Hindawi
Journal of Mathematics
Volume 2023, Article ID 1416097, 15 pages
https://doi.org/10.1155/2023/1416097

Research Article

Exploring the Analytical Solutions to the Economical Model via Three Different Methods

M. Raheel,[1,2] Khalid K. Ali,[3] Asim Zafar,[2] Ahmet Bekir,[4] Omar Abu Arqub,[5] and Marwan Abukhaled[6]

[1]*Department of Mathematics and Statistics, Institute of Southern Punjab, Multan, Pakistan*
[2]*Department of Mathematics, CUI, Vehari Campus, Vehari, Pakistan*
[3]*Mathematics Department, Faculty of Science, Al-Azhar University, Nasr, Cairo, Egypt*
[4]*Neighbourhood of Akcaglan, Imarli Street, Number: 28/4, Eskisehir 26030, Turkey*
[5]*Department of Mathematics, Faculty of Science, Al-Balqa Applied University, As-Salt, Jordan*
[6]*Department of Mathematics and Statistics, American University of Sharjah, Sharjah, UAE*

Correspondence should be addressed to Ahmet Bekir; bekirahmet@gmail.com

Received 9 November 2022; Revised 22 January 2023; Accepted 5 April 2023; Published 22 April 2023

Academic Editor: Francisco J. Garcia Pacheco

In this article, the analytical solutions of economically important model named as the Ivancevic option pricing model (IOPM) along new definition of derivative have been explored. For this purpose, $\exp_a$ function, extended sinh-Gordon equation expansion (EShGEE) and extended (G'/G)-expansion methods have been utilized. The resulting solutions are dark, bright, dark-bright, periodic, singular, and other kinds of solutions. These solutions are obtained and also verified by a Mathematica tool. Some of the gained results are explained by 2-D, 3-D, and contour plots.

1. Introduction

Many mathematical models have been developed in many areas of sciences in the form of nonlinear partial differential equations (NLPDEs). Numerous techniques are made to gain exact solutions of NLPDEs such as generalized exponential rational function scheme (GERFS) [1–4], $(m + 1/G)$-expansion and Adomian decomposition schemes [5], new generalized expansion method [6], simplest equation and Kudryashov's new function techniques [7], modified simple equation scheme [8], modified Kudryashov simple equation technique [9], first integral technique [10], Bäcklund transformation scheme [11], extended jacobi elliptic function expansion technique [12], and extended (G/G)-expansion and improved (G'/G)-expansion schemes [13].

In modern century, one of the most studied fields from all over the world is the economy or finance. Therefore, such problems were studied to be explained and investigated by using scientific norms. Thus, such works introduce more intellectual ways for the user. Therefore, to observe financial market is highly important. Deeper properties of the modeling of a global financial market produce global informative systems. Especially, these dynamical systems can be used for deep investigation of the productions. The first step is to treatment its mathematical models being either complex-valued or real values with wave function. Therefore, many models were developed by experts in extracting their wave distributions in today and future direction. In our study, we use three methods $\exp_a$ function, extended sinh-Gordon equation expansion (EShGEE), and extended (G'/G)-expansion methods. These methods have various applications. Likely, some new kind of analytical results of perturbed Gerdjikov–Ivanov model (pGIM) have been achieved by using $\exp_a$ function and extended tanh function expansion methods in [14]. By applying $\exp_a$ function and hyperbolic function methods, various types of wave solutions of two nonlinear Schrödinger equations are gained in [15]. New trigonometry and hyperbolic function type soliton solutions of $(2+1)$-dimensional hyperbolic and cubic-quintic nonlinear Schrödinger equations are achieved by applying extended sinh-Gordon equation expansion technique in [16]. Bright, dark, and bright-dark soliton solutions of generalized nonlinear Schrödinger equation have been determined by utilizing extended sinh-Gordon equation expansion approach in [17]. Some exact solitons of $(2+1)$-dimensional that improved the Eckhaus equation are calculated by using extended (G'/G)-expansion technique in [18]. Different types of exact solitons of time-fractional parabolic equations are obtained by using extended (G'/G)-expansion scheme in [19].

Our considering model is one of the important and interesting economical models, namely, the Ivancevic option pricing model (IOPM). It can be possible to derive the Ivancevic option pricing model by using the Brownian movement like the Black–Scholes option pricing model. The Ivancevic option pricing model is an adaptive-wave model which is a nonlinear wave alternative for the standard Black–Scholes option-pricing model, representing controlled Brownian behavior of financial markets, which is formally defined by adaptive nonlinear Schrodinger (NLS) equations, defining the option-pricing wave function in terms of the stock price and time. In the literature, few techniques have been used on this model to get different exact solutions. For example, new solutions have been achieved in this model by applying the fractionally reduced differential transform technique in [20]. Dark, bright, dark-bright, complex, travelling, periodic, trigonometric, and hyperbolic function solutions have been achieved by applying rational sine-Gordon expansion scheme and modified exponential method in [21]. Rogue wave and dark wave solitons of the Ivancevic option pricing equation have been obtained by using the trial function method in [22].

The fundamental purpose of the work is to explore analytical solutions of the truncated M-fractional Ivancevic option pricing model based on $\exp_a$ function, extended sinh-Gordon equation expansion, and extended (G'/G)-expansion methods.

The paper is structured as follows: The brief introduction of model has been given in Section 2, together with other useful properties and characterizations. Section 3 contains the description of methodologies. The mathematical analysis of model and its analytical solutions have been provided in Section 4. Section 5 some solutions have been represented through different types of graphics. Finally, Section 6 contains some discussion about the graphs and conclusion of our research.

2. Model Description

Scholes and Robert Merton published their now-well-known option pricing formula which would have an important effect on the development of quantitative finance. In their model, typically known as Black–Scholes, the value of an option depends on the future volatility of a stock rather than on its expected return. The Ivancevic option pricing model is an adaptive-wave model which is a nonlinear wave alternative for the standard Black–Scholes option-pricing model, representing controlled Brownian behavior of financial markets, which is formally defined by adaptive nonlinear Schrodinger (NLS) equations, defining the option-pricing wave.

Let's assume the M-fractional Ivancevic option pricing model (IOPM) [22] given as follows:

$$\imath D_{M,t}^{\epsilon,\varrho} q + \frac{\delta}{2} D_{M,2s}^{2\epsilon,\varrho} q + \Omega q|q|^2 = 0, \imath = \sqrt{-1}, \tag{1}$$

where $D_{M,t}^{\varepsilon,\varrho} q = \lim_{\tau \longrightarrow 0} q(t E_{\varrho}(\tau t^{1-\varepsilon})) - q(t)/\tau, 0 < \varepsilon < 1, \varrho > 0$.

This model was first developed by Ivancevic [23] to fulfill both behavioral and efficient markets. Here, $q = q(s,t)$ describes the option price wave profile. While t is time variable and s is asset price of the model. Parameter δ represents the volatility which shows either stochastic process itself or only a constant. Where $\Omega = \Omega(r,\omega)$ is called Landau coefficient which describes adaptive market potential. In nonadaptive simplest case, Ω and r become equal which shows the interest rate while in adaptive case, $\Omega(r,\omega)$ may be connected to market temperature and it depends on the set of tractable parameters $\{W_i\}$. In third term, $|q|^2$ shows the probability density function which denotes the potential field.

3. Description of Methodologies

3.1. Summary of $\exp_a$ *Function Scheme.* Here, we will give complete concept of this scheme.

Assuming the nonlinear partial differential equation (PDE),

$$G\left(q, q^2 q_t, q_x, q_{tt}, q_{xx}, q_{xt}, \ldots\right) = 0. \tag{2}$$

Equation (2) transformed into nonlinear partial differential equation as follows:

$$\Lambda\left(Q, Q', Q'', \ldots,\right) = 0. \tag{3}$$

By using following transformations,

$$q(x,y,t) = Q(\zeta), \zeta = ax + by + rt. \tag{4}$$

Considering root of equation (3) is shown in [24–27].

$$Q(\zeta) = \frac{\alpha_0 + \alpha_1 d^{\zeta} + \ldots + \alpha_m \mathrm{d}^{m\zeta}}{\beta_0 + \beta_1 d^{\zeta} + \cdots + \beta_m d^{m\zeta}}, d \neq 0, 1, \tag{5}$$

where α_i and $\beta_i (0 \le i \le m)$ are undetermined. Positive integral value of m is calculated by utilizing homogenous balance technique in equation (3). Putting equation (5) into equation (3) gives

$$\wp\left(d^{\zeta}\right) = \ell_0 + \ell_1 d^{\zeta} + \cdots + \ell_t d^{t\zeta} = 0. \tag{6}$$

Taking ℓ_i $(0 \le i \le t)$ in equation (6) equal to 0, a set of algebraic equations is gained which is given as

$$\ell_i = 0, \text{where } i = 0, \ldots, t. \tag{7}$$

By using the got roots, we attain analytical results of equation (2).

3.2. Detail of Extended Sinh-Gordon Equation Expansion Method (EShGEEM). Here, we will describe main steps of this technique.

Step 1:

Let a nonlinear partial differential equation be given as

$$Z\left(f, D_{M,t}^{\alpha,\gamma} f^2, f^2 f_x, f_y, f_{yy}, f_{xx}, f_{xy}, f_{xt}, \ldots\right) = 0, \tag{8}$$

where $f = f(x, y, t)$ denotes the wave function.

Assuming the travelling wave transformation,

$$f(x, y, t) = F(\xi), \xi = x - \nu y + \frac{\Gamma(\gamma+1)}{\alpha}(\kappa t^{\alpha}). \tag{9}$$

Inserting equation (9) into equation (8), we attain the nonlinear ODE given as

$$Z\left(F(\xi), F^2(\xi)F'(\xi), F''(\xi), \cdots\right) = 0. \tag{10}$$

Step 2:

Assuming the results of equation (9) in the series form,

$$F(p) = \alpha_0 + \sum_{i=1}^{m} \left(\beta_i \sinh(p) + \alpha_i \cosh(p)\right)^i. \tag{11}$$

Here, α_0, α_i, and β_i $(i = 1, 2, 3, \cdots, m)$ are unknowns. Consider a function p of ξ that satisfies the following equation:

$$\frac{\mathrm{d}p}{\mathrm{d}\zeta} = \sinh(p). \tag{12}$$

Natural number m can be attained with the use of homogenous balance approach. Equation (12) is gained from sinh-Gordon equation as shown as

$$q_{xt} = \kappa \sinh(\nu). \tag{13}$$

By what being present in [28], we get the results of equation (13) given as follows:

$$\sin h p(\xi) = \pm \csc h(\xi) \text{ or } \cosh p(\xi) = \pm \coth(\xi). \tag{14}$$

In addition,

$$\sinh p(\xi) = \pm \imath \sec h(\xi) \text{ or } \cosh p(\xi) = \pm \tanh(\xi), \tag{15}$$

where $\imath^2 = -1$.

Step 3:

Using equation (11) with equation (13) into equation (10), we get the algebraic equations involving $p'^k(\xi)\sinh^l p(\xi)\cosh^m p(\xi)$ $(k = 0, 1;\ l = 0, 1;\ m = 0, 1, 2, \ldots)$. We take the every coefficient of $p'^k(\zeta)\sinh^l p(\zeta)\cosh^m p(\zeta)$ equal to 0, to attain system of algebraic equations having $\nu, \kappa, \alpha_0, \alpha_i$ and β_i $(i = 1, 2, 3, \ldots, m)$.

Step 4:

By solving the obtained system of algebraic equations, one may obtain value of $\nu, \kappa, \alpha_0, \alpha_i$ and β_i.

Step 5:

By achieved solutions, equations (14) and (15), we get the wave solitons of equation (10) shown as

$$F(\xi) = \alpha_0 + \sum_{i=1}^{m} \left(\pm \beta_i \operatorname{csch}(\xi) \pm \alpha_i \coth(\xi)\right)^i, \tag{16}$$

$$F(\xi) = \alpha_0 + \sum_{i=1}^{m} \left(\pm \imath \beta_i \sec h(\xi) \pm \alpha_i \tanh(\xi)\right)^i. \tag{17}$$

3.3. Explanation of Extended (G'/G)–Expansion Method. Here, we will represent some main steps of method given in [13].

Step 1: Considering the nonlinear PDE,

$$Z\left(f, D_{M,t}^{\alpha,\gamma} f, f^2 f_x, f_y, f_{yy}, f_{xx}, f_{xy}, f_{xt}, \ldots\right) = 0, \tag{18}$$

where $f = f(x, y, t)$ denotes the wave profile.

Considering the travelling wave transformations:

Step 2:

$$f(x, y, t) = F(\xi), \xi = x - \nu y + \frac{\Gamma(\gamma+1)}{\alpha}(\kappa t^{\alpha}). \tag{19}$$

Using equation (19) along equation (18), we attain the nonlinear ODE as follows:

$$Z\left(F(\xi), F^2(\xi)F'(\xi), F''(\xi), \ldots\right) = 0. \tag{20}$$

Step 3:

Assuming the results of equation (20) in the series form given as

$$F(\xi) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i. \tag{21}$$

In equation (21), α_0 and α_i, $(i = \pm 1, \pm 2, \pm 3, \ldots, \pm m)$ are undetermined and $\alpha_i \neq 0$. Applying homogenous balance technique into equation (20), natural number m can be obtained.

Function $G = G(\xi)$ satisfies the Riccati differential equation.

$$dGG'' - aG^2 - bGG' - c\left(G'\right)^2 = 0, \tag{22}$$

where a, b, c, and d are constants.

Step 4:

Considering equation (22) has roots in the form:

When $b \neq 0$ and $b^2 + 4a(d-c) > 0$, then

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{b}{2(d-c)} + \frac{\sqrt{-4a(c-d)+b^2}}{2(d-c)}\left(\frac{C_1 \sinh\left(\xi\sqrt{-4ac+4ad+b^2}/2d\right) + C_2 \cosh\left(\xi\sqrt{-4a(c-d)+b^2}/2d\right)}{C_1 \cosh\left(\xi\sqrt{-4ac+4ad+b^2}/2d\right) + C_2 \sinh\left(\xi\sqrt{-4ac+4ad+b^2}/2d\right)}\right). \tag{23}$$

When $b \neq 0$ and $b^2 + 4ad - 4ac < 0$, then

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{b}{2(d-c)} + \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2 \cos\left(\xi\sqrt{4ac-4ad-b^2}/2d\right) - C_1 \sin\left(\xi\sqrt{4ac-4ad-b^2}/2d\right)}{C_1 \cos\left(\xi\sqrt{4ac-4ad-b^2}/2d\right) + C_2 \sin\left(\xi\sqrt{4ac-4ad-b^2}/2d\right)}\right). \tag{24}$$

When $b \neq 0$ and $b^2 + 4a(d-c) = 0$, then

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{b}{2(d-c)} + \frac{dD}{(d-c)(C-D\xi)}. \tag{25}$$

When $b = 0$ and $a(d-c) > 0$, then

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1 \sinh(\xi\sqrt{ad-ac}/d) + C_2 \cosh(\xi\sqrt{a(d-c)}/d)}{C_1 \cosh(\xi\sqrt{ad-ac}/d) + C_2 \sinh(\xi\sqrt{a(d-c)}/d)}\right). \tag{26}$$

When $b = 0$ and $a(d-c) < 0$, then

$$\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\sqrt{a(c-d)}}{d-c}\left(\frac{C_2 \cos(\xi\sqrt{ac-ad}/d) - C_1 \sin(\xi\sqrt{a(c-d)}/d)}{C_1 \cos(\xi\sqrt{ac-ad}/d) + C_2 \sin(\xi\sqrt{a(c-d)}/d)}\right), \tag{27}$$

where a, b, c, d, C_1, and C_2 are constants.

Step 5:

Putting equation (21) with equation (22) into equation (20) and collecting coefficients of each power of $(G'(\xi)/G(\xi))$. Taking every coefficient equal to 0, we attain a set of algebraic equations involving ν, κ, α_i, $(i = 0, \pm 1, \pm 2, \ldots, \pm m)$, and other parameters.

Step 6:

Finding the gain system of equations with the use of the tool.

Step 7:

Putting the attained solutions into the equation (21) and we gain analytical solutions of equation (18).

4. Mathematical Treatment of Model

Suppose the travelling wave transform given as

$$q(s,t) = Q(\zeta) \times \exp\left(\iota\left(\frac{\Gamma(\varrho+1)}{\epsilon}\left(\mu s^{\epsilon} + \rho t^{\epsilon}\right)\right)\right), \zeta = \frac{\Gamma(\varrho+1)}{\epsilon}\left(\lambda s^{\epsilon} + \tau t^{\epsilon}\right), \tag{28}$$

where $Q(\zeta)$ shows the amplitude of wave function while ρ and τ represent the time velocity. Parameters μ and λ are obtaining from asset price of the product.

Using equation (28) into equation (1), we gain real part and imaginary part given as

Real part:

$$2\Omega Q^3 + \delta\lambda^2 Q'' - \left(\delta\mu^2 + 2\rho\right)Q = 0. \tag{29}$$

Imaginary part:

$$(\delta\mu\lambda + \tau)Q' = 0. \tag{30}$$

From equation (30), we get the velocity of wave function given as follows:

$$\tau = -\delta\mu\lambda. \tag{31}$$

Utilizing the homogenous balance method into equation (29), we achieve $m = 1$

Now, we will gain the exact solutions of equation (29) by using three abovementioned methods.

4.1. Analytical Solutions via $\exp_a$ *Function Technique.* Equation (5) changes into the following for $m = 1$:

$$Q(\zeta) = \frac{\alpha_0 + \alpha_1 d^{\zeta}}{\beta_0 + \beta_1 d^{\zeta}}. \tag{32}$$

Inserting equation (32) into equation (29), a system of equations is achieved. By solving the system, we obtain different solution sets given as follows:

Set 1:

$$\left\{\alpha_0 = -\frac{\iota\beta_0\sqrt{\delta}\,\lambda\log(d)}{2\sqrt{\Omega}}, \alpha_1 = \frac{i\beta_1\sqrt{\delta}\,\lambda\log(d)}{2\sqrt{\Omega}}, \rho = -\frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\right\}. \tag{33}$$

From equations (28), (32) and (33), we get

$$\begin{aligned} q(s,t) &= -\frac{\iota\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}\left(\frac{\beta_0 - \beta_1 d^{\zeta}}{\beta_0 + \beta_1 d^{\zeta}}\right) \\ &\quad \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\varepsilon}s^{\varepsilon} - \frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\frac{\Gamma(\varrho+1)}{\varepsilon}t^{\varepsilon}\right)\right). \end{aligned} \tag{34}$$

Set 2:

$$\left\{\alpha_0 = \frac{\iota\beta_0\sqrt{\delta}\,\lambda\log(d)}{2\sqrt{\Omega}}, \alpha_1 = -\frac{i\beta_1\sqrt{\delta}\,\lambda\log(d)}{2\sqrt{\Omega}}, \rho = -\frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\right\}. \tag{35}$$

From equations (28), (32) and (35), we get

$$q(s,t) = \frac{\iota\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}\left(\frac{\beta_0 - \beta_1 d^{\zeta}}{\beta_0 + \beta_1 d^{\zeta}}\right) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\varepsilon}s^{\varepsilon} - \frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\frac{\Gamma(\varrho+1)}{\varepsilon}t^{\varepsilon}\right)\right). \tag{36}$$

Where $\zeta = \lambda\Gamma(\varrho+1)/\varepsilon(s^{\varepsilon} - \delta\mu t^{\varepsilon})$.

4.2. Exact Solutions through EShGEEM. For $m = 1$, equations (9), (16) and (17) and become

$$Q(\zeta) = \alpha_0 \pm \beta_1\operatorname{csch}(\zeta) \pm \alpha_1\coth(\zeta), \tag{37}$$

$$Q(\zeta) = \alpha_0 \pm \iota\beta_1\operatorname{sech}(\zeta) \pm \alpha_1\tanh(\zeta), \tag{38}$$

$$Q(\zeta) = \alpha_0 + \beta_1\sinh(p) + \alpha_1\cosh(p). \tag{39}$$

Here, $\alpha_0, \alpha_1,$ and β_1 are undetermined. Utilizing equation (39) into equation (29), we attain algebraic equations containing $\alpha_0, \alpha_1, \beta_1$ and other parameters. By using the Mathematica tool, we get different solution sets given as

Set 1:

$$\left\{\alpha_0 = 0, \alpha_1 = -\frac{\iota\sqrt{\delta}\lambda}{\sqrt{\Omega}}, \beta_1 = 0, \rho = -\frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\right\}. \tag{40}$$

From equations (28), (37) and (40), we get

$$q_1(s,t) = \mp\frac{\iota\sqrt{\delta}\lambda}{\sqrt{\Omega}}\coth(\zeta) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{41}$$

From equations (28), (38), and (40), we get

$$q_2(s,t) = \mp\frac{\iota\sqrt{\delta}\lambda}{\sqrt{\Omega}}\tanh(\zeta) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{42}$$

Set 2:

$$\left\{\alpha_0 = 0, \alpha_1 = \frac{\iota\sqrt{\delta}\lambda}{\sqrt{\Omega}}, \beta_1 = 0, \rho = -\frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\right\}. \tag{43}$$

From equations (28), (37), and (43), we get

$$q_1(s,t) = \pm\frac{\iota\sqrt{\delta}\lambda}{\sqrt{\Omega}}\coth(\zeta) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{44}$$

From equations (28), (38), and (43), we get

$$q_2(s,t) = \pm\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\tanh(\zeta) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{45}$$

Set 3:

$$\left\{\alpha_0 = 0, \alpha_1 = -\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}, \beta_1 = -\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}, \rho = -\frac{1}{4}\delta\left(\lambda^2 + 2\mu^2\right)\right\}. \tag{46}$$

From equations (28), (37), and (46) we get

$$q_1(s,t) = \mp\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\coth(\zeta) + \operatorname{csch}(\zeta)) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2 + 2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{47}$$

From equations (28), (38), and (46), we get

$$q_2(s,t) = \mp\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\iota\operatorname{sech}(\zeta) + \tanh(\zeta)) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2 + 2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{48}$$

Set 4:

$$\left\{\alpha_0 = 0, \alpha_1 = \frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}, \beta_1 = -\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}, \rho = -\frac{1}{4}\delta\left(\lambda^2 + 2\mu^2\right)\right\}. \tag{49}$$

From equations (28), (37), and (49), we get

$$q_1(s,t)=\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\coth(\zeta)\mp\operatorname{csch}(\zeta))\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{50}$$

From equations (28), (38), and (49), we get

$$q_2(s,t)=\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\tanh(\zeta)\mp\iota\ \operatorname{sech}(\zeta))\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{51}$$

Set 5:

$$\left\{\alpha_0=0,\alpha_1=-\frac{\iota\sqrt{\delta}\,\lambda}{2\sqrt{\Omega}},\beta_1=\frac{i\sqrt{\delta}\,\lambda}{2\sqrt{\Omega}},\rho=-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\right\}. \tag{52}$$

From equations (28), (37), and (52), we get

$$q_1(s,t)=-\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\coth(\zeta)\mp\operatorname{csch}(\zeta))\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{53}$$

From equations (28), (38), and (52), we get

$$q_2(s,t)=-\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\tanh(\zeta)\mp\iota\operatorname{sech}(\zeta))\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{54}$$

Set 6:

$$\left\{\alpha_0=0,\alpha_1=\frac{\iota\sqrt{\delta}\,\lambda}{2\sqrt{\Omega}},\beta_1=\frac{\iota\sqrt{\delta}\,\lambda}{2\sqrt{\Omega}},\rho=-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\right\}. \tag{55}$$

From equations (28), (37), and (55), we get

$$q_1(s,t)=\pm\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\coth(\zeta)+\operatorname{csch}(\zeta))\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{56}$$

From equations (28), (38), and (55), we get

$$q_2(s,t)=\pm\frac{\iota\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\iota\operatorname{sech}(\zeta)+\tanh(\zeta))\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{57}$$

Set 7:

$$\left\{\alpha_0=0,\alpha_1=0,\beta_1=-\frac{\iota\sqrt{\delta}\,\lambda}{\sqrt{\Omega}},\rho=\frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\right\}. \tag{58}$$

By using equations (28), (37), and (58), we obtain

$$q_1(s,t)=\mp\frac{\iota\lambda\sqrt{\delta}}{\sqrt{\Omega}}\operatorname{csch}(\zeta)\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}+\frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}.t^{\epsilon}\right)\right). \tag{59}$$

From equations (28), (38) and (58), we get

$$q_2(s,t)=\pm\frac{\sqrt{\delta}\lambda}{\sqrt{\Omega}}\operatorname{sech}(\zeta)\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}+\frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{60}$$

Set 8:

$$\left\{\alpha_0=0,\alpha_1=0,\beta_1=\frac{\iota\sqrt{\delta}\,\lambda}{\sqrt{\Omega}},\rho=\frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\right\}. \tag{61}$$

From equations (28), (37) and (61), we get

$$q_1(s,t)=\pm\frac{\iota\sqrt{\delta}\lambda}{\sqrt{\Omega}}\operatorname{csch}(\zeta)\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}+\frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{62}$$

From equations (28), (38), and (61), we get

$$q_2(s,t)=\mp\frac{\sqrt{\delta}\lambda}{\sqrt{\Omega}}\operatorname{sech}(\zeta)\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}+\frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right), \tag{63}$$

where $\zeta=\lambda\Gamma(\varrho+1)/\epsilon(s^{\epsilon}-\delta\mu)t^{\epsilon}$.

4.3. Analytical Solutions via Extended (G'/G)-Expansion Technique. Equation (21) changes into following form for $m=1$:

$$Q(\zeta) = \alpha_{-1}\left(\frac{G'(\zeta)}{G(\zeta)}\right)^{-1} + \alpha_0 + \alpha_1\left(\frac{G'(\zeta)}{G(\zeta)}\right), \quad (64)$$

where α_{-1}, α_0 and α_1 are undetermined.

Inserting equation (64) along equation (22) into equation (29) and manipulating the set having $\alpha_{-1}, \alpha_0, \alpha_1$ and other parameters, we gain different sets of solutions given as

Set 1:

$$\left\{\alpha_{-1} = -\frac{\iota a\sqrt{\delta}\lambda}{d\sqrt{\Omega}}, \alpha_0 = -\frac{\iota b\sqrt{\delta}\lambda}{2d\sqrt{\Omega}}, \alpha_1 = 0, \rho = -\frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\right\}. \quad (65)$$

From equations (23), (28), (64), and (65), we get

$$\begin{aligned} q(s,t) = {} & -\frac{\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2} + a\left(\frac{b}{2(d-c)}\right.\right. \\ & \left.\left. + \frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right) + C_2\cosh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)}{C_1\cosh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right) + C_2\sinh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)}\right)\right)^{-1}\right) \\ & \times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \end{aligned} \quad (66)$$

From equations (24), (28), (64), and (65), we get

$$\begin{aligned} q(s,t) = {} & -\frac{\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2} + a\left(\frac{b}{2(d-c)}\right.\right. \\ & \left.\left. + \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\zeta\sqrt{4ac-4ad-b^2}/2d\right) - C_1\sin\left(\zeta\sqrt{4ac-4ad-b^2}/2d\right)}{C_1\cos\left(\zeta\sqrt{4ac-4ad-b^2}/2d\right) + C_2\sin\left(\zeta\sqrt{4ac-4ad-b^2}/2d\right)}\right)\right)^{-1}\right) \\ & \times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \end{aligned} \quad (67)$$

From equations (26), (28), (64), and (65), we get

$$\begin{aligned} q(s,t) = {} & -\frac{\iota a\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\zeta\sqrt{ad-ac}/d\right) + C_2\cosh\left(\zeta\sqrt{ad-ac}/d\right)}{C_1\cosh\left(\zeta\sqrt{ad-ac}/d\right) + C_2\sinh\left(\zeta\sqrt{ad-ac}/d\right)}\right)\right)^{-1} \\ & \times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \end{aligned} \quad (68)$$

From equations (27), (28), (64), and (65), we get

$$q(s,t) = -\frac{\iota a\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos(\zeta\sqrt{ac-ad}/d) - C_1\sin(\zeta\sqrt{ac-ad}/d)}{C_1\cos(\zeta\sqrt{ac-ad}/d) + C_2\sin(\zeta\sqrt{ac-ad}/d)}\right)\right)^{-1} \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \quad (69)$$

Set 2:

$$\left\{\alpha_{-1} = \frac{\iota a\sqrt{\delta}\lambda}{d\sqrt{\Omega}}, \alpha_0 = \frac{\iota b\sqrt{\delta}\lambda}{2d\sqrt{\Omega}}, \alpha_1 = 0, \rho = -\frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\right\}. \quad (70)$$

From equations (23), (28), (64), and (70), we get

$$q(s,t) = \frac{\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2} + a\left(\frac{b}{2(d-c)} + \frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh(\zeta\sqrt{-4ac+4ad+b^2}/2d) + C_2\cosh(\zeta\sqrt{-4ac+4ad+b^2}/2d)}{C_1\cosh(\zeta\sqrt{-4ac+4ad+b^2}/2d) + C_2\sinh(\zeta\sqrt{-4ac+4ad+b^2}/2d)}\right)\right)^{-1}\right) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \quad (71)$$

From equations (24), (28), (64), and (70), we get

$$q(s,t) = \frac{\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2} + a\left(\frac{b}{2(d-c)} + \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos(\zeta\sqrt{4ac-4ad-b^2}/2d) - C_1\sin(\zeta\sqrt{4ac-4ad-b^2}/2d)}{C_1\cos(\zeta\sqrt{4ac-4ad-b^2}/2d) + C_2\sin(\zeta\sqrt{4ac-4ad-b^2}/2d)}\right)\right)^{-1}\right) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \quad (72)$$

From equations (26), (28), (64), and (70), we get

$$q(s,t) = \frac{\iota a\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh(\zeta\sqrt{ad-ac}/d) + C_2\cosh(\zeta\sqrt{ad-ac}/d)}{C_1\cosh(\zeta\sqrt{ad-ac}/d) + C_2\sinh(\zeta\sqrt{ad-ac}/d)}\right)\right)^{-1} \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \quad (73)$$

From equations (27), (28), (64), and (70), we get

$$q(s,t)=\frac{\iota a\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos(\zeta\sqrt{ac-ad}/d)-C_1\sin(\zeta\sqrt{ac-ad}/d)}{C_1\cos(\zeta\sqrt{ac-ad}/d)+C_2\sin(\zeta\sqrt{ac-ad}/d)}\right)\right)^{-1}$$
$$\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{74}$$

Set 3:

$$\left\{\alpha_{-1}=0,\alpha_0=-\frac{\iota b\sqrt{\delta}\lambda}{2d\sqrt{\Omega}},\alpha_1=-\frac{\iota\sqrt{\delta}\lambda(c-d)}{d\sqrt{\Omega}},\rho=-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\right\}. \tag{75}$$

From equations (23), (28), (64), and (75), we get

$$q(s,t)=\frac{-\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2}-\left(\frac{b}{2}\right.\right.$$
$$\left.\left.+\frac{\sqrt{-4ac+4ad+b^2}}{2}\left(\frac{C_1\sinh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)+C_2\cosh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)}{C_1\cosh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)+C_2\sinh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)}\right)\right)\right)$$
$$\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{76}$$

From equations (24), (28), (64), and (75), we get

$$q(s,t)=-\frac{\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2}\right.$$
$$\left.+(-d+c)\left(\frac{b}{2(-c+d)}+\frac{\sqrt{-4ad+4ac-b^2}}{2(-c+d)}\left(\frac{C_2\cos\left(\zeta\sqrt{-4ad+4ac-b^2}/2d\right)-C_1\sin\left(\zeta\sqrt{-4ad+4ac-b^2}/2d\right)}{C_1\cos\left(\zeta\sqrt{-4ad+4ac-b^2}/2d\right)+C_2\sin\left(\zeta\sqrt{-4ad+4ac-b^2}/2d\right)}\right)\right)\right)$$
$$\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{77}$$

From equations (26), (28), (64), and (75), we get

$$q(s,t) = -\frac{\iota(c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{a(d-c)}}{(d-c)}\left(\frac{C_1\sinh(\zeta\sqrt{ad-ac}/d)+C_2\cosh(\zeta\sqrt{ad-ac}/d)}{C_1\cosh(\zeta\sqrt{ad-ac}/d)+C_2\sinh(\zeta\sqrt{ad-ac}/d)}\right)\right)$$
$$\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \quad (78)$$

From equations (27), (28), (64), and (75), we get

$$q(s,t) = -\frac{\iota(c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos(\zeta\sqrt{ac-ad}/d)-C_1\sin(\zeta\sqrt{ac-ad}/d)}{C_1\cos(\zeta\sqrt{ac-ad}/d)+C_2\sin(\zeta\sqrt{ac-ad}/d)}\right)\right)$$
$$\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \quad (79)$$

Set 4:

$$\left\{\alpha_{-1}=0, \alpha_0=\frac{\iota b\sqrt{\delta}\lambda}{2d\sqrt{\Omega}}, \alpha_1=\frac{\iota\sqrt{\delta}\lambda(c-d)}{d\sqrt{\Omega}}, \rho=-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\right\}. \quad (80)$$

From equations (23), (28), (64), and (80), we get

$$q(s,t)=\frac{\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2}+(c-d)\left(\frac{b}{2(d-c)}\right.\right.$$
$$\left.\left.+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)+C_2\cosh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)}{C_1\cosh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)+C_2\sinh\left(\zeta\sqrt{-4ac+4ad+b^2}/2d\right)}\right)\right)\right)$$
$$\times\exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \quad (81)$$

From equations (24), (28), (64), and (80), we get

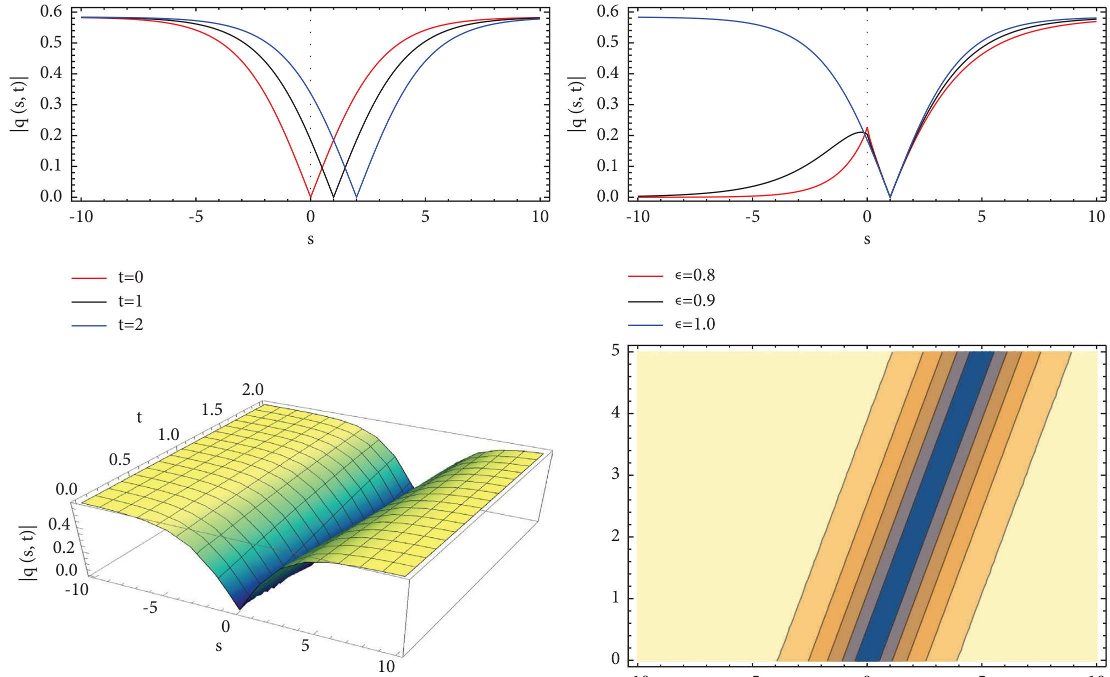

Figure 1: Structure of (34) for $\delta = 0.5, \lambda = 0.3, \varrho = 0.1, \mu = 2, \Omega = 0.7, \beta_0 = 0.1, \beta_1 = 0.1, d = 0.1, \epsilon = 1$.

$$q(s,t) = \frac{\iota\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2} + (c-d)\left(\frac{b}{2(d-c)} + \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\zeta\sqrt{4ac-4ad-b^2}/2d\right) - C_1\sin\left(\zeta\sqrt{4ac-4ad-b^2}/2d\right)}{C_1\cos\left(\zeta\sqrt{-4ad+4ac-b^2}/2d\right) + C_2\sin\left(\zeta\sqrt{-4ad+4ac-b^2}/2d\right)}\right)\right)\right) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(-c+d) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{82}$$

From equations (26), (28), (64), and (80), we get

$$q(s,t) = \frac{\iota(c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh(\zeta\sqrt{ad-ac}/d) + C_2\cosh(\zeta\sqrt{ad-ac}/d)}{C_1\cosh(\zeta\sqrt{ad-ac}/d) + C_2\sinh(\zeta\sqrt{ad-ac}/d)}\right)\right) \times \exp\left(\iota\left(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right)\right). \tag{83}$$

From equations (27), (28), (64), and (80), we get

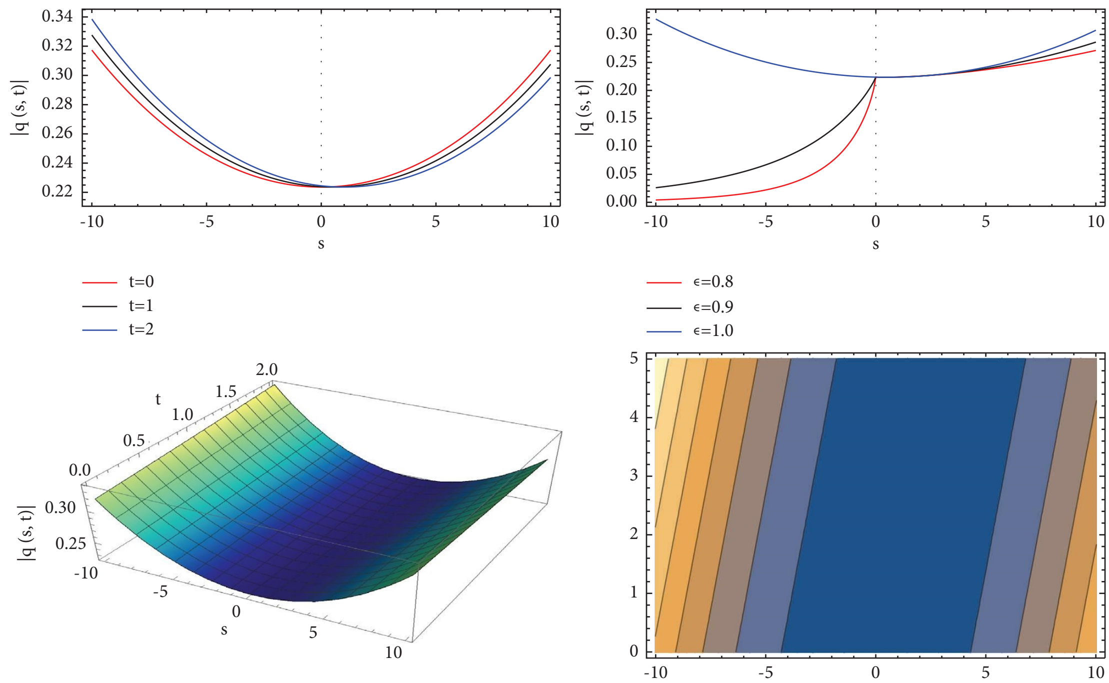

Figure 2: Structure of (41) for $\delta = 0.5, \lambda = 0.1, \varrho = 0.5, \mu = 1, \Omega = 0.1, \epsilon = 1$.

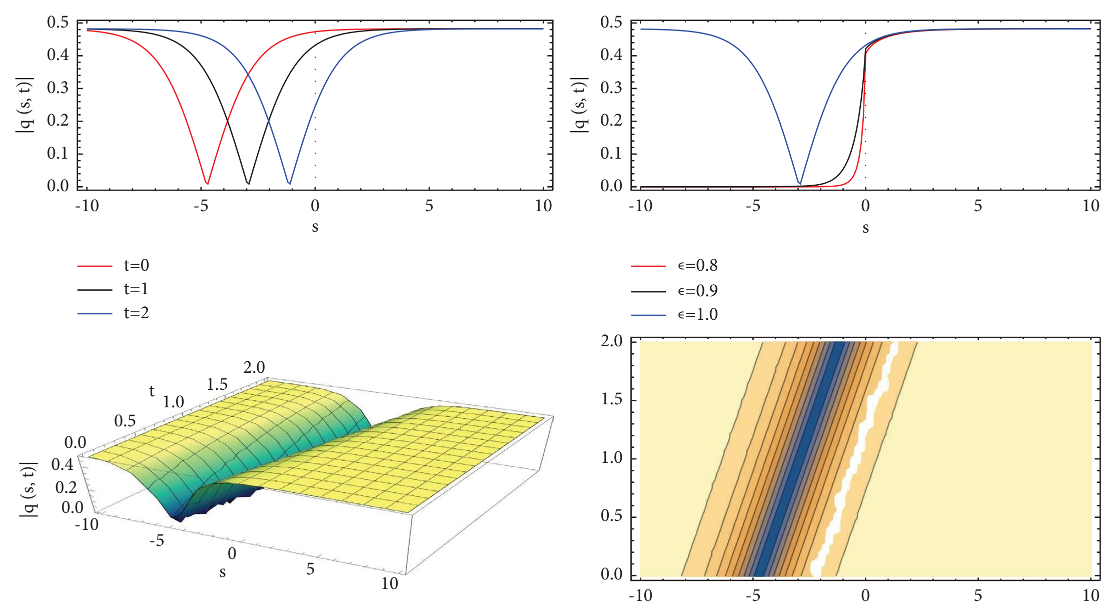

Figure 3: Structure of (66) for $\delta = 0.3, \lambda = 0.4, \varrho = 0.5, \mu = 6, \Omega = 0.4, d = 0.17, a = 0.1, c = 0.01, b = 0.4, C_1 = 0.4, C_2 = 0.5$.

$$q(s,t) = \frac{\iota (c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}} \left(\frac{\sqrt{ac-ad}}{d-c} \left(\frac{C_2 \cos\left(\zeta \sqrt{ac-ad}/d\right) - C_1 \sin\left(\zeta\sqrt{ac-ad}/d\right)}{C_1 \cos\left(\zeta\sqrt{ac-ad}/d\right) + C_2 \sin\left(\zeta\sqrt{ac-ad}/d\right)} \right) \right) \times \exp\left(\iota \left(\mu \frac{\Gamma(\varrho+1)}{\epsilon} s^{\epsilon} - \frac{\delta\left(4a\lambda^2 (d-c) + 2d^2\mu^2\right)}{4d^2} \frac{\Gamma(\varrho+1)}{\epsilon} t^{\epsilon} \right) \right). \tag{84}$$

Here, $\zeta = \lambda\Gamma(\varrho+1)/\varepsilon(s^{\varepsilon} - \delta\mu t^{\varepsilon})$ for all above-mentioned solutions.

5. Illustrations with Graphics

In this portion, we will represent some 2-D, 3-D, and contour structures that help us to classify the type of results. Figures 1–3 show some of the analytical solutions. In Figure 1, we apply our technique to represent the plot of (34) for $\delta = 0.5, \lambda = 0.3, \varrho = 0.1, \mu = 2, \quad \Omega = 0.7, \beta_0 = 0.1, \beta_1 = 0.1, d = 0.1, \epsilon = 1$. Furthermore, Figure 2 denotes the plot of (41) $\delta = 0.5, \lambda = 0.1, \varrho = 0.5, \mu = 1, \Omega = 0.1, \epsilon = 1$. Finally, the plot of (66) for $\delta = 0.3, \lambda = 0.4, \varrho = 0.5, \mu = 6, \Omega = 0.4, d = 0.17, a = 0.1, c = 0.01, b = 0.4, C_1 = 0.4, C_2 = 0.5$ is presented in Figure 3. We see that the wave retains its shape over time, moves to the right, and breaks by changing the value of ϵ.

Through our analysis of the forms presented in the previous section, we can reach important results as follows: First, in Figure 1, we apply the $\exp_a$ function technique to represent the plot of (34) at $\delta = 0.5, \lambda = 0.3, \varrho = 0.1, \mu = 2, \Omega = 0.7, \beta_0 = 0.1, \beta_1 = 0.1, d = 0.1, \epsilon = 1$. Further, Figure 2 denotes the plot of (41) at $\delta = 0.5, \lambda = 0.1, \varrho = 0.5, \mu = 1, \Omega = 0.1, \epsilon = 1$ using EShGEE technique. Finally, the plot of (66) for $\delta = 0.3, \lambda = 0.4, \varrho = 0.5, \mu = 6, \Omega = 0.4, d = 0.17, a = 0.1, c = 0.01, b = 0.4, C_1 = 0.4, C_2 = 0.5$ presented in Figure 3 using the extended (G'/G)-expansion technique.

6. Conclusion

In this article, we obtain modernistic analytical solutions to the Ivancevic option pricing model along M-fractional derivative by utilizing $\exp_a$ function, extended sinh-Gordon equation expansion, and extended (G'/G)-expansion methods. The achieved results are also verified and demonstrated with different plots by Mathematica tool. The obtained results are also explained graphically by 2-dimensional, 3-dimensional, and contour plots. Finally, it is suggested that to deal with the other fractional nonlinear PDEs, the $\exp_a$ function, extended sinh-Gordon equation expansion, and extended (G'/G)-expansion methods are very helpful, reliable, and straight forward. The results achieved in this paper may be useful for the progress in the supplementary analyzing of this model.

Data Availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

The work in this study was supported, in part, by the Open Access Program from the American University of Sharjah.

References

[1] B. Ghanbari, M. S. Osman, and D. Baleanu, "Generalized exponential rational function method for extended Zakharov–Kuzetsov equation with conformable derivative," *Modern Physics Letters A*, vol. 34, no. 20, Article ID 1950155, 2019.

[2] A. Neirameh and M. Eslami, "New solitary wave solutions for fractional Jaulent–Miodek hierarchy equation," *Modern Physics Letters B*, vol. 36, no. 7, Article ID 2150612, 2022.

[3] R. I. Nuruddeen, K. S. Aboodh, and K. K. Ali, "Analytical Investigation of Soliton Solutions to Three quantum zakharov-kuznetsov Equations," *Communications in Theoretical Physics*, vol. 70, no. 4, pp. 405–412, 2018.

[4] A. Saha, K. K. Ali, H. Rezazadeh, and Y. Ghatani, "Analytical optical pulses and bifurcation analysis for the traveling optical pulses of the hyperbolic nonlinear Schrödinger equation," *Optical and Quantum Electronics*, vol. 53, no. 3, p. 150, 2021.

[5] H. F. Ismael, İ. Okumuş, T. Aktürk, H. Bulut, and M. S. Osman, "Analyzing study for the 3D potential Yu–Toda–Sasa–Fukuyama equation in the two-layer liquid medium," *Journal of Ocean Engineering and Science*, 2022.

[6] A. S. Muhannad, K. K. Ali, K. R. Raslan, H. Rezazadeh, and A. Bekir, "Exact solutions of the conformable fractional EW and MEW equations by a new generalized expansion method," *Journal of Ocean Engineering and Science*, vol. 5, no. 3, pp. 223–229, 2020.

[7] S. El-Ganaini, M. O. Al-Amr, and O. Mohammed, "New abundant solitary wave structures for a variety of some nonlinear models of surface wave propagation with their geometric interpretations," *Mathematical Methods in the Applied Sciences*, vol. 45, no. 11, pp. 7200–7226, 2022.

[8] M. O. Al-Amr, H. Rezazadeh, K. K. Ali, A. Korkmazki, and A. Korkmaz, "N1-soliton solution for Schrödinger equation with competing weakly nonlocal and parabolic law nonlinearities," *Communications in Theoretical Physics*, vol. 72, no. 6, Article ID 65503, 2020.

[9] N. M. Rasheed, M. O. Al-Amr, E. A. Az-Zo'bi et al., "Stable optical solitons for the Higher-order Non-Kerr NLSE via the modified simple equation method," *Mathematics*, vol. 9, no. 16, p. 1986, 2021.

[10] M. Eslami and H. Rezazadeh, "The first integral method for Wu–Zhang system with conformable time-fractional derivative," *Calcolo*, vol. 53, no. 3, pp. 475–485, 2016.

[11] H. Rezazadeh, D. Kumar, A. Neirameh, M. Eslami, and M. Mirzazadeh, "Applications of three methods for obtaining optical soliton solutions for the Lakshmanan–Porsezian–Daniel model with Kerr law nonlinearity," *Pramana*, vol. 94, no. 1, p. 39, 2020.

[12] A. Zafar, M. Raheel, M. Mirzazadeh, and M. Eslami, "Different soliton solutions to the modified equal-width wave equation with Beta-time fractional derivative via two different methods," *Revista Mexicana de Física*, vol. 68, no. 1, p. 1, 2021.

[13] S. Sahoo, S. Saha Ray, and M. A. Abdou, "New exact solutions for time-fractional Kaup-Kupershmidt equation using improved (G′/G)- expansion and extended (G′/G)- expansion methods," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3105–3110, 2020.

[14] A. Zafar, K. K. Ali, M. N. Raheel, K. S. Nisar, and A. Bekir, "Abundant M-fractional optical solitons to the pertubed

Gerdjikov–Ivanov equation treating the mathematical nonlinear optics," *Optical and Quantum Electronics*, vol. 54, no. 1, p. 25, 2022.
[15] A. Zafar, A. Bekir, M. Raheel, and H. Rezazadeh, "Investigation for optical soliton solutions of two nonlinear Schrödinger equations via two concrete finite series methods," *International Journal of Algorithms, Computing and Mathematics*, vol. 6, no. 3, p. 65, 2020.
[16] A. R. Seadawy, D. Kumar, and A. K. Chakrabarty, "Dispersive optical soliton solutions for the hyperbolic and cubic-quintic nonlinear Schrödinger equations via the extended sinh-Gordon equation expansion method," *The European Physical Journal Plus*, vol. 133, no. 5, p. 182, 2018.
[17] A. Safaei Bezgabadi and M. A. Bolorizadeh, "Analytic combined bright-dark, bright and dark solitons solutions of generalized nonlinear Schrödinger equation using extended sinh-Gordon equation expansion method," *Results in Physics*, vol. 30, Article ID 104852, 2021.
[18] N. Taghizadeh, M. Noori, R. Seyyedeh, M. Noori, and B. Seyyedeh, "Application of the extended (G'/G)-expansion method to the improved Eckhaus equation, applications and applied mathematics," *International Journal*, vol. 9, no. 1, p. 24, 2014.
[19] M. Ekici, "Soliton and other solutions of nonlinear time fractional parabolic equations using extended (G'/G)-expansion method," *Optik*, vol. 130, pp. 1312–1319, 2017.
[20] R. M. Jena, S. Chakraverty, and D. Baleanu, "A novel analytical technique for the solution of time-fractional Ivancevic option pricing model," *Physica A: Statistical Mechanics and Its Applications*, vol. 550, Article ID 124380, 2020.
[21] Q. Chen, H. M. Baskonus, W. Gao, and E. Ilhan, "Soliton theory and modulation instability analysis: the Ivancevic option pricing model in economy," *Alexandria Engineering Journal*, vol. 61, no. 10, pp. 7843–7851, 2022.
[22] Y.-Q. Chen, Y.-H. Tang, J. Manafian, H. Rezazadeh, and M. S. Osman, "Dark wave, rogue wave and perturbation solutions of Ivancevic option pricing model," *Nonlinear Dynamics*, vol. 105, no. 3, pp. 2539–2548, 2021.
[23] V. G. Ivancevic, "Adaptive-wave alternative for the black-scholes option pricing model," *Cogn. Comput*, vol. 2, no. 1, pp. 17–30, 2010.
[24] A. T. Ali and E. R. Hassan, "General Exp_a-function method for nonlinear evolution equations," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 451–459, 2010.
[25] E. M. E. Zayed and A. G. Al-Nowehy, "Generalized kudryashov method and general $\exp_a$ function method for solving a high order nonlinear Schrödinger equation," *J. Space Explor*, vol. 6, pp. 1–26, 2017.
[26] K. Hosseini, Z. Ayati, and R. Ansari, "New exact solutions of the Tzitzéica-type equations in non-linear optics using the $\exp_a$ function method," *Journal of Modern Optics*, vol. 65, no. 7, pp. 847–851, 2018.
[27] A. Zafar, "The $\exp_a$ function method and the conformable time-fractional KdV equations," *Nonlinear Engineering*, vol. 8, no. 1, pp. 728–732, 2019.
[28] X. L. Yang and J. S. Tang, "Travelling wave solutions for Konopelchenko-Dubrovsky equation using an extended sinh-Gordon equation expansion method," *Communications in Theoretical Physics*, vol. 50, Article ID 10471051, 2008.