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Research Article

Dual Toeplitz Operators on the Orthogonal Complement of the Fock–Sobolev Space

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In this paper, we consider the dual Toeplitz operators on the orthogonal complement of the Fock–Sobolev space and characterize their boundedness and compactness. It turns out that the dual Toeplitz operator S_f is bounded if and only if $f \in L^\infty$, and $\|S_f\| = \|f\|_\infty$. We also obtain that the dual Toeplitz operator with L^∞ symbol on orthogonal complement of the Fock–Sobolev space is compact if and only if the corresponding symbol is equal to zero almost everywhere.

1. Introduction

Let $\mathbb{C}$ denote the set of complex numbers and fix a positive integer n. Let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}. \tag{1}$$

denote the Euclidean space of dimension n. For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$\begin{aligned} z \cdot \overline{w} &= \langle z, w \rangle = z_1 \cdot \overline{w_1} + \cdots + z_n \cdot \overline{w_n}, \\ |z|^2 &= \langle z, z \rangle = |z_1|^2 + \cdots + |z_n|^2. \end{aligned} \tag{2}$$

Let dv be the Lebesgue measure on $\mathbb{C}^n$ and $dv_\alpha(z) = (\alpha/\pi)^n \cdot e^{-\alpha|z|^2} dv(z)$ be the Gaussian measure on $\mathbb{C}^n$, where $\alpha > 0$. Let $\mathbb{B}_n(w, s) = \{z \in \mathbb{C}^n: |z - w| < s\}$ be the ball with center w with radius s. Let $H(\mathbb{C}^n)$ be the set of the holomorphic functions on $\mathbb{C}^n$.

The Fock space F_α^2 consists of all entire functions f on $\mathbb{C}^n$ such that

$$f(z) \cdot e^{-\frac{\alpha}{2}|z|^2} \in L^2(\mathbb{C}^n, dv), \tag{3}$$

or equivalently, $F_\alpha^2 = L^2(\mathbb{C}^n, dv) \cap H(\mathbb{C}^n)$, with the norm

$$\|f\|_2 = \left[\int_{\mathbb{C}^n} |f(z)|^2 dv_\alpha(z)\right]^{1/2}. \tag{4}$$

Then, F_α^2 is a Hilbert space with the inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}^n} f(z)\overline{g(z)} dv_\alpha(z). \tag{5}$$

For any multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, write $|\gamma| = \gamma_1 + \cdots + \gamma_n, \gamma! = \gamma_1! \cdots \gamma_n!$, and $z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$. For $m \in \mathbb{N}$, the Fock–Sobolev space of order m is defined by

$$F_\alpha^{2,m} = \left\{f \in H(\mathbb{C}^n): D^\gamma f(z) \in F_\alpha^2 \text{ for} |\gamma| \leq m\right\}, \tag{6}$$

where $D^\gamma f(z) = \partial_{z_1}^{\gamma_1} \cdots \partial_{z_n}^{\gamma_n} f$.

The Fock–Sobolev space was introduced in [1], where they proved that $f \in F_\alpha^{2,m}$ if and only if the function $z^\gamma f(z)$ is in F_α^2 for all multi-indexes γ with $|\gamma| = m$. This shows that the Fock–Sobolev space can also be equivalently defined as

$$F_\alpha^{2,m} = \left\{f \in H(\mathbb{C}^n): z^\gamma f(z) \in F_\alpha^2 \text{ for} |\gamma| = m\right\}, \tag{7}$$

with the norm

$$\|f\|_{\alpha,2,m} = \left[c(\alpha, 2, m)\int_{\mathbb{C}^n} \left| |z|^m f(z) e^{-\frac{\alpha}{2}|z|^2}\right|^2 dv(z)\right]^{1/2}, \tag{8}$$

where $c(\alpha,2,m)=1/\sum_{|\gamma|=m} m!/\alpha^m(\pi/\alpha)^n$.

Moreover, $F_\alpha^{2,m}$ is a Hilbert space with the corresponding inner product

$$\begin{aligned}\langle f,g\rangle_{F_\alpha^{2,m}} &= \left(\frac{\alpha}{\pi}\right)^n \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n}|z|^{2m} f(z)\overline{g(z)}e^{-\alpha|z|^2}dv(z)\\ &= \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n}|z|^{2m}f(z)\overline{g(z)}dv_\alpha(z).\end{aligned} \tag{9}$$

Write $L^2(\mathbb{C}^n,|z|^{2m}dv_\alpha)$ as the L^2-integrable space under the measure $|z|^{2m}dv_\alpha$ on $\mathbb{C}^n$, with the norm

$$\|f\|_{L^2}=\left[\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n}|f(z)|^2|z|^{2m}dv_\alpha(z)\right]^{1/2}, \tag{10}$$

for $f\in L^2(\mathbb{C}^n,|z|^{2m}dv_\alpha)$. Obviously, $F_\alpha^{2,m}$ is a closed subspace of $L^2(\mathbb{C}^n,|z|^m dv_\alpha)$.

It is not difficult to check that

$$e_k(z)=\sqrt{\frac{\alpha^{|k|}(m+n-1)!(n+|k|-1)!}{k!(n-1)!(m+n+|k|-1)!}}\cdot z^k \tag{11}$$

forms an orthonormal basis for $F_\alpha^{2,m}$, where $k=(k_1,\ldots,k_n)$ runs over all multi-indexes of nonnegative integers (see [2, 3] for more details). Direct calculation reveals that the reproducing kernel of $F_\alpha^{2,m}$ is

$$\begin{aligned}K^\alpha(z,w) &= \sum_k e_k(z)\overline{e_k(w)}\\ &= \frac{(m+n-1)!}{(n-1)!}\sum_{k\in N^n}\frac{(n+|k|-1)!(\alpha z\overline{w})^k}{k!(m+n+|k|-1)!}\\ &= \frac{(m+n-1)!}{(n-1)!}\cdot\frac{d^{n-1}}{d\lambda^{n-1}}\left\{\frac{e^{\alpha z\overline{w}}-P_{n-1+m}(\alpha z\overline{w})}{(\alpha z\overline{w})^m}\right\}\\ &= \frac{(m+n-1)!}{(n-1)!}\mathscr{D}_m k_z^\alpha(w),\end{aligned} \tag{12}$$

where $P_{n-1+m}(\alpha z\overline{w})$ is the Taylor polynomial of $e^{\alpha z\overline{w}}$ of order $n-1+m$ and $k_z^\alpha(w)$ is the normalized reproducing kernel at z.

Let P_α be the orthogonal projection from $L^2(\mathbb{C}^n,|\cdot|^{2m}dv_\alpha)\longrightarrow F_\alpha^{2,m}$, that is,

$$\begin{aligned}P_\alpha f(z) &= \langle f,K_z^\alpha\rangle_{F_\alpha^{2,m}}=\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n}|z|^{2m}\\ &\quad f(z)\overline{K_z^\alpha(w)}dv_\alpha(z).\end{aligned} \tag{13}$$

For $f\in L^\infty$, the Toeplitz operator with symbol f from $F_\alpha^{2,m}\longrightarrow F_\alpha^{2,m}$ is defined by

$$T_f g=P_\alpha(fg). \tag{14}$$

The Hankel operator with symbol f from $F_\alpha^{2,m}\longrightarrow(F_\alpha^{2,m})^\perp$ is defined by

$$H_f g=Q_\alpha(fg)=(I-P_\alpha)(fg), \tag{15}$$

where Q_α is the orthogonal projection from $L^2(\mathbb{C}^n,dv_\alpha)\longrightarrow(F_\alpha^{2,m})^\perp$ and for arbitrary $u\in(F_\alpha^{2,m})^\perp$,

$$\begin{aligned}\left(H_{\overline{f}}^*\right)u(z) &= \langle H_{\overline{f}}^*u,K_z^\alpha\rangle_{F_\alpha^{2,m}}\\ &= \langle u,H_{\overline{f}}K_z^\alpha\rangle_{F_\alpha^{2,m}}\\ &= \left(\frac{\alpha}{\pi}\right)^n\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n}|w|^{2m}u(w)\overline{H_{\overline{f}}K_z^\alpha(w)}e^{-\alpha|w|^2}dv(w)\\ &= \left(\frac{\alpha}{\pi}\right)^n\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n}|w|^{2m}u(w)\overline{(I-P_\alpha)\left(\overline{f}K_z^\alpha\right)(w)}e^{-\alpha|w|^2}dv(w)\end{aligned}$$

$$= \left(\frac{\alpha}{\pi}\right)^n \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n} |w|^{2m} u(w)\overline{\left[\overline{f}K_z^\alpha - P_\alpha\left(\overline{f}K_z^\alpha\right)\right](w)}e^{-\alpha|w|^2}dv(w)$$

$$= \left(\frac{\alpha}{\pi}\right)^n \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n} |w|^{2m} u(w)\overline{\overline{f(w)}K_z^\alpha(w)}e^{-\alpha|w|^2}dv(w)$$

$$-\langle u, P_\alpha\left(\overline{f}K_z^\alpha\right)\rangle_{F_\alpha^{2,m}}$$

$$= \left(\frac{\alpha}{\pi}\right)^n \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n} |w|^{2m} u(w)\overline{\overline{f(w)}K_z^\alpha(w)}e^{-\alpha|w|^2}dv(w)$$

$$= \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n} |w|^{2m} u(w) f(w)\overline{K_z^\alpha(w)}dv_\alpha(w). \tag{16}$$

The dual Toeplitz operator with symbol f from $(F_\alpha^{2,m})^\perp \longrightarrow (F_\alpha^{2,m})^\perp$ is defined by

$$S_f u = Q_\alpha(fu) = (I - P_\alpha)(fu). \tag{17}$$

As we know, operator theory has developed rapidly since the beginning of the last century. It is closely related to function theory, topology, and other mathematical branches on function space. The application of operator theory is also gradually widespread, and it has been deeply applied into the field of other disciplines, such as quantum physics. The Toeplitz operators, dual Toeplitz operators, Hankel operators, and dual Hankel operators are widely studied classes of operators on function spaces, which have profound influence on operator theory, operator algebra, and complex analysis. For example, see [4–11].

In recent decades, the dual Toeplitz operators on the orthogonal complement of classical analytic function spaces have received much attention and have been well studied. On the setting of the orthogonal complement of the Bergman space, Stroethoff and Zheng [12] first characterized (semi-)commuting dual Toeplitz operators on the unit disk and their results were extended to the unit ball or unit polydisk as in [7, 13–16] and reference therein. Later, the corresponding problems have been studied on the Dirichlet spaces and Hardy–Sobolev spaces of the unit disk or unit ball as in [17–21].

Motivated by these results, we in this paper focus on the dual Toeplitz operators on the orthogonal complement of the Fock–Sobolev space and characterize their boundedness and compactness. We obtain that the dual Toeplitz operator S_f is bounded if and only if $f \in L^\infty$ with $\|S_f\| = \|f\|_\infty$, and S_f with L^∞ symbol is compact if and only if the corresponding symbol is equal to zero almost everywhere. The results of the dual Toeplitz operator on Fock space of the complex plane are extended and generalized.

2. The Proof of Main Result

For $f, g \in L^\infty(\mathbb{C}^n)$, it is well known that

$$\begin{aligned} T_{fg} &= T_f T_g + H_{\overline{f}}^* H_g, \\ H_{fg} &= H_f T_g + S_f H_g, \\ S_{fg} &= S_f S_g + H_f H_{\overline{g}}^*. \end{aligned} \tag{18}$$

For $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $0 < s < 1$, we define

$$g_{w,s}(z) = \overline{(z_1 - w_1)}e^{\alpha|z|^2}\chi_{\mathbb{B}_n}(w,s)(z)|z|^{-m}, \tag{19}$$

where $\chi_{\mathbb{B}_n}(w,s)(z)$ is the characteristic function on $\mathbb{B}_n(w,s)$.

To get the main proofs, we still need to prove some lemmas at first.

Lemma 1. *For $w \in \mathbb{C}^n$, $g_{w,s}(z) \in (F_\alpha^{2,m})^\perp$.*

Proof. Suppose $z \in \mathbb{C}^n$, and j is a nonnegative multi-index; then,

$$\begin{aligned} \langle z^j, g_{w,s}\rangle_{F_\alpha^{2,m}} &= \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n} |z|^{2m} z^j \overline{g_{w,s}(z)}dv_\alpha(z) \\ &= \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{B}_n(w,s)} |z|^{2m} z^j \\ &[middot](z_1 - w_1)e^{\alpha|z|^2}|z|^{-m}dv_\alpha(z) \\ &= \frac{\alpha^{m+2n-1}}{\pi^n(m+n-1)!}\int_{\mathbb{B}_n(w,s)} z^j(z_1 - w_1)|z|^m dv(z) \\ &= \frac{\alpha^{m+2n-1}}{\pi^n(m+n-1)!}\int_{|z|<s} (z+w)^j z_1 |z+w|^m dv(z) \\ &= 0. \end{aligned} \tag{20}$$

This indicates $z^j \perp g_{w,s}$, which implies that $g_{w,s} \in (F_\alpha^{2,m})^\perp$. ☐

Lemma 2. *For arbitrary $m \in \mathbb{N}$ and $z \in \mathbb{C}^n$, we have*

$$\|K_z^\alpha\|_{\alpha,2,m}^2 \approx \frac{e^{\alpha|z|^2}}{1 + \alpha^m|z|^{2m}}. \tag{21}$$

Proof. The following results can be obtained by using similar methods in Lemma 2.6 in [22].

We suppose that $\lambda = \alpha|z|^2 \geq 0$.Put

$$P_m(\lambda) = \sum_{j=n-1+m}^{\infty} \frac{\lambda^{j-m}}{j!} = \frac{e^{\lambda} - Q_{n-1+m}(\lambda)}{\lambda^m},$$
$$P_m^{(n-1)}(\lambda) = \frac{d^{n-1}}{d\lambda^{n-1}} \left\{ \frac{e^{\lambda} - Q_{n-1+m}(\lambda)}{\lambda^m} \right\}, \tag{22}$$

where $Q_0 = 0$ and Q_j is the Taylor polynomial of e^{λ} of order $j-1$ for $j \geq 1$. Then, $P_m^{(n-1)}(\lambda)$ is a power series with positive coefficient. So, it is monotonically increasing, that is, for each $\lambda \geq 0$, we have

$$P_m^{(n-1)}(\lambda) \geq P_m^{(n-1)}(0) = \frac{(n-1)!}{(n-1+m)!}. \tag{23}$$

When $0 \leq \lambda \leq M$, we have

$$\left| P_m^{(n-1)}(\lambda) \right| \approx \frac{e^{\lambda}}{1+\lambda^m} \tag{24}$$

When $\lambda > 0$,we have

$$P_m(\lambda) = \frac{e^{\lambda}}{\lambda^m} - \sum_{l=0}^{n+m-2} \frac{\lambda^{l-m}}{l!}, \tag{25}$$

So,

$$P_m^{(n-1)}(\lambda) = \frac{e^{\lambda}}{\lambda^m} + \sum_{j=1}^{n-1} \frac{e^{\lambda}}{\lambda^m} \cdot \frac{c_j}{\lambda_j} + \sum_{j=n}^{n+m-1} \frac{c_j}{\lambda_j}, \tag{26}$$

where c_j represents the real coefficient.

Thus,

$$\left| P_m^{(n-1)}(\lambda) \right| \approx \frac{e^{\lambda}}{1+\lambda^m} \tag{27}$$

as $\lambda > 0$ and λ is big enough.

According to the above analysis, for arbitrary $\lambda \geq 0$, $|P_m^{n-1}(\lambda)| \approx e^{\lambda}/1+\lambda^m$,that is,

$$\|K_m(z,.)\|_{\alpha,2,m}^2 = K_m(z,z) = \frac{(m+n-1)!}{(n-1)!} P_m^{(n-1)}\left(\alpha |z|^2\right) \approx \frac{e^{\alpha|z|^2}}{1+\alpha^m |z|^{2m}}. \tag{28}$$

□

For arbitrary $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n, 0 < s < 1$, we define

$$\phi_{w,s}(z) = \frac{g_{w,s}(z)}{\|g_{w,s}\|_{L^2}}, \tag{29}$$

where $\|g_{w,s}\|_{L^2}$ is the norm on $L^2(\mathbb{C}^n, |\cdot|^{2m} dv_{\alpha})$.

Lemma 3. *If $f \in L^2(\mathbb{C}^n, |\cdot|^{2m} dv_{\alpha}), w \in \mathbb{C}^n$, then*

$$\lim_{s \longrightarrow 0} \left\| H_{\bar{f}}^{*} \phi_{w,s} \right\|_{\alpha,2,m} = 0. \tag{30}$$

Proof. Since

$$\begin{aligned}
\left| H_{\bar{f}}^{*} \phi_{w,s}(z) \right| &= \frac{\alpha^{m+n-1}}{(m+n-1)!} \int_{\mathbb{C}^n} |w|^{2m} \phi_{w,s}(w) f(w) \overline{K_z^{\alpha}(w)} dv_{\alpha}(w) \\
&\leq \frac{\alpha^{m+n-1}}{(m+n-1)!} \int_{B_n(w,s)} \left| |w|^{2m} \phi_{w,s}(w) f(w) \overline{K_z^{\alpha}(w)} \right| dv_{\alpha}(w) \\
&\leq \|\phi_{w,s}\|_{L^2} \sqrt{\frac{\alpha^{m+n-1}}{(m+n-1)!}} \left(\int_{B_{n(w,s)}} |w|^{2m} |f(w)|^2 \left| K_z^{\alpha}(w) \right|^2 dv_{\alpha}(w) \right)^{1/2} \\
&= \sqrt{\frac{\alpha^{m+n-1}}{(m+n-1)!}} \left(\int_{B_{n(w,s)}} |w|^{2m} |f(w)|^2 \left| K_z^{\alpha}(w) \right|^2 dv_{\alpha}(w) \right)^{1/2},
\end{aligned} \tag{31}$$

we have

$$
\begin{aligned}
\left\|H_{\bar{f}}^{*}\phi_{w,s}\right\|_{\alpha,2,m}^{2} &= \frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{C}^n}\left|H_{\bar{f}}^{*}\phi_{w,s}(z)\right|^{2}|z|^{2m}dv_{\alpha}(z)\\
&\leq \frac{\alpha^{2(m+n-1)}}{[(m+n-1)!]^{2}}\int_{\mathbb{C}^n}\int_{B_{n(w,s)}}|w|^{2m}|f(w)|^{2}\left|K_{z}^{\alpha}(w)\right|^{2}dv_{\alpha}(w)|z|^{2m}dv_{\alpha}(z)\\
&= \frac{\alpha^{2(m+n-1)}}{[(m+n-1)!]^{2}}\int_{B_{n(w,s)}}|w|^{2m}|f(w)|^{2}dv_{\alpha}(w)\int_{\mathbb{C}^n}\left|K_{z}^{\alpha}(w)\right|^{2}|z|^{2m}dv_{\alpha}(z)\\
&= \frac{\alpha^{m+n-1}}{(m+n-1)!}\|K_{w}^{\alpha}\|_{\alpha,2,m}^{2}\int_{B_{n(w,s)}}|w|^{2m}|f(w)|^{2}dv_{\alpha}(w)\\
&\leq \frac{\alpha^{m+n-1}}{(m+n-1)!}\frac{e^{\alpha|z|^{2}}}{1+\alpha^{m}|z|^{2m}}\int_{B_{n(w,s)}}|w|^{2m}|f(w)|^{2}dv_{\alpha}(w),
\end{aligned}
\tag{32}
$$

where the last inequality follows from Lemma 2.

Notice that

$$
\lim_{s\longrightarrow 0}\int_{\mathbb{B}_n(w,s)}|w|^{2m}|f(w)|^{2}dv_{\alpha}(w)=0. \tag{33}
$$

When $\mathbb{B}_n(w,s)$ is small enough by the absolute continuity of the integral, we get the desired result. □

Lemma 4. *If* $f\in L^{2}(\mathbb{C}^n,|\cdot|^{2m}dv_{\alpha}), w\in\mathbb{C}^n$, *then*

$$
|f(w)|=\lim_{s\longrightarrow 0}\left\|S_{f}\phi_{w,s}\right\|_{L^2}. \tag{34}
$$

Proof. For the multiplication operator M_f, it is obvious that

$$
M_f\left(\phi_{w,s}\right)=S_f\left(\phi_{w,s}\right)+H_{\bar{f}}^{*}\left(\phi_{w,s}\right), \tag{35}
$$

where $S_f(\phi_{w,s})\perp H_{\bar{f}}^{*}(\phi_{w,s})$.Then,

$$
\left\|M_f\phi_{w,s}\right\|_{L^2}^{2}=\left\|S_f\phi_{w,s}\right\|_{L^2}^{2}+\left\|H_{\bar{f}}^{*}\left(\phi_{w,s}\right)\right\|_{\alpha,2,m}^{2}, \tag{36}
$$

and Lemma 3 gives that

$$
\lim_{s\longrightarrow 0}\left\|M_f\phi_{w,s}\right\|_{L^2}^{2}=\lim_{s\longrightarrow 0}\left\|S_f\phi_{w,s}\right\|_{L^2}^{2}. \tag{37}
$$

Notice that

$$
\|M_f\phi_{w,s}\|_{L^2}^{2}=\frac{\int_{|z-w|<s}|f(z)|^{2}\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}{\int_{|z-w|<s}\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}, \tag{38}
$$

and we just need to show

$$
\lim_{s\longrightarrow 0}\frac{\int_{|z-w|<s}|f(z)|^{2}\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}{\int_{|z-w|<s}\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}=|f(w)|^{2}. \tag{39}
$$

Direct calculation indicates

$$
\begin{aligned}
&\left|\frac{\int_{|z-w|<s}|f(z)|^{2}\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}{\int_{|z-w|<s}\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}-|f(w)|^{2}\right|\\
&=\left|\frac{\int_{|z-w|<s}\left(|f(z)|^{2}-|f(w)|^{2}\right)\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}{\int_{|z-w|<s}\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)}\right|.
\end{aligned}
\tag{40}
$$

For $0<s<1$, assume $L=\inf\left\{e^{-\alpha|z|^{2}}\mid z\in\mathbb{B}_n(w,s)\right\}$. Clearly, $L>0$. Set $M=1/L$. Then, $e^{\alpha|z|^{2}}\in[1,M]$ for arbitrary $z\in\mathbb{B}_n(w,s)$, which implies that

$$
\begin{aligned}
&\int_{|z-w|<s}\left|z_1-w_1\right|e^{2\alpha|z|^{2}}dv_{\alpha}(z)\\
&=\frac{\alpha^{n}}{\pi^{n}}\int_{|z-w|<s}\left|z_1-w_1\right|e^{\alpha|z|^{2}}dv(z)\\
&\geq\frac{\alpha^{n}}{\pi^{n}}\int_{|z-w|<s}\left|z_1-w_1\right|dv(z)\\
&=\frac{\alpha^{n}}{\pi^{n}}s^{2n+2}\int_{|z|<1}|z|^{2}dv(z)=\frac{\alpha^{n}s^{2n+2}}{(n+1)!}\\
&=\frac{\alpha^{n}s^{2}s^{2n}}{(n+1)n!}=\frac{\alpha^{n}s^{2}}{\pi^{n}(n+1)}V\left(\mathbb{B}_n(w,s)\right),
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
&\int_{|z-w|<s}\left(\left|f(z)^{2}\right|-|f(w)|^{2}\right)\left|z_1-w_1\right|^{2}e^{2\alpha|z|^{2}}dv_{\alpha}(z)\\
&\leq\frac{\alpha^{n}}{\pi^{n}}\int_{|z-w|<s}\left||f(z)|^{2}-|f(w)|^{2}\right|\left|z_1-w_1\right|^{2}e^{\alpha|z|^{2}}dv(z)\\
&\leq\frac{\alpha^{n}Ms^{2}}{\pi^{n}}\int_{|z-w|<s}\left||f(z)|^{2}-|f(w)|^{2}\right|dv(z).
\end{aligned}
\tag{42}
$$

By combining the inequalities (41) and (42), we have

$$\left|\frac{\int_{|z-w|<s}|f(z)|^2|z_1-w_1|^2e^{2\alpha|z|^2}dv_\alpha(z)}{\int_{|z-w|<s}|z_1-w_1|^2e^{2\alpha|z|^2}dv_\alpha(z)}-|f(w)|^2\right| \le \frac{(n+1)M}{V(\mathbb{B}_n(w,s))}\int_{\mathbb{B}_n(w,s)}\left||f(z)|^2-|f(w)|^2\right|dv(z) \longrightarrow 0, \tag{43}$$

as $s \longrightarrow 0$ by using Theorem 8.8 in [23]. This ends the proof. □

Theorem 1. *Suppose $f \in L^2(\mathbb{C}^n, |\cdot|^{2m}dv_\alpha)$; then, dual Toeplitz operator S_f is bounded if and only if $f \in L^\infty$, and $\|S_f\| = \|f\|_\infty$.*

Proof. If $f \in L^\infty$, then for arbitrary $u \in (F_\alpha^{2,m})^\perp$, we have

$$\left\|S_f(u)\right\|_{L^2} = \|(I-P_\alpha)uf\|_{L^2} \le \left\|uf\right\|_{L^2} \le \|f\|_\infty\left\|u\right\|_{L^2}. \tag{44}$$

This implies that S_f is bounded and

$$\left\|S_f\right\| \le \left\|f\right\|_\infty. \tag{45}$$

Conversely, if S_f is bounded, then (45) holds and

$$\left\|S_f\phi_{w,s}\right\|_{L^2} \le \left\|S_f\right\|. \tag{46}$$

Lemma 4 shows that $|f(w)| = \lim_{s\longrightarrow 0}\|S_f\phi_{w,s}\|_{L^2}$ and then

$$|f(w)| \le \left\|S_f\right\|. \tag{47}$$

Taking the upper bound of the above equation indicates

$$\|f\|_\infty \le \left\|S_f\right\|. \tag{48}$$

By combining (45) and (48), we get

$$\left\|S_f\right\| = \left\|f\right\|_\infty. \tag{49}$$

□

Lemma 5. *For $w \in \mathbb{C}^n$, $\phi_{w,s}$ is weakly convergent to 0 on $(F_\alpha^{2,m})^\perp$ as $s \longrightarrow 0$.*

Proof. Suppose $w \in \mathbb{C}^n$; for arbitrary $f \in (F_\alpha^{2,m})^\perp$, Hölder inequality implies that

$$\begin{aligned}
\left|\langle\phi_{w,s}, f\rangle_{L^2}\right| &= \frac{\alpha^{m+n-1}}{(m+n-1)!}\left|\int_{\mathbb{C}^n}|z|^{2m}\phi_{w,s}(z)\overline{f(z)}dv_\alpha(z)\right| \\
&= \frac{\alpha^{m+n-1}}{(m+n-1)!}\left|\int_{\mathbb{B}_n(w,s)}|z|^{2m}\phi_{w,s}(z)\overline{f(z)}dv_\alpha(z)\right| \\
&\le \left[\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{B}_n(w,s)}|z|^{2m}\left|\phi_{w,s}(z)\right|^2dv_\alpha(z)\right]^{1/2} \\
&\quad\cdot\left[\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{B}_n(w,s)}|z|^{2m}|f(z)|^2dv_\alpha(z)\right]^{1/2} \\
&= \left\|\phi_{w,s}\right\|_{L^2}\cdot\left[\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{B}_n(w,s)}|z|^{2m}|f(z)|^2dv_\alpha(z)\right]^{1/2} \\
&= \left[\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{B}_n(w,s)}|z|^{2m}|f(z)|^2dv_\alpha(z)\right]^{1/2}.
\end{aligned} \tag{50}$$

Note that $f \in L^2(\mathbb{C}^n, |\cdot|^{2m}dv_\alpha)$, and we have

$$\left[\frac{\alpha^{m+n-1}}{(m+n-1)!}\int_{\mathbb{B}_n(w,s)}|z|^{2m}|f(z)|^2dv_\alpha(z)\right]^{1/2} \longrightarrow 0 \tag{51}$$

as $s \longrightarrow 0$. This shows that $|\langle\phi_{w,s}, f\rangle_{L^2}| \longrightarrow 0$ for arbitrary $f \in (F_\alpha^{2,m})^\perp$, and then $\phi_{w,s}$ is weakly convergent to 0 on $(F_\alpha^{2,m})^\perp$. □

Theorem 2. *If $f \in L^\infty$, then S_f is compact if and only if $\|f\|_\infty = 0$.*

Proof. If $\|f\|_\infty = 0$, then Theorem 1 gives that S_f is bounded and $\|S_f\| = \|f\|_\infty = 0$. Thus, $S_f = 0$ must be compact.

Conversely, if S_f is compact, then

$$f(w) = \lim_{s\longrightarrow 0}\left\|S_f\phi_{w,s}\right\|_2 = 0 \tag{52}$$

by Lemma 4. Thus, $\|f\|_\infty = 0$. This completes the proof. □

Data Availability

No data were used to support this study.

Disclosure

Li He and Biqian Wu are co-first authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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