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Research Article

Applications of n^{th} Power Root Fuzzy Sets in Multicriteria Decision Making

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An n^{th} power root fuzzy set is a useful extension of a fuzzy set for expressing uncertain data. Because of their wider range of showing membership grades, n^{th} power root fuzzy sets can cover more ambiguous situations than intuitionistic fuzzy sets. In this article, we present several novel operations on n^{th} power root fuzzy sets, as well as their various features. Besides, we develop a new weighted aggregated operator, namely, n^{th} power root fuzzy weighted power average (nPR-FWPA) over n^{th} power root fuzzy sets to deal with choice information and show some of their basic properties. In addition, we define a scoring function for n^{th} power root fuzzy sets ranking. Furthermore, we use this operator to determine the optimal location for constructing a home and demonstrate how we may choose the best alternative by comparing aggregate outputs using score values. Finally, we compare the nPR-FWPA operator outcomes to those of other well-known operators.

1. Introduction

Making a decision is the process of selecting the best option/options from a set of possibilities. Humans make numerous decisions throughout the course of their daily lives. There is no need to make a decision if there is just one alternative, but it is beneficial when there are two or more options. Multicriteria decision-making (MCDM) is a type of operational research that deals with one kind of outcomes by evaluating viable alternatives against a set of criteria in decision-making that is inconsistent. It is a far-fetched assumption that perfect numerical data are necessary to replicate real-world decision-making methods, which are characterized by intrinsic ambiguity in human judgments. Therefore, to cope with imprecise data, Zadeh [1] introduced the notion of fuzzy sets, and several studies on generalizations of the concept of fuzzy set were conducted after that. Generalization of fuzzy sets begun by Atanassov [2] who described the intuitionistic fuzzy sets as a fascinating generalization of fuzzy sets and explored essential features. Intuitionistic fuzzy sets have a wide range of applications in different fields including reservoir flood control operation, image fusion [3], pattern recognition [4], medical diagnosis, optimization issues [5], group theory [6, 7], and decision-making [8, 9]. Then, Yager [10] explored Pythagorean fuzzy sets (PFSs) as a model for dealing with imprecise data, and Zhang and Xu [11] introduced the concept of a Pythagorean fuzzy number. Garg [12] examined the use of PFSs in decision-making situations. Senapati and Yager [13] introduced Fermatean fuzzy sets and basic processes on them, as well as a Fermatean fuzzy TOPSIS technique for solving multiple criteria decision-making problems. To broaden the scope of membership and nonmembership degrees, Yager [14] put forward the idea of q-rung orthopair fuzzy sets (q-ROFSs), where $q \geq 1$.

Recently, it has been suggested different approaches to deal with the input data inspired by the fact that the significance of membership and nonmembership degrees need not to be equal in general cases. These approaches are useable to describe some real-life issues and enlarge the spaces of data under study. In this regard, Ibrahim et al. [15] defined the (3,2)-fuzzy sets as another type of generalized Pythagorean fuzzy set. Al-Shami et al. [16] introduced a new type of fuzzy set known as the SR-fuzzy set and studied its features in depth. Then, Al-Shami [17] displayed the idea of (2,1)-fuzzy sets and furnished its basic set of operations. At the beginning of the year 2023, Al-Shami and Mhemdi [18] offered the concept of (m, n)-fuzzy sets as a generalized frame for these types of fuzzy sets. They introduced different types of operations and aggregation operators via the environment of (m, n)-fuzzy sets. Gao and Zhang [19] provided the concept of linear orthopair fuzzy sets to address some empirical problems of vagueness.

Multiple attribute decision-making (MADM) is a strategy that takes into account the best possible alternatives. To deal with the complications and complexity of MADM problems, a variety of helpful mathematical methods, such as soft sets and fuzzy sets, were improved. MADM is a procedure that may produce ranking outcomes for finite alternatives based on their attribute values, and it is an important part of decision sciences. The concept of intuitionistic fuzzy weighted averaging operators was proposed by Xu [20], and Xu and Yager [21] proposed geometric weighted and geometric hybrid operators in the context of intuitionistic fuzzy sets. To cope with Pythagorean fuzzy MCDM difficulties, Yager [22] devised a useful decision technique based on Pythagorean fuzzy aggregation operators. Senapati and Yager [23] developed the Fermatean fuzzy weighted power average operator over Fermatean fuzzy sets, as well as their attributes. Al-Shami et al. [16] proposed the SR-fuzzy weighted power average operator and used it to choose the best university. Akram et al. [24] discussed a new approach to opt for the optimal alternative(s) using the 2-tuple linguistic T-spherical fuzzy numbers. Ambrin et al. [25] studied TOPSIS method in the frame of picture hesitant fuzzy sets utilizing linguistic variables. Jana et al. [26] discussed multiple attribute decision-making methods under Pythagorean fuzzy information.

The notion of n^{th} power root fuzzy sets was created by Al-Shami et al. [27], and they are more likely to be employed in uncertain situations than other forms of fuzzy sets due of their larger range of displaying membership grades. They also looked into the idea of topology for n^{th} power root fuzzy sets. In this context, we continue to investigate some concepts and notions inspired by this type of extension of fuzzy sets and show how this class of extension of fuzzy sets we can enable to evaluate the input data with different significance for grades of membership and nonmembership, which is appropriate for some real-life issues.

The layout of this manuscript is as follows. In Section 2, we survey orthopairs in the light of fuzzy computing with an illustrative example. In Section 3, we introduce a series of operations for the n^{th} power root fuzzy set and investigate their major characteristics. In Section 4, we display the concept of a weighted power average operator defined over the class of n^{th} power root fuzzy set. Then, we go into the MADM issues that can arise when using this operator and provide an empirical example. It can be seen that the primary advantage of n^{th} power root fuzzy sets is that they can be applied to a wide variety of decision-making scenarios. In Section 5, we supply a comparison analysis of the proposed nPR-FWPA operator with other well-known operators and compared the current operator with SR-FWPA [16] and FFWPA operators [23]. Finally, in Section 6, we summarize the paper's major accomplishments and suggest some future research.

2. Preliminaries

In this section, we recall some relevant definitions related to this paper.

Definition 1. Let W be the universal set and let $\varpi_\hbar, \omega_\hbar: W \longrightarrow [0,1]$ be the functions that, respectively, determine the degrees of membership and nonmembership for every $w \in W$. Then, the triplet $\hbar = \{\langle w, \varpi_\hbar(w), \omega_\hbar(w)\rangle : w \in W\}$ is called the following:

(i) An intuitionistic fuzzy set (IFS) [2] if $0 \leq \varpi_\hbar(w) + \omega_\hbar(w) \leq 1$

(ii) A Pythagorean fuzzy set (PFS) [10] if $0 \leq (\varpi_\hbar(w))^2 + (\omega_\hbar(w))^2 \leq 1$

(iii) A Fermatean fuzzy set (FFS) [13] if $0 \leq (\varpi_\hbar(w))^3 + (\omega_\hbar(w))^3 \leq 1$

(iv) A q-rung orthopair fuzzy set (q-ROFS), where $q \geq 1$, [14] if $0 \leq (\varpi_\hbar(w))^q + (\omega_\hbar(w))^q \leq 1$

Definition 2 (see [17]). The (2,1)-FS defined over the universal set W is represented for each $q \geq 1$ as follows.

$\hbar = \{\langle w, \varpi_\hbar(w), \omega_\hbar(w)\rangle : w \in W\}$, where $\varpi_\hbar, \omega_\hbar: W \longrightarrow [0,1]$ are functions that respectively determine the degrees of membership and nonmembership for every $w \in W$ under the constraint $0 \leq (\varpi_\hbar(w))^2 + (\omega_\hbar(w)) \leq 1$.

Definition 3 (see [27]). Let $\mathbb{N}$ be a set of all natural numbers and W be a universal set. An n^{th} power root fuzzy set (briefly, nPR-FS) $\hbar$ which is a set of ordered pairs over W is defined as following:

$$\hbar = \{\langle w, \varpi_\hbar(w), \omega_\hbar(w)\rangle : w \in W\}, \tag{1}$$

where $\varpi_\hbar(w)$ (resp. $\omega_\hbar(w)$): $W \longrightarrow [0,1]$ is the degree of membership (resp. nonmembership) of $w \in W$ to $\hbar$, such that

$$0 \leq (\varpi_\hbar(w))^n + \sqrt[n]{\omega_\hbar(w)} \leq 1, \frac{n \in \mathbb{N}}{\{1\}}. \tag{2}$$

For the sake of simplicity, we shall mention the symbol $\hbar = (\varpi_\hbar, \omega_\hbar)$ for the nPR-FS $\hbar = \{\langle w, \varpi_\hbar(w), \omega_\hbar(w)\rangle : w \in W\}$.

The spaces of some kinds of nPR-fuzzy membership grades are displayed in Figure 1.

Remark 1. From Figure 2, we get that

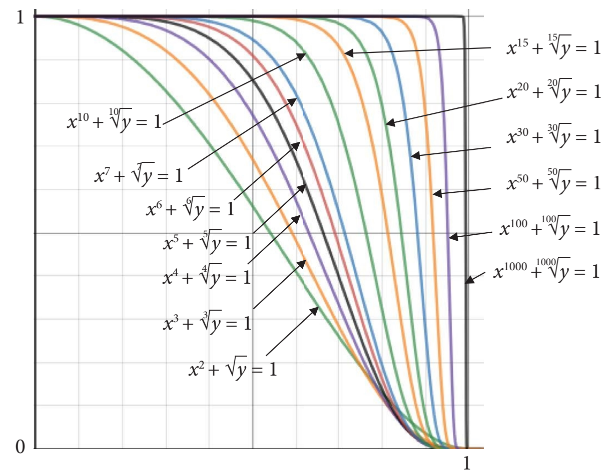

Figure 1: Grades spaces of some kinds of nPR-fuzzy sets.

(1) the space of 4-rung orthopair fuzzy membership grades is larger than the space of 4PR-fuzzy membership grades
(2) $\hbar = (\varpi_\hbar \approx 0.724, \omega_\hbar \approx 0.276)$ is a point of intersection between 4PR-fuzzy and intuitionistic fuzzy sets
(3) for $\varpi_\hbar \in (0, 0.724)$ and $\omega_\hbar \in (0.276, 1)$, the space of 4PR-fuzzy membership grades starts to be larger than the space of intuitionistic membership grades
(4) for $\varpi_\hbar \in (0.724, 1)$ and $\omega_\hbar \in (0, 0.276)$, the space of 4PR-fuzzy membership grades starts to be smaller than the space of intuitionistic membership grades
(5) $\hbar = (\varpi_\hbar = \sqrt{\sqrt{5} - 1/2}, \omega_\hbar = 7 - 3\sqrt{5}/2)$ is a point of intersection between 4PR-fuzzy and SR-fuzzy
(6) for $\varpi_\hbar \in (0, \sqrt{\sqrt{5} - 1/2})$ and $\omega_\hbar \in (7 - 3\sqrt{5}/2, 1)$, the space of 4PR-fuzzy membership grades starts to be larger than the space of SR-fuzzy membership grades
(7) for $\varpi_\hbar \in (\sqrt{\sqrt{5} - 1/2}, 1)$ and $\omega_\hbar \in (0, 7 - 3\sqrt{5}/2)$, the space of 4PR-fuzzy membership grades starts to be smaller than the space of SR-fuzzy membership grades
(8) $\hbar = (\varpi_\hbar \approx 0.819, \omega_\hbar \approx 0.091)$ is a point of intersection between 4PR-fuzzy and CR-fuzzy sets
(9) for $\varpi_\hbar \in (0, 0.819)$ and $\omega_\hbar \in (0.091, 1)$, the space of 4PR-fuzzy membership grades starts to be larger than the space of CR-fuzzy membership grades
(10) for $\varpi_\hbar \in (0.819, 1)$ and $\omega_\hbar \in (0, 0.091)$, the space of 4PR-fuzzy membership grades starts to be smaller than the space of CR-fuzzy membership grades

Remark 2. It is clear that for any nPR-FS $\hbar = (\varpi_\hbar, \omega_\hbar)$, we have

$$0 \le \varpi_\hbar^n + \omega_\hbar^n \le \varpi_\hbar^n + \sqrt[n]{\omega_\hbar} \le 1, \tag{3}$$

then $\hbar$ is an n-rung orthopair fuzzy set. Therefore, every nPR-fuzzy set is an n-rung orthopair fuzzy set.

Definition 4 (see [27]). Let $\hbar = (\varpi_\hbar, \omega_\hbar)$, $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$, and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ be three nPR-FSs, then

(1) $\hbar_1 \cap \hbar_2 = (\min\{\varpi_{\hbar_1}, \varpi_{\hbar_2}\}, \max\{\omega_{\hbar_1}, \omega_{\hbar_2}\})$
(2) $\hbar_1 \cup \hbar_2 = (\max\{\varpi_{\hbar_1}, \varpi_{\hbar_2}\}, \min\{\omega_{\hbar_1}, \omega_{\hbar_2}\})$
(3) $\hbar^c = (\sqrt[n^2]{\omega_\hbar}, (\varpi_\hbar)^{n^2})$

Definition 5 (see [27]). Let $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$ and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ be two nPR-FSs, then

(1) $\hbar_1 = \hbar_2 \Leftrightarrow \varpi_{\hbar_1} = \varpi_{\hbar_2}$ and $\omega_{\hbar_1} = \omega_{\hbar_2}$
(2) $\hbar_1 \ge \hbar_2 \Leftrightarrow \varpi_{\hbar_1} \ge \varpi_{\hbar_2}$ and $\omega_{\hbar_1} \le \omega_{\hbar_2}$
(3) $\hbar_2 \subset \hbar_1$ or $\hbar_1 \supset \hbar_2$ if $\hbar_1 \ge \hbar_2$

3. Some Operations via nPR-Fuzzy Sets

In this section, we propose various new operations on nPR-fuzzy sets and discuss some of their features in detail. In the entire work, we employ only three decimal places for computations.

Definition 6. Let $\hbar = (\varpi_\hbar, \omega_\hbar)$, $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$, and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ be three nPR-FSs and ε be a positive real number ($\varepsilon > 0$), then their operations are defined as follows:

(1) $\hbar_1 \oplus \hbar_2 = (\sqrt[n]{\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n}, \omega_{\hbar_1}\omega_{\hbar_2})$
(2) $\hbar_1 \otimes \hbar_2 = (\varpi_{\hbar_1}\varpi_{\hbar_2}, (\sqrt{[n]}\,\omega_{\hbar_1} + \sqrt{[n]}\,\omega_{\hbar_2} - \sqrt{[n]}\,\omega_{\hbar_1}\sqrt{[n]}\,\omega_{\hbar_2})^n)$
(3) $\varepsilon\hbar = (\sqrt{[n]}\,1 - (1 - \varpi_\hbar^n)^\varepsilon, \omega_\hbar^\varepsilon)$
(4) $\hbar^\varepsilon = (\varpi_\hbar^\varepsilon, (1 - (1 - \sqrt{[n]}\,\omega_\hbar)^\varepsilon)^n)$

Example 1. Consider the 4PR-FSs $\hbar_1 = (0.61, 0.51)$ and $\hbar_2 = (0.52, 0.62)$ for $W = \{w\}$. Then,

(1) $\hbar_1 \oplus \hbar_2 = (\sqrt{[n]}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n, \omega_{\hbar_1}\omega_{\hbar_2})$ $= (\sqrt{[4]}\,0.61^4 + 0.52^4 - (0.61)^4(0.52)^4,$ $(0.51)(0.62)) \approx (0.670, 0.316)$
(2) $\hbar_1 \otimes \hbar_2 = (\varpi_{\hbar_1}\varpi_{\hbar_2}, (\sqrt{[n]}\,\omega_{\hbar_1} + \sqrt{[n]}\,\omega_{\hbar_2} - \sqrt{[n]}\,\omega_{\hbar_1}\sqrt{[n]}\,\omega_{\hbar_2})^n) = ((0.61)(0.52),$

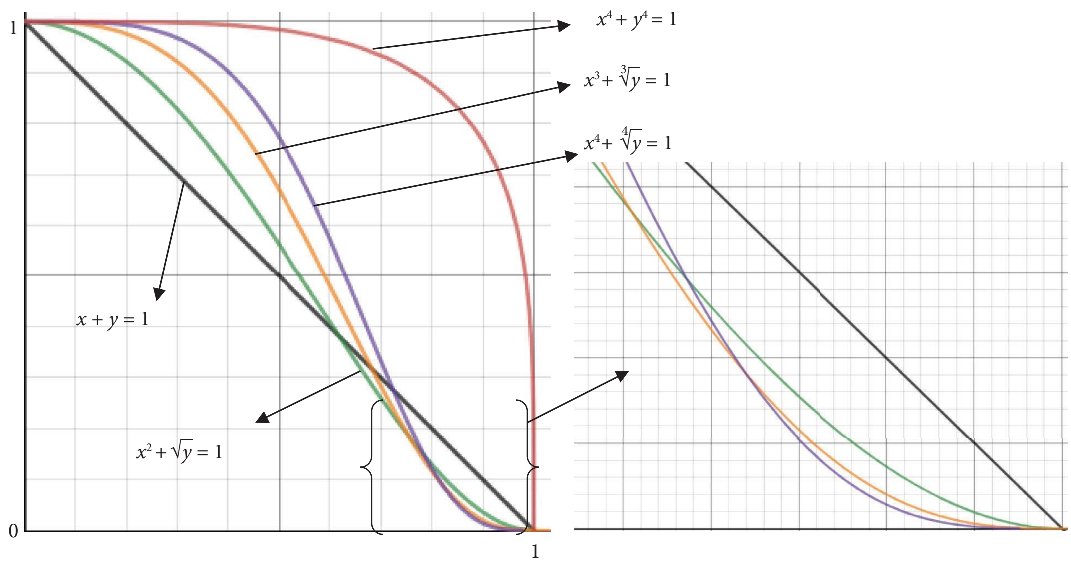

FIGURE 2: Some comparisons between some kinds of nPR-fuzzy sets and other generalizations of IFSs.

$(\sqrt[4]{0.51} + \sqrt[4]{0.62} - \sqrt[4]{0.51}\sqrt[4]{0.62})^4) = (0.317, 0.932)$

(3) $\varepsilon\hbar_1 = (\sqrt[n]{1-(1-\varpi_{\hbar_1}^n)^\varepsilon}, \omega_{\hbar_1}^\varepsilon) = (\sqrt[4]{1-(1-0.61^4)^2}, 0.51^2) \approx (0.713, 0.260)$, for $\varepsilon = 2$

(4) $\hbar_1^\varepsilon = (\varpi_{\hbar_1}^\varepsilon, (1-(1-\sqrt[n]{\omega_{\hbar_1}})^\varepsilon)^n) = (0.61^2, (1-(1-\sqrt[4]{0.51})^2)^4) = (0.372, 0.907)$, for $\varepsilon = 2$

Theorem 1. *If $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$ and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ are two nPR-FSs, then $\hbar_1 \oplus \hbar_2$ and $\hbar_1 \otimes \hbar_2$ are also nPR-FSs.*

Proof. For nPR-FSs $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$ and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ the following relations are evident:

$$\begin{aligned} &0 \le \varpi_{\hbar_1}^n \le 1, 0 \le \sqrt[n]{\omega_{\hbar_1}} \le 1, 0 \le \left(\varpi_{\hbar_1}\right)^n + \sqrt[n]{\omega_{\hbar_1}} \le 1, \\ &0 \le \varpi_{\hbar_2}^n \le 1, 0 \le \sqrt[n]{\omega_{\hbar_2}} \le 1, 0 \le \left(\varpi_{\hbar_2}\right)^n + \sqrt[n]{\omega_{\hbar_2}} \le 1. \end{aligned} \tag{4}$$

Then, we have

$$\begin{aligned} &\varpi_{\hbar_1}^n \ge \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n, \varpi_{\hbar_2}^n \ge \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n, 1 \ge \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n \ge 0, \\ &\sqrt[n]{\omega_{\hbar_1}} \ge \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}}, \sqrt[n]{\omega_{\hbar_2}} \ge \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}}, 1 \ge \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}} \ge 0, \end{aligned} \tag{5}$$

which indicates that

$$\begin{aligned} &\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n \ge 0 \text{ implies } \sqrt[n]{\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n} \ge 0, \\ &\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_2}} - \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}} \ge 0 \text{ implies } \left(\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_2}} - \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}}\right)^n \ge 0. \end{aligned} \tag{6}$$

Since $\varpi_{\hbar_2}^n \le 1$ and $0 \le 1 - \varpi_{\hbar_1}^n$, then $\varpi_{\hbar_2}^n(1-\varpi_{\hbar_1}^n) \le (1-\varpi_{\hbar_1}^n)$ and we get $\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n \le 1$ and hence $\sqrt[n]{\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n} \le 1$.

Similarly, we can obtain the following equation:

$$\left(\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_2}} - \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}}\right)^n \le 1. \tag{7}$$

It is obvious that

$$\begin{aligned} &0 \le \sqrt[n]{\omega_{\hbar_1}} \le 1 - \varpi_{\hbar_1}^n, \\ &0 \le \sqrt[n]{\omega_{\hbar_2}} \le 1 - \varpi_{\hbar_2}^n, \end{aligned} \tag{8}$$

then we can obtain the following equation:

$$\begin{aligned} &\left(\sqrt[n]{\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n}\right)^n + \sqrt[n]{\omega_{\hbar_1}\omega_{\hbar_2}} \\ &\le \varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n + \left(1-\varpi_{\hbar_1}^n\right)\left(1-\varpi_{\hbar_2}^n\right) = 1. \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} &0 \le \sqrt[n]{\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n} \le 1, \\ &0 \le \omega_{\hbar_1}\omega_{\hbar_2} \le 1, \\ &0 \le \left(\sqrt[n]{\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n \varpi_{\hbar_2}^n}\right)^n + \sqrt[n]{\omega_{\hbar_1}\omega_{\hbar_2}} \le 1. \end{aligned} \tag{10}$$

Similarly, we have

$$0 \le \varpi_{\hbar_1}\varpi_{\hbar_2} \le 1, 0 \le \left(\sqrt[n]{}\,\omega_{\hbar_1} + \sqrt[n]{}\,\omega_{\hbar_2} - \sqrt[n]{}\,\omega_{\hbar_1}\sqrt[n]{}\,\omega_{\hbar_2}\right)^n \le 1,$$
$$0 \le \left(\varpi_{\hbar_1}\varpi_{\hbar_2}\right)^n + \sqrt[n]{}\left(\sqrt[n]{}\,\omega_{\hbar_1} + \sqrt[n]{}\,\omega_{\hbar_2} - \sqrt[n]{}\,\omega_{\hbar_1}\sqrt[n]{}\,\omega_{\hbar_2}\right)^n \le 1. \quad (11)$$

Thus, $\hbar_1 \oplus \hbar_2$ and $\hbar_1 \otimes \hbar_2$ are nPR-FSs. □

Theorem 2. *Let $\hbar = (\varpi_\hbar, \omega_\hbar)$ be a nPR-FS and $\varepsilon > 0$. Then, $\varepsilon\hbar$ and $\hbar^\varepsilon$ are nPR-FSs.*

Proof. Since $0 \le \varpi_\hbar^n \le 1$, $0 \le \sqrt[n]{}\omega_\hbar \le 1$ and $0 \le (\varpi_\hbar)^n + \sqrt[n]{}\omega_\hbar \le 1$, then

$$\begin{aligned} &0 \le \sqrt[n]{}\omega_\hbar \le 1 - \varpi_\hbar^n \\ &\Rightarrow 0 \le \left(1 - \varpi_\hbar^n\right)^\varepsilon \\ &\Rightarrow 1 - \left(1 - \varpi_\hbar^n\right)^\varepsilon \le 1 \\ &\Rightarrow 0 \le \sqrt[n]{}1 - (1 - \varpi_\hbar^n)^\varepsilon \le \sqrt[n]{}1 = 1. \end{aligned} \quad (12)$$

It is obvious that $0 \le \omega_\hbar^\varepsilon \le 1$, then we can obtain the following equation:

$$\begin{aligned} 0 &\le \left(\sqrt[n]{}\,1 - (1 - \varpi_\hbar^n)^\varepsilon\right)^n + \sqrt[n]{}\omega_\hbar^\varepsilon \\ &\le 1 - \left(1 - \varpi_\hbar^n\right)^\varepsilon + \left(1 - \varpi_\hbar^n\right)^\varepsilon = 1. \end{aligned} \quad (13)$$

Similarly, we can also obtain the following equation:

$$0 \le (\varpi_\hbar^\varepsilon)^n + \sqrt[n]{}\left(1 - \left(1 - \sqrt[n]{}\,\omega_\hbar\right)^\varepsilon\right)^n \le 1. \quad (14)$$

Therefore, $\varepsilon\hbar$ and $\hbar^\varepsilon$ are nPR-FSs. □

Theorem 3. *Let $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$ and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ be two nPR-FSs. Then,*

(1) $\hbar_1 \oplus \hbar_2 = \hbar_2 \oplus \hbar_1$

(2) $\hbar_1 \otimes \hbar_2 = \hbar_2 \otimes \hbar_1$

Proof

(1) $\hbar_1 \oplus \hbar_2 = (\sqrt[n]{}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n, \omega_{\hbar_1}\omega_{\hbar_2})$ $(\sqrt[n]{}\,\varpi_{\hbar_2}^n + \varpi_{\hbar_1}^n - \varpi_{\hbar_2}^n\varpi_{\hbar_1}^n, \omega_{\hbar_2}\omega_{\hbar_1}) = \hbar_2 \oplus \hbar_1$.

(2) $\hbar_1 \otimes \hbar_2 = (\varpi_{\hbar_1}\varpi_{\hbar_2}, (\sqrt[n]{}\,\omega_{\hbar_1} + \sqrt[n]{}\,\omega_{\hbar_2} - \sqrt[n]{}\,\omega_{\hbar_1}\sqrt[n]{}\,\omega_{\hbar_2})^n) = (\varpi_{\hbar_2}\varpi_{\hbar_1}, (\sqrt[n]{}\,\omega_{\hbar_2} + \sqrt[n]{}\,\omega_{\hbar_1} - \sqrt[n]{}\,\omega_{\hbar_2}\sqrt[n]{}\,\omega_{\hbar_1})^n) = \hbar_2 \otimes \hbar_1$ □

Theorem 4. *Let $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$, $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ and $\hbar_3 = (\varpi_{\hbar_3}, \omega_{\hbar_3})$ be three nPR-FSs, then*

(1) $\hbar_1 \oplus \hbar_2 \oplus \hbar_3 = \hbar_1 \oplus \hbar_3 \oplus \hbar_2$

(2) $\hbar_1 \otimes \hbar_2 \otimes \hbar_3 = \hbar_1 \otimes \hbar_3 \otimes \hbar_2$

Proof

(1) $\hbar_1 \oplus \hbar_2 \oplus \hbar_3 = (\varpi_{\hbar_1}, \omega_{\hbar_1}) \oplus (\varpi_{\hbar_2}, \omega_{\hbar_2}) \oplus (\varpi_{\hbar_3}, \omega_{\hbar_3}) = (\sqrt[n]{}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n, \omega_{\hbar_1}\omega_{\hbar_2}) \oplus (\varpi_{\hbar_3}, \omega_{\hbar_3}) = (\sqrt[n]{}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n + \varpi_{\hbar_3}^n - \varpi_{\hbar_3}^n(\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n), \omega_{\hbar_1}\omega_{\hbar_2}\omega_{\hbar_3}) = (\sqrt[n]{}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n + \varpi_{\hbar_3}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_3}^n - \varpi_{\hbar_2}^n\varpi_{\hbar_3}^n + \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n\varpi_{\hbar_3}^n, \omega_{\hbar_1}\omega_{\hbar_2}\omega_{\hbar_3}) = (\sqrt[n]{}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_3}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_3}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_2}^n(\varpi_{\hbar_1}^n + \varpi_{\hbar_3}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_3}^n), \omega_{\hbar_1}\omega_{\hbar_2}\omega_{\hbar_3}) = (\sqrt[n]{}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_3}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_3}^n, \omega_{\hbar_1}\omega_{\hbar_3}) \oplus (\varpi_{\hbar_2}, \omega_{\hbar_2}) = \hbar_1 \oplus \hbar_3 \oplus \hbar_2$

(2) Following similar technique given in (1) □

Theorem 5. *Let $\hbar = (\varpi_\hbar, \omega_\hbar)$, $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$, and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ be three nPR-FSs, then*

(1) $\varepsilon(\hbar_1 \oplus \hbar_2) = \varepsilon\hbar_1 \oplus \varepsilon\hbar_2$, for $\varepsilon > 0$

(2) $(\varepsilon_1 + \varepsilon_2)\hbar = \varepsilon_1\hbar \oplus \varepsilon_2\hbar$, for $\varepsilon_1, \varepsilon_2 > 0$

(3) $(\hbar_1 \otimes \hbar_2)^\varepsilon = \hbar_1^\varepsilon \otimes \hbar_2^\varepsilon$, for $\varepsilon > 0$

(4) $\hbar^{\varepsilon_1} \otimes \hbar^{\varepsilon_2} = \hbar^{(\varepsilon_1 + \varepsilon_2)}$, for $\varepsilon_1, \varepsilon_2 > 0$

Proof

(1) $\varepsilon(\hbar_1 \oplus \hbar_2) = \varepsilon(\sqrt[n]{}\,\varpi_{\hbar_1}^n + \varpi_{\hbar_2}^n - \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n, \omega_{\hbar_1}\omega_{\hbar_2}) = (\sqrt[n]{}\,1 - (1 - \varpi_{\hbar_1}^n - \varpi_{\hbar_2}^n + \varpi_{\hbar_1}^n\varpi_{\hbar_2}^n)^\varepsilon, (\omega_{\hbar_1}\omega_{\hbar_2})^\varepsilon) = (\sqrt[n]{}\,1 - (1 - \varpi_{\hbar_1}^n)^\varepsilon(1 - \varpi_{\hbar_2}^n)^\varepsilon, \omega_{\hbar_1}^\varepsilon\omega_{\hbar_2}^\varepsilon)$. And $\varepsilon\hbar_1 \oplus \varepsilon\hbar_2 = (\sqrt[n]{}\,1 - (1 - \varpi_{\hbar_1}^n)^\varepsilon, \omega_{\hbar_1}^\varepsilon) \oplus (\sqrt[n]{}\,1 - (1 - \varpi_{\hbar_2}^n)^\varepsilon, \omega_{\hbar_2}^\varepsilon) = (\sqrt[n]{}\,1 - (1 - \varpi_{\hbar_1}^n)^\varepsilon + 1 - (1 - \varpi_{\hbar_2}^n)^\varepsilon - (1 - (1 - \varpi_{\hbar_1}^n)^\varepsilon)(1 - (1 - \varpi_{\hbar_2}^n)^\varepsilon), \omega_{\hbar_1}^\varepsilon\omega_{\hbar_2}^\varepsilon) = (\sqrt[n]{}\,1 - (1 - \varpi_{\hbar_1}^n)^\varepsilon(1 - \varpi_{\hbar_2}^n)^\varepsilon, \omega_{\hbar_1}^\varepsilon\omega_{\hbar_2}^\varepsilon) = \varepsilon(\hbar_1 \oplus \hbar_2)$

(2) $(\varepsilon_1 + \varepsilon_2)\hbar = (\varepsilon_1 + \varepsilon_2)(\varpi_\hbar, \omega_\hbar) = (\sqrt[n]{}\,1 - (1 - \varpi_\hbar^n)^{\varepsilon_1 + \varepsilon_2}, \omega_\hbar^{\varepsilon_1 + \varepsilon_2}) = (\sqrt[n]{}\,1 - (1 - \varpi_\hbar^n)^{\varepsilon_1}(1 - \varpi_\hbar^n)^{\varepsilon_2}, \omega_\hbar^{\varepsilon_1 + \varepsilon_2}) = (\sqrt[n]{}\,1 - (1 - \varpi_\hbar^n)^{\varepsilon_1} + 1 - (1 - \varpi_\hbar^n)^{\varepsilon_2} - (1 - (1 - \varpi_\hbar^n)^{\varepsilon_1})(1 - (1 - \varpi_\hbar^n)^{\varepsilon_2}), \omega_\hbar^{\varepsilon_1}\omega_\hbar^{\varepsilon_2}) = (\sqrt[n]{}\,1 - (1 - \varpi_\hbar^n)^{\varepsilon_1}, \omega_\hbar^{\varepsilon_1}) \oplus (\sqrt[n]{}\,1 - (1 - \varpi_\hbar^n)^{\varepsilon_2}, \omega_\hbar^{\varepsilon_2}) = \varepsilon_1\hbar \oplus \varepsilon_2\hbar$

(3) $(\hbar_1 \otimes \hbar_2)^\varepsilon = (\varpi_{\hbar_1}\varpi_{\hbar_2}, (\sqrt[n]{}\,\omega_{\hbar_1} + \sqrt[n]{}\,\omega_{\hbar_2} - \sqrt[n]{}\,\omega_{\hbar_1}\sqrt[n]{}\,\omega_{\hbar_2})^n)^\varepsilon = ((\varpi_{\hbar_1}\varpi_{\hbar_2})^\varepsilon, (1 - (1 - \sqrt[n]{}\,\omega_{\hbar_1} - \sqrt[n]{}\,\omega_{\hbar_2} + \sqrt[n]{}\,\omega_{\hbar_1}\sqrt[n]{}\,\omega_{\hbar_2})^\varepsilon)^n) = (\varpi_{\hbar_1}^\varepsilon\varpi_{\hbar_2}^\varepsilon, (1 - (1 - \sqrt[n]{}\,\omega_{\hbar_1})^\varepsilon(1 - \sqrt[n]{}\,\omega_{\hbar_2})^\varepsilon)^n) = (\varpi_{\hbar_1}^\varepsilon, (1 - (1 - \sqrt[n]{}\,\omega_{\hbar_1})^\varepsilon)^n) \otimes (\varpi_{\hbar_2}^\varepsilon, (1 - (1 - \sqrt[n]{}\,\omega_{\hbar_2})^\varepsilon)^n) = \hbar_1^\varepsilon \otimes \hbar_2^\varepsilon$

(4) $\hbar^{\varepsilon_1} \otimes \hbar^{\varepsilon_2} = (\varpi_\hbar^{\varepsilon_1}, (1 - (1 - \sqrt[n]{}\,\omega_\hbar)^{\varepsilon_1})^n) \otimes (\varpi_\hbar^{\varepsilon_2}, (1 - (1 - \sqrt[n]{}\,\omega_\hbar)^{\varepsilon_2})^n) = (\varpi_\hbar^{\varepsilon_1 + \varepsilon_2}, 1 - (1 - \sqrt[n]{}\,\omega_\hbar)^{\varepsilon_1} + 1 - (1 - \sqrt[n]{}\,\omega_\hbar)^{\varepsilon_2} - (1 - (1 - \sqrt[n]{}\,\omega_\hbar)^{\varepsilon_1})(1 - (1 - \sqrt[n]{}\,\omega_\hbar)^{\varepsilon_2})) = (\varpi_\hbar^{\varepsilon_1 + \varepsilon_2}, (1 - (1 - \sqrt[n]{}\,\omega_\hbar)^{\varepsilon_1 + \varepsilon_2})^n) = \hbar^{(\varepsilon_1 + \varepsilon_2)}$ □

Theorem 6. *Let $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$, $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$, and $\hbar_3 = (\varpi_{\hbar_3}, \omega_{\hbar_3})$ be three nPR-FSs, then*

(1) $(\hbar_1 \cap \hbar_2)\oplus\hbar_3 = (\hbar_1\oplus\hbar_3) \cap (\hbar_2\oplus\hbar_3)$
(2) $(\hbar_1 \cup \hbar_2)\oplus\hbar_3 = (\hbar_1\oplus\hbar_3) \cup (\hbar_2\oplus\hbar_3)$
(3) $(\hbar_1 \cap \hbar_2)\otimes\hbar_3 = (\hbar_1\otimes\hbar_3) \cap (\hbar_2\otimes\hbar_3)$
(4) $(\hbar_1 \cup \hbar_2)\otimes\hbar_3 = (\hbar_1\otimes\hbar_3) \cup (\hbar_2\otimes\hbar_3)$

Proof. From Definitions 6 and 4, we have

(1) $(\hbar_1 \cap \hbar_2)\oplus\hbar_3 = (\min\{\varpi_{\hbar_1}, \varpi_{\hbar_2}\}, \max\{\omega_{\hbar_1}, \omega_{\hbar_2}\})\oplus(\varpi_{\hbar_3}, \omega_{\hbar_3}) = (\sqrt[n]{\min\{\varpi^n_{\hbar_1}, \varpi^n_{\hbar_2}\} + \varpi^n_{\hbar_3} - \varpi^n_{\hbar_3}\min\{\varpi^n_{\hbar_1}, \varpi^n_{\hbar_2}\}}, \max\{\omega_{\hbar_1}, \omega_{\hbar_2}\}\omega_{\hbar_3}) = (\sqrt[n]{(1-\varpi^n_{\hbar_3})\min\{\varpi^n_{\hbar_1}, \varpi^n_{\hbar_2}\} + \varpi^n_{\hbar_3}}, \max\{\omega_{\hbar_1}\omega_{\hbar_3}, \omega_{\hbar_2}\omega_{\hbar_3}\})$. And, $(\hbar_1\oplus\hbar_3) \cap (\hbar_2\oplus\hbar_3) = (\sqrt[n]{\varpi^n_{\hbar_1} + \varpi^n_{\hbar_3} - \varpi^n_{\hbar_1}\varpi^n_{\hbar_3}}, \omega_{\hbar_1}\omega_{\hbar_3}) \cap (\sqrt[n]{\varpi^n_{\hbar_2} + \varpi^n_{\hbar_3} - \varpi^n_{\hbar_2}\varpi^n_{\hbar_3}}, \omega_{\hbar_2}\omega_{\hbar_3}) = (\min\{\sqrt[n]{\varpi^n_{\hbar_1} + \varpi^n_{\hbar_3} - \varpi^n_{\hbar_1}\varpi^n_{\hbar_3}}, \sqrt[n]{\varpi^n_{\hbar_2} + \varpi^n_{\hbar_3} - \varpi^n_{\hbar_2}\varpi^n_{\hbar_3}}\}, \max\{\omega_{\hbar_1}\omega_{\hbar_3}, \omega_{\hbar_2}\omega_{\hbar_3}\}) = (\min\{\sqrt[n]{(1-\varpi^n_{\hbar_3})\varpi^n_{\hbar_1} + \varpi^n_{\hbar_3}}, \sqrt[n]{(1-\varpi^n_{\hbar_3})\varpi^n_{\hbar_2} + \varpi^n_{\hbar_3}}\}, \max\{\omega_{\hbar_1}\omega_{\hbar_3}, \omega_{\hbar_2}\omega_{\hbar_3}\}) = (\sqrt[n]{(1-\varpi^n_{\hbar_3})\min\{\varpi^n_{\hbar_1}, \varpi^n_{\hbar_2}\} + \varpi^n_{\hbar_3}}, \max\{\omega_{\hbar_1}\omega_{\hbar_3}, \omega_{\hbar_2}\omega_{\hbar_3}\})$.

Thus, $(\hbar_1 \cap \hbar_2)\oplus\hbar_3 = (\hbar_1\oplus\hbar_3) \cap (\hbar_2\oplus\hbar_3)$.

(2) Following similar technique given in (1).

(3) $(\hbar_1 \cap \hbar_2)\otimes\hbar_3 = (\min\{\varpi_{\hbar_1}, \varpi_{\hbar_2}\}, \max\{\omega_{\hbar_1}, \omega_{\hbar_2}\})\otimes\hbar_3 = (\min\{\varpi_{\hbar_1}, \varpi_{\hbar_2}\}\varpi_{\hbar_3}, (\max\{\sqrt[n]{\omega_{\hbar_1}}, \sqrt[n]{\omega_{\hbar_2}}\} + \sqrt[n]{\omega_{\hbar_3}} - \sqrt[n]{\omega_{\hbar_3}}\max\{\sqrt[n]{\omega_{\hbar_1}}, \sqrt[n]{\omega_{\hbar_2}}\})^n) = (\min\{\varpi_{\hbar_1}\varpi_{\hbar_3}, \varpi_{\hbar_2}\varpi_{\hbar_3}\}, ((1-\sqrt[n]{\omega_{\hbar_3}})\max\{\sqrt[n]{\omega_{\hbar_1}}, \sqrt[n]{\omega_{\hbar_2}}\} + \sqrt[n]{\omega_{\hbar_3}})^n)$. And, $(\hbar_1\otimes\hbar_3) \cap (\hbar_2\otimes\hbar_3) = (\varpi_{\hbar_1}\varpi_{\hbar_3}, (\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_3}} - \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_3}})^n) \cap (\varpi_{\hbar_2}\varpi_{\hbar_3}, (\sqrt[n]{\omega_{\hbar_2}} + \sqrt[n]{\omega_{\hbar_3}} - \sqrt[n]{\omega_{\hbar_2}}\sqrt[n]{\omega_{\hbar_3}})^n) = (\varpi_{\hbar_1}\varpi_{\hbar_3}, ((1-\sqrt[n]{\omega_{\hbar_3}})\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_3}})^n) \cap (\varpi_{\hbar_2}\varpi_{\hbar_3}, ((1-\sqrt[n]{\omega_{\hbar_3}})\sqrt[n]{\omega_{\hbar_2}} + \sqrt[n]{\omega_{\hbar_3}})^n) = (\min\{\varpi_{\hbar_1}\varpi_{\hbar_3}, \varpi_{\hbar_2}\varpi_{\hbar_3}\}, \max\{((1-\sqrt[n]{\omega_{\hbar_3}})\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_3}})^n, ((1-\sqrt[n]{\omega_{\hbar_3}})\sqrt[n]{\omega_{\hbar_2}} + \sqrt[n]{\omega_{\hbar_3}})^n\}) = (\min\{\varpi_{\hbar_1}\varpi_{\hbar_3}, \varpi_{\hbar_2}\varpi_{\hbar_3}\}, ((1-\sqrt[n]{\omega_{\hbar_3}})\max\{\sqrt[n]{\omega_{\hbar_1}}, \sqrt[n]{\omega_{\hbar_2}}\} + \sqrt[n]{\omega_{\hbar_3}})^n)$.

Thus, $(\hbar_1 \cap \hbar_2)\otimes\hbar_3 = (\hbar_1\otimes\hbar_3) \cap (\hbar_2\otimes\hbar_3)$.

(4) Following similar technique given in (3). □

Theorem 7. *Let $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$ and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ be two nPR-FSs and $\varepsilon > 0$, then*

(1) $\varepsilon(\hbar_1 \cup \hbar_2) = \varepsilon\hbar_1 \cup \varepsilon\hbar_2$
(2) $(\hbar_1 \cup \hbar_2)^\varepsilon = \hbar_1^\varepsilon \cup \hbar_2^\varepsilon$

Proof. From Definitions 4 and 6, we have

(1) $\varepsilon(\hbar_1 \cup \hbar_2) = \varepsilon(\max\{\varpi_{\hbar_1}, \varpi_{\hbar_2}\}, \min\{\omega_{\hbar_1}, \omega_{\hbar_2}\}) = (\sqrt[n]{1-(1-\max\{\varpi^n_{\hbar_1}, \varpi^n_{\hbar_2}\})^\varepsilon}, \min\{\omega^\varepsilon_{\hbar_1}, \omega^\varepsilon_{\hbar_2}\})$.

And, $\varepsilon\hbar_1 \cup \varepsilon\hbar_2 = (\sqrt[n]{1-(1-\varpi^n_{\hbar_1})^\varepsilon}, \omega^\varepsilon_{\hbar_1}) \cup (\sqrt[n]{1-(1-\varpi^n_{\hbar_2})^\varepsilon}, \omega^\varepsilon_{\hbar_2}) = (\max\{\sqrt[n]{1-(1-\varpi^n_{\hbar_1})^\varepsilon}, \sqrt[n]{1-(1-\varpi^n_{\hbar_2})^\varepsilon}\}, \min\{\omega^\varepsilon_{\hbar_1}, \omega^\varepsilon_{\hbar_2}\}) = (\sqrt[n]{1-(1-\max\{\varpi^n_{\hbar_1}, \varpi^n_{\hbar_2}\})^\varepsilon}, \min\{\omega^\varepsilon_{\hbar_1}, \omega^\varepsilon_{\hbar_2}\}) = \varepsilon(\hbar_1 \cup \hbar_2)$.

(2) It can be proved similar to (1). □

Theorem 8. *Let $\hbar = (\varpi_\hbar, \omega_\hbar)$, $\hbar_1 = (\varpi_{\hbar_1}, \omega_{\hbar_1})$, and $\hbar_2 = (\varpi_{\hbar_2}, \omega_{\hbar_2})$ be three nPR-FSs, and $\varepsilon > 0$, then*

(1) $(\hbar_1\oplus\hbar_2)^c = \hbar_1^c \otimes \hbar_2^c$
(2) $(\hbar_1\otimes\hbar_2)^c = \hbar_1^c \oplus \hbar_2^c$
(3) $(\hbar^c)^\varepsilon = (\varepsilon\hbar)^c$
(4) $\varepsilon(\hbar)^c = (\hbar^\varepsilon)^c$

Proof. From Definitions 6 and 4 (3), we have

(1) $(\hbar_1\oplus\hbar_2)^c = (\sqrt[n]{\varpi^n_{\hbar_1} + \varpi^n_{\hbar_2} - \varpi^n_{\hbar_1}\varpi^n_{\hbar_2}}, \omega_{\hbar_1}\omega_{\hbar_2})^c = (\sqrt[n^2]{\omega_{\hbar_1}\omega_{\hbar_2}}, (\sqrt[n]{\varpi^n_{\hbar_1} + \varpi^n_{\hbar_2} - \varpi^n_{\hbar_1}\varpi^n_{\hbar_2}})^{n^2}) = (\sqrt[n^2]{\omega_{\hbar_1}}\sqrt[n^2]{\omega_{\hbar_2}}, (\varpi^n_{\hbar_1} + \varpi^n_{\hbar_2} - \varpi^n_{\hbar_1}\varpi^n_{\hbar_2})^n) = (\sqrt[n^2]{\omega_{\hbar_1}}, (\varpi_{\hbar_1})^{n^2}) \otimes (\sqrt[n^2]{\omega_{\hbar_2}}, (\varpi_{\hbar_2})^{n^2}) = \hbar_1^c \otimes \hbar_2^c$

(2) $(\hbar_1\otimes\hbar_2)^c = (\varpi_{\hbar_1}\varpi_{\hbar_2}, (\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_2}} - \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}})^n)^c = (\sqrt[n^2]{(\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_2}} - \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}})^n}, (\varpi_{\hbar_1}\varpi_{\hbar_2})^{n^2}) = (\sqrt[n]{(\sqrt[n]{\omega_{\hbar_1}} + \sqrt[n]{\omega_{\hbar_2}} - \sqrt[n]{\omega_{\hbar_1}}\sqrt[n]{\omega_{\hbar_2}})}, (\varpi_{\hbar_1})^{n^2}(\varpi_{\hbar_2})^{n^2}) = (\sqrt[n^2]{\omega_{\hbar_1}}, (\varpi_{\hbar_1})^{n^2}) \oplus (\sqrt[n^2]{\omega_{\hbar_2}}, (\varpi_{\hbar_2})^{n^2}) = \hbar_1^c \oplus \hbar_2^c$

(3) $(\hbar^c)^\varepsilon = (\sqrt[n^2]{\omega_\hbar}, (\varpi_\hbar)^{n^2})^\varepsilon = ((\sqrt[n^2]{\omega_\hbar})^\varepsilon, (1-(1-\varpi^n_\hbar)^\varepsilon)^n) = (\sqrt[n]{1-(1-\varpi^n_\hbar)^\varepsilon}, \omega^\varepsilon_\hbar)^c = (\varepsilon\hbar)^c$

(4) $\varepsilon(\hbar)^c = \varepsilon(\sqrt[n^2]{\omega_\hbar}, (\varpi_\hbar)^{n^2}) = (\sqrt[n]{1-(1-\sqrt[n]{\omega_\hbar})^\varepsilon}, (((\varpi_\hbar)^{n^2})^\varepsilon) = (\varpi^\varepsilon_\hbar, (1-(1-\sqrt[n]{\omega_\hbar})^\varepsilon)^n)^c = (\hbar^\varepsilon)^c$ □

4. nPR-Fuzzy Weighted Power Average Inspired by the Class of nPR-Fuzzy Sets

In this section, we put forth the operator of nPR-fuzzy weighted power average and evince its main characterizations. In particular, we prove the properties of boundedness, monotonicity, and idempotency for this operator. Then, we illustrate how nPR-FWPA operator is applied to evaluate options of an MCDM problem with nPR-fuzzy data.

Definition 7. Let $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})\,(i = 1, 2, \ldots, k)$ be the values of nPR-FSs and $\eta = (\eta_1, \eta_2, \ldots, \eta_k)^T$ be weight vector of $\hbar_i$ with $\eta_i > 0$, $\sum_{i=1}^k \eta_i = 1$ and $n > 1$. Then, an nPR-fuzzy weighted power average (nPR-FWPA) operator is a function nPR-FWPA: $\hbar^k \longrightarrow \hbar$, where

$$\text{nPR}-\text{FWPA}(\hbar_1, \hbar_2, \ldots, \hbar_k) = \left(\left(\sum_{i=1}^k \eta_i \varpi^n_{\hbar_i}\right)^{1/n}, \left(\sum_{i=1}^k \eta_i \omega^{1/n}_{\hbar_i}\right)^n\right). \tag{15}$$

Example 2. Consider $\hbar_1 = (0.5, 0.5), \hbar_2 = (0.2, 0.7), \hbar_3 = (0.6, 0.1), \hbar_4 = (0.4, 0.3)$ and $\hbar_5 = (0.2, 0.9)$ are five nPR-fuzzy sets and let $\eta = (0.2, 0.4, 0.2, 0.1, 0.1)^T$ be a weight vector of $\hbar_i$ $(i = 1, 2, \ldots, 5)$. Then,

$$\text{nPR} - \text{FWPA}(\hbar_1, \hbar_2, \ldots, \hbar_5) = \begin{pmatrix} (0.5^n \times 0.2 + 0.2^n \times 0.4 + 0.6^n \times 0.2 + 0.4^n \times 0.1 + 0.2^n \times 0.1)^{1/n}, \\ (0.5^{1/n} \times 0.2 + 0.7^{1/n} \times 0.4 + 0.1^{1/n} \times 0.2 + 0.3^{1/n} \times 0.1 + 0.9^{1/n} \times 0.1)^n \end{pmatrix} \approx \begin{cases} (0.428, 0.457), & \text{if } n = 3, \\ (0.470, 0.442), & \text{if } n = 5, \\ (0.505, 0.433), & \text{if } n = 8, \\ (0.519, 0.430), & \text{if } n = 10, \\ (0.541, 0.426), & \text{if } n = 15, \\ (0.563, 0.423), & \text{if } n = 25, \\ (0.586, 0.420), & \text{if } n = 70, \\ (0.590, 0.419), & \text{if } n = 100. \end{cases} \tag{16}$$

Theorem 9. *Let $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})\,(i = 1, 2, \ldots, k)$ be a value of nPR-FSs and $\eta = (\eta_1, \eta_2, \ldots, \eta_k)^T$ be weight vector of $\hbar_i$ with $\eta_i > 0$ and $\sum_{i=1}^k \eta_i = 1$. Then, nPR-FWPA $(\hbar_1, \hbar_2, \ldots, \hbar_k)$ is an nPR-FS.*

Proof. For any nPR-FS $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})$, we have

$$\begin{aligned} &0 \leq \varpi_{\hbar_i}^n \leq 1, \\ &0 \leq \omega_{\hbar_i}^{1/n} \leq 1, \\ &0 \leq \varpi_{\hbar_i}^n + \omega_{\hbar_i}^{1/n} \leq 1. \end{aligned} \tag{17}$$

Then, we obtain the following equation:

$$\begin{aligned} &0 \leq \eta_1 \varpi_{\hbar_1}^n + \eta_1 \omega_{\hbar_1}^{1/n} \leq \eta_1, \\ &0 \leq \eta_2 \varpi_{\hbar_2}^n + \eta_2 \omega_{\hbar_2}^{1/n} \leq \eta_2, \\ &0 \leq \eta_k \varpi_{\hbar_k}^n + \eta_k \omega_{\hbar_k}^{1/n} \leq \eta_k, \end{aligned} \tag{18}$$

and so

$$\begin{aligned} 0 \leq &\left(\eta_1 \varpi_{\hbar_1}^n + \eta_1 \omega_{\hbar_1}^{1/n}\right) + \left(\eta_2 \varpi_{\hbar_2}^n + \eta_2 \omega_{\hbar_2}^{1/n}\right) \\ &+ \cdots + \left(\eta_k \varpi_{\hbar_k}^n + \eta_k \omega_{\hbar_k}^{1/n}\right) \leq \eta_1 + \eta_2 + \cdots + \eta_k, \end{aligned} \tag{19}$$

implies that

$$0 \leq \sum_{i=1}^k \eta_i \varpi_{\hbar_i}^n + \sum_{i=1}^k \eta_i \omega_{\hbar_i}^{1/n} \leq \sum_{i=1}^k \eta_i = 1. \tag{20}$$

Therefore,

$$\begin{aligned} 0 &\leq \left(\left(\left(\sum_{i=1}^k \eta_i \varpi_{\hbar_i}^n\right)^{1/n}\right)^n + \left(\left(\sum_{i=1}^k \eta_i \omega_{\hbar_i}^{1/n}\right)^n\right)^{1/n}\right) \\ &= \sum_{i=1}^k \eta_i \varpi_{\hbar_i}^n + \sum_{i=1}^k \eta_i \omega_{\hbar_i}^{1/n} \leq 1. \end{aligned} \tag{21}$$

It is obvious that,

$$\begin{aligned} &0 \leq \left(\sum_{i=1}^k \eta_i \varpi_{\hbar_i}^n\right)^{1/n} \leq 1, \\ &0 \leq \left(\sum_{i=1}^k \eta_i \omega_{\hbar_i}^{1/n}\right)^n \leq 1. \end{aligned} \tag{22}$$

Then, nPR-FWPA $(\hbar_1, \hbar_2, \ldots, \hbar_k)$ is an nPR-FS. □

Theorem 10. *Let $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})\,(i = 1, 2, \ldots, k)$ be a value of nPR-FSs, $\hbar = (\varpi_{\hbar}, \omega_{\hbar})$ be nPR-FS and $\eta = (\eta_1, \eta_2, \ldots, \eta_k)^T$ be a weight vector of $\hbar_i$ with $\sum_{i=1}^k \eta_i = 1$. Then,* $\text{nPR} - \text{FWPA}(\hbar_1 \oplus \hbar, \hbar_2 \oplus \hbar, \ldots, \hbar_k \oplus \hbar) \geq \text{nPR} - \text{FWPA}(\hbar_1 \otimes \hbar, \hbar_2 \otimes \hbar, \ldots, \hbar_k \otimes \hbar)$.

Proof. For any $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})\,(i = 1, 2, \ldots, k)$ and $\hbar = (\varpi_{\hbar}, \omega_{\hbar})$, we have

$$\begin{aligned} &\varpi_{\hbar_i}^n + \varpi_{\hbar}^n - \varpi_{\hbar_i}^n \varpi_{\hbar}^n \geq 2\varpi_{\hbar_i}^n \varpi_{\hbar}^n - \varpi_{\hbar_i}^n \varpi_{\hbar}^n = \varpi_{\hbar_i}^n \varpi_{\hbar}^n, \\ &\omega_{\hbar_i}^{1/n} + \omega_{\hbar}^{1/n} - \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n} \geq 2\omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n} - \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n} = \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n}, \end{aligned} \tag{23}$$

that is,

(1) $\sum_{i=1}^k \eta_i (\varpi_{\hbar_i}^n + \varpi_{\hbar}^n - \varpi_{\hbar_i}^n \varpi_{\hbar}^n) \geq \sum_{i=1}^k \eta_i \varpi_{\hbar_i}^n \varpi_{\hbar}^n \Rightarrow (\sum_{i=1}^k \eta_i (\varpi_{\hbar_i}^n + \varpi_{\hbar}^n - \varpi_{\hbar_i}^n \varpi_{\hbar}^n))^{1/n} \geq (\sum_{i=1}^k \eta_i \varpi_{\hbar_i}^n \varpi_{\hbar}^n)^{1/n}$

(2) $\sum_{i=1}^k \eta_i (\omega_{\hbar_i}^{1/n} + \omega_{\hbar}^{1/n} - \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n}) \geq \sum_{i=1}^k \eta_i \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n}$. $\Rightarrow (\sum_{i=1}^k \eta_i (\omega_{\hbar_i}^{1/n} + \omega_{\hbar}^{1/n} - \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n}))^n \geq (\sum_{i=1}^k \eta_i \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n})^n$

Hence, we have

$$\begin{aligned}
&\mathrm{nPR-FWPA}\left(\hbar_1\oplus\hbar,\hbar_2\oplus\hbar,\ldots,\hbar_k\oplus\hbar\right)=\left(\left(\sum_{i=1}^{k}\eta_i\left(\varpi_{\hbar_i}^{n}+\varpi_{\hbar}^{n}-\varpi_{\hbar_i}^{n}\varpi_{\hbar}^{n}\right)\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\omega_{\hbar}^{1/n}\right)^{n}\right),\\
&\mathrm{nPR-FWPA}\left(\hbar_1\otimes\hbar,\hbar_2\otimes\hbar,\ldots,\hbar_k\otimes\hbar\right)=\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\varpi_{\hbar}^{n}\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\left(\omega_{\hbar_i}^{1/n}+\omega_{\hbar}^{1/n}-\omega_{\hbar_i}^{1/n}\omega_{\hbar}^{1/n}\right)\right)^{n}\right).
\end{aligned}\tag{24}$$

Then, from (1) and (2), we obtain the following equation:

$$\mathrm{nPR-FWPA}\left(\hbar_1\oplus\hbar,\hbar_2\oplus\hbar,\ldots,\hbar_k\oplus\hbar\right)\geq\mathrm{nPR-FWPA}\left(\hbar_1\otimes\hbar,\hbar_2\otimes\hbar,\ldots,\hbar_k\otimes\hbar\right).\tag{25}$$

□

Theorem 11. *Let $\hbar_i=(\varpi_{\hbar_i},\omega_{\hbar_i})\,(i=1,2,\cdots,k)$ be a value of nPR-FSs, $\hbar=(\varpi_{\hbar},\omega_{\hbar})$ be nPR-FS and $\eta=(\eta_1,\eta_2,\ldots,\eta_k)^T$ be weight vector of $\hbar_i$ with $\sum_{i=1}^{k}\eta_i=1$, then*

(1) nPR-FWPA $(\hbar_1\oplus\hbar,\hbar_2\oplus\hbar,\cdots,\hbar_k\oplus\hbar)\geq$ nPR-FWPA $(\hbar_1,\hbar_2,\cdots,\hbar_k)\otimes\hbar$

(2) nPR-FWPA $(\hbar_1,\hbar_2,\cdots,\hbar_k)\oplus\hbar\geq$ nPR-FWPA $(\hbar_1,\hbar_2,\cdots,\hbar_k)\otimes\hbar$

Proof. We will give the proof of (1). The other claim is proven in a similar manner. Since for any $\hbar_i=(\varpi_{\hbar_i},\omega_{\hbar_i})\,(i=1,2,\cdots,k)$ and $\hbar=(\varpi_{\hbar},\omega_{\hbar})$, we have

$$\left(\sum_{i=1}^{k}\eta_i\left(\varpi_{\hbar_i}^{n}+\varpi_{\hbar}^{n}-\varpi_{\hbar_i}^{n}\varpi_{\hbar}^{n}\right)\right)^{1/n}\geq\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\varpi_{\hbar}^{n}\right)^{1/n}=\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n}\varpi_{\hbar}.\tag{26}$$

Similarly,

$$\left(\sum_{i=1}^{k}\eta_i\left(\omega_{\hbar_i}^{1/n}+\omega_{\hbar}^{1/n}-\omega_{\hbar_i}^{1/n}\omega_{\hbar}^{1/n}\right)\right)^{n}\geq\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\omega_{\hbar}^{1/n}\right)^{n}=\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\right)^{n}\omega_{\hbar}.\tag{27}$$

Therefore, we have

$$\begin{aligned}
&\mathrm{nPR-FWPA}\left(\hbar_1\oplus\hbar,\hbar_2\oplus\hbar,\ldots,\hbar_k\oplus\hbar\right)\\
&=\left(\left(\sum_{i=1}^{k}\eta_i\left(\varpi_{\hbar_i}^{n}+\varpi_{\hbar}^{n}-\varpi_{\hbar_i}^{n}\varpi_{\hbar}^{n}\right)\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\omega_{\hbar}^{1/n}\right)^{n}\right),\\
&\mathrm{nPR-FWPA}\left(\hbar_1,\hbar_2,\ldots,\hbar_k\right)\otimes\hbar\\
&=\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\right)^{n}\right)\otimes\left(\varpi_{\hbar},\omega_{\hbar}\right)\\
&=\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n}\varpi_{\hbar},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}+\omega_{\hbar}^{1/n}-\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\omega_{\hbar}^{1/n}\right)^{n}\right).
\end{aligned}\tag{28}$$

Then, from (1) and (2) we obtain the following equation:

$$\text{nPR} - \text{FWPA}\left(\hbar_1 \oplus \hbar, \hbar_2 \oplus \hbar, \ldots, \hbar_k \oplus \hbar\right) \geq \text{nPR} - \text{FWPA}\left(\hbar_1, \hbar_2, \ldots, \hbar_k\right) \otimes \hbar. \tag{29}$$

□

Theorem 12. *Let $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})$ and $L_i = (\varpi_{L_i}, \omega_{L_i})$ $(i = 1, 2, \ldots, k)$ be two values of nPR-FSs, and $\eta = (\eta_1, \eta_2, \ldots, \eta_k)^T$ be a weight vector of them with $\sum_{i=1}^{k} \eta_i = 1$. Then,*

(1) nPR-FWPA $(\hbar_1 \oplus L_1, \hbar_2 \oplus L_2, \ldots, \hbar_k \oplus L_k) \geq$ nPR-FWPA $(\hbar_1 \otimes L_1, \hbar_2 \otimes L_2, \ldots, \hbar_k \otimes L_k)$

(2) nPR-FWPA $(\hbar_1, \hbar_2, \ldots, \hbar_k) \oplus$ nPR-FWPA $(L_1, L_2, \ldots, L_k) \geq$ nPR-FWPA $(\hbar_1, \hbar_2, \ldots, \hbar_k) \otimes$ nPR-FWPA $(L_1, L_2, \ldots, L_k)$

Proof. For any $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})$ and $L_i = (\varpi_{L_i}, \omega_{L_i})$ $(i = 1, 2, \ldots, k)$, we have

(1) $\varpi_{\hbar_i}^n + \varpi_{L_i}^n - \varpi_{\hbar_i}^n \varpi_{L_i}^n \geq 2\varpi_{\hbar_i}^n \varpi_{L_i}^n - \varpi_{\hbar_i}^n \varpi_{L_i}^n = \varpi_{\hbar_i}^n \varpi_{L_i}^n$

$\omega_{\hbar_i}^{1/n} + \omega_{L_i}^{1/n} - \omega_{\hbar_i}^{1/n}\omega_{L_i}^{1/n} \geq 2\omega_{\hbar_i}^{1/n}\omega_{L_i}^{1/n} - \omega_{\hbar_i}^{1/n}\omega_{L_i}^{1/n} = \omega_{\hbar_i}^{1/n}\omega_{L_i}^{1/n}$

that is,

(1) $\sum_{i=1}^{k} \eta_i (\varpi_{\hbar_i}^n + \varpi_{L_i}^n - \varpi_{\hbar_i}^n \varpi_{L_i}^n) \geq \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \varpi_{L_i}^n \Rightarrow (\sum_{i=1}^{k} \eta_i (\varpi_{\hbar_i}^n + \varpi_{L_i}^n - \varpi_{\hbar_i}^n \varpi_{L_i}^n))^{1/n} \geq (\sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \varpi_{L_i}^n)^{1/n}$

(2) $\sum_{i=1}^{k} \eta_i (\omega_{\hbar_i}^{1/n} + \omega_{L_i}^{1/n} - \omega_{\hbar_i}^{1/n}\omega_{L_i}^{1/n}) \geq \sum_{i=1}^{k} \eta_i \omega_{\hbar_i}^{1/n}\omega_{L_i}^{1/n} \Rightarrow (\sum_{i=1}^{k} \eta_i (\omega_{\hbar_i}^{1/n} + \omega_{L_i}^{1/n} - \omega_{\hbar_i}^{1/n}\omega_{L_i}^{1/n}))^n \geq (\sum_{i=1}^{k} \eta_i \omega_{\hbar_i}^{1/n} \omega_{\hbar}^{1/n})^n$

Therefore, we have

$$\begin{aligned}
&\text{nPR} - \text{FWPA}\left(\hbar_1 \oplus L_1, \hbar_2 \oplus L_2, \ldots, \hbar_k \oplus L_k\right) \\
&= \left(\left(\sum_{i=1}^{k} \eta_i\left(\varpi_{\hbar_i}^n + \varpi_{L_i}^n - \varpi_{\hbar_i}^n \varpi_{L_i}^n\right)\right)^{1/n}, \left(\sum_{i=1}^{k} \eta_i \omega_{\hbar_i}^{1/n} \omega_{L_i}^{1/n}\right)^n\right), \\
&\text{nPR} - \text{FWPA}\left(\hbar_1 \otimes L_1, \hbar_2 \otimes L_2, \ldots, \hbar_k \otimes L_k\right) \\
&= \left(\left(\sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \varpi_{L_i}^n\right)^{1/n}, \left(\sum_{i=1}^{k} \eta_i\left(\omega_{\hbar_i}^{1/n} + \omega_{L_i}^{1/n} - \omega_{\hbar_i}^{1/n} \omega_{L_i}^{1/n}\right)\right)^n\right).
\end{aligned} \tag{30}$$

Thus, from (1) and (2) we obtain the following equation:

$$\text{nPR} - \text{FWPA}\left(\hbar_1 \oplus L_1, \hbar_2 \oplus L_2, \ldots, \hbar_k \oplus L_k\right) \geq \text{nPR} - \text{FWPA}\left(\hbar_1 \otimes L_1, \hbar_2 \otimes L_2, \ldots, \hbar_k \otimes L_k\right). \tag{31}$$

(2) Since,

$$\begin{aligned}
\sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n &\geq \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n, \\
\sum_{i=1}^{k} \eta_i \varpi_{L_i}^n &\geq \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n,
\end{aligned} \tag{32}$$

so

$$\begin{aligned}
\sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n + \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n \geq{}& \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n \\
&+ \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n,
\end{aligned} \tag{33}$$

implies that

$$\begin{aligned}
&\sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n + \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n - \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n \\
&\geq \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n,
\end{aligned} \tag{34}$$

and hence

$$\begin{aligned}
&\sqrt[n]{} \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n + \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n - \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n \\
&\geq \sqrt[n]{} \sum_{i=1}^{k} \eta_i \varpi_{\hbar_i}^n \sum_{i=1}^{k} \eta_i \varpi_{L_i}^n.
\end{aligned} \tag{35}$$

Similarly,

$$\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}+\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n}-\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n}\right)^n\geq\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n}\right)^n. \quad (36)$$

Therefore, we have

$$\begin{aligned}
&\mathrm{nPR-FWPA}(\hbar_1,\hbar_2,\ldots,\hbar_k)\oplus\mathrm{nPR-FWPA}(L_1,L_2,\ldots,L_k)\\
&=\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\right)^{n}\right)\oplus\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{L_i}^{n}\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n}\right)^{n}\right)\\
&=\left(\sqrt{[n]}\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}+\sum_{i=1}^{k}\eta_i\varpi_{L_i}^{n}-\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\sum_{i=1}^{k}\eta_i\varpi_{L_i}^{n},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\right)^{n}\left(\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n}\right)^{n}\right),\\
&\mathrm{nPR-FWPA}(\hbar_1,\hbar_2,\ldots,\hbar_k)\otimes\mathrm{nPR-FWPA}(L_1,L_2,\ldots,L_k)\\
&=\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\right)^{n}\right)\otimes\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{L_i}^{n}\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{L_i}^{\frac{1}{n}}\right)^{n}\right)\\
&=\left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n}\left(\sum_{i=1}^{k}\eta_i\varpi_{L_i}^{n}\right)^{1/n},\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}+\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n}-\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n}\right)^{n}\right).
\end{aligned} \quad (37)$$

Thus, from (1) and (2) we obtain the following equation:

$$\begin{aligned}
&\mathrm{nPR-FWPA}(\hbar_1,\hbar_2,\ldots,\hbar_k)\oplus\mathrm{nPR-FWPA}(L_1,L_2,\ldots,L_k)\geq\\
&\mathrm{nPR-FWPA}(\hbar_1,\hbar_2,\ldots,\hbar_k)\otimes\mathrm{nPR-FWPA}(L_1,L_2,\ldots,L_k)
\end{aligned} \quad (38)$$

□

Theorem 13. *(Boundedness) Let* $\hbar_i=(\varpi_{\hbar_i},\omega_{\hbar_i})(i=1,2,\ldots,k)$ *be a number of nPR-FSs, and* $\eta=(\eta_1,\eta_2,\ldots,\eta_k)^T$ *be a weight vector of* $\hbar_i$ *with* $\sum_{i=1}^{k}\eta_i=1$. *Suppose that* $\varpi_{\hbar}^{*}=\min_{1\leq i\leq k}\{\varpi_{\hbar_i}\}$, $\varpi_{\hbar}^{\star}=max_{1\leq i\leq k}\{\varpi_{\hbar_i}\}$, $\omega_{\hbar}^{*}=\min_{1\leq i\leq k}\{\omega_{\hbar_i}\}$ *and* $\omega_{\hbar}^{\star}=\max_{1\leq i\leq k}\{\omega_{\hbar_i}\}$. *Then,*

$$(\varpi_{\hbar}^{*},\omega_{\hbar}^{\star})\leq\mathrm{nPR-FWPA}(\hbar_1,\hbar_2,\ldots,\hbar_k)\leq(\varpi_{\hbar}^{\star},\omega_{\hbar}^{*}). \quad (39)$$

Proof. For any $\hbar_i=(\varpi_{\hbar_i},\omega_{\hbar_i})(i=1,2,\ldots,k)$, we can get $\varpi_{\hbar}^{*}\leq\varpi_{\hbar_i}\leq\varpi_{\hbar}^{\star}$ and $\omega_{\hbar}^{*}\leq\omega_{\hbar_i}\leq\omega_{\hbar}^{\star}$. Then, the inequalities for membership value are

$$\varpi_{\hbar}^{*}=\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar}^{*n}\right)^{1/n}\leq\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n}\leq\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar}^{\star n}\right)^{1/n}=\varpi_{\hbar}^{\star}. \quad (40)$$

Similarly, for nonmembership value

$$\omega_{\hbar}^{*}=\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar}^{*(1/n)}\right)^{n}\leq\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\right)^{n}\leq\left(\sum_{i=1}^{k}\eta_i\omega_{\hbar}^{\star}1/n\right)^{n}=\omega_{\hbar}^{\star}. \quad (41)$$

Therefore, $(\varpi_{\hbar}^{*},\omega_{\hbar}^{\star})\leq$ nPR-FWPA $(\hbar_1,\hbar_2,\ldots,\hbar_k)\leq(\varpi_{\hbar}^{\star},\omega_{\hbar}^{*})$. □

Theorem 14. *(Monotonicity) Let* $\hbar_i=(\varpi_{\hbar_i},\omega_{\hbar_i})$ *and* $L_i=(\varpi_{L_i},\omega_{L_i})(i=1,2,\ldots,k)$ *be two numbers of nPR-FSs. If* $\varpi_{\hbar_i}\leq\varpi_{L_i}$ *and* $\omega_{\hbar_i}\geq\omega_{L_i}\forall i$, *then*

$$\begin{aligned}
&\mathrm{nPR-FWPA}(\hbar_1,\hbar_2,\ldots,\hbar_k)\\
&\leq\mathrm{nPR-FWPA}(L_1,L_2,\ldots,L_k).
\end{aligned} \quad (42)$$

Proof. Since for all i we have $\varpi_{\hbar_i}\leq\varpi_{L_i}$ and $\omega_{\hbar_i}\geq\omega_{L_i}$, then $(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n})^{1/n}\leq(\sum_{i=1}^{k}\eta_i\varpi_{L_i}^{n})^{1/n}$ and $(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n})^{n}\geq(\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n})^{n}$, therefore nPR-FWPA $(\hbar_1,\hbar_2,\ldots,\hbar_k)=((\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n})^{1/}$ $n,(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n})^{n})\leq((\sum_{i=1}^{k}\eta_i\varpi_{L_i}^{n})^{1/n},(\sum_{i=1}^{k}\eta_i\omega_{L_i}^{1/n})^{n})=$ nPR-FWPA $(L_1,L_2,\ldots,L_k)$. □

Theorem 15. *(Idempotency) Let* $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})(i = 1, 2, \ldots, k)$ *be a number of nPR-FSs such that* $\hbar_i = \hbar = (\varpi_{\hbar}, \omega_{\hbar})$ *and* $\eta = (\eta_1, \eta_2, \ldots, \eta_k)^T$ *be a weight vector of* $\hbar_i$ *with* $\sum_{i=1}^{k}\eta_i = 1$, *then nPR-FWPA* $(\hbar_1, \hbar_2, \ldots, \hbar_k) = \hbar$.

Proof. Since $\hbar_i = \hbar = (\varpi_{\hbar}, \omega_{\hbar})(i = 1, 2, \ldots, k)$, then nPR-FWPA $(\hbar_1, \hbar_2, \ldots, \hbar_k) = \left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar_i}^{n}\right)^{1/n}, \left(\sum_{i=1}^{k}\eta_i\omega_{\hbar_i}^{1/n}\right)^{n}\right) = \left(\left(\sum_{i=1}^{k}\eta_i\varpi_{\hbar}^{n}\right)^{1/n}, \left(\sum_{i=1}^{k}\eta_i\omega_{\hbar}^{1/n}\right)^{n}\right) = (\varpi_{\hbar}, \omega_{\hbar}) = \hbar$.

We introduce the score and accuracy functions of the nPR-FS in order to rank nPR-FSs. □

Definition 8

(1) The score function of an nPR-FS $\hbar = (\varpi_{\hbar}, \omega_{\hbar})$ is defined as $\overline{s}(\hbar) = \varpi_{\hbar}^{n} - \sqrt{[n]}\omega_{\hbar}$

(2) The accuracy function of an nPR-FS $\hbar = (\varpi_{\hbar}, \omega_{\hbar})$ is defined as $\overline{a}(\hbar) = \varpi_{\hbar}^{n} + \sqrt{[n]}\omega_{\hbar}$

Example 3. Consider $\hbar = (0.2, 0.9)$ is nPR-FS, then

$$\overline{a}(\hbar) \approx \begin{cases} 0.992, & \text{if } n = 13, \\ 0.988, & \text{if } n = 9, \\ 0.983, & \text{if } n = 6, \\ 0.973, & \text{if } n = 3, \end{cases} \qquad \overline{s}(\hbar) \approx \begin{cases} -0.992 & \text{if } n = 13, \\ -0.988 & \text{if } n = 9, \\ -0.983 & \text{if } n = 6, \\ -0.957 & \text{if } n = 3. \end{cases} \tag{43}$$

Theorem 16. *Let* $\hbar = (\varpi_{\hbar}, \omega_{\hbar})$ *be any nPR-FS, then*

(1) $\overline{s}(\hbar) \in [-1, 1]$

(2) $\overline{a}(\hbar) \in [0, 1]$

Proof

(1) For any nPR-FS $\hbar$, we have $\varpi_{\hbar}^{n} + \omega_{\hbar}^{1/n} \leq 1$. Hence, $\varpi_{\hbar}^{n} - \omega_{\hbar}^{1/n} \leq \varpi_{\hbar}^{n} \leq 1$ and $\varpi_{\hbar}^{n} - \omega_{\hbar}^{1/n} \geq -\omega_{\hbar}^{1/n} \geq -1$. Thus, $-1 \leq \varpi_{\hbar}^{n} - \omega_{\hbar}^{1/n} \leq 1$, namely $\overline{s}(\hbar) \in [-1, 1]$. In particular, if $\hbar = (0, 1)$, then $\overline{s}(\hbar) = -1$ and if $\hbar = (1, 0)$, then $\overline{s}(\hbar) = 1$.

(2) The proof is obvious. □

Note 1. For any nPR-FSs $\hbar_i = (\varpi_{\hbar_i}, \omega_{\hbar_i})$, the comparison technique is supposed as follows:

(1) if $\overline{s}(\hbar_1) < \overline{s}(\hbar_2)$, then $\hbar_1 \prec \hbar_2$

(2) if $\overline{s}(\hbar_1) > \overline{s}(\hbar_2)$, then $\hbar_1 \succ \hbar_2$

(3) if $\overline{s}(\hbar_1) = \overline{s}(\hbar_2)$, then

(a) if $\overline{a}(\hbar_1) < \overline{a}(\hbar_2)$, then $\hbar_1 \prec \hbar_2$

(b) if $\overline{a}(\hbar_1) > \overline{a}(\hbar_2)$, then $\hbar_1 \succ \hbar_2$

(c) if $\overline{a}(\hbar_1) = \overline{a}(\hbar_2)$, then $\hbar_1 \approx \hbar_2$

In what follows, we will use an nPR-FWPA operator to MCDM issues in order to evaluate options with nPR-fuzzy data. The proposed method, in general, intertwines the following steps:

Step 1. We formulate the nPR-fuzzy decision matrix $R = (a_{ij})_{m_2 \times m_1}$ for an MCDM problem with values of nPR-FSs, where the elements $a_{ij}\,(j = 1, 2, \ldots, m_1, i = 1, 2, \ldots, m_2)$ are the appraisals of the alternative $L_i \in W$ regarding the criterion $K_j \in K$

Step 2. Convert the nPR-fuzzy decision matrix $R = (a_{ij})_{m_2 \times m_1}$ into the normalized nPR-fuzzy decision matrix

Step 3. To compute alternative preference values with related weights, we use the proposed nPR-FWPA operator

Step 4. Calculate the scores and accuracy of the nPR-FSs values obtained in Step 3

Step 5. By using Note 1, determine the best ranking order for the alternatives and identify the best option

Step 6. End

In order to exemplify the proposed method, we will show a realistic example of evaluating specific locations using nPR-fuzzy data.

Example 4. Every family on the planet fantasizes about having their own home. It is assumed that a family wishes to build their home at a specific location. They go to five different places: L_1, L_2, L_3, L_4, and L_5 and establish the following five criteria for selecting a house-building site:

Accessibility and location (K_1): make an effort to learn the location's address as well as any other pertinent information. Is it possible to locate the site using Google Maps? Is it simple to get to? You will have a leg up on the competition if you can find answers to questions like these.

Access to raw resources and utility services (K_2): any construction or building project must be carried out in an area with easy access to infrastructure and utilities in order to be successful. Water, electricity, shopping mall, a good waste disposal system, and healthcare, among other things, should all be available.

Shape and size (K_3): both of these aspects must be taken into account. Knowing the shape and size of your home will help you find a layout that is ideal for you. The site should be large enough to accommodate future expansion, and the shape should be even and free of sharp corners.

The nature of the neighborhood and security (K_4): in any residential area, the protection of lives and property is critical. As a result, this element should not be taken lightly. Before you start anything, conduct a thorough investigation of the security system in place at the location and its environs.

Table 1: nPR-fuzzy values.

Places	K_1	K_2	K_3	K_4	K_5
L_1	(0.29, 0.79)	(0.61, 0.35)	(0.55, 0.46)	(0.67, 0.27)	(0.63, 0.32)
L_2	(0.32, 0.80)	(0.65, 0.29)	(0.56, 0.45)	(0.69, 0.26)	(0.64, 0.31)
L_3	(0.37, 0.71)	(0.69, 0.25)	(0.58, 0.44)	(0.71, 0.24)	(0.65, 0.28)
L_4	(0.24, 0.85)	(0.43, 0.65)	(0.53, 0.51)	(0.61, 0.36)	(0.61, 0.34)
L_5	(0.35, 0.69)	(0.67, 0.26)	(0.57, 0.45)	(0.70, 0.25)	(0.65, 0.30)

Table 2: Aggregated nPR-fuzzy information matrix.

Operators	L_1	L_2	L_3	L_4	L_5
3PR-FWPA	(0.609, 0.364)	(0.630, 0.342)	(0.654, 0.311)	(0.542, 0.478)	(0.643, 0.322)
4PR-FWPA	(0.614, 0.362)	(0.635, 0.340)	(0.658, 0.310)	(0.549, 0.476)	(0.648, 0.321)
5PR-FWPA	(0.617, 0.361)	(0.639, 0.339)	(0.662, 0.309)	(0.556, 0.475)	(0.651, 0.320)
10PR-FWPA	(0.629, 0.360)	(0.650, 0.337)	(0.674, 0.307)	(0.574, 0.473)	(0.663, 0.318)
15PR-FWPA	(0.636, 0.359)	(0.657, 0.336)	(0.680, 0.306)	(0.584, 0.472)	(0.669, 0.317)
20PR-FWPA	(0.640, 0.359)	(0.661, 0.336)	(0.684, 0.306)	(0.590, 0.472)	(0.673, 0.317)

Table 3: Ranking using score value.

n	$\bar{s}(L_1)$	$\bar{s}(L_2)$	$\bar{s}(L_3)$	$\bar{s}(L_4)$	$\bar{s}(L_5)$	Ranking
3	0.488	−0.450	−0.398	−0.623	−0.420	$L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$
4	−0.634	−0.601	−0.559	−0.740	−0.576	$L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$
5	−0.726	−0.699	−0.664	−0.809	−0.679	$L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$
10	−0.893	−0.883	−0.869	−0.924	−0.875	$L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$
15	−0.933	−0.928	−0.921	−0.951	−0.924	$L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$
20	−0.950	−0.947	−0.942	−0.963	−0.944	$L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$

Table 4: Comparison analysis.

	SR-FWPA	FFWPA
L_1	(0.602, 0.367)	(0.609, 0.437)
$\bar{s}(L_1)$	−0.243	0.142
L_2	(0.624, 0.345)	(0.630, 0.428)
$\bar{s}(L_2)$	−0.198	0.172
L_3	(0.648, 0.315)	(0.654, 0.389)
$\bar{s}(L_3)$	−0.141	0.221
L_4	(0.532, 0.483)	(0.542, 0.549)
$\bar{s}(L_4)$	−0.412	−0.006
L_5	(0.637, 0.325)	(0.643, 0.390)
$\bar{s}(L_5)$	−0.164	0.207

Knowing the area crime rate allows you to make informed decisions and take preventative measures to safeguard yourself, your family, employees, and property.

The neighborhood's nature is also highly important. Are your neighbors pleasant? Is there a lot of toxins in the environment? Is there any kind of contamination at the location that could endanger people's health?

Recognize the soil type (K_5): on a given site, various varieties of soil can be found. As a result, you must pay close attention to the soil available on your site and assess whether it is suitable for construction.

It is assumed that $L = \{L_1, L_2, L_3, L_4, L_5\}$ is a set of alternatives (places), and $K = \{K_1, K_2, K_3, K_4, K_5\}$ is a set of criteria for the selection of places. Table 1 shows how to build the nPR-fuzzy set decision-making matrices, could a chance to be demonstrated that the degree to which the area L_i fulfills those criteria K_i is ϖ_{L_i} and the level with which the area L_i dissatisfies those criteria K_i is ω_{L_i} such that $0 \leq (\varpi_{L_i})^n + \omega_{L_i}^{1/n} \leq 1$ for $\varpi_{L_i}, \omega_{L_i} \in [0, 1]$. The weight vector of the criteria was established by the family as follows: $\eta = (0.09, 0.24, 0.17, 0.31, 0.19)^T$. They place a lower priority on K_1 and a higher priority on K_4.

Now, using weight vectors $\eta = (0.09, 0.24, 0.17, 0.31, 0.19)^T$ and $n = 3, 4, 5, 10, 15, 20$, we apply the nPR-FWPA operator as follows in Table 2.

Now, as shown in Table 3, we calculate the score value of each choice as well as their ranking.

We used different values of n to rank the options to explain the effect of the parameter n on MADM end findings. Table 3 shows the results of the ranking order of the alternatives based on the nPR-FWPA operator. When $n = 3, 4, 5, 10, 15, 20$, we obtained a rank of alternatives as $L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$, here, L_3 is the best choice.

5. Comparison Analysis

This section gives the comparison analysis of the proposed nPR-FWPA operator under nPR-fuzzy numbers with other well-known operators. We compared the results of nPR-FWPA operator with SR-FWPA [16] and FFWPA operators [23]. The following is a summary of the findings, which can be found in Table 4.

The most ideal ranking order of the five areas is $L_3 \succ L_5 \succ L_2 \succ L_1 \succ L_4$, if we utilize SR-fuzzy weighted power average (SR-FWPA) and Fermatean fuzzy weighted power average (FFWPA) operators for aggregating the distinctive options. Along these lines, the best option is L_3, which is same as that of the suggested operator. As a result, our proposed method is more adaptable than other methods already in use.

6. Conclusions

In this study, a set of operations via the class of nPR-fuzzy sets have been studied and their relationship have been illustrated with the assistance of suitable examples. Then, we have presented a new weighted aggregated operator over nPR-fuzzy sets and discussed their properties in details. In addition, with one fully practical example, we have demonstrated this procedure. Finally, the findings of the nPR-FWPA operator have been compared to the outcomes of other well-known operators.

On the one side, the proposed type of fuzzy sets enables us to evaluate the input data with different significance for grades of membership and nonmembership, which is appropriate for some real-life issues. In contrast, the different values estimated for the nonmembership and membership spaces require a comprehensive realization of the situations by the experts they are in charge of to evaluate the inputs of the case under study. This procedure is not required in the previous types of extensions of IFSs inspired by the same values of nonmembership and membership spaces.

In future works, it is possible that other uses of nPR-fuzzy sets will be investigated, for example, construct abstract structures like those given in [28]. Furthermore, over nPR-FSs, we will try to provide several different types of weighted aggregated operators and study novel MCDM methods depending on these operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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