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Research Article

Formulas for the Number of Weak Homomorphisms from Paths to Ladder Graphs and Stacked Prism Graphs

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Let G and H be graphs. A mapping f from $V(G)$ to $V(H)$ is called a weak homomorphism from G to H if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. A ladder graph is the Cartesian product of two paths, where one of the paths has only one edge. A stacked prism graph is the Cartesian product of a path and a cycle. In this paper, we provide a formula to determine the number of weak homomorphisms from paths to ladder graphs and a formula to determine the number of weak homomorphisms from paths to stacked prism graphs.

1. Introduction

Let G and H be graphs. A mapping $f\colon V(G) \longrightarrow V(H)$ is a homomorphism from G to H if f preserves the edges, i.e., if $\{f(x), f(y)\} \in E(H)$, whenever $\{x, y\} \in E(G)$. We denote the set of homomorphisms from G to H by $\mathrm{Hom}(G, H)$. Let P_n denote a path of order n such that $V(P_n) = \{0, 1, \ldots, n-1\}$ and $E(P_n) = \{\{i, i+1\} \mid \quad i = 0, 1, \ldots, n-2\}$. Let C_n denote a cycle of order n $(n \geqslant 3)$ such that $V(C_n) = \{0, 1, \ldots, n-1\}$ and $E(C_n) = \{\{i, i+1\} \mid \quad i = 0, 1, \ldots, n-1\}$, where $+$ is the addition modulo n. Furthermore, we will refer to [1, 2] for more information about graphs and algebraic graphs.

The formula for the number of homomorphism from P_n to P_n itself, $\mathrm{End}(P_n)$, was stated by Arworn [3] in 2009. Arworn [3] transformed the problem into counting the numbers of shortest paths from the point $(0, 0)$ to any point (i, j) in an r-ladder square lattice and obtained a concise formula.

In general, a homomorphism from G to G itself is called an endomorphism on G. Clearly, the set of endomorphisms on G forms a monoid under a composition of mappings.

For a mapping $f\colon V(G) \longrightarrow V(H)$, we say that f contracts an edge $\{x, y\} \in E(G)$ if $f(x) = f(y)$. A mapping $f\colon V(G) \longrightarrow V(H)$ is called a weak homomorphism from a graph G to a graph H (also called an egamorphism) if f contracts or preserves the edges; i.e., if $\{x, y\} \in E(G)$, then $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$. A weak homomorphism from G to G itself is called a weak endomorphism on G. We denote the set of weak homomorphisms from G to H by $\mathrm{WHom}(G, H)$ and the set of weak endomorphisms on G by $\mathrm{WEnd}(G)$. Clearly, $\mathrm{WEnd}(G)$ forms a monoid under a composition of mappings.

The composition of (weak) homomorphisms is also a (weak) homomorphism. When we have a collection of objects and morphisms between them, satisfying certain properties such as composition and identity, we can define a category. In this context, the category consists of graphs as objects and (weak) homomorphisms as morphisms, where the composition of (weak) homomorphisms and the identity (weak) homomorphism for each graph form the necessary structure [1]. It provides a structured way to study and analyze relationships between graphs, allowing for a wide range of applications, including graph database querying, graph theory research, and network analysis. The choice between strong (graph) homomorphisms and weak graph homomorphisms in the category can lead to different ways of capturing and studying relationships between graphs, depending on the specific requirements of the problem at hand.

Given a graph product $\circledast$, the cancellation problem for the product is the conditions under which $G \circledast K \cong H \circledast K$ implies $G \cong H$. The problem is simple when $\circledast$ is the Cartesian product $\square$. Consequently, we assert that cancellation holds for the Cartesian product. It is much more complicated for the direct product $\times$ and the strong product $\boxtimes$. In the case of the direct product of graphs, if there exist homomorphisms from G to K and from H to K, then $G \cong H$ [4]. By utilizing the fact that $|\mathrm{WHom}(X, A \boxtimes B)| = |\mathrm{WHom}(X, A)||\mathrm{WHom}(X, B)|$ for all finite simple graphs A and B, and for all finite graphs X where loops are admitted, cancellation also holds for the strong product of graphs [4].

In 2010, Sirisathianwatthana and Pipattanajinda [5] provided the number of weak homomorphisms of cycles WHom (C_m, C_n) in terms of the collection of $\mathrm{WHom}^i_j(P_{m-1}, C_n)$ where $\mathrm{WHom}^i_j(P_{m-1}, C_n)$ is a set of weak homomorphisms from P_{m-1} to C_n, where $f(0) = i$ and $f(m-1) = j$.

Motivated by Arworn's work [3], in 2018, Knauer and Pipattanajinda [6] used a cubic lattice and an r-ladder cubic lattice to construct the number of weak endomorphisms on paths $\mathrm{WEnd}(P_n)$. Moreover, they provided formulas for the number of shortest paths from the point $(0, 0, 0)$ to any point (i, j, k), as in Proposition 1. Figures 1 and 2 represent the cubic lattice and the 2-ladder cubic lattice when $i = 6$, $j = 4$, and $k = 4$, respectively.

Proposition 1 (see [6]). *The numbers $M(i, j, k)$ and $M_r(i, j, k)$ of the shortest paths from the point $(0, 0, 0)$ to any point (i, j, k) in the cubic lattice and in the r-ladder cubic lattice are as follows:*

$$M(i,j,k) = \binom{i+j+k}{i,\mathrm{j},\mathrm{k}},$$
$$M_r(i,j,k) = \left[\binom{i+j+k}{i,\mathrm{j},\mathrm{k}} - \binom{i+j+k}{j-r-1, i+r+1, k}\right], \tag{1}$$

respectively.

For any two graphs G_1 and G_2, the Cartesian product of G_1 and G_2 is the graph $G_1 \square G_2$ with vertices $V(G_1 \square G_2) = V(G_1) \times V(G_2)$, for which $\{(a, u), (b, v)\}$ is an edge if $a = b$ and $\{u, v\} \in E(G_2)$, or $\{a, b\} \in E(G_1)$ and $u = v$. The ladder graph L_n is the Cartesian product of P_n and P_2. The stacked prism graph $Y_{n,m}$ is the Cartesian product of P_n and C_m.

We see that a mapping $f\colon V(P_n) \longrightarrow V(G_1 \square G_2)$ is a homomorphism if and only if $f(0), f(1), \ldots, f(n-1)$ is a walk in $G_1 \square G_2$. We thus get a one-one correspondence between the set of homomorphisms $f\colon P_n \longrightarrow G_1 \square G_2$ and the set of walks of n vertices in $G_1 \square G_2$. In the same manner, we can see that there is a one-one correspondence between the set WHom $(P_n, G_1 \square G_2)$ and the set of partial walks of n vertices in $G_1 \square G_2$, where the partial walk is a sequence obtained by joining q walks $W_1, W_2, \ldots, W_q$ for some q with the ending vertex of W_i being the starting vertex of W_{i+1} for all $i = 1, 2, \ldots, q-1$.

In this paper, we are interested in finding the number of weak homomorphisms from paths to ladder graphs, WHom (P_n, L_n), and from paths to stacked prism graphs, WHom $(P_n, Y_{n,m})$. These give the numbers of partial walks of n vertices in L_n and $Y_{n,m}$. Here, we generalize the original cubic lattice by adding double edges and triple edges in the backward directions and obtain a double-bridge cubic lattice and a triple-bridge cubic lattice as in Figures 3 and 4, respectively. Similarly, we obtain an r-ladder double-bridge cubic lattice and an r-ladder triple-bridge cubic lattice (see Figures 5 and 6).

The number of shortest paths from the point $(0, 0, 0)$ to any point (i, j, k) in a double-bridge cubic lattice and in an r-ladder double-bridge cubic lattice is as follows.

Proposition 2. *The numbers $M^2(i, j, k)$ and $M^2_r(i, j, k)$ of shortest paths from the point $(0, 0, 0)$ to any point (i, j, k) in the double-bridge cubic lattice and in the r-ladder double-bridge cubic lattice are as follows:*

$$M^2(i,j,k) = 2^k\binom{i+j+k}{i,\mathrm{j},\mathrm{k}},$$
$$M^2_r(i,j,k) = 2^k\left[\binom{i+j+k}{i,\mathrm{j},\mathrm{k}} - \binom{i+j+k}{j-r-1, i+r+1, k}\right], \tag{2}$$

respectively.

Proof. To obtain the shortest path, at any instance, from the point (x, y, z), one can only go to $(x+1, y, z)$, $(x, y+1, z)$, or $(x, y, z+1)$. Since there are double bridges, there are two possible ways to visit $(x, y, z+1)$, namely, through the dashed line and through the dotted arc. We denote the move to the right by $\mathbf{R}$, the move up by $\mathbf{U}$, the move to the back through the dashed line by $\mathbf{B}_1$, and the move to the back through the dotted arc by $\mathbf{B}_2$. The number of shortest paths from the point $(0, 0, 0)$ to any point (i, j, k) in the double-bridge cubic lattice is equal to the number of different arrangements of i $\mathbf{R}$'s, j $\mathbf{U}$'s, u $\mathbf{B}_1$'s, and v $\mathbf{B}_2$'s, where $u + v = k$. Thus,

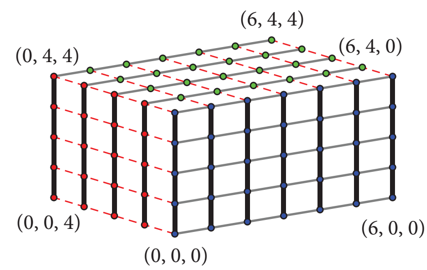

FIGURE 1: Cubic lattice.

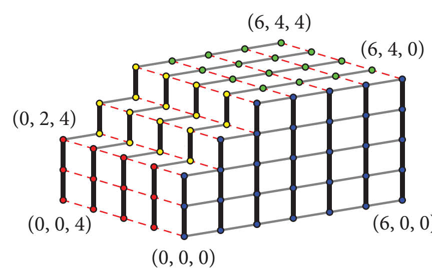

FIGURE 2: 2-ladder cubic lattice.

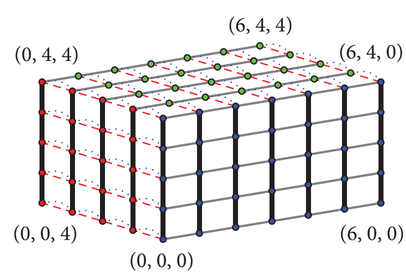

FIGURE 3: Double-bridge cubic lattice.

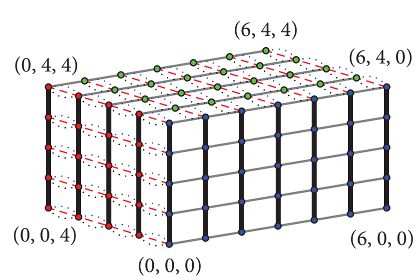

FIGURE 4: Triple-bridge cubic lattice.

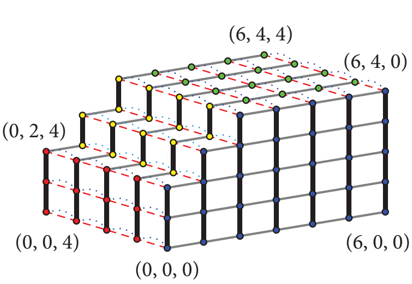

FIGURE 5: 2-ladder double-bridge cubic lattice.

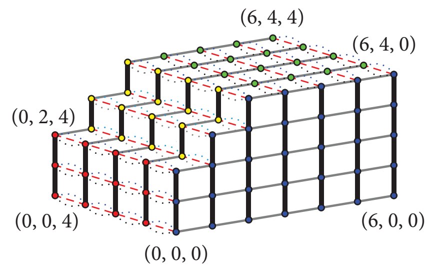

Figure 6: 2-ladder triple-bridge cubic lattice.

$$M^2(i,j,k) = \frac{(i+j+u+v)!}{i!j!u!v!} = \sum_{v=0}^{k} \frac{(i+j+k)!}{i!j!(k-v)!v!} = \frac{(i+j+k)!}{i!j!k!} \sum_{v=0}^{k} \frac{k!}{(k-v)!v!} = \binom{i+j+k}{i,j,k} \sum_{v=0}^{k} \binom{k}{v} = 2^k \binom{i+j+k}{i,j,k}.$$

While the number of shortest paths from the point $(0,0,0)$ to any point (i,j,k) in the r-ladder double-bridge cubic lattice is equal to the number of different arrangements of $i\,\mathbf{R}$'s, $j\,\mathbf{U}$'s, $u\,\mathbf{B}_1$'s, and $v\,\mathbf{B}_2$'s, where $u+v=k$, and $t^{\text{th}}\,\mathbf{R}$ always appears before $(r+t)^{\text{th}}\,\mathbf{U}$. Therefore,

$$\begin{aligned} M_r^2(i,j,k) &= \frac{(i+j+u+v)!}{i!j!u!v!} - \frac{(i+j+u+v)!}{(i+r+1)!(j-r-1)!u!v!} \\ &= \sum_{v=0}^{k} \left[\frac{(i+j+k)!}{i!j!(k-v)!v!} - \frac{(i+j+k)!}{(i+r+1)!(j-r-1)!(k-v)!v!} \right] \\ &= \left[\frac{(i+j+k)!}{i!j!k!} - \frac{(i+j+k)!}{(i+r+1)!(j-r-1)!k!} \right] \sum_{v=0}^{k} \frac{k!}{(k-v)!v!} \\ &= \left[\frac{(i+j+k)!}{i!j!k!} - \frac{(i+j+k)!}{(i+r+1)!(j-r-1)!k!} \right] \sum_{v=0}^{k} \binom{k}{v} \\ &= 2^k \left[\frac{(i+j+k)!}{i!j!k!} - \frac{(i+j+k)!}{(i+r+1)!(j-r-1)!k!} \right]. \end{aligned}$$

Similarly, in a triple-bridge cubic lattice, there are three possible ways to move from the point (x,y,z) to the point $(x,y,z+1)$. Thus, we obtain the following proposition analogously. □

Proposition 3. *The numbers $M^3(i,j,k)$ and $M_r^3(i,j,k)$ of the shortest paths from the point $(0,0,0)$ to any point (i,j,k) in the triple-bridge cubic lattice and in the r-ladder triple-bridge cubic lattice are as follows:*

$$M^3(i,j,k) = 3^k \binom{i+j+k}{i,j,k},$$
$$M_r^3(i,j,k) = 3^k \left[\binom{i+j+k}{i,j,k} - \binom{i+j+k}{j-r-1, i+r+1, k} \right], \quad (3)$$

respectively.

2. The Number of Weak Homomorphisms from Paths to Ladder Graphs

In this section, we provide the formula for finding the number of weak homomorphisms from paths P_n to ladder graphs L_n. We denote the set of weak homomorphisms from P_n to L_n, which maps 0 to (j,i) by $\mathrm{WHom}^{ji}(P_n, L_n)$. By the symmetry of L_n, we obtain the following lemma.

Lemma 4. *Let j and n be integers such that $0 \le j < n$.*

(1) $|WHom^{j0}(P_n, L_n)| = |WHom^{(n-j-1)0}(P_n, L_n)| = |WHom^{j1}(P_n, L_n)| = |WHom^{(n-j-1)1}(P_n, L_n)|$.

(2) $|WHom(P_{2n}, L_{2n})| = 4\sum_{j=0}^{n-1} |WHom^{j0}(P_{2n}, L_{2n})|$.

(3) $|WHom(P_{2n+1}, L_{2n+1})| = 4\sum_{j=0}^{n-1} |WHom^{j0}(P_{2n+1}, L_{2n+1})| + 2|WHom^{n0}(P_{2n+1}, L_{2n+1})|$.

To gain insight into the main theorem, we begin by observing a simple example. In this step, we aim to visualize weak homomorphisms. Figure 7 shows the possible weak homomorphisms from P_4 to L_4, which map 0 to (0, 0). The numbers on the top are elements of the domain set $V(P_4)$, and the tuples on the left are elements of the image set $V(L_4)$.

The mapping $f_1, f_2 \in \mathrm{WHom}^{00}(P_4, L_4)$ with $f_1(0) = (0,0), f_1(1) = (0,1), f_1(2) = (0,0), f_1(3) = (0,1)$ and $f_2(0) = (0,0), f_2(1) = (1,0), f_2(2) = (2,0), f_2(3) = (3,0)$ is represented by the dotted arcs on the top and black line (see Figure 8). We noted that normal lines represent the change in the first coordinate, dotted arcs represent the change in the second coordinate, and dashed lines indicate no change in coordinates.

Figure 9 visualizes weak homomorphisms using the double-bridge cubic lattice. There are multiple cases to be considered. First, $f(x+1) = f(x) + (1,0)$ correspond to moving from (i,j,k) to $(i+1,j,k)$. Second, $f(x+1) = f(x) - (1,0)$ correspond to moving from (i,j,k) to $(i,j+1,k)$. For the remaining cases, $f(x+1) = f(x) \pm (0,1)$ correspond to moving from (i,j,k) to $(i,j,k+1)$ through the dotted arc and $f(x+1) = f(x)$ correspond to moving from (i,j,k) to $(i,j,k+1)$ through the dashed line. Therefore, the mapping f_1, f_2 is represented by a shortest path from $(0,0,0)$ to $(0,0,3)$ and $(3,0,0)$ in the 0-ladder double-bridge cubic lattice, respectively.

Cardinality $|\mathrm{WHom}^{00}(P_4, L_4)|$ is the summation of $M^2(i,j,k)$ and $M_0^2(i,j,k)$, where $i+j+k=3$ (large black points). Based on Figure 10, if $j \le 0$ and $i \le 3$, we use $M^2(i,j,k)$; otherwise, $M_0^2(i,j,k)$:

$$\begin{aligned}
\left|\mathrm{WHom}^{00}(P_4, L_4)\right| &= M^2(3,0,0) + M_0^2(2,1,0) + M^2(2,0,1) + M_0^2(1,1,1) \\
&\quad + M^2(1,0,2) + M^2(0,0,3) \\
&= 2^0\binom{3}{3,0,0} + 2^0\left[\binom{3}{2,1,0} - \binom{3}{0,3,0}\right] + 2^1\binom{3}{2,0,1} \\
&\quad + 2^1\left[\binom{3}{1,1,1} - \binom{3}{0,2,1}\right] + 2^2\binom{3}{1,0,2} + 2^3\binom{3}{0,0,3} \\
&= 35.
\end{aligned} \quad (4)$$

Similar to the above example, Figure 11 visualizes all possible weak homomorphisms of the path P_4 to L_4 that map 0 to (1, 0).

Cardinality $|\mathrm{WHom}^{10}(P_4, L_4)|$ is the summation of $M^2(i,j,k)$ and $M_1^2(i,j,k)$, where $i+j+k=3$ (large black points). Based on Figure 12, if $j \le 1$ and $i \le 2$, we use $M^2(i,j,k)$; otherwise, $M_1^2(i,j,k)$:

$$\begin{aligned}
\left|\mathrm{WHom}^{10}(P_4, L_4)\right| &= M^2(2,1,0) + M_1^2(1,2,0) + M^2(2,0,1) + M^2(1,1,1) \\
&\quad + M^2(1,0,2) + M^2(0,1,2) + M^2(0,0,3) \\
&= 2^0\binom{3}{2,1,0} + 2^0\left[\binom{3}{1,2,0} - \binom{3}{0,3,0}\right] + 2^1\binom{3}{2,0,1} \\
&\quad + 2^1\binom{3}{1,1,1} + 2^2\binom{3}{1,0,2} + 2^2\binom{3}{0,1,2} + 2^3\binom{3}{0,0,3} \\
&= 55.
\end{aligned} \tag{5}$$

In general, to find $|\mathrm{Whom}^{r0}(P_n, L_n)|$, we use $M_r^2(i,j,k)$ to compute the number of shortest paths from (0, 0, 0) to (i,j,k) when $j > r$, and we use $M_{n-r-1}^2(j,i,k)$ when $i > n-r-1$; otherwise, we use $M^2(i,j,k)$.

Theorem 5. *Let n be a positive integer and j be a nonnegative integer such that $j < n/2 - 1$. It follows that*

$$\begin{aligned}
\left|\mathrm{WHom}^{j0}(P_n, L_n)\right| &= \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} 2^{i_0+j_0}\binom{n-1}{i-i_0, j-j_0, i_0+j_0} \\
&\quad + \sum_{s=1}^{\lfloor i/2\rfloor}\sum_{i_0=0}^{i-2s} 2^{i_0}\left[\binom{n-1}{i-s-i_0, j+s, i_0} - \binom{n-1}{s-1, n-s-i_0, i_0}\right] \\
&\quad + \sum_{s=1}^{\lfloor j/2\rfloor}\sum_{j_0=0}^{j-2s} 2^{j_0}\left[\binom{n-1}{j-s-j_0, i+s, j_0} - \binom{n-1}{s-1, n-s-j_0, j_0}\right],
\end{aligned} \tag{6}$$

where $n-1 = i+j$.

Proof. Let $i = n-j-1$. To find $|\mathrm{WHom}^{j0}(P_n, L_n)|$, we count the number of shortest paths from the point $(0,0,0)$ to any point (i_0, j_0, k_0), where $i_0 + j_0 + k_0 = n-1$ in the j-ladder double-bridge cubic lattice. We consider the following three cases corresponding to the value of j_0. □

If $j_0 > j$, then for each $j_0 = j+t$, there are $\sum_{i_0=t}^{i-t} M_j^2(i_0, j_0, k_0)$ shortest paths. Figure 13 displays possible end points $(i_0, j_0 = j+t, k_0)$ when $t=1$ by big circles, while small circles stand for all possible origins of dashed lines with an end point (i_0, j_0, k_0).

Since $t \le i/2$, we obtain the following equation:

$$\sum_{t=1}^{\lfloor i/2\rfloor}\sum_{i_0=t}^{i-t} M_j^2(i_0, j+t, k_0) = \sum_{t=1}^{\lfloor i/2\rfloor}\sum_{i_0=t}^{i-t} 2^{k_0}\left[\binom{n-1}{i_0, j+t, k_0} - \binom{n-1}{i_0+j+1, t-1, k_0}\right]. \tag{7}$$

We replace t and i_0 with s and $i-i_0-s$, respectively, and the total number of shortest paths is as follows:

$$\sum_{s=1}^{\lfloor i/2\rfloor}\sum_{i_0=0}^{i-2s} 2^{i_0}\left[\binom{n-1}{i-s-i_0, j+s, i_0} - \binom{n-1}{s-1, n-s-i_0, i_0}\right]. \tag{8}$$

If $j_0 \le j$ and $i_0 \le i$, then for each i_0, j_0, there are $M^2(i_0, j_0, k_0)$ shortest paths.

This contributes $\sum_{i_0=0}^{i}\sum_{j_0=0}^{j} M^2(i_0, j_0, k_0) = \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} 2^{k_0}\binom{n-1}{i_0, j_0, k_0}$. Figure 14 displays a possible end point (i_0, j_0, k_0) by the big circle, while small circles stand for all possible origins of dashed lines with an end point (i_0, j_0, k_0).

We replace i_0 and j_0 with $i-i_0$ and $j-j_0$, respectively, and the total number of shortest paths is as follows:

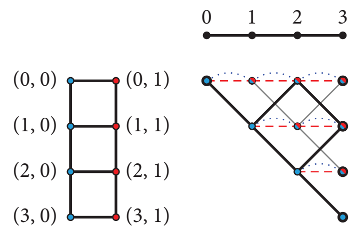

FIGURE 7: Graphical presentation of the domain and image of all possible weak homomorphisms $f\colon P_4 \longrightarrow L_4$, where $f(0) = (0,0)$.

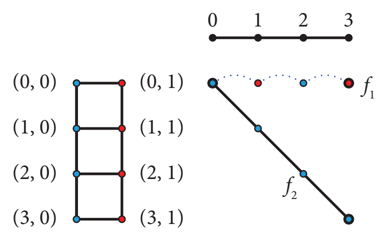

FIGURE 8: Graphical presentation of the domain and image of f_1 and f_2.

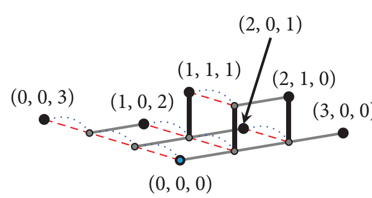

FIGURE 9: Double-bridge cubic lattice presentation of all possible weak homomorphisms $f\colon P_4 \longrightarrow L_4$, where $f(0) = (0,0)$.

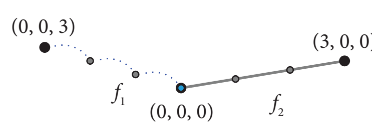

FIGURE 10: Double-bridge cubic lattice presentation of f_1 and f_2.

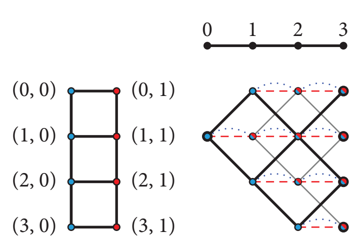

FIGURE 11: Graphical presentation of the domain and image of all possible weak homomorphisms $f\colon P_4 \longrightarrow L_4$, where $f(0) = (1,0)$.

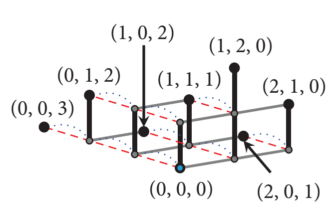

FIGURE 12: Double-bridge cubic lattice presentation of all possible weak homomorphisms $f\colon P_4 \longrightarrow L_4$, where $f(0) = (1,0)$.

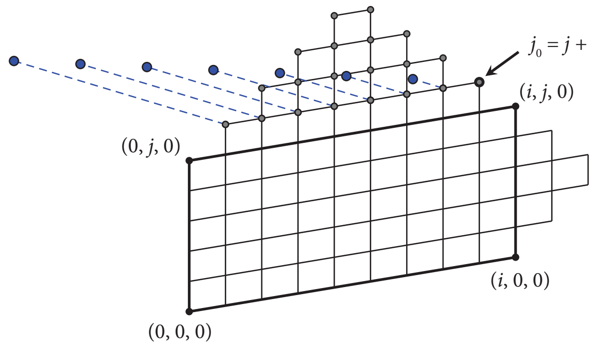

Figure 13: Possible end points in the j-ladder double-bridge cubic lattice when $j_0 > j$.

$$\sum_{i_0=0}^{i}\sum_{j_0=0}^{j} 2^{i_0+j_0}\binom{n-1}{i-i_0, i-j_0, i_0+j_0}. \tag{9}$$

If $j_0 \leq j$ and $i_0 \geq i+1$, then for each $i_0 = i + t'$, there are $\sum_{j_0=t'}^{j-t'} M_i^2(j_0, i_0, k_0)$ shortest paths. This can be obtained by flipping the cubic lattice diagonally. Figure 15 displays possible end points $(i_0 = i + t', j_0, k_0)$ when $t' = 1$ by big circles, while small circles stand for all possible origins of dashed lines with an end point (i_0, j_0, k_0).

Since $t' \leq j/2$, we obtain the following equation:

$$\sum_{t'=1}^{\lfloor j/2 \rfloor}\sum_{j_0=t'}^{j-t'} M_i^2\left(j_0, i+t', k_0\right) = \sum_{t'=1}^{\lfloor j/2 \rfloor}\sum_{j_0=t'}^{j-t'} 2^{k_0}\left[\binom{n-1}{j_0, i+t', k_0} - \binom{n-1}{j_0+i+1, t'-1, k_0}\right]. \tag{10}$$

We replace t' and j_0 with s and $j - j_0 - s$, respectively, and the total number of shortest paths is as follows:

$$\sum_{s=1}^{\lfloor j/2 \rfloor}\sum_{j_0=0}^{j-2s} 2^{j_0}\left[\binom{n-1}{j-s-j_0, i+s, j_0} - \binom{n-1}{s-1, n-s-j_0, j_0}\right]. \tag{11}$$

Adding up over all cases, $|\mathrm{WHom}^{j0}(P_n, L_n)|$ is as desired.

3. The Number of Weak Homomorphisms from Paths to Stacked Prism Graphs

In this section, we provide the formula for finding the number of weak homomorphisms from paths P_n to stacked prism graphs $Y_{n,m}$. We denote the set of weak homomorphisms from P_n to $Y_{n,m}$, which maps 0 to (j, i) by $\mathrm{WHom}^{ji}(P_n, Y_{n,m})$. By the symmetry of $Y_{n,m}$, we obtain the following lemma.

Lemma 6. *Let i and n be integers such that* $0 \leq j < n$, *and let* $m > 2$ *be a positive integer.*

(1) $|WHom^{j0}(P_n, Y_{n,m})| = |WHom^{(n-j-1)0}(P_n, Y_{n,m})| = |WHom^{ji}(P_n, Y_{n,m})| = |WHom^{(n-j-1)i}(P_n, Y_{n,m})|$, *for all* $i \in \{0, 1, \ldots, m-1\}$.

(2) $|WHom(P_{2n}, Y_{2n,m})| = 2m\sum_{j=0}^{n-1}|WHom^{j0}(P_{2n}, Y_{2n,m})|$.

(3) $|WHom(P_{2n+1}, Y_{2n+1,m})| = 2m\sum_{j=0}^{n-1}|WHom^{j0}(P_{2n+1}, Y_{2n+1,m})| + m|WHom^{n0}(P_{2n+1}, Y_{2n+1,m})|$.

Similar to the previous section, we provide some insight via examples. Figure 16 shows all the possible weak homomorphisms from P_4 to $Y_{4,3}$, which map 0 to (0, 0). The numbers on the top are elements of the domain set $V(P_4)$, and the tuples on the left are elements of the image set $V(Y_{4,3})$. Figure 17 visualizes weak homomorphisms using the triple-bridge cubic lattice, where the move from (i, j, k) to the next point is depicted as follows:

(1) To $(i+1, j, k)$, if $f(x+1) = f(x) + (1, 0)$.

(2) To $(i, j+1, k)$, if $f(x+1) = f(x) - (1, 0)$.

(3) To $(i, j, k+1)$ through the dashed line, if $f(x) = f(x)$.

(4) To $(i, j, k+1)$ through the dotted upper arc, if $f(x+1) = f(x) + (0, 1)$.

(5) To $(i, j, k+1)$ through the dotted lower arc, if $f(x+1) = f(x) - (0, 1)$.

Note that + is the addition modulo m for the second coordinate of images. Again, normal lines in Figure 16 represent the change in the first coordinate, dotted arcs represent the change in the second coordinate, and dashed lines indicate no change in coordinates.

Therefore, the mappings $f_3, f_4 \in \mathrm{WHom}^{00}(P_4, Y_{4,3})$ with $f_3(0) = (0,0), f_3(1) = (0,1), f_3(2) = (0,0), f_3(3) = (0,1)$, and $f_4(0) = (0,0), f_4(1) = (1,0), f_4(2) = (2,0), f_4(3) = (3,0)$ are represented by the dotted arcs on the top and black line, respectively (see Figure 18). The mapping f_3, f_4 is represented by the shortest path from $(0,0,0)$ to $(0,0,3)$ and (3, 0, 0) in the 0-ladder triple-bridge cubic lattice, respectively (see Figure 19).

Cardinality $|\mathrm{WHom}^{00}(P_4, Y_{4,3})|$ is the summation of $M^3(i, j, k)$ and $M_0^3(i, j, k)$, where $i + j + k = 3$ (large black points). Based on Figure 17, if $j \leq 0$ and $i \leq 3$, we use $M^3(i, j, k)$; otherwise, $M_0^3(i, j, k)$:

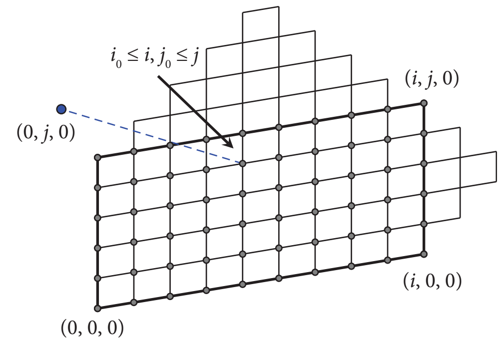

FIGURE 14: Possible end points in the j-ladder double-bridge cubic lattice when $j_0 \le j$ and $i_0 \le i$.

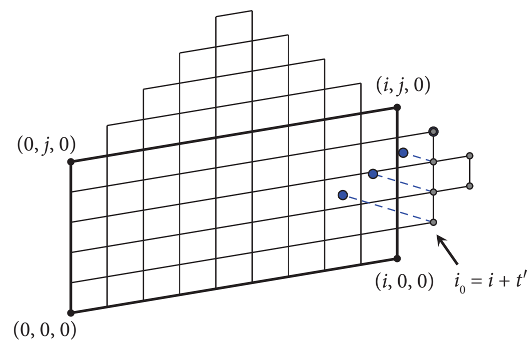

FIGURE 15: Possible end points in the j-ladder double-bridge cubic lattice when $j_0 \le j$ and $i_0 > i$.

$$\begin{aligned}\left|\mathrm{WHom}^{00}\left(P_4, Y_{4,3}\right)\right| &= M^3(3,0,0) + M_0^3(2,1,0) + M^3(2,0,1) \\ &\quad + M_0^3(1,1,1) + M^3(1,0,2) + M^3(0,0,3) \\ &= 3^0\binom{3}{3,0,0} + 3^0\left[\binom{3}{2,1,0} - \binom{3}{0,3,0}\right] + 3^1\binom{3}{2,0,1} \\ &\quad + 3^1\left[\binom{3}{1,1,1} - \binom{3}{0,2,1}\right] + 3^2\binom{3}{1,0,2} + 3^3\binom{3}{0,0,3} \\ &= 75.\end{aligned} \tag{12}$$

Similar to the abovementioned example, Figures 20 and 21 visualize the possible weak homomorphisms of the path P_4 to $Y_{4,3}$, which map 0 to (1, 0). We then use the 1-ladder cubic lattice, since $f(0) = (1, i)$ for some $i \in \{0, 1, \ldots, m-1\}$.

Cardinality $|\mathrm{WHom}^{10}(P_4, Y_{4,3})|$ is the summation of $M^3(i, j, k)$ and $M_1^3(i, j, k)$, where $i + j + k = 3$ (large black points). Based on Figure 21, if $j \le 1$ and $i \le 2$, we use $M^3(i, j, k)$; otherwise, $M_1^3(i, j, k)$:

$$\begin{aligned}\left|\mathrm{WHom}^{10}\left(P_4, Y_{4,3}\right)\right| &= M^3(2,1,0) + M_1^3(1,2,0) + M^3(2,0,1) + M^3(1,1,1) \\ &\quad + M^3(1,0,2) + M^3(0,1,2) + M^3(0,0,3) \\ &= 3^0\binom{3}{2,1,0} + 3^0\left[\binom{3}{1,2,0} - \binom{3}{0,3,0}\right] + 3^1\binom{3}{2,0,1} + 3^1\binom{3}{1,1,1} \\ &\quad + 3^2\binom{3}{1,0,2} + 3^2\binom{3}{0,1,2} + 3^3\binom{3}{0,0,3} \\ &= 113.\end{aligned} \tag{13}$$

In general, to find $|\mathrm{Whom}^{r0}(P_n, Y_{n,m})|$, we use $M_r^3(i,j,k)$ to compute the number of shortest paths from (0, 0, 0) to (i,j,k) when $j > r$, and we use $M_{n-r-1}^3(j,i,k)$ when $i > n-r-1$; otherwise, we use $M^3(i,j,k)$.

Theorem 7. *Let m, n be positive integers and j be a nonnegative integer such that $m \geq 3$ and $j < n/2 - 1$. It follows that*

$$\begin{aligned}\left|\mathrm{WHom}^{j0}\left(P_n, Y_{n,m}\right)\right| &= \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} 3^{i_0+j_0}\binom{n-1}{i-i_0, j-j_0, i_0+j_0} \\ &+ \sum_{s=1}^{\lfloor i/2\rfloor}\sum_{i_0=0}^{i-2s} 3^{i_0}\left[\binom{n-1}{i-s-i_0, j+s, i_0} - \binom{n-1}{s-1, n-s-i_0, i_0}\right] \\ &+ \sum_{s=1}^{\lfloor j/2\rfloor}\sum_{j_0=0}^{j-2s} 3^{j_0}\left[\binom{n-1}{j-s-j_0, i+s, j_0} - \binom{n-1}{s-1, n-s-j_0, j_0}\right],\end{aligned} \tag{14}$$

where $n-1 = i+j$.

Proof. Analogously to the proof of Theorem 5, we replace $M^2(i_0, j_0, k_0)$ and $M_j^2(i_0, j_0, k_0)$ with $M^3(i_0, j_0, k_0)$ and $M_j^3(i_0, j_0, k_0)$, respectively. The theorem is proved. □

4. Main Results

From Lemma 4 and Theorem 5, we obtain the theorem as follows.

Theorem 8. *The cardinalities $|WHom(P_n, L_n)|$ of weak homomorphisms from undirected paths P_n to ladder graphs L_n are as follows:*

$$\begin{aligned}\left|\mathrm{WHom}(P_n, L_n)\right| &= 4\sum_{j=0}^{\lfloor n/2\rfloor-1}\left|\mathrm{WHom}^{j0}(P_n, L_n)\right| \\ &+ \left[1-(-1)^n\right]\left|\mathrm{WHom}^{\lfloor n/2\rfloor 0}(P_n, L_n)\right|,\end{aligned} \tag{15}$$

where

$$\begin{aligned}\left|\mathrm{WHom}^{j0}(P_n, L_n)\right| &= \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} 2^{i_0+j_0}\binom{n-1}{i-i_0, j-j_0, i_0+j_0} \\ &+ \sum_{s=1}^{\lfloor i/2\rfloor}\sum_{i_0=0}^{i-2s} 2^{i_0}\left[\binom{n-1}{i-s-i_0, j+s, i_0} - \binom{n-1}{s-1, n-s-i_0, i_0}\right] \\ &+ \sum_{s=1}^{\lfloor j/2\rfloor}\sum_{j_0=0}^{j-2s} 2^{j_0}\left[\binom{n-1}{j-s-j_0, i+s, j_0} - \binom{n-1}{s-1, n-s-j_0, j_0}\right],\end{aligned} \tag{16}$$

with $i = n-1-j$.

From Lemma 6 and Theorem 7, we obtain the theorem as follows.

Theorem 9. *The cardinalities $|WHom(P_n, Y_{n,m})|$ of weak homomorphisms from undirected paths P_n to stacked prism graphs $Y_{n,m}$ are as follows:*

$$\begin{aligned}\left|\mathrm{WHom}\left(P_n, Y_{n,m}\right)\right| &= 2m\sum_{j=0}^{\lfloor n/2\rfloor-1}\left|\mathrm{WHom}^{j0}\left(P_n, Y_{n,m}\right)\right| \\ &+ \frac{m}{2}\left[1-(-1)^n\right]\left|\mathrm{WHom}^{\lfloor n/2\rfloor 0}\left(P_n, Y_{n,m}\right)\right|,\end{aligned} \tag{17}$$

where

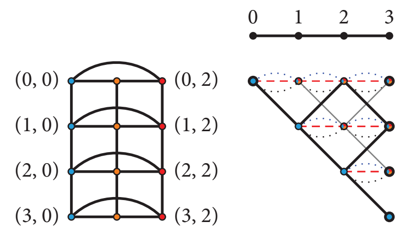

FIGURE 16: Graphical presentation of the domain and image of all possible weak homomorphisms $f\colon P_4 \longrightarrow Y_{4,3}$, where $f(0) = (0,0)$.

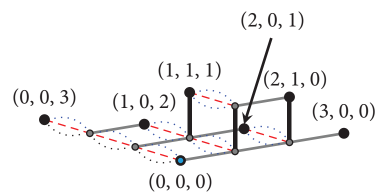

FIGURE 17: Triple-bridge cubic lattice presentation of all possible weak homomorphisms $f\colon P_4 \longrightarrow Y_{4,3}$, where $f(0) = (0,0)$.

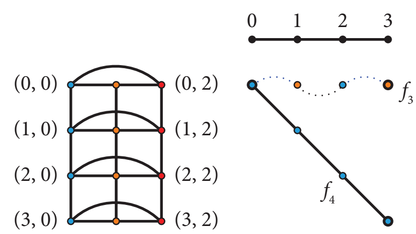

FIGURE 18: Graphical presentation of the domain and image of f_3 and f_4.

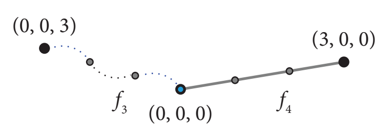

FIGURE 19: Triple-bridge cubic lattice presentation of f_3 and f_4.

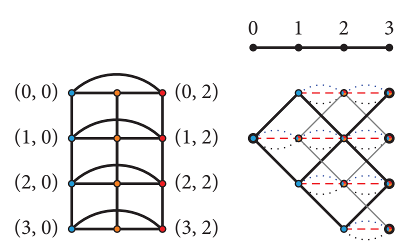

FIGURE 20: Graphical presentation of the domain and image of all possible weak homomorphisms $f\colon P_4 \longrightarrow Y_{4,3}$, where $f(0) = (1,0)$.

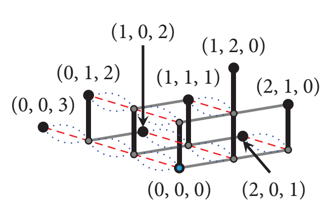

FIGURE 21: Triple-bridge cubic lattice presentation of all possible weak homomorphisms $f\colon P_4 \longrightarrow Y_{4,3}$, where $f(0) = (1,0)$.

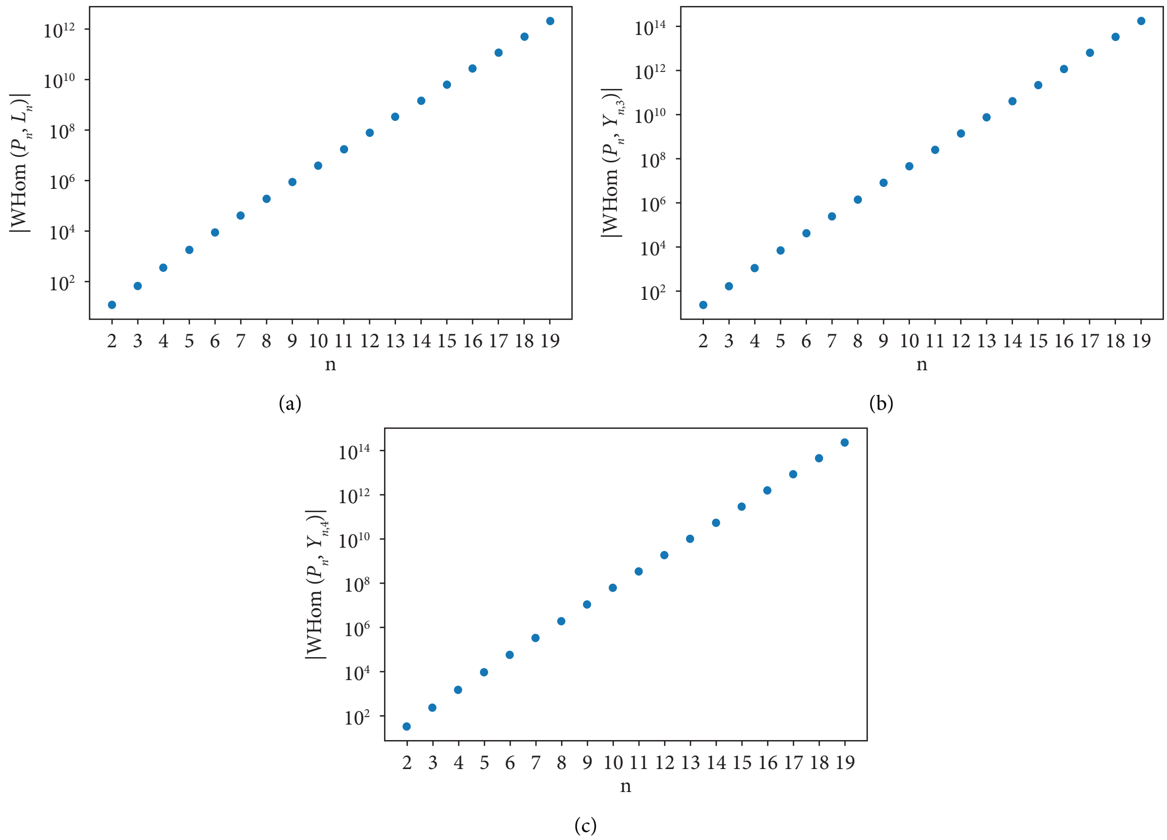

Figure 22: The size of $|\mathrm{WHom}(P_n, G)|$ on a logarithmic scale for $2 \leq n \leq 19$. (a) $G = L_n$. (b) $G = Y_{n,3}$. (c) $G = Y_{n,4}$.

$$\begin{aligned}\left|\mathrm{WHom}^{j0}\left(P_n, Y_{n,m}\right)\right| = & \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} 3^{i_0+j_0}\binom{n-1}{i-i_0,\, j-j_0,\, i_0+j_0} \\ & + \sum_{s=1}^{\lfloor i/2\rfloor}\sum_{i_0=0}^{i-2s} 3^{i_0}\left[\binom{n-1}{i-s-i_0,\, j+s,\, i_0} - \binom{n-1}{s-1,\, n-s-i_0,\, i_0}\right] \\ & + \sum_{s=1}^{\lfloor j/2\rfloor}\sum_{j_0=0}^{j-2s} 3^{j_0}\left[\binom{n-1}{j-s-j_0,\, i+s,\, j_0} - \binom{n-1}{s-1,\, n-s-j_0,\, j_0}\right],\end{aligned} \tag{18}$$

with $i = n - 1 - j$.

The algorithmic complexity of the evaluation of the formulas is $O(n^4)$ for memoryless computation. This complexity can be toned down to $O(n^3)$ using linear space memory. The computed $|\mathrm{WHom}(P_n, L_n)|$ and $|\mathrm{WHom}(P_n, Y_{n,m})|$ for $2 \leq n \leq 19$ and $3 \leq m \leq 4$ on a logarithmic scale are presented in Figure 22. Although the number of weak homomorphisms from paths to ladder graphs and stacked prism graphs is not bounded, one can conjecture that the asymptotic behaviour of the formulas, depending on n, is in exponential form.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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