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Research Article

A Generalization of Hermite–Hadamard–Fejer Type Inequalities for the p-Convex Function via α-Generator

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In the 17th century, I. Newton and G. Leibniz found independently each other the basic operations of calculus, i.e., differentiation and integration. And this development broke new ground in mathematics. From 1967 to 1970, Michael Grossman and Robert Katz gave definitions of a new kind of derivative and integral, converting the roles of subtraction and addition into division and multiplication, respectively. And then they generalised this operation. Later, they named this analysis non-Newtonian calculus. This calculus is basically generated by generators. So, in this article, first, we give the definition of the p-convex function due to α-generator. Second, we obtain some new theorems for this function with respect to the α-generator. Third, we get some new theorems using Hermite–Hadamard–Fejer inequality for the α p-convex function. Finally, we show that our obtained results are reduced to the classical case in the special conditions.

1. Introduction

Inequalities play an important role in almost all areas of mathematics. The first basic work on inequalities was the book “Inequalities” written by Hardy et al. [1]. This is the first reference book on pure inequalities and includes many new inequalities and their applications. The second book “Inequalities” was written by Beckenbach and Bellman [2], which contains some interesting results on inequalities in the period 1934–1960. Mitrinovic’s book [3], “Analytic Inequalities,” published in 1970, contains new topics that were not included in the two books mentioned above. Besides these three main sources, Mitrinovic et al. [4], “Classical and New Inequalities in Analysis” by Pachpatte, and “Mathematical Inequalities” by Pachpatte [5], in the recent years, many books and articles on inequalities have been published by researchers such as S. S. Dragomir, V. Lakshmikantham, R. P. Agarwal, M. E. Özdemir, E. Set, İşcan, M. Z. Sarkaya, and A. O. Akdemir.

Although the history of convex functions is very old, its beginning can be shown as the end of the 19th century. The basis of such functions is mentioned in Hadamard’s work in 1893 [6], although it is not explicitly stated. After this date, although results implying convex functions were encountered in the literature, convex functions were first systematically used in 1905 and 1906 by J. L. W. V. Jensen. It is accepted that the theory of convex functions has developed rapidly since Jensen’s pioneering work. Many researchers such as Beckenbach and Bellman [2] and Mitrinovic [3] have discussed the issue of inequalities for convex functions in their books. Also, Roberts and Varberg [7], Pecaric et al. [8], and Niculescu and Persson [9] have done many studies on inequalities on convex functions. Integral inequalities constitute some of these studies.

In the literature, well-known inequalities related to the integral mean of a convex function f are the Hermite–Hadamard inequalities [6] or its weighted versions, namely, Hermite–Hadamard–Fejer inequality [10]. Also, there are many kinds of convex functions. p-convex functions were first defined by Kunt and İşcan [11] for Hermite–Hadamard–Fejer inequality.

Sir I. Newton and G. Leibniz developed mathematics by establishing differentiation and integration in the 17th century. These developments were very useful for mathematics and related sciences. From 1967 to 1970, Grossman and Katz gave new definitions for derivative and integral, converting the roles of subtraction and addition into division and multiplication, respectively. Then, they established geometric, bigeometric, harmonic, biharmonic, quadratic, and biquadratic calculus, and they named these calculi non-Newtonian calculi. Grossman and Katz published the first book concerning with non-Newtonian calculus in 1972 [12] and then wrote nine books related to the non-Newtonian calculi.

In geometric calculus and bigeometric calculus from within these calculi, the derivative and integral are both multiplicative. The geometric derivative and the bigeometric derivative are closely related to the well-known logarithmic derivative and elasticity, respectively. Also, the linear functions of classical calculus are the functions which have a constant derivative, and besides the exponential functions in the geometric calculus which have a constant derivative, the power functions in the bigeometric calculus are the functions which have a constant derivative. Among the non-Newtonian calculi, geometric and bigeometric calculi have been often used.

Since these calculi emerged, it has become a serious alternative to the classical analysis developed by Newton and Leibniz. Just like the classical analysis, non-Newtonian calculi have many varieties as a derivative, an integral, a natural average, a special class of functions having a constant derivative, and two fundamental theorems which reveal that the derivative and integral are inversely related. However, the results obtained by non-Newtonian calculus are also significantly different from the classical analysis. For example, infinitely, many non-Newtonian calculi have a nonlinear derivative or integral.

The non-Newtonian calculi are useful mathematical tools in science, engineering, and mathematics and provide a wide variety of possibilities, as a different perspective. Specific fields of application include fractal theory, image analysis (e.g., in biomedicine), growth/decay processes (e.g., in economic growth, bacterial growth, and radioactive decay), finance (e.g., rates of return), the theory of elasticity in economics, marketing, the economics of climate change, atmospheric temperature, wave theory in physics, quantum physics and the Gauge theory, signal processing, information technology, pathogen counts in treated water, actuarial science, tumor therapy in medicine, materials science/engineering, demographics, and differential equations [13–29].

2. Preliminaries

Hermite–Hadamard inequality is one of the well-known inequalities concern with the integral mean for convex functions. And the other one is Hermite–Hadamard–Fejer inequality [10]. It is the weighted version of Hermite–Hadamard inequality. If $f\colon [a,b] \longrightarrow \mathrm{R}$ is a convex function, $g\colon [a,b] \longrightarrow \mathrm{R}$ function is integrable on $[a,b]$, nonnegative, and symmetric to $(a+b/2)$; then, the following inequality holds for all $x \in [a,b]$.

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\mathrm{d}x. \quad (1)$$

p-convex function was firstly defined on $I \subset \mathrm{R}$ by Zhang and Wan [30]. And İşcan also defined the p-convex function on $(0,+\infty)$ in a different way as follows.

Definition 1 (see [31]). Let $I \subset (0,\infty)$ be a real interval and $p \in \mathrm{R}/\{0\}$. A function $f\colon I \longrightarrow \mathrm{R}$ is said to be a p-convex function if

$$f\left(\left(tx^p + (1-t)y^p\right)^{1/p}\right) \leq t f(x) + (1-t) f(y), \quad (2)$$

for all $x, y \in I$ and $t \in [0,1]$. If inequality (2) is reversed, then f is said to be p-concave.

Theorem 1 (see [31]). *Let $f\colon I \subset (0,\infty) \longrightarrow \mathrm{R}$ be a p-convex function, $p \in \mathrm{R}/\{0\}$, $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then we have*

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \quad (3)$$

Now, we give about a short brief of non-Newtonian calculus [12].

2.1. Systems of Arithmetic. Arithmetic is any system that satisfies the whole of the ordered field axiom whose domain is a subset of R. There are many types arithmetic, all of which are isomorphic, that is, structurally equivalent.

A generator α is a one-to-one function whose domain is R and whose range is a subset R_α of R, where $\mathrm{R}_\alpha = \{\alpha(x)\colon x \in \mathrm{R}\}$. Each generator generates exactly one arithmetic, and conversely, each arithmetic is generated by exactly one generator.

Throughout this article, the identity function is denoted by Id because we do not want to mix an interval. When we take identity and exponential functions as a generator, then $\alpha = \mathrm{Id}$ and $\alpha = \exp$, and α generates the classical and geometric arithmetic, respectively.

2.1.1. α-Arithmetics. We suppose that α be a generator and $x, y \in \mathrm{R}_\alpha$. Then, by α-arithmetic, we mean the arithmetic whose domain is R and whose operations are defined as follows:

$$\alpha - \text{addition}, x \dot{+} y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\},$$

$$\alpha - \text{subtraction}, x \dot{-} y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\},$$

$$\alpha - \text{multiplication}, x \dot{\times} y = \alpha\left\{\alpha^{-1}(x) \times \alpha^{-1}(y)\right\}, \quad (4)$$

$$\alpha - \text{division}, x / y = \alpha\left\{\frac{\alpha^{-1}(x)}{\alpha^{-1}(y)}\right\},$$

$$\alpha - \text{order}, x \dot{<} y \Longleftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y).$$

As a generator, we choose the exp function acting from R into the set $R_{\exp} = (0, \infty)$ as follows:

$$\begin{aligned} \alpha &: R \longrightarrow R_{\exp}, \\ x &\longrightarrow y = \alpha(x) = e^{x}. \end{aligned} \quad (5)$$

It is obvious that α arithmetic reduces to the geometric arithmetic as follows:

$$\text{geometricaddition}, x \dot{+} y = e^{\{\ln x + \ln y\}} = x.y,$$

$$\text{geometricsubtraction}, x \dot{-} y = e^{\{\ln x - \ln y\}} = \frac{x}{y},$$

$$\text{geometricmultiplication}, x \dot{\times} y = e^{\{\ln x \times \ln y\}} = x^{\ln y} = y^{\ln x},$$

$$\text{geometricdivision}, x / y = e^{\{\ln x / \ln y\}} = x^{1/\ln y},$$

$$\text{geometricorder}, x \dot{<} y \Longleftrightarrow \ln(x) < \ln(y). \quad (6)$$

Definition 2 (see [12]). Let $\dot{-}\dot{n} = \dot{0} \dot{-} \dot{n} = \alpha(-n)$ for all $n \in Z$. Set of α-integers is defined and denoted by Z_α as can be seen in the figure below:

$$\begin{aligned} Z_\alpha &= \left\{\ldots, \dot{-}2, \dot{-}1, 0, \dot{1}, \dot{2}, \ldots\right\} \\ &= \{\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots\}. \end{aligned} \quad (7)$$

Namely, $Z_\alpha = \{\dot{n}\colon \dot{n} = \alpha(n), n \in R\}$.

Remark 1 (see [28]). Let α be the generator. Then, $I_\alpha \subset R_\alpha$ is said to be an α-interval on R_α if for all $x, y \in I_\alpha$;

(1) $(x, y)\colon = \{z \in I_\alpha\colon x \dot{<} z \dot{<} y\} \subset I_\alpha$
(2) $(x, y]\colon = \{z \in I_\alpha\colon x \dot{<} z \dot{\leq} y\} \subset I_\alpha$
(3) $[x, y)\colon = \{z \in I_\alpha\colon x \dot{\leq} z \dot{<} y\} \subset I_\alpha$
(4) $[x, y]\colon = \{z \in I_\alpha\colon x \dot{\leq} z \dot{\leq} y\} \subset I_\alpha$
(5) $(x, \dot{+}\dot{\infty})\colon = \{z \in I_\alpha\colon x \dot{<} z \dot{<} \dot{+}\dot{\infty}\} \subset I_\alpha$
(6) $(\dot{-}\dot{\infty}, y)\colon = \{z \in I_\alpha\colon \dot{-}\dot{\infty} \dot{<} z \dot{<} y\} \subset I_\alpha$
(7) $[x, \dot{+}\dot{\infty})\colon = \{z \in I_\alpha\colon x \dot{\leq} z \dot{<} \dot{+}\dot{\infty}\} \subset I_\alpha$
(8) $(\dot{-}\dot{\infty}, y]\colon = \{z \in I_\alpha\colon \dot{-}\dot{\infty} \dot{<} z \dot{\leq} y\} \subset I_\alpha$

We can also show the α-intervals as follows:

$$(x, y]_\alpha, [x, y)_\alpha, [x, y]_\alpha, (x, +\infty)_\alpha, (-\infty, y)_\alpha, [x, +\infty)_\alpha, (-\infty, y]_\alpha. \quad (8)$$

Remark 2 (see [28]). Afterwards, $[x, y]_\alpha$ and $(x, y)_\alpha$ are said to be α-closed interval and α-open interval, respectively.

Definition 3 (see [32]). Let $[a, b]_\alpha \subset R_\alpha$ be an α-closed interval and $f\colon [a, b]_\alpha \longrightarrow R$ be a function. If the following inequality holds for all $x, y \in [a, b]_\alpha$ and $t \in [0, 1]$,

$$f(\alpha(t) \dot{\times} x \dot{+} \alpha(1 - t) \dot{\times} y) \leq t f(x) + (1 - t) f(y), \quad (9)$$

we say that f is an α-convex function.

Definition 4 (see [33]). Let $I_\alpha \subset R_\alpha$ be an α-interval. A function $f\colon I_\alpha \longrightarrow R$ is said to be α-harmonically convex if the following inequality holds for all $a, b \in I_\alpha$ and $t \in [0, 1.]$

$$f\left(\frac{a \dot{\times} b}{\alpha(t) \dot{\times} a \dot{+} \alpha(1 - t) \dot{\times} b}\right) \leq t f(b) + (1 - t) f(a). \quad (10)$$

Definition 5 (see [33]). Let $g\colon [a, b]_\alpha \longrightarrow R$ be a function. If the function g holds the following equality

$$g\left(\frac{a \dot{\times} b}{x}\right) = g\left(\frac{a \dot{\times} b}{a \dot{+} b \dot{-} x}\right), \quad (11)$$

then we say that the function g is an α-symmetric according to $(a \dot{+} b / \alpha(2) \cdot)$.

Definition 6 (see [33]). A function $g\colon [a, b]_\alpha \subseteq R_\alpha / \{\dot{0}\} \longrightarrow R$ is said to be α-harmonically symmetric with respect to $\alpha(2) \dot{\times} a \dot{\times} b / a \dot{+} b \cdot$ if

$$g(x) = g\left(\frac{\alpha(1)}{\alpha(1)/a \cdot \dot{+} \alpha(1)/b \cdot \dot{-} \alpha(1)/x \cdot}\right), \quad (12)$$

holds for all $x \in [a, b]_\alpha$.

Theorem 2 (see [28]). *Let I_α be an α-closed interval in R_α, and $f\colon I_\alpha \longrightarrow R$ also be any α-convex function. Then, the following double inequality holds for all $a, b \in I_\alpha$:*

$$f\left(\alpha\left(\frac{1}{2}\right)\dot{\times}\,(a\dot{+}b)\right) \leq \int_0^1 f\,(\alpha(t)\,\dot{\times}\,a\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,b)\mathrm{d}t \leq \frac{f\,(a)+f\,(b)}{2}. \tag{13}$$

3. Main Results

Definition 7. Let $I_\alpha \subset (0,\infty)_\alpha$ be an α-interval and $p \in R\backslash\{0\}$. A function $f\colon I_\alpha \longrightarrow R$ is said to be p_α-convex if

$$f\left(\left(\alpha(t)\,\dot{\times}\,x^{\dot{p}}\ \dot{+}\,\alpha(1-t)\,\dot{\times}\,y^{\dot{p}}\right)^{\dot{1}/\dot{p}.}\right) \leq t f\,(x) + (1-t) f\,(y), \tag{14}$$

inequality holds for all $x, y \in I_\alpha$ and $t \in [0,1]$.

Definition 8. Let $p \in R\backslash\{0\}$. A function $w\colon [a,b]_\alpha \subset (0,\infty)_\alpha \longrightarrow R$ is said to be p_α-symmetric with respect to $(a^{\dot{p}}\dot{+}b^{\dot{p}}/\alpha(2))^{\dot{1}/\dot{p}.}$ if the relation $w(x) = w(a^{\dot{p}}\dot{+}b^{\dot{p}}\dot{-}x^{\dot{p}})^{\dot{1}/\dot{p}.}$ holds for all $x \in [a,b]_\alpha$.

Theorem 3. *Let $f\colon [a,b]_\alpha \subset R_\alpha \longrightarrow R$ be an α-convex function and $g\colon [a,b]_\alpha \subset R_\alpha \longrightarrow R$ is a nonnegative, integrable, and non-Newtonian-symmetric to $(a\dot{+}b/\alpha(2)\cdot)$. In this case, the following inequality holds:*

$$f\left(\frac{a\dot{+}b}{\dot{2}}\cdot\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g\circ\alpha)(x)\mathrm{d}x \leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (f\circ\alpha)(x)(g\circ\alpha)(x)\mathrm{d}x \leq \frac{f\,(a)+f\,(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g\circ\alpha)(x)\mathrm{d}x. \tag{15}$$

Proof. According to the claim of the theorem, for all $t \in [0,1]$, we can write the below inequality:

$$f\left(\frac{a\dot{+}b}{\dot{2}}\cdot\right) = f\left(\frac{\alpha(t)\,\dot{\times}\,a\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,b\,\dot{+}\,\alpha(t)\,\dot{\times}\,b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a}{\dot{2}}\right) \leq \frac{f\,(\alpha(t)\,\dot{\times}\,a\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,b)}{2} + \frac{f\,(\alpha(t)\,\dot{\times}\,b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a)}{2}. \tag{16}$$

Multiplying both sides of (16) by $g\,(\alpha(t)\,\dot{\times}\,b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a)$ and integrating to t over $[0,1]$, we have the following inequality:

$$\begin{aligned}
\int_0^1 f\left(\frac{a\dot{+}b}{\dot{2}}\cdot\right) g\,(\alpha(t)\,\dot{\times}\,b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a)\mathrm{d}t &\leq \int_0^1\left[\frac{f\,(\alpha(t)\,\dot{\times}\,a\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,b)}{2} g\,(\alpha(t)\,\dot{\times}\,b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a)\mathrm{d}t\right] \\
&\quad + {}^{1}\!\!\int 0\left[\frac{f\,(\alpha(t)\times b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a)}{2} g\,(\alpha(t)\,\dot{\times}\,b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a)\mathrm{d}t\right] \\
&\Longrightarrow f\left(\frac{a\dot{+}b}{\dot{2}}\right)\int_0^1 g\,(\alpha(t)\,\dot{\times}\,b\,\dot{+}\,\alpha(1-t)\,\dot{\times}\,a)\mathrm{d}t
\end{aligned}$$

$$\leq \int_0^1 \frac{(f\circ\alpha)\left(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)\right)(g\circ\alpha)\left(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\right)}{2}\mathrm{d}t$$
$$+\int_0^1 \frac{(f\circ\alpha)\left(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\right)(g\circ\alpha)\left(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\right)}{2}\mathrm{d}t. \tag{17}$$

Then, g function is α-symmetric according to $(a \dot{+} b/\dot{2}.)$, and inequality (12) becomes as follows for $x := t\alpha^{-1}(a) + (1-t)\alpha^{-1}(b)$:

$$f\left(\frac{a\dot{+}b}{\dot{2}}.\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(g\circ\alpha)(x)\mathrm{d}x \leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(x)(g\circ\alpha)(x)}{2}\mathrm{d}x + \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(x)(g\circ\alpha)(x)}{2}\mathrm{d}x$$
$$\Longrightarrow f\left(\frac{a\dot{+}b}{\dot{2}}.\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(g\circ\alpha)(x)\mathrm{d}x \leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(f\circ\alpha)(x)(g\circ\alpha)(x)\mathrm{d}x. \tag{18}$$

So, the proof of the left hand of (15) is completed. Now, we prove the right hand. Since f is an α-convex function, we can write the below inequality:

$$f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b) + f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a) \leq f(a) + f(b). \tag{19}$$

Multiplying both sides of (19) by $g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)$ and integrating to t over $[0,1]$, we get the following inequalities:

$$\int_0^1 f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)\mathrm{d}t$$
$$+\int_0^1 f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)\mathrm{d}t \tag{20}$$
$$\leq \int_0^1 [f(a)+f(b)]g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)\mathrm{d}t,$$

$$\int_0^1 (f\circ\alpha)\left(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)\right)(g\circ\alpha)\left(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\right)\mathrm{d}t$$
$$+\int_0^1 (f\circ\alpha)\left(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\right)(g\circ\alpha)\left(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\right)\mathrm{d}t \tag{21}$$
$$\leq \int_0^1 [f(a)+f(b)](g\circ\alpha)\left(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)\right)\mathrm{d}t.$$

Similarly, g function is α-symmetric according to $(a\dot{+}b/\dot{2}\cdot)$, and inequality (21) becomes as follows for $x := t\alpha^{-1}(a) + (1-t)\alpha^{-1}(b)$:

$$\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (f \circ \alpha)(x)(g \circ \alpha)(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g \circ \alpha)(x)\mathrm{d}x. \tag{22}$$

Then, we have

$$\begin{aligned} f\left(\frac{a \dot{+} b}{\dot{2}}\dot{}\right) \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g \circ \alpha)(x)\mathrm{d}x &\leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (f \circ \alpha)(x)(g \circ \alpha)(x)\mathrm{d}x \\ &\leq \frac{f(a)+f(b)}{2} \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g \circ \alpha)(x)\mathrm{d}x. \end{aligned} \tag{23}$$

Consequently, the proof is completed. □

Corollary 1. *If we take* $\alpha = \mathrm{Id}$ *in Theorem 3, then we obtain the Hermite–Hadamard–Fejer inequality in [10], i.e.,*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)\mathrm{d}x. \tag{24}$$

Corollary 2. *If we take* $\alpha = \exp$ *in Theorem 3, then we obtain the following inequality:*

$$f(\sqrt{ab}) \int_{\ln a}^{\ln b} w(e^x)\mathrm{d}x \leq \int_{\ln a}^{\ln b} f(e^x)w(e^x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \int_{\ln a}^{\ln b} w(e^x)\mathrm{d}x. \tag{25}$$

Corollary 3. *If we take* $e^x = u$ *in Theorem 3, then we obtain the inequality in [27]:*

$$f(\sqrt{ab}) \int_a^b \frac{g(x)}{x}\mathrm{d}x \leq \int_a^b \frac{f(x)g(x)}{x}\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \int_a^b \frac{g(x)}{x}\mathrm{d}x. \tag{26}$$

Theorem 4. *Let* $f\colon [a,b]_\alpha \subset \mathrm{R}_\alpha/\{\dot{0}\} \longrightarrow \mathrm{R}$ *be an* α*-harmonically convex function. Then, for all* $x \in [a,b]_\alpha$, *we have*

$$f\left(\frac{\alpha(2) \dot{\times} a \dot{\times} b}{a \dot{+} b}\dot{}\right) \leq \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{\alpha^{-1}(b) - \alpha^{-1}(a)} \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f \circ \alpha)(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{27}$$

Proof. Since f is an α-harmonically convex function, for all $t \in [0,1]$ and $u, v \in \mathrm{R}_\alpha$, we can write

$$f\left(\frac{u \dot{\times} v}{\alpha(t) \dot{\times} u \dot{+} \alpha(1-t) \dot{\times} v}\dot{}\right) \leq t f(v) + (1-t) f(u). \tag{28}$$

If we choose $t = 1/2$ in (28), then we get the below inequality:

$$f\left(\frac{\dot{2} \dot{\times} u \dot{\times} v}{u \dot{+} v}\dot{}\right) \leq \frac{f(u)+f(v)}{2}. \tag{29}$$

In other words, if we choose

$$u = \frac{a \dot{\times} b}{\alpha(t) \dot{\times} a \dot{+} \alpha(1-t) \dot{\times} b},$$
$$v = \frac{a \dot{\times} b}{\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a}, \tag{30}$$

in (29), then we have the following inequality:

$$f\left(\frac{\dot{2} \dot{\times} u \dot{\times} v}{u \dot{+} v}\dot{}\right) = f\left(\frac{\dot{2} \dot{\times} a \dot{\times} b}{a \dot{+} b}\dot{}\right) \leq \frac{1}{2}\left[f\left(\frac{a \dot{\times} b}{\alpha(t) \dot{\times} a \dot{+} \alpha(1-t) \dot{\times} b}\dot{}\right) + f\left(\frac{a \dot{\times} b}{\alpha(t) \dot{\times} b \dot{+} \alpha(1-t) \dot{\times} a}\dot{}\right)\right]. \tag{31}$$

If we integrate to t over $[0,1]$ of (31), then we have

$$\int_0^1 f\left(\frac{\dot{2} \dot{\times} u \dot{\times} v}{u \dot{+} v}\dot{}\right) \leq \frac{1}{2}\left[\int_0^1 (f \circ \alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a) + (1-t)\alpha^{-1}(b)}\right)\mathrm{d}t + \int_0^1 (f \circ \alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)}\right)\mathrm{d}t\right]. \tag{32}$$

We obtain the following inequality for $x := \alpha^{-1}(a)\alpha^{-1}(b)/t\alpha^{-1}(a) + (1-t)\alpha^{-1}(b)$ in (32):

$$f\left(\frac{\dot{2} \dot{\times} u \dot{\times} v}{u \dot{+} v}\dot{}\right) \leq \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{\alpha^{-1}(b) - \alpha^{-1}(a)} \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f \circ \alpha)(x)}{x^2}\mathrm{d}x. \tag{33}$$

And then we can write

$$f\left(\frac{\alpha(2) \dot{\times} a \dot{\times} b}{a \dot{+} b}\dot{}\right) \leq \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{\alpha^{-1}(b) - \alpha^{-1}(a)} \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f \circ \alpha)(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \tag{34}$$

This completes the proof. □

Corollary 4. *If we take $\alpha = \mathrm{Id}$ in Theorem 4, then we obtain the inequality in [34]:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{35}$$

Corollary 5. *If we take $\alpha = \exp$ in Theorem 3, then we obtain the following inequality:*

$$f\left(e^{\ln 2 \ln a \ln b / \ln ab}\right) \leq \frac{\ln a \ln b}{\ln b - \ln a}\int_{\ln a}^{\ln b} \frac{f(e^x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{36}$$

Theorem 5. *Let* $f\colon [a,b]_\alpha \subset \mathrm{R}_\alpha/\{\dot{0}\} \longrightarrow \mathrm{R}$ *be an* α*-harmonically convex function. And* $f \in L[a,b]_\alpha$ *and* $w\colon [a,b]_\alpha \subset \mathrm{R}_\alpha/\{\dot{0}\} \longrightarrow \mathrm{R}$ *nonnegative, integrable, and* α*-symmetric to* $[\alpha(2)\dot{\times} a \dot{\times} b/a \dot{+} b]$*. Then, we have*

$$\begin{aligned} f\left(\frac{\alpha(2)\dot{\times} a \dot{\times} b}{a \dot{+} b}\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^2}\mathrm{d}x &\le \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(x)(w\circ\alpha)(x)}{x^2}\mathrm{d}x \\ &\le \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^2}\mathrm{d}x. \end{aligned} \tag{37}$$

Proof. Because f is an α-harmonically convex function, we can write the following inequality:

$$f\left(\frac{\dot{2}\dot{\times} u \dot{\times} v}{u \dot{+} v}\right) \le \frac{f(u)+f(v)}{2}. \tag{38}$$

In other words, if we choose

$$\begin{aligned} u &= \frac{a \dot{\times} b}{\alpha(t)\dot{\times} a \dot{+} \alpha(1-t)\dot{\times} b}, \\ v &= \frac{a \dot{\times} b}{\alpha(t)\dot{\times} b \dot{+} \alpha(1-t)\dot{\times} a}, \end{aligned} \tag{39}$$

in (38), then we have the following inequality:

$$f\left(\frac{\dot{2}\dot{\times} u \dot{\times} v}{u \dot{+} v}\right) = f\left(\frac{\dot{2}\dot{\times} a \dot{\times} b}{a \dot{+} b}\right) \le \frac{1}{2}\left[f\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} a \dot{+} \alpha(1-t)\dot{\times} b}\right) + f\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} b \dot{+} \alpha(1-t)\dot{\times} a}\right)\right]. \tag{40}$$

Multiplying both sides of (40) by

$$w\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} b \dot{+} \alpha(1-t)\dot{\times} a}\right), \tag{41}$$

and integrating to t over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 f\left(\frac{\dot{2}\dot{\times} a \dot{\times} b}{a \dot{+} b}\right) w\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} b \dot{+} \alpha(1-t)\dot{\times} a}\right)\mathrm{d}t \\ &\le \frac{1}{2}\left[\int_0^1 f\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} a \dot{+} \alpha(1-t)\dot{\times} b}\right) w\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} b \dot{+} \alpha(1-t)\dot{\times} a}\right)\mathrm{d}t \right. \\ &\quad \left. + \int_0^1 f\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} a \dot{+} \alpha(1-t)\dot{\times} b}\right) w\left(\frac{a \dot{\times} b}{\alpha(t)\dot{\times} b \dot{+} \alpha(1-t)\dot{\times} a}\right)\mathrm{d}t\right], \end{aligned} \tag{42}$$

$$\begin{aligned} &f\left(\frac{\dot{2}\dot{\times} a \dot{\times} b}{a \dot{+} b}\right)\int_0^1 (w\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)\mathrm{d}t \\ &\le \frac{1}{2}\left[\int_0^1 (f\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}\right)(w\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)\mathrm{d}t \right. \\ &\quad \left. + \int_0^1 (f\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)(w\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)\mathrm{d}t\right]. \end{aligned} \tag{43}$$

In this case, we can write the following inequality for $x = (\alpha^{-1}(a)\alpha^{-1}(b)/t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a))$ in (43):

$$f\left(\frac{\alpha(2)\dot{\times} a \dot{\times} b}{a \dot{+} b}\cdot\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^2}\mathrm{d}x \leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(x)(w\circ\alpha)(x)}{x^2}\mathrm{d}x. \tag{44}$$

Then, we obtain

$$\begin{aligned} f\left(\frac{\alpha(2)\dot{\times} a \dot{\times} b}{a \dot{+} b}\cdot\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^2}\mathrm{d}x &\leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(x)(w\circ\alpha)(x)}{x^2}\mathrm{d}x \\ &\leq \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^2}\mathrm{d}x. \end{aligned} \tag{45}$$

This completes the proof. □

Corollary 6. *If we take* $\alpha = \mathrm{Id}$ *in Theorem 5, then we obtain the inequality in [35]:*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \leq \int_a^b \frac{f(x)w(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x. \tag{46}$$

Corollary 7. *If we take* $\alpha = \exp$ *in Theorem 5, then we obtain the following inequality:*

$$f\left(e^{\ln 2 \ln a \ln b/\ln ab}\right)\int_{\ln a}^{\ln b} \frac{w(e^x)}{x^2}\mathrm{d}x \leq \int_{\ln a}^{\ln b} \frac{f(e^x)w(e^x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_{\ln a}^{\ln b} \frac{w(e^x)}{x^2}\mathrm{d}x. \tag{47}$$

Theorem 6. *Let* $f\colon I_\alpha \subset (0,\infty)_\alpha \longrightarrow \mathrm{R}$ *be a* p_α*-convex function,* $p \in \mathrm{R}/\{0\}$, $a, b \in I_\alpha$ *with* $a \dot{<} b$. *If* $f \in L[a,b]_\alpha$ *and* $w\colon [a,b]_\alpha \longrightarrow \mathrm{R}$ *is nonnegative, integrable, and* p_α*-symmetric with respect to* $[a^p \dot{+} b^p/\dot{2}\cdot]$, *then the following inequality holds:*

$$\begin{aligned} f\left(\left[\frac{a^{\dot{p}} \dot{+} b^{\dot{p}}}{\dot{2}}\cdot\right]^{\dot{1}/p\cdot}\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^{1-p}}\mathrm{d}x &\leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(w\circ\alpha)(x)}{x^{1-p}}\mathrm{d}x \\ &\leq \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^{1-p}}\mathrm{d}x. \end{aligned} \tag{48}$$

Proof. Since $f\colon I_\alpha \subset (0,\infty)_\alpha \longrightarrow \mathrm{R}$ is a p_α-convex function, we can write the following inequality for all $x, y \in I_\alpha$, and for $t = 1/2$ in (9):

$$f\left(\left[\frac{x^{\dot{p}} \dot{+} y^{\dot{p}}}{\dot{2}}\right]^{\dot{1}/p\cdot}\right) \leq \frac{f(x)+f(y)}{2}. \tag{49}$$

If we choose

$$\begin{aligned} x &= \left[\alpha(t) \dot{\times} a^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} b^{\dot{p}}\right]^{\dot{1}/\dot{p}}, \\ y &= \left[\alpha(t) \dot{\times} b^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} a^{\dot{p}}\right]^{\dot{1}/\dot{p}}, \end{aligned} \tag{50}$$

in (49), then we have

$$f\left(\left[\frac{a^{\dot{p}} \dot{+} b^{\dot{p}}}{\dot{2}}\right]^{\dot{1}/\dot{p}}\right) \leq \frac{f\left(\left[\alpha(t) \dot{\times} a^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} b^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right)}{2} + \frac{f\left(\left[\alpha(t) \dot{\times} b^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} a^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right)}{2}. \tag{51}$$

Multiplying both sides of (51) by $w([\alpha(t) \dot{\times} a^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} b^{\dot{p}}]^{\dot{1}/\dot{p}})$ and integrating with respect to t over $[0, 1]$, we get below the inequality:

$$\begin{aligned} &\int_0^1 f\left(\left[\frac{a^{\dot{p}} \dot{+} b^{\dot{p}}}{\dot{2}}\right]^{\dot{1}/\dot{p}}\right) w\left(\left[\alpha(t) \dot{\times} a^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} b^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right) \mathrm{d}t \\ &\leq \int_0^1 \frac{f\left(\left[\alpha(t) \dot{\times} a^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} b^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right) w\left(\left[\alpha(t) \dot{\times} a^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} b^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right)}{2} \mathrm{d}t \\ &\quad + \int_0^1 \frac{f\left(\left[\alpha(t) \dot{\times} b^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} a^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right) w\left(\left[\alpha(t) \dot{\times} a^{\dot{p}} \dot{+} \alpha(1-t) \dot{\times} b^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right)}{2} \mathrm{d}t, \end{aligned} \tag{52}$$

$$\begin{aligned} &f\left(\left[\frac{a^{p} \dot{+} b^{p}}{\dot{2}}\right]^{\dot{1}/\dot{p}}\right) \int_0^1 (w \circ \alpha)\left(\left[t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}\right]^{1/\alpha^{-1}(p)}\right) \mathrm{d}t \\ &\leq \frac{1}{2} \int_0^1 (f \circ \alpha)\left(\left[t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}\right]^{1/\alpha^{-1}(p)}\right) \times (w \circ \alpha)\left(\left[t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}\right]^{1/\alpha^{-1}(p)}\right) \mathrm{d}t \\ &\quad + \frac{1}{2} \int_0^1 (f \circ \alpha)\left(\left[t\alpha^{-1}(b)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(a)^{\alpha^{-1}(p)}\right]^{1/\alpha^{-1}(p)}\right) \times (w \circ \alpha)\left(\left[t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}\right]^{1/\alpha^{-1}(p)}\right) \mathrm{d}t. \end{aligned} \tag{53}$$

If we take $x = [t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}]^{1/\alpha^{-1}(p)}$ in (53), then we can write below the inequality:

$$f\left(\left[\frac{a^{\dot{p}} \dot{+} b^{\dot{p}}.}{\dot{2}}\right]^{\dot{1}/\dot{p}}\right) \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w \circ \alpha)(x)}{x^{1-p}} \mathrm{d}x \leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f \circ \alpha)(w \circ \alpha)(x)}{x^{1-p}} \mathrm{d}x. \tag{54}$$

So, the proof of the left hand of (31) is completed. Now, we prove the right hand. We know that the following inequality holds:

$$\frac{f\left(\left[\alpha(t)\dot{\times}a^{\dot{p}}\dot{+}\alpha(1-t)\dot{\times}b^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right)}{2}+\frac{f\left(\left[\alpha(t)\dot{\times}b^{\dot{p}}\dot{+}\alpha(1-t)\dot{\times}a^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right)}{2}\leq\frac{f(a)+f(b)}{2}. \tag{55}$$

Multiplying both sides of (55) by

$$w\left(\left[\alpha(t)\dot{\times}a^{\dot{p}}\dot{+}\alpha(1-t)\dot{\times}b^{\dot{p}}\right]^{\dot{1}/\dot{p}}\right), \tag{56}$$

integrating respect to t over $[0,1]$ and

$$x=\left[t\alpha^{-1}(a)^{\alpha^{-1}(p)}+(1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}\right]^{1/\alpha^{-1}(p)}, \tag{57}$$

changing variables, we have the following inequality:

$$\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(w\circ\alpha)(x)}{x^{1-p}}\mathrm{d}x\leq\frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^{1-p}}\mathrm{d}x. \tag{58}$$

So, the proof is completed. □

Corollary 8. *If we take* $\alpha=\mathrm{Id}$ *in Theorem 6, then we obtain the inequality in [36]:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\leq\frac{f(a)+f(b)}{2}. \tag{59}$$

Corollary 9. *If we take* $\alpha=\exp$ *in Theorem 6, then we obtain the following inequality:*

$$f\left(\left[(a+b)\ln p\right]^{1/p}\right)\int_{\ln a}^{\ln b}\frac{w(e^x)}{x^{1-p}}\mathrm{d}x\leq\int_{\ln a}^{\ln b}\frac{f(e^x)w(e^x)}{x^{1-p}}\mathrm{d}x\leq\frac{f(a)+f(b)}{2}\int_{\ln a}^{\ln b}\frac{w(e^x)}{x^{1-p}}\mathrm{d}x. \tag{60}$$

Corollary 10. *If we take* $\alpha=\mathrm{Id}$ *and* $w=\mathrm{Id}$ *in Theorem 6, then we obtain the following inequality:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\leq\frac{f(a)+f(b)}{2}. \tag{61}$$

Remark 3. In inequality (48), one can see the following inequalities:

(1) If one takes $p=1$ and $(w\circ\alpha)(x)=1$, one has (8).

(2) If one takes $p=1$, one has (10).

(3) If one takes $p=-1$ and $(w\circ\alpha)(x)=1$, one has (27).

(4) If one takes $p=-1$, one has (24).

(5) If one takes $\alpha=\mathrm{Id}$ and $p=1$, one has (26).

(6) If one takes $\alpha=\mathrm{Id}$, $p=-1$ and $w(x)=1$, one has (35).

(7) If one takes $\alpha=\mathrm{Id}$ and $p=-1$, one has (46).

(8) If one takes $\alpha=\mathrm{Id}$ and $w(x)=1$, one has (3).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2 edition, 1952.

[2] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, Germany, 1961.

[3] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Berlin, Germany, 1970.

[4] D. S. Mitrinovic, J. E. Pecaric, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, London, UK, 1993.

[5] B. G. Pachpatte, *Mathematical Inequalities*, Elsevier B.V, Amsterdam, The Netherlands, 2005.

[6] J. Hadamard, "Etude sur les propri´et´es des fonctions enti'eres et en particulier d'une fonction consider´ee par Riemann," *Journal de Mathematiques Pures et Appliquees*, vol. 58, pp. 171–215, 1893.

[7] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, NY, USA, 1973.

[8] J. Pecaric, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, NY, USA, 1992.

[9] C. Niculescu and L. E. Persson, *Convex Functions and Their Applications, A Contemporary Approach*, Springer Science+Business Media, Inc, Berlin, Germany, 2006.

[10] L. Fejer, "Über die fourierreihen II," *Math. Naturwiss. Anz Ungar. Akad., Wiss*, vol. 24, pp. 369–390, 1906.

[11] M. Kunt and İ. İşcan, "Hermite–Hadamard–Fejér type inequalities forp-convex functions," *Arab Journal of Mathematical Sciences*, vol. 23, no. 2, pp. 215–230, 2017.

[12] M. Grosmann and R. Katz, *Non-Newtonian Calculus*, Lee Press Pigeon Cove, Massachusetts, MA, USA, 1972.

[13] U. Kadak, *Non-Newtonian Calculus and its Applications, PhD*, Gazi University, Ankara, Turkey, 2015.

[14] D. Aniszewska, "Multiplicative Runge–Kutta methods," *Nonlinear Dynamics*, vol. 50, no. 1-2, pp. 265–272, 2007.

[15] A. E. Bashirov, E. M. Kurpınar, and A. Özyapıcı, "Multiplicative calculus and its applications," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 36–48, 2008.

[16] M. Rıza, A. Özyapıcı, and E. Mısırlı, "Multiplicative finite difference methods," *Quarterly of Applied Mathematics*, vol. 67, no. 4, pp. 745–754, 2009.

[17] A. Uzer, "Multiplicative type complex calculus as an alternative to the classical calculus," *Computers and Mathematics with Applications*, vol. 60, no. 10, pp. 2725–2737, 2010.

[18] E. Mısırlı and Y. Gürefe, "Multiplicative adams bashforth moulton methods," *Numerical Algorithms*, vol. 57, no. 4, pp. 425–439, 2011.

[19] A. E. Bashirov and M. Rıza, "On complex multiplicative differentiation," *TWMS Journal of Applied and Engineering Mathematics*, vol. 1, no. 1, pp. 75–85, 2011.

[20] A. F. Çakmak and F. Başar, "Some new results on sequence spaces with respect to non-Newtonian calculus," *Journal of Inequalities and Applications*, vol. 2012, no. 1, p. 228, 2012.

[21] S. Tekin and F. Başar, "Some basic results on the sets of sequences with geometric calculus," *AIP Conference Proceedings*, vol. 1470, pp. 95–98, 2012.

[22] C. Türkmen and F. Başar, "Some basic results on the geometric calculus," *Communications Faculty of Sciences University of Ankara Series A1*, vol. 61, no. 2, pp. 17–34, 2012.

[23] S. Tekin and F. Başar, "Certain sequence spaces over the non-Newtonian complex Field," *Abstract and Applied Analysis*, vol. 2013, Article ID 739319, 11 pages, 2013.

[24] U. Kadak and H. Efe, "Matrix transformations between certain sequence spaces over the non-newtonian complex field," *The Scientific World Journal*, vol. 2014, Article ID 705818, 12 pages, 2014.

[25] U. Kadak and H. Efe, "The construction of hilbert spaces over the non-Newtonian field," *International Journal of Analysis*, vol. 2014, Article ID 746059, 10 pages, 2014.

[26] U. Kadak, M. Kirişci, and A. F. Çakmak, "On the classical paranormed sequence spaces and related duals over the non-Newtonian complex field," *Journal of Function Spaces*, vol. 2015, Article ID 416906, 11 pages, 2015.

[27] M. A. Latif, S. S. Dragomir, and E. Momoniat, "Some Fejér type integral inequalities for geometrically-arithmetically-convex functions with applications," *Filomat*, vol. 32, no. 6, pp. 2193–2206, 2018.

[28] E. Ünlüyol and S. Salaş, "Convexity and hermite-hadamard type inequality via non-Newtonian calculus," *Konuralp Journal of Mathematics*, vol. 7, no. 2, pp. 352–358, 2019.

[29] E. Ünlüyol, Y. Erdaş, and S. Salaş, "$*$-Hermite-Hadamard-Fejer inequality and some new inequality via $*$-Calculus," *Sigma Journal of Engineering and Natural Sciences*, vol. 10, no. 3, pp. 287–299, 2019.

[30] K. S. Zhang and J. P. Wan, "p-Convex functions and their properties," *Pure and Applied Mathematics Journal*, vol. 23, no. 1, pp. 130–133, 2007.

[31] İ. İşcan, "Ostrowski type inequalities for p-convex functions," *New Trends in Mathematical Sciences*, vol. 3, pp. 140–150, 2016.

[32] S. Salaş, *Some Types of Convexity and Hermite-Hadamard Type Inequalities in Non-newtonian Analysis, PhD*, Ordu University, Ordu, Turkey, 2022.

[33] Y. Erdaş, *Hermite-Hadamard-Fejer Inequality and Some Varios of Convexity Obtained in Non-newtonian Calculus, PhD*, Ordu University, Ordu, Turkey, 2022.

[34] İ. İşcan, "Hermite-Hadamard type inequalities for harmonically convex functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 6, pp. 935–942, 2014.

[35] F. Chen and S. Wu, "Fej´er and Hermite-Hadamard type inequalities for harmonically convex functions," *Journal of Applied Mathematics*, vol. 2014, Article ID 386806, 6 pages, 2014.

[36] İ. İşcan, "Hermite-Hadamard type inequalities for p-convex functions," *International Journal of Analysis and Applications*, vol. 11, no. 2, pp. 137–145, 2016.