}^2) \times \mathfrak{D}(\mathbb{R}^2_*)$ such that

$$\text{For all } \xi \in \mathbb{R}^2, \chi(\xi) + \sum_{q \in \mathbb{N}} \psi(2^q \xi) = 1. \tag{9}$$

For every $f \in \mathcal{S}'(\mathbb{R}^d)$, we defined the operators $\dot{\Delta}_\varsigma$, and $\dot{S}_\varsigma$ as follows. Let $\varsigma \in \mathbb{Z}$

$$\begin{aligned} \dot{\Delta}_\varsigma b &\triangleq \psi(2^{-\varsigma}\partial)b, \\ \dot{S}_\varsigma b &\triangleq \chi(2^{-\varsigma}\partial)b, \end{aligned} \tag{10}$$

and Δ_ς as follows:

$$\Delta_{-1} b \triangleq \chi(\partial) f, \text{and for all } \varsigma \geq 0, \Delta_\varsigma b \triangleq \psi(2^{-\varsigma}\partial)b. \tag{11}$$

By the definition of localization operators above, we have some interesting properties as follows:

(1)The formal Littlewood–Paley decomposition:

$$b = \sum_{\varsigma \in \mathbb{Z}} \dot{\Delta}_\varsigma b, \text{for all } b \in \mathcal{S}'(\mathbb{R}^d). \tag{12}$$

(2)The quasiorthogonality:

(i) If $|\varsigma - \kappa| \geq 2$, then $\dot{\Delta}_\kappa \dot{\Delta}_\varsigma b = 0$;
(ii) if $|\varsigma - \kappa| \geq 5$, then $\dot{\Delta}_\kappa (\dot{S}_\varsigma b \dot{\Delta}_\varsigma b) = 0$.

The paradifferential calculus plays a key role in the proof of the existence result, and its given by the following definition:

(i) Bony decomposition

For $a, b \in \mathcal{S}'(\mathbb{R}^d)$, we have the following equation:

$$ab = \dot{\mathfrak{T}}_a b + \dot{\mathfrak{T}}_b a + \dot{\mathfrak{R}}(a, b), \tag{13}$$

with

$$\dot{T}_a b = \sum_q \dot{S}_{q-1} a \dot{\Delta}_q b, \dot{\mathfrak{R}}(a, b) = \sum_q \dot{\Delta}_q \tilde{\dot{\Delta}}_q b \text{ with } \tilde{\dot{\Delta}}_q = \dot{\Delta}_{q-1} + \dot{\Delta}_q + \dot{\Delta}_{q+1}. \tag{14}$$

The next result is the famous Bernstein inequality, for the proof see [29].

Proposition 1. *Let $1 \leq \theta \leq \vartheta \leq \infty$. Assume $H \in L^\theta$, then for every $\iota \in \mathbb{N}^d$, there exists constants M_1, M_2 such that*

$$\begin{aligned} &\operatorname{supp} \widehat{H} \subset \{|\xi| \leq A_0 2^q\} \Rightarrow \|\partial^\iota H\|_{L^\vartheta} \leq M_1 2^{q(|\iota| + d(1/\theta - 1/\vartheta))} \|H\|_{L^\theta}, \\ &\operatorname{supp} \widehat{H} \subset \{A_1 2^q \leq |\xi| \leq A_0 2^q\} \Rightarrow \|H\|_{L^\theta} \leq M_2 2^{-qk} \sup_{|\iota| = k} \partial^k \|H\|_{L^\theta}. \end{aligned} \tag{15}$$

Now, we recall the definitions of homogeneous and nonhomogeneous Besov spaces.

Definition 2. For $(\eta, \theta) \in \mathbb{R} \times [1, +\infty]$ and $1 \leq v < \infty$. The homogeneous Besov space $\dot{B}^\eta_{\theta,v}$ is defined as the set of all tempered distributions $a \in \mathcal{S}'(\mathbb{R}^d)$ such that:

$$\|a\|_{\dot{B}^\eta_{\theta,v}} \triangleq \left(2^{q\eta} \left\|\dot{\Delta}_q a\right\|_{L^\theta}\right)_{\ell^v(\mathbb{Z})} < \infty. \tag{16}$$

Besides, *if* $v = \infty$

$$\|a\|_{\dot{B}^\eta_{\theta,\infty}} \triangleq \sup_{q \in \mathbb{Z}} \left(2^{q\eta} \left\|\dot{\Delta}_q a\right\|_{L^\theta}\right) < \infty. \tag{17}$$

Definition 3. For $(\eta, \theta) \in \mathbb{R} \times [1, +\infty]$ and $1 \leq v < \infty$. The nonhomogeneous Besov space $B^\eta_{\theta,v}$ is defined as the set of all tempered distributions $a \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|a\|_{B^\eta_{\theta,v}} \triangleq \left(2^{q\eta} \left\|\Delta_q a\right\|_{L^\theta}\right)_{\ell^v(\mathbb{Z})} < \infty. \tag{18}$$

Besides, *if* $v = \infty$

$$\|a\|_{B^\eta_{\theta,\infty}} \triangleq \sup_{q \in \mathbb{N} \cup \{-1\}} \left(2^{q\eta} \left\|\Delta_q a\right\|_{L^\theta}\right) < \infty. \tag{19}$$

The following spaces are introduced by Chemin and Lerner in [30]:

Definition 4. We assume that $\tau > 0$ and $\zeta \geq 1$, the space $L^\zeta_T B^\eta_{\theta,v}$ is the set of $a \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\left\|a_{L_\tau^\zeta \dot{B}_{\theta,v}^{\eta}} \triangleq \left\|\left(2^{q\eta}\left\|\dot{\Delta}_q a\right\|_{L^\theta}\right)_{\ell^r(\mathbb{Z})}\right.\right\|_{L_\tau^\zeta} < \infty, \tag{20}$$

and the space $\widetilde{L}_\tau^\zeta \dot{B}_{\theta,\infty}^{\eta}$ is the space $a \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|a\|_{\widetilde{L}_\tau^\zeta \dot{B}_{\theta,v}^{\eta}} \triangleq \left(2^{q\eta}\left\|\dot{\Delta}_q a\right\|_{L_\tau^\zeta L^\theta}\right)_{\ell^r(\mathbb{Z})} < \infty. \tag{21}$$

As a consequence of the last definition, we have the following embedding. Let $\epsilon > 0$, then we have the following equation:

$$\begin{aligned} &L_\tau^\zeta B_{\theta,v}^{\eta} \rightarrow \widetilde{L}_\tau^\zeta \dot{B}_{\theta,v}^{\eta}, \forall v \geq \zeta, \\ &\widetilde{L}_\tau^\zeta \dot{B}_{\theta,v}^{\eta} \rightarrow L_\tau^\zeta \dot{B}_{\theta,v}^{\eta}, \forall \zeta \geq v. \end{aligned} \tag{22}$$

The following Lemma is useful in our contribution, for more details see [31].

Lemma 1. *(i) Let* $\nu_1, \nu_2 > 0$ *satisfying* $\max\{\nu_1, \nu_2\} > 1$. *Then*

$$\int_0^{\tau'} \left(1 + \tau' - \tau\right)^{-\nu_1} (1+\tau)^{-\nu_2} d\tau \leq M_{\nu_1,\nu_2}\left(1+\tau'\right)^{-\min\{\nu_1,\nu_2\}}, \quad \tau' > 0. \tag{23}$$

(ii) Let $\nu_1, \nu_2 > 0$ *and* $b \in L^\infty((0,\infty))$. *Then*

$$\int_0^{\tau'} \left(1 + \tau' - \tau\right)^{-\nu_1} (1+\tau)^{-\nu_2} b(\tau) d\tau \leq M_{\nu_1,\nu_2}\left(1+\tau'\right)^{-\min\{\nu_1,\nu_2\}} \int_0^{\tau'} |b(\tau) d\tau|, \tau' > 0. \tag{24}$$

We finished this section by establishing a product estimate.

Lemma 2. *Let* $\sigma \in (1,2), v \in [1,\infty]$ *and* $u, v \in \dot{B}_{2,v}^{(d/2)-\sigma+1} \cap \dot{B}_{2,v}^{(d/2)+1}$. *Then* $uv \in \dot{B}_{2,v}^{(d/2)-\sigma+2}$ *and we have the following equation:*

$$\|uv\|_{\dot{B}_{2,v}^{(d/2)-\sigma+2}} \leq C\left(\|u\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}}\|v\|_{\dot{B}_{2,v}^{(d/2)+1}} + \|v\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}}\|u\|_{\dot{B}_{2,v}^{(d/2)+1}}\right). \tag{25}$$

Proof. By using the Bony decomposition (13), we split the term uv as follows:

$$uv = \dot{\mathfrak{T}}_u v + \dot{\mathfrak{T}}_v u + \dot{\mathfrak{R}}(u,v). \tag{26}$$

From the property (2) part (ii), we have the following equation:

$$\begin{aligned} \left\|\dot{\Delta}_\kappa(uv)\right\|_{L^2} = &\sum_{|\varsigma-\kappa|\leq 4}\left\|\dot{\Delta}_\kappa\left(\dot{S}_{\varsigma-1}u\dot{\Delta}_\varsigma v\right)\right\|_{L^2} + \sum_{|\varsigma-\kappa|\leq 4}\left\|\dot{\Delta}_\kappa\left(\dot{S}_{\varsigma-1}v\dot{\Delta}_\varsigma u\right)\right\|_{L^2} \\ &+ \sum_{\substack{l\varsigma\geq\kappa-3 \\ |\iota|\leq 1}}\left\|\dot{\Delta}_\kappa\left(\dot{\Delta}_\varsigma u\dot{\Delta}_{\varsigma-\iota}v\right)\right\|_{L^2}. \end{aligned} \tag{27}$$

Multiplying equation (27) by $2^{\kappa((d/2)+2-\sigma)}$ and taking the ℓ^v norm we find out that

$$\begin{aligned} \left\|uv_{\dot{B}_{2,v}^{(d/2)-\sigma+2}} \leq \right\|&\left(2^{k((d/2)-\sigma+2)}\sum_{|\varsigma-\kappa|\leq 4}\left\|\dot{\Delta}_\kappa\left(\dot{S}_{\varsigma-1}u\dot{\Delta}_\varsigma v\right)\right\|_{L^2}\right)_{\kappa\in\mathbb{Z}}\ell^v(\mathbb{Z}) \\ &+\left\|\left(2^{\kappa((d/2)-\sigma+2)}\sum_{|\varsigma-\kappa|\leq 4}\left\|\dot{\Delta}_\kappa\left(\dot{S}_{\varsigma-1}v\dot{\Delta}_\varsigma u\right)\right\|_{L^2}\right)_{\kappa\in\mathbb{Z}}\right\|_{\ell^v(\mathbb{Z})} \\ &+\left\|\left(2^{\kappa((d/2)-\sigma+2)}\sum_{\varsigma\geq\kappa-3|\iota|\leq 1}\left\|\dot{\Delta}_\kappa\left(\dot{\Delta}_\varsigma u\dot{\Delta}_{\varsigma-\iota}v\right)\right\|_{L^2}\right)\kappa\in\mathbb{Z}\right\|_{\ell^v(\mathbb{Z})} \\ &\triangleq \mathbb{K}_1 + \mathbb{K}_2 + \mathbb{K}_3. \end{aligned} \tag{28}$$

For the first term $\mathbb{K}_1$ we used Hölder inequality and Young inequality and obtained the following equation:

$$\begin{aligned}\mathbb{K}_1 &\lesssim \left\| \left(2^{\kappa((d/2)-\sigma+2)} \sum_{\kappa' \leq \kappa-2} \|\dot{\Delta}_{\kappa'} u\|_{L^\infty} \|\dot{\Delta}_\kappa v\|_{L^2} \right)_{\kappa\in\mathbb{Z}} \right\|_{\ell^\upsilon(\mathbb{Z})} \\ &\leq \left\| \left(\sum_{\kappa' \leq \kappa-2} 2^{(\kappa'-\kappa)(\sigma-1)} 2^{\kappa'(1-\sigma)} \|\dot{\Delta}_{\kappa'} u\|_{L^\infty} 2^{\kappa((d/2)+1)} \|\dot{\Delta}_\kappa v\|_{L^2} \right)_{\kappa\in\mathbb{Z}} \right\|_{\ell^\upsilon(\mathbb{Z})} \\ &\leq \|u_{\dot{B}^{1-\sigma}_{\infty,\infty}}\| v_{\dot{B}^{(d/2)+1}_{2,\upsilon}}.\end{aligned} \tag{29}$$

Similarly, we obtain the following equation:

$$\mathbb{K}_2 \lesssim \|u\|_{\dot{B}^{(d/2)+1}_{2,\upsilon}} \|v\|_{\dot{B}^{1-\sigma}_{\infty,\infty}}. \tag{30}$$

Taking advantage of Young inequality and the Hölder inequality, it holds that

$$\begin{aligned}\mathbb{K}_3 &\lesssim \left\| \left(2^{\kappa((d/2)-\sigma+2)} \sum_{\substack{\varsigma \geq \kappa-3 \\ |\iota| \leq 1}} \|\dot{\Delta}_\varsigma u\|_{L^\infty} \|\dot{\Delta}_{\varsigma-\iota} v\|_{L^2} \right)_{\kappa\in\mathbb{Z}} \right\|_{\ell^\upsilon(\mathbb{Z})} \\ &\lesssim \left\| \left(\sum_{\varsigma \geq \kappa-3 |\iota| \leq 1} 2^{(1-\sigma)\varsigma} \|\dot{\Delta}_\varsigma u\|_{L^\infty} 2^{((d/2)+1)(\varsigma-\iota)} \|\dot{\Delta}_{\varsigma-\iota} v\|_{L^2} 2^{((d/2)+2-\sigma)(\kappa-\varsigma)} \right)_{\kappa\in\mathbb{Z}} \right\|_{\ell^\upsilon(\mathbb{Z})} \\ &\lesssim \|u\|_{\dot{B}^{1-\sigma}_{\infty,\infty}} \|v\|_{\dot{B}^{(d/2)+1}_{2,\upsilon}}.\end{aligned} \tag{31}$$

By the Besov embedding $\dot{B}^{(d/2)-\sigma+1}_{2,\upsilon} \to \dot{B}^{-\sigma+1}_{\infty,\infty}$, we get the desired result. □

3. Reformulation of System and the a Priori Estimate

In this paragraph, we reformulated the original system (1), where we will assume (without loss of generality) that the equilibrium state $\overline{p} = 1$ and we set $\pi = \widetilde{p} - 1$, the system becomes as follows:

$$\begin{cases} \partial_t \pi + \Lambda^\sigma \pi = \operatorname{div}(\pi q) + \operatorname{div} q & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t q + \Lambda^\sigma q = \nabla\left(\pi + |q|^2\right) & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (\pi, q)_{|t=0} = (\pi_0, q_0). \end{cases} \tag{32}$$

We will study the a priori estimates for linearized system (32) with general source term as follows:

$$\begin{cases} \partial_t \pi + \Lambda^\sigma \pi - \operatorname{div} q = F & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t q + \Lambda^\sigma q - \nabla \pi = H & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (\pi, q)_{|t=0} = (\pi_0, q_0). \end{cases} \tag{33}$$

We note that system (33) is given by the following equation:

$$\begin{cases} \partial_t \mathbb{V} + L_\sigma \mathbb{V} = \mathbb{G} & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \mathbb{V}_{|t=0} = \mathbb{V}_0, \end{cases} \tag{34}$$

where $\mathbb{V}(t) = (\pi(t), q(t))^{\mathbf{T}}, \mathbb{G} = (F, H)^{\mathbf{T}}$ and

$$L_\sigma = \begin{pmatrix} \Lambda^\sigma & -\operatorname{div} \\ -\nabla & \Delta^\sigma \end{pmatrix}. \tag{35}$$

Next, we prove the A priori estimate for system (32).

3.1. A Priori Estimate

Proposition 2. *We assume that (π, q) is a regular solution for the system (32), then for all $t \in [0,T)$,*

$$\|\pi\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}} + \|q\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}} + \int_0^t \|\pi(\tau)\|_{\dot{B}^{(d/2)+1}_{2,v}} + \|q(\tau)\|_{\dot{B}^{(d/2)+1}_{2,v}} d\tau$$
$$\lesssim \|\pi_0\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}} + \|q_0\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}}$$
$$+ \|F\|_{L^1_t \dot{B}^{(d/2)-\sigma+1}_{2,v}} + \|H\|_{L^1_t \dot{B}^{(d/2)-\sigma+1}_{2,v}}. \tag{36}$$

Proof. Let $k \in \mathbb{Z}$ and we set $(\pi_k, q_k) = (\dot{\Delta}_k \pi, \dot{\Delta}_k q)$ and $(F_k, H_k) = (\dot{\Delta}_k(\operatorname{div}\pi q), \dot{\Delta}_k(\nabla(|q|^2)))$.

We observed that (π_k, q_k) solves the following system:

$$\begin{cases} \partial_t \pi_k + \Lambda^\sigma \pi_k = \operatorname{div} q_k + F_k \\ \partial_t q_k + \Lambda^\sigma q_k = \nabla \pi_k + H_k. \end{cases} \tag{37}$$

Taking the L^2-scalar product of equation (37), we found out that

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|\pi_k(t)\|^2_{L^2} + \|\Lambda^{\sigma/2}\pi_k(t)\|^2_{L^2} = (\operatorname{div} q_k, \pi_k) + (F_k, \pi_k) \\[2ex] \dfrac{1}{2}\dfrac{d}{dt}\|q_k(t)\|^2_{L^2} + \|\Lambda^{\sigma/2} q_k(t)\|^2_{L^2} = (\nabla \pi_k, q_k) + (H_k, q_k). \end{cases} \tag{38}$$

For simplicity, we set $X^2(t) \triangleq \|\pi_k(t)\|^2_{L^2} + \|q_k(t)\|^2_{L^2}$. According to the identity $(\nabla \pi_k, q_k) = (\operatorname{div} q_k, \pi_k)$ and Bernstein's Lemma, we get the following equation:

$$\frac{1}{2}\frac{d}{dt}X^2(t) + 2^{\sigma k}X^2(t) \leq \left(\|F_k\|_{L^2} + \|H_k\|_{L^2}\right)X(t). \tag{39}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}X(t) + 2^{\sigma k}X(t) \leq \|F_k\|_{L^2} + \|H_k\|_{L^2}. \tag{40}$$

Multiplying equation (40) by $2^{k((d/2)-\sigma+1)}$ and taking the ℓ^v norm over $k \in \mathbb{Z}$, we obtained the following equation:

$$\frac{1}{2}\frac{d}{dt}\left(\|\pi\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}} + \|q\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}}\right) + \|\pi(t)\|_{\dot{B}^{(d/2)+1}_{2,v}}$$
$$+ \|q(t)\|_{\dot{B}^{(d/2)+1}_{2,v}} d\tau \lesssim \|F\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}} + \|H\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}}. \tag{41}$$

Integrating the above estimate with respect to time, hence, we get the desired result. □

4. Existence

For the proof of the existence part, we are going to use the classical Friedrichs' regularization method combined with the energy method. First, we define the spectral cutoff as follows. Let $\mu > 0$

$$\widehat{\mathbb{J}_\mu a}(\omega) \triangleq \mathbf{1}_{B_\mu}\widehat{a}(\omega), \tag{42}$$

where $B_\mu = \{x \in \mathbb{R}^d / |x| \leq \mu\}$ and $\mathbf{1}_{B_\mu}$ denotes the characteristic function on the ball B_μ. We define the following equation:

$$L^2_\mu \triangleq \{a \in L^2 \colon \operatorname{supp}\widehat{a} \subset B(0,\mu)\}. \tag{43}$$

We consider the following approximate system:

$$\begin{cases} \partial_t \pi_\mu + \Lambda^\sigma \pi_\mu = F_\mu & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t q_\mu + \Lambda^\sigma q_\mu = \nabla \pi_\mu + H_\mu & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (\pi_\mu, q_\mu)_{|t=0} = (\mathbb{J}_\mu \pi_0, \mathbb{J}_\mu q_0), \end{cases} \tag{44}$$

where $F_\mu = \mathbb{J}_\mu(\operatorname{div}(\pi q))$ and $H_\mu = \mathbb{J}_\mu(\nabla(|q|^2))$.

The usual (Cauchy Lipschitz) theorem guarantees that system (44) admits a unique regular solution $C([0, T^*_\mu)\ L^2_\mu)$.

Denote

$$\|(\pi,q)\|_{\mathcal{X}^{(d/2),\sigma}_t} \triangleq \|(\pi,q)\|_{L^\infty_t \dot{B}^{(d/2)-\sigma+1}_{2,v}} + \|(\pi,q)\|_{L^1_t \dot{B}^{(d/2)+1}_{2,v}}. \tag{45}$$

Let

$$T_\mu \triangleq \sup\left\{T \in (0, T^*_\mu) \colon \left\|(\pi_\mu, q_\mu)\right\|_{\mathcal{X}^{(d/2),\sigma}_t} \leq M\widetilde{C}\|(\pi_0, q_0)\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}}\right\}, \tag{46}$$

where $\widetilde{C} \geq 2$ and $M = (1/2C\widetilde{C}^2\|(\pi_0,q_0)\|_{\dot{B}^{(d/2)-\sigma+1}_{2,v}})$, thus we have $T_\mu > 0$. From Lemma 2.5, we deduce that

$$\begin{aligned} \left\|F_\mu\right\|_{\widetilde{L}^1_t \dot{B}^{(d/2)+1-\sigma}_{2,v}} &\leq C\left\|\pi_\mu q_\mu\right\|_{\widetilde{L}^1_t \dot{B}^{(d/2)+2-\sigma}_{2,v}} \\ &\leq C\left(\left\|q_\mu\right\|_{\widetilde{L}^\infty_t \dot{B}^{(d/2)-\sigma+1}_{2,v}}\left\|\pi_\mu\right\|_{\widetilde{L}^1_t \dot{B}^{(d/2)+1}_{2,v}} + \left\|\pi_\mu\right\|_{\widetilde{L}^\infty_t \dot{B}^{(d/2)-\sigma+1}_{2,v}}\left\|q_\mu\right\|_{\widetilde{L}^1_t \dot{B}^{(d/2)+1}_{2,v}}\right) \\ &\leq C\left\|(\pi_\mu, q_\mu)\right\|_{\widetilde{L}^\infty_t \dot{B}^{(d/2)-\sigma+1}_{2,v}}\left\|(\pi_\mu, q_\mu)\right\|_{\widetilde{L}^1_t \dot{B}^{(d/2)+1}_{2,v}}. \end{aligned} \tag{47}$$

In a similar way,

$$\begin{aligned}\left\|H_\mu\right\|_{\widetilde{L}_t^1 \dot{B}_{2,v}^{(d/2)+1-\sigma}} &\le C\left\|q_\mu q_\mu\right\|_{\widetilde{L}_t^1 \dot{B}_{2,v}^{(d/2)+2-\sigma}}\\ &\le C\left(\left\|q_\mu\right\|_{\widetilde{L}_t^\infty \dot{B}_{2,v}^{(d/2)-\sigma+1}}\left\|q_\mu\right\|_{\widetilde{L}_1^1 \dot{B}_{2,v}^{(d/2)+1}}\right).\end{aligned} \tag{48}$$

Hence,

$$\begin{aligned}\left\|\left(\pi_\mu, q_\mu\right)\right\|_{\mathcal{X}_t^{(d/2),\sigma}} &\le \widetilde{C}\left(\left\|\left(\pi_0, q_0\right)\right\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}} + C\left\|\left(\pi_\eta, q_\eta\right)\right\|^2_{\mathcal{X}_t^{(d/2),\sigma}}\right)\\ &\le \widetilde{C}\left\|\left(\pi_0, q_0\right)\right\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}}\left(1 + C\left(M\widetilde{C}\right)^2\left\|\left(\pi_0, q_0\right)\right\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}}\right),\end{aligned} \tag{49}$$

where $C \ge 2$. We choose $\|(\pi_0, q_0)\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}} < 1/4\widetilde{C}^2 C$. Therefore, $1 + M^2\widetilde{C}^2 C\|(\pi_0, q_0)\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}} < M$. Then, for any $T < T_\mu$, we have $\|(\pi_\mu, q_\mu)\|_{\mathcal{X}_t^{(d/2),\sigma}} \le \widetilde{C}M\|(\pi_0, q_0)\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}} \le (1/4)$. Our next aim is to prove that $T_\mu = T_\mu^\star$; previously, we showed that $\|(\pi_\mu, q_\mu)\|_{\mathcal{X}_t^{(d/2),\sigma}} \le M\widetilde{C}\|(\pi_0, q_0)\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}}$ for all $T_\mu < T_\mu^\star$. By the continuity argument, we can get $\|(\pi_\mu, q_\mu)\|_{\mathcal{X}_{t+\delta}^{(d/2),\sigma}} \le M\widetilde{C}\|(\pi_0, q_0)\|_{\dot{B}_{2,v}^{(d/2)-\sigma+1}}$, for a sufficiently small constant $\delta > 0$, which contradicts the definition of T_μ. Next, we will show that system (44) admits global solutions. For this aim, we assumed that $T^\star < +\infty$, we have $\|(\pi_\mu, q_\mu)_{\mathcal{X}_t^{(d/2),\sigma}} \le M\widetilde{C}\|(\pi_0, q_0)_{\dot{B}_{2,v}^{(d/2)-\sigma+1}}$. On the other hand, we have $\pi_\mu, q_\mu \in \widetilde{L}^\infty([0, T_\mu^\star); \dot{B}_{2,v}^{(d/2)-\sigma+1})$, then we have $\|(\pi_\mu, q_\mu)\|_{L^\infty([0,T_\mu^\star);L_\mu^2)} < +\infty$. By the Cauchy-Lipschitz Theorem we can continue the solution beyond the time $T_\mu^\star$ and this contradicts the definition of $T_\mu^\star$. Hence, $T_\mu^\star = +\infty$. By standard arguments, we can show that $(\pi_\mu, q_\mu)_{\mu>0}$ converges to the solution (π, q) which solves system (32) and here we omit the details.

5. Optimal Decay of Solutions

To know more about the asymptotic behavior of the solutions obtained in the previous results, we studied in this section the optimal temporal decay of perturbed solution in terms of $B_{2,1}^{(d/2)-\sigma+1}$– norm, where we will estimate low frequencies and high frequencies separately. For the low frequencies, we use the good behavior of the semigroup $\mathcal{A}_\sigma(\rho)$. Regarding the high frequencies, we make use of the Fourier localization method.

Proof of Theorem 2. We assume that $\mathbb{V}(t) \triangleq (\pi(t), q(t))^{\mathrm{T}}$ is a smooth solution for system (33) and $(F, H) \triangleq (\operatorname{div}(\pi q), \nabla(|q|^2))^{\mathrm{T}}$, Then, employing Duhamel's principle, we found out that

$$\mathbb{V}(\rho) = \mathcal{A}_\sigma(\rho)\mathbb{V}_0 + \int_0^\rho \mathcal{A}_\sigma(\rho)\mathbb{G}(\mathbb{V}(\gamma))d\gamma, \tag{50}$$

where $\mathbb{V}_0 = (\pi_0, q_0)^T$, $\mathbb{G}(\mathbb{V}(\rho)) = (F, H)$ and $\mathcal{A}_\sigma(\rho)$ is the semigroup associated with the LHS of system (33), which is given by the following equation:

$$\mathcal{A}_\sigma(\rho)b = F^{-1}\left(e^{\widehat{L}_\sigma(\xi)}\widehat{b}\right), \tag{51}$$

where

$$\widehat{L}_\sigma(\xi) = \begin{pmatrix} |\xi|^\sigma & -i\xi \\ -\xi^{\mathrm{T}} & |\xi|^\sigma I_d \end{pmatrix}, \tag{52}$$

where I_d represents the identity matrix of the d dimension. For convenience, we denoted the following equation:

$$\begin{aligned} W_{\text{low}}(t) &\triangleq \sup_{\gamma\in[0,t]}(1+\gamma)^{d/2\sigma}\left\|\Delta_{-1}\mathbb{V}(\gamma)\right\|_{L^2}\\ W_{\text{high}}(t) &\triangleq \sup_{\gamma\in[0,t]}(1+\gamma)^{d/2\sigma}\sum_{k\ge 0}2^{d/2-\sigma+1}\left\|\Delta_k\mathbb{V}(\gamma)\right\|_{L^2},\end{aligned} \tag{53}$$

therefore,

$$\|\mathbb{V}(t)\|_{B_{2,1}^{(d/2)-\sigma+1}} \le (1+\gamma)^{-d/2\sigma}\left(W_{low}(t) + W_{\text{high}}(t)\right). \tag{54}$$

Since Δ_k coincides with $\dot{\Delta}_k$ for all $k \ge 0$, then

$$\begin{aligned} W_{\text{low}}(t) &\le \sup_{\gamma\in[0,t]}(1+t)^{d/2\sigma}\left\|\mathcal{A}_\sigma(\gamma)\Delta_{-1}\mathbb{V}_0\right\|_{L^2} + \sup_{\gamma\in[0,t]}(1+t)^{d/2\sigma}\int_0^\gamma\left\|\mathcal{A}_\sigma\left(\gamma-\gamma'\right)\Delta_{-1}G\left(\mathbb{V}\left(\gamma'\right)\right)\right\|_{L^2},\\ W_{\text{high}}(t) &\le \sup_{\gamma\in[0,t]}(1+t)^{d/2\sigma}\sum_{k\ge0}2^{d/2-\sigma+1}\left\|\mathcal{A}_\sigma(\tau)\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2} + \sup_{\gamma\in[0,t]}(1+t)^{d/2\sigma}\sum_{k\ge0}2^{d/2-\sigma+1}\left\|\mathcal{A}_\sigma\left(\tau-\tau'\right)\dot{\Delta}_k\mathbb{G}\left(\mathbb{V}\left(\tau'\right)\right)\right\|_{L^2}.\end{aligned} \tag{55}$$

By virtue of [1], Lemma 2.4, we have the following equation:

$$\left\|\mathscr{A}_\sigma(t)\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2} = \left\|\left(\widehat{\mathscr{A}_\sigma(t)\dot{\Delta}_k}\mathbb{V}_0\right)_{L^2} \le Ce^{-c_0 t 2^{\sigma k}}\left\|\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2}. \tag{56}$$

We also have the following equation:

$$\begin{aligned}\left\|\mathscr{A}_\sigma(t)\Delta_{-1}\mathbb{V}_0\right\|_{L^2} &\le \sum_{k'\le 0}\left\|\mathscr{A}_\sigma(t)\dot{\Delta}_{k'}\Delta_{-1}\mathbb{V}_0\right\|_{L^2}\\ &\le \sum_{k'\le 0}\left\|\mathscr{A}_\sigma(t)\dot{\Delta}_{k'}\mathbb{V}_0\right\|_{L^2}.\end{aligned} \tag{57}$$

In view of Bernstein's Lemma, we obtained the following equation:

$$\left\|\mathscr{A}_\sigma(t)\Delta_{-1}\mathbb{V}_0\right\|_{L^2} \le \sum_{k'\le 0} 2^{k'd/2}e^{-c_0 t 2^{\sigma k'}}\left\|\dot{\Delta}_{k'}\mathbb{V}_0\right\|_{L^1}. \tag{58}$$

Multiplying both sides by $t^{d/2\sigma}$, according to [1], Lemma 2.35, we concluded that

$$\begin{aligned}t^{d/2\sigma}\left\|\mathscr{A}_\sigma(t)\Delta_{-1}\mathbb{V}_0\right\|_{L^2} &\le C\left\|\mathbb{V}_0\right\|_{\dot{B}^1_{1,\infty}}\sum_{k'\in\mathbb{Z}} t^{d/2\sigma}2^{k'd/2}e^{-c_0 t 2^{\sigma k'}}\\ &\le C\left\|\mathbb{V}_0\right\|_{\dot{B}^1_{1,\infty}}.\end{aligned} \tag{59}$$

In a similar way, we also get the following equation:

$$\begin{aligned}\left\|\mathscr{A}_\sigma(t)\Delta_{-1}\mathbb{V}_0\right\|_{L^2} &\le \sum_{k'\le 0} 2^{k'd/2}e^{-c_0 t 2^{\sigma k'}}\left\|\dot{\Delta}_{k'}\mathbb{V}_0\right\|_{L^1}\\ &\le \left\|\mathbb{V}_0\right\|_{\dot{B}^1_{1,\infty}}\sum_{k'\le 0}2^{k'd/2}\\ &\le C\left\|\mathbb{V}_0\right\|_{\dot{B}^1_{1,\infty}}.\end{aligned} \tag{60}$$

Summing up equations (59) and (60), we deduced the following equation:

$$\left\|\mathscr{A}_\sigma(t)\Delta_{-1}\mathbb{V}_0\right\|_{L^2} \le C(1+t)^{-d/2\sigma}\left\|\mathbb{V}_0\right\|_{\dot{B}^0_{1,\infty}}. \tag{61}$$

For the high frequencies, we have the following equation:

$$\begin{aligned}t^{d/2\sigma}\sum_{k\ge 0}\left\|\mathscr{A}_\sigma(t)\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2} &\le C\sum_{k\ge 0}t^{d/2\sigma}e^{-c_0 t 2^{\sigma k}}\left\|\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2}\\ &\le C\left\|\mathbb{V}_0\right\|_{\dot{B}^{d/2-\sigma+1}_{2,1}}.\end{aligned} \tag{62}$$

Similarly, we get the following equation:

$$\begin{aligned}\sum_{k\ge 0}\left\|\mathscr{A}_\sigma(t)\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2} &\le C\sum_{k\ge 0}e^{-c_0 t 2^{\sigma k}}\left\|\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2}\\ &\le C\left\|\mathbb{V}_0\right\|_{\dot{B}^{d/2-\sigma+1}_{2,1}}.\end{aligned} \tag{63}$$

Consequently,

$$\sum_{k\ge 0}\left\|\mathscr{A}_\sigma(t)\dot{\Delta}_k\mathbb{V}_0\right\|_{L^2} \le C(1+t)^{d/2\sigma}\left\|\mathbb{V}_0\right\|_{\dot{B}^{d/2-\sigma+1}_{2,1}}. \tag{64}$$

According to Hölder inequality, we obtained the following equation:

$$\begin{aligned}\|\mathbb{G}(\mathbb{V})\|_{L^1} &\le \left\|\operatorname{div}(\pi q)\right\|_{L^1} + \left\|\nabla\left(|q|^2\right)\right\|_{L^1}\\ &\le \left\|\nabla\pi\right\|_{L^2}\left\|q\right\|_{L^2} + \left\|\pi_{L^2}\right\|\operatorname{div}q\|_{L^2} + \left\|\nabla q\right\|_{L^2}\left\|q\right\|_{L^2}\\ &\le \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3.\end{aligned} \tag{65}$$

By using the Besov embedding ($B^s_{2,1} \longrightarrow L^2$) for $s \ge 0$, we found that

$$\begin{aligned}\mathbb{J}_1 &\le \|q\|_{\dot{B}^{(d/2)-\sigma+1}_{2,1}}\left(\left\|\Delta_{-1}\nabla\pi\right\|_{L^2} + \left\|\sum_{\kappa\ge 0}\Delta_\kappa\nabla\pi\right\|_{B^{(d/2)}_{2,1}}\right)\\ &\le \|q\|_{B^{(d/2)-\sigma+1}_{2,1}}\left(\|\pi\|_{B^{(d/2)-\sigma+1}_{2,1}} + \|\pi\|_{\dot{B}^{(d/2)+1}_{2,1}}\right)\\ &\le \|\mathbb{V}\|_{B^{(d/2)-\sigma+1}_{2,1}}\left(\|\mathbb{V}\|_{B^{(d/2)-\sigma+1}_{2,1}} + \|\mathbb{V}\|_{\dot{B}^{(d/2)+1}_{2,1}}\right).\end{aligned} \tag{66}$$

In a similar way, we bound $\mathbb{J}_2, \mathbb{J}_3$. Thus, we get the following equation:

$$\left\|\mathbb{G}(\mathbb{V})_{L^1} \le \right\|\mathbb{V}_{B^{(d/2)-\sigma+1}_{2,1}}\left(\|\mathbb{V}\|_{B^{(d/2)-\sigma+1}_{2,1}} + \|\mathbb{V}\|_{\dot{B}^{(d/2)+1}_{2,1}}\right). \tag{67}$$

According to equations (61) and (64) and Lemma 1, we infer that

$$\begin{aligned}\int_0^\gamma\left\|\mathscr{A}_\sigma(\gamma-z)\Delta_{-1}\mathbb{G}(\mathbb{V}(z))_{L^2}dz\right. &\le \int_0^\gamma(1+\gamma-z)^{-d/2\sigma}\|\mathbb{G}(\mathbb{V}(z))\|_{\dot{B}^0_{\infty,1}}dz\\ &\le \int_0^\gamma(1+\gamma-z)^{-d/2\sigma}\|\mathbb{G}(\mathbb{V}(z))\|_{L^1}dz\\ &\le \int_0^\gamma(1+\gamma-z)^{-d/2\sigma}(1+z)^{-d/2\sigma}W(\gamma)\left((1+z)^{-d/2\sigma}W(\tau) + \|\mathbb{V}\|_{\dot{B}^{d/2+1}_{2,1}}\right)dz\\ &\lesssim (1+\gamma)^{-d/2\sigma}W(\tau)\|\mathbb{V}\|_{L^1_t\dot{B}^{d/2+1}_{2,1}} + (1+\gamma)^{-d/2\sigma}W^2(\tau).\end{aligned} \tag{68}$$

We used equations (47) and (48) with $(v = 1)$ and obtained the following equation:

$$\begin{aligned}\|\mathbb{G}(\mathbb{V}(z))\|_{\dot{B}^{(d/2)-\sigma+1}_{2,1}} &\le \|\operatorname{div}(\pi q)\|_{\dot{B}^{(d/2)-\sigma+1}_{2,1}} + \left\|\nabla\left(|q|^2\right)\right\|_{\dot{B}^{(d/2)-\sigma+1}_{2,1}}\\ &\lesssim \|\mathbb{V}\|_{B^{(d/2)-\sigma+1}_{2,1}}\|\mathbb{V}\|_{\dot{B}^{(d/2)+1}_{2,1}},\end{aligned} \tag{69}$$

where we have used also the embedding $\dot{B}_{2,1}^{(d/2)-\sigma+1} \longrightarrow B_{2,1}^{(d/2)-\sigma+1}$. According to equation (64) and Lemma 1, we found out that

$$\begin{aligned}\sum_{k\geq 0} 2^{k((d/2)-\sigma+1)} \int_0^\gamma \left\|\mathscr{A}_\sigma(\gamma-z)\Delta_k \mathbb{G}(\mathbb{V}(z))\right\|_{L^2} ds &\leq \int_0^\gamma (1+\gamma-z)^{-(d/2\sigma)} \|\mathbb{G}(\mathbb{V}(z))\|_{\dot{B}_{2,1}^{(d/2)-\sigma+1}} dz \\ &\leq \int_0^\gamma (1+\gamma-z)^{-(d/2\sigma)} (1+z)^{-(d/2\sigma)} W(\gamma) \|\mathbb{V}(z)\|_{\dot{B}_{2,1}^{(d/2)+1}} dz \\ &\leq (1+\gamma)^{-(d/2\sigma)} W(\gamma) \|\mathbb{V}(z)\|_{L_t^1 \dot{B}_{2,1}^{(d/2)+1}}.\end{aligned} \tag{70}$$

In view of equations (61) and (64), we deduced the following equation:

$$\begin{aligned}W(t) \leq C\left\|\mathbb{V}_0\right\|_{\dot{B}_{\infty,1}^0 \cap \dot{B}_{2,1}^{(d/2)-\sigma+1}} &+ \sup_{\gamma\in[0,t]} \int_0^\gamma \left\|\Delta_{-1}\mathscr{A}_\sigma(\gamma-z)\mathbb{G}(\mathbb{V}(z))\right\|_{L^2} dz \\ &+ \sup_{\gamma\in[0,t]} \int_0^\gamma \sum_{k\geq 0} 2^{k((d/2)-\sigma+1)} \int_0^\gamma \left\|\mathscr{A}_\sigma(\gamma-z)\Delta_k \mathbb{G}(\mathbb{V}(z))\right\|_{L^2} dz.\end{aligned} \tag{71}$$

Putting equations (68) and (70) into equation (71), we obtained the following equation:

$$W(t) \lesssim \left\|\mathbb{V}_0\right\|_{\dot{B}_{\infty,1}^0 \cap \dot{B}_{2,1}^{(d/2)-\sigma+1}} + W(t)\left\|V_0\right\|_{\dot{B}_{\infty,1}^0 \cap \dot{B}_{2,1}^{(d/2)-\sigma+1}} + W^2(t). \tag{72}$$

A bootstrapping argument implies that there is $\epsilon > 0$ such that, if $\|\mathbb{V}_0\|_{\dot{B}_{\infty,1}^0 \cap \dot{B}_{2,1}^{(d/2)-\sigma+1}} < \epsilon$, then for all $t \geq 0$,

$$W(t) \leq M_0 \epsilon, \tag{73}$$

for some positive constant M_0. Then, the Proof of Theorem 2 is now achieved. □

6. Conclusions

More or less recently, several methodologies have been proposed to describe behaviors of some complex world problems emerging in several applications, especially in molecular biology. The chemotaxis model gains increasing interest from mathematicians; so far, the problem of existence and uniqueness of classical solutions to system (1) (with fractional dissipation Λ^σ or classical dissipation $-\Delta$) in multidimensions $d > 1$ remains an open problem. In this work, we were able to give a positive answer regarding the existence of classical solutions with small initial data lying in the Besov spaces; also, we managed to get the optimal temporal decay of strong solutions. In the future direction, it will be interesting to study the current model on recent fractional derivatives, then demonstrate the effect of the fractional order through some simulations.

Data Availability

No underlying data was collected or produced in this study.

Disclosure

This work was conducted during our work at Hodeidah University.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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