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Research Article

Multiplicative Generalized Reverse *CE-Derivations Acting on Rings with Involution

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Let S be a ring with involution having a nontrivial symmetric idempotent element e. If Ω is any appropriate multiplicative generalized reverse *CE-derivation of S with involution $*$, then under some suitable restrictions on S, Ω is centrally-extended additive.

1. Introduction

In [1], Bell and Daif introduced the notion of centrally-extended derivations as follows. Let S be a ring with center $Z(S)$, a map δ of S is called a centrally-extended derivation (CE-derivation) if for each $r, s \in S, \delta(r+s) - \delta(r) - \delta(s) \in Z(S)$ and $\delta(rs) - \delta(r)s - r\delta(s) \in Z(S)$. They discussed the existence of such map which is not a derivation and gave some commutativity results. In [2], the authors generalized this notion to other kinds of maps and extended some results due to Bell and Daif. Recently, in [3], the authors gave the notion of Jordan CE-derivations and, under some conditions, they proved that every Jordan CE-derivation of a prime ring S is a CE-derivation.

Martindale [4] has asked the following question: When is a multiplicative mapping additive? He answered his question for a multiplicative isomorphism of a ring S. In [5], Daif has given an answer to that question when the mapping is a multiplicative derivation on S. Also, in [6–8], a generalization of this question can be found for the case of multiplicative generalized derivations, multiplicative generalized reverse $*-$ derivations, and multiplicative left centralizers.

In this article, we generalized the idea of Martindale [4] and Daif [5] for the notion of the multiplicative generalized reverse *CE-derivation.

2. Preliminaries

In this note, we introduce the notion of the multiplicative generalized reverse *CE-derivation of a ring S with involution $*$ to be a mapping ω of S into S such that $\omega(rs) - \omega(s)r^* - s^*\delta(r) \in Z(S)$, for all $r, s \in S$, where δ is a reverse *CE-derivation from S into S; i.e., for all $r, s \in S$, $\delta(r+s) - \delta(r) - \delta(s) \in Z(S)$ and $\delta(rs) - \delta(s)r^* - s^*\delta(r) \in Z(S)$. In other words, we can write the maps ω and δ by $\omega(rs) = \omega(s)r^* + s^*\delta(r) + \phi(r,s)$ and $\delta(rs) = \delta(s)r^* + s^*\delta(r) + \psi(r,s)$, where $\phi(r,s)$ and $\psi(r,s)$ are central elements depend on the choice of r and s and related to the mappings ω and δ, respectively.

Here, we ask the following question: When is a multiplicative generalized reverse *CE-derivation a *CE-additive? Under suitable conditions, we give an answer for this question.

As in [9], let $e \in S$ be a nontrivial symmetric idempotent element so that $e \neq 1, e \neq 0$, and $e^* = e$ (S need not have an identity). We will formally set $e_1 = e$ and $e_2 = 1 - e$. The two-sided Peirce decomposition of S relative to the idempotents e_1 and e_2 takes the form $S = e_1Se_1 \oplus e_1Se_2 \oplus e_2Se_1 \oplus e_2Se_2$. So letting $S_{ij} = e_iSe_j$: $i, j \in \{1, 2\}$, we may write $S = S_{11} \oplus S_{12} \oplus S_{21} \oplus S_{22}$. An element of the subring S_{ij} will be denoted by s_{ij}. If $\mu = \mu_{11} + \mu_{12} + \mu_{21} + \mu_{22} \in Z(S)$, since $e_1\mu = \mu e_1$, then

$\mu_{12} = \mu_{21} = 0$, so we conclude that $Z(S) \subset S_{11} \oplus S_{22}$. Also, we formally use the symbol Z_{ii} for referring to the subring $S_{ii} \cap Z(S)$.

By the definition of δ, we note that $\delta(0) = \psi(0,0) \in Z(S)$. However, since $*$ is bijection and, for all $x \in S$, $\delta(0) = \delta(0x^*) = x^{**}\delta(0) + z_1 = x\delta(0) + z_1 \in Z(S)$. So that $S\delta(0) = \delta(0)S = \delta(0)S^* = S^*\delta(0) \subset Z(S)$. Then, we have $\delta(0)S^* = \delta(0)S = \{\delta(0)s^*\colon s^* \in S^*\}$, which is a two sided central ideal in S. Since $e\delta(0) \in Z(S)$, if $\psi(0,0) = \psi_{11}(0,0) + \psi_{22}(0,0)$, then $e\delta(0) = \psi_{11}(0,0) \in Z_{11}$ and this gives also $\psi_{22}(0,0) \in Z_{22}$. Similarly, $\omega(0)S^*$ is a two sided central ideal in S and $\phi_{11}(0,0) \in Z_{11}$ and $\phi_{22}(0,0) \in Z_{22}$.

Moreover,
$\delta(e) = \delta(e^2) = \delta(e)e^* + e^*\delta(e) + \pi$, for $\pi = \psi(e,e) \in Z(S)$. If we express $\delta(e) = a_{11} + a_{12} + a_{21} + a_{22}$ and use the two expressions of $\delta(e)$, we get $a_{22} = \pi_{22}$ and $a_{11} = -\pi_{11}$. Consequently, we have the following equation:

$$\delta(e) = a_{12} + a_{21} - \pi_{11} + \pi_{22}. \tag{1}$$

By the same manner, if $\omega\colon S \longrightarrow S$ is a multiplicative generalized reverse *CE-derivation associated with a reverse *CE-derivation δ,then $\omega(e) = \omega(e^2) = \omega(e)e^* + e^*\,\delta(e) + \xi$, where $\xi = \phi(e,e) \in Z(S)$ and we can write $\omega(e) = b_{11} + b_{12} + b_{21} + b_{22}$, and using the values of $\omega(e)$ and $\delta(e)$. we conclude that $\xi_{11} = \pi_{11}$, $b_{22} = \xi_{22}$ and $b_{12} = a_{12}$ so,

$$\omega(e) = b_{11} + a_{12} + b_{21} + \xi_{22}. \tag{2}$$

In our work we will need the following facts.

Proposition 1 [7]. *Let* $s \in S$ $(s_{ij} \in S_{ij}$ *where* $i, j \in \{1,2\})$. *Then,* $s^*_{ij} = r_{ji}$, *where* $r = s^* \in S$. *Moreover,* $s_{ij} = r^*_{ji}$.

Lemma 2. $\xi_{ii} \in Z_{ii}$ *and* $\pi_{ii} \in Z_{ii}$, *where* $i \in \{1,2\}$.

Proof. For any element $s \in S$, by expanding both sides of $\omega(es) = \omega(e(es))$, we get the following equation:

$$s^*\delta(e) + \phi(e,s) = s^*\delta(e)e + s^*e\delta(e) + \phi(e,s)e + \phi(e,es). \tag{3}$$

Now using equation (1) in equation (3), we get $\pi s^* = \phi(e,s)e + \phi(e,es) - \phi(e,s)$ and since $\phi(e,s), \phi(e,es) \in Z(S) \subset S_{11} \oplus S_{22}$, this means

$$\pi s^* \in S_{11} \oplus S_{22}. \tag{4}$$

Now, Since $*$ is bijection, there exist $r \in S$ such that $r = s^*$, so we can rewrite $s^* = r = r_{11} + r_{12} + r_{21} + r_{22}$ and using that $\pi \in Z(S)$, we get $\pi_{11}r_{11} = r_{11}\pi_{11}$ and $\pi_{22}r_{22} = r_{22}\pi_{22}$ which implies $\pi_{11} \in Z(S_{11})$ and $\pi_{22} \in Z(S_{22})$. And again equation (4) gives $\pi_{11}r_{12} + \pi_{22}r_{21} = 0$ and $r_{12}\pi_{22} + r_{21}\pi_{11} = 0$, which gives $\pi_{11}r_{12} = \pi_{22}r_{21} = 0$ and $r_{12}\pi_{22} = r_{21}\pi_{11} = 0$, this means that π is a left and right annihilator of the two subrings S_{12} and S_{21}. Now, for any $r \in S$, $\pi_{11}r = \pi_{11}r_{11} = r_{11}\pi_{11} = r\pi_{11}$, which gives $\pi_{11} \in Z(S)$. Since $\pi_{22} = \pi - \pi_{11}$, then $\pi_{22} \in Z(S)$. Also, since $\xi_{11} = \pi_{11}$, we get $\xi_{11} \in Z(S)$ and $\xi_{22} = (\xi - \xi_{11}) \in Z(S)$.

To achieve our main result, we assume that the ring S endowed with an involution $*$ contains a nontrivial symmetric idempotent e and satisfies the following conditions:

(C_1) $tSe \subset Z(S)$ implies $t \in Z(S)$. And $tS \subset Z(S)$ implies $t \in Z(S)$.

(C_2) $teS(1-e) \subset Z(S)$ implies $t \in Z(S)$.

And ω is any multiplicative generalized reverse *CE-derivation of S associated with a reverse *CE-derivation δ of S.

The following lemma is fruitful in our proofs: □

Lemma 3. *The ideals* $S^*\xi$, $S^*\xi_{ii}$, $S^*\pi$, $S^*\pi_{ii}$, *and* $S^*\overline{\pi}$ *are central ideals in* S, *where* $\psi = \phi(e,e) \in Z(S)$, $\pi = \psi(e,e) \in Z(S)$, $\overline{\pi} = \pi_{22} - \pi_{11} \in Z(S)$, *and* $i \in \{1,2\}$.

Proof. First, using Lemma 2, for any $s^*_{11} \in S_{11}$, we get $\xi s^*_{11}s_{12} = s^*_{11}\xi s_{12} = 0 \in Z(S)$ and using condition (C_2), we get $s^*_{11}\xi = \xi s^*_{11} \in Z(S)$. Secondly, assume that $\delta(s_{22}) = c_{11} + c_{12} + c_{21} + c_{22}$ and since $\omega(s_{22}e) = \omega(0) \in Z(S)$, so using equation (2), we have $\omega(0) = \omega(e)s^*_{22} + e\delta(s_{22}) + \phi(s_{22},e) = a_{12}s^*_{22} + \xi_{22}s^*_{22} + c_{11} + c_{12} + \phi(s_{22},e)$ and this gives $a_{12}s^*_{22} + c_{12} = 0$ and $\xi_{22}s^*_{22} = \beta - c_{11}$, where $\beta = (\omega(0) - \phi(s_{22},e)) \in Z(S)$. Now, using Lemma 2, for any $r \in S$, we get $\xi_{22}s^*_{22}r = s^*_{22}\xi_{22}r_{22} = \xi_{22}s^*_{22}r_{22} = (\beta - c_{11})r_{22} = \beta r_{22} = r_{22}\beta = r_{22}(\beta - c_{11}) = r_{22}\xi_{22}s^*_{22} = r\xi_{22}s^*_{22} = rs^*_{22}\xi_{22}$ and this gives $s^*_{22}\xi_{22} \in Z(S)$. Also, if $s, r \in S$, then $rs^*\xi = r(s^*_{11}\xi + s^*_{22}\xi) = rs^*_{11}\xi + rs^*_{22}\xi = s^*_{11}\xi r + s^*_{22}\xi r = (s^*_{11} + s^*_{22})\xi r = s^*\xi r$. By a similar method one can prove the other cases. □

Remark 4. An example of a reverse *CE-derivation, if a is any fixed element in S, the map $\delta_a\colon S \longrightarrow S$ which satisfies $\delta_a(r) - [r^*, a] \in K$ where K is a central ideal, we can call it an inner reverse *CE-derivation. Now, using Lemma 3 we can show that the map δ_1 given by $\delta_1(s) = [s^*, a_{12} - a_{21}] + \overline{\pi}$ is a reverse *CE-derivation and with equation (1), we get the following equation:

$$\delta_1(e) = a_{12} + a_{21} + \overline{\pi} = \delta(e). \tag{5}$$

Remark 5. An example of a generalized reverse *CE-derivation is if a and b are any two fixed elements in S, the map $\omega_{(a,b)}\colon S \longrightarrow S$ which satisfies $\omega_{(a,b)}(r) - ar^* - r^*b \in L$, where L is a central ideal, we can call it an inner generalized reverse *CE-derivation associated with the inner reverse *CE-derivation δ_b which is given by $\delta_b - [s^*, b] \in L$.

Again, using Lemma 3, we can show that the map ω_1 given by $\omega_1(s) = (b_{11} + b_{21} - \xi_{11})s^* + s^*(a_{12} - a_{21}) + \xi$ is a generalized reverse *CE-derivation associated with the inner reverse *CE-derivation δ_1 and with equation (2) we get the following equation:

$$\omega_1(e) = b_{11} + b_{21} + a_{12} + \xi_{22} = \omega(e). \tag{6}$$

Remark 6. For simplification, we will replace, without loss of generality, the reverse *CE-derivation δ by the reverse *CE-derivation $\Delta = \delta - \delta_1$ which by using equation (5) bring us to $\Delta(e) = 0$ and the multiplicative generalized reverse *CE-derivation ω by the multiplicative generalized reverse *CE-derivation $\Omega = \omega - \omega_1$ with $\Omega(e) = 0$ by equation (6). Also, $\Delta(0) = \delta(0) - \delta_1(0) = \delta(0) - \overline{\pi} = \theta \in Z(S)$ and $\Omega(0) = \omega(0) - \omega_1(0) = \omega(0) - \xi = \alpha \in Z(S)$. One can easily show that both of θ and α generates a two sided central ideal in S.

To prove our main theorem we need the following lemmas.

Lemma 7. *For any element $s_{ij} \in S_{ij}$, there exists $r_{ji} \in S_{ji}$ and $\rho_{ii}, \sigma_{ii} \in Z_{ii}$; $i, j \in \{1,2\}$ such that*

(1) $\Delta(s_{ii}) = r_{ii} + \rho_{jj}, \quad i \neq j$,

(2) $\Delta(s_{ij}) = r_{ji} + \rho_{ii} + \sigma_{jj}, \quad i \neq j$.

Proof. For (1), we have to prove two separable cases:

(a) Let s_{11} be an arbitrary element of S_{11} and let $\Delta(s_{11}) = r_{11} + r_{12} + r_{21} + r_{22}$. Then, $\Delta(s_{11}) = \Delta(es_{11}) = \Delta(s_{11})e + s_{11}^*\Delta(e) + \rho$; $\rho \in Z(S)$, which gives $r_{12} = 0, \rho_{11} = 0$ and $r_{22} = \rho_{22} \in Z_{22}$, so we get $\Delta(s_{11}) = r_{11} + r_{21} + \rho_{22}$. Similarly, $\Delta(s_{11}) = \Delta(s_{11}e) = \Delta(e)s_{11}^* + e\Delta(s_{11}) + \gamma$; $\gamma \in Z(S)$ which means $r_{21} = 0$ and we get $\Delta(s_{11}) = r_{11} + \rho_{22}$.

(b) Assume that $s_{22} \in S_{22}$, write $\Delta(s_{22}) = r_{11} + r_{12} + r_{21} + r_{22}$ so $\theta = \Delta(es_{22}) = \Delta(s_{22})e + s_{22}^*\Delta(e) + \gamma_1 = r_{11} + r_{21} + \gamma_1$; $\gamma_1 \in Z(S)$, so $r_{11} + r_{21} = \theta - \gamma_1 \in Z(S)$ which means $r_{21} = 0$ and $r_{11} \in Z_{11}$. Likewise, $\theta = \Delta(s_{22}e) = r_{11} + r_{12} + \gamma_2$; $\gamma_2 \in Z(S)$, so $r_{11} + r_{12} = \theta - \gamma_2 \in Z(S)$, so that $r_{12} = 0$ and thus $\Delta(s_{22}) = r_{11} + r_{22}$, where $r_{11} \in Z_{11}$.

Also, For (2) , we have to prove two separable cases:

(a) Assume that $\Delta(s_{12}) = r_{11} + r_{12} + r_{21} + r_{22}$, so that $\Delta(s_{12})e = r_{11} + r_{21}$. Also, we have $\Delta(s_{12}) = \Delta(es_{12}) = r_{11} + r_{21} + \sigma$; $\sigma \in Z(S)$ which gives $\Delta(s_{12})e = r_{11} + r_{21} + \sigma_{11}$. Comparing between the two values of $\Delta(s_{12})e$, we get $\sigma_{11} = 0$ and having $\sigma = \sigma_{22} \in Z_{22}$ and we get $\Delta(s_{12}) = r_{11} + r_{21} + \sigma_{22}$. Now, $\theta = \Delta(s_{12}e) = e\Delta(s_{12}) + \mu$; $\mu \in Z(S)$, hence $e\Delta(s_{12}) = (\theta - \mu) = \eta \in Z(S)$ and this gives $e\Delta(s_{12}) = r_{11} + r_{12} = \eta \in Z(S)$ which means $r_{12} = 0$ and $r_{11} = \eta_{11} \in Z_{11}$. So, we arrive to $\Delta(s_{12}) = r_{21} + \eta_{11} + \sigma_{22}$.

(b) Assume that $\Delta(s_{21}) = r_{11} + r_{12} + r_{21} + r_{22}$, so that $e\Delta(s_{21}) = r_{11} + r_{12}$. Also, we have $\Delta(s_{21}) = \Delta(s_{21}e) = r_{11} + r_{12} + \kappa$; $\kappa \in Z(S)$. which gives $e\Delta(s_{21}) = r_{11} + r_{12} + \kappa_{11}$. Comparing the two expressions of $e\Delta(s_{21})$, we get $\kappa_{11} = 0$, $\kappa = \kappa_{22} \in Z_{22}$ and we get $\Delta(s_{21}) = r_{11} + r_{12} + \kappa_{22}$. Now, $\theta = \Delta(es_{21}) = \Delta(s_{21})e + \nu$; $\nu \in Z(S)$, hence $\Delta(s_{21})e = (\theta - \nu) = \zeta \in Z(S)$ which means $r_{11} = \zeta_{11} \in Z_{11}$ and we have $\Delta(s_{21}) = r_{12} + \zeta_{11} + \kappa_{22}$. □

Lemma 8. *For any element $s_{11} \in S_{11}$, we have $\Omega(s_{11}) = r_{11} + \varphi_{22}$ for some $r_{11} \in S_{11}$ and $\varphi_{22} \in Z_{22}$.*

Proof. Since $\Omega(rs) = \Omega(s)r^* + s^*\Delta(r) + \gamma$, for every $r, s \in S$ and $\gamma \in Z(S)$ it follows that for every $s_{11} \in S_{11}$, we have $\Omega(s_{11}) = \Omega(s_{11}e) = e\Delta(s_{11}) + \gamma_1$; $\gamma_1 \in Z(S)$ because $\Omega(e) = 0$ and by Lemma 7 $\Delta(S_{11}) \subset S_{11} + Z(S)$ and $Z(S) \subset S_{11} + S_{22}$, so we have that $\Omega \mid S_{11} \subset S_{11} + Z(S)$. Now, assume that $\Omega(s_{11}) = a_{11} + \varphi, \varphi \in Z(S)$, then $\Omega(s_{11}) = \Omega\ (es_{11}) = \Omega(s_{11})e + \gamma_2$, $\gamma_2 \in Z(S)$ which gives $\Omega(s_{11}) - \Omega(s_{11})e = a_{11} + \varphi - a_{11} - \varphi_{11} \in Z(S)$. We conclude that $\varphi_{22} \in Z_{22}$ and $\Omega(s_{11}) = a_{11} + \varphi = a_{11} + \varphi_{11} + \varphi_{22} = r_{11} + \varphi_{22}$ with $r_{11} = a_{11} + \varphi_{11} \in S_{11}$ and $\varphi_{22} \in Z_{22}$ as required. □

Lemma 9. *For any $s_{12} \in S_{12}, \Omega(s_{12}) = r_{11} + r_{21} + \gamma_{22}$ for some $r_{11} \in S_{11}, r_{21} \in S_{21}$ and $\gamma_{22} \in Z_{22}$.*

Proof. If $s_{12} \in S_{12}$ with $\Omega(s_{12}) = r_{11} + r_{12} + r_{21} + r_{22}$, then $\Omega(s_{12}) = \Omega(es_{12}) = \Omega(s_{12})e + \gamma$; $\gamma \in Z(S)$, so $\Omega(s_{12}) = r_{11} + r_{21} + \gamma$ for some $\gamma \in Z(S)$. Also, $e\Omega(s_{12}) = r_{11} + r_{12} = r_{11} + \gamma_{11}$ which gives $\gamma_{11} = 0$ and $\gamma = \gamma_{22} \in Z(S)$, hence $\Omega(s_{12}) = r_{11} + r_{21} + \gamma_{22}$. □

Lemma 10. *For any $s_{21} \in S_{21}$, we have $\Omega(s_{21}) = r_{12} + \chi_{11} + \mu_{22}$, for some $r_{12} \in S_{12}, \chi_{11} \in Z_{11}$ and $\mu_{22} \in Z_{22}$.*

Proof. For $s_{21} \in S_{21}$, using Lemma 7, we have $\Omega(s_{21}) = \Omega(s_{21}e) = e\Delta(s_{21}) + \mu = r_{12} + \kappa_{11} + \mu$; $r_{12} \in S_{12}, \kappa_{11} \in Z_{11}$ and $\mu \in Z(S)$. Also, we have $\Omega(0) = \Omega(es_{21}) = \Omega(s_{21})e + \zeta = \kappa_{11} + \mu_{11} + \zeta$; $\zeta \in Z(S)$, which gives $\mu_{11} \in Z_{11}$, and hence $\mu_{22} = \mu - \mu_{11} \in Z_{22}$. So, we arrive to $\Omega(s_{21}) = r_{12} + \chi_{11} + \mu_{22}$, where $\chi_{11} = \kappa_{11} + \mu_{11} \in Z_{11}$ which is required. □

Lemma 11. *For any element $t \in (S_{11} + S_{12}), \Omega(t) = r_{11} + r_{21} + \gamma_{22}$, for some $r_{11} \in S_{11}$, $r_{21} \in S_{21}$, and $\gamma_{22} \in Z_{22}$.*

Proof. Assuming that $t \in (S_{11} + S_{12})$ and $\Omega(t) = r_{11} + r_{12} + r_{21} + r_{22}$, then $\Omega(t) = \Omega(s_{11} + s_{12}) = \Omega[e(s_{11} + s_{12})] = \Omega(s_{11} + s_{12})e + \gamma = r_{11} + r_{21} + \gamma$; $\gamma \in Z(S)$. This gives $r_{12} = 0$, $\gamma_{11} = 0$, and $\gamma_{22} = \gamma \in Z(S)$, and hence $\Omega(t) = r_{11} + r_{21} + \gamma_{22}$. □

Lemma 12. *Ω is centrally-extended additive on S_{11}.*

Proof. Assuming that r_{11} and $s_{11} \in S_{11}$, then $\Omega(r_{11} + s_{11}) = \Omega((r_{11} + s_{11})e) = e\Delta(r_{11} + s_{11}) + \gamma_1 = \Delta\ [(r_{11} + s_{11})e] - \Delta(e)(r_{11} + s_{11})^* + \gamma_2 = \Delta(r_{11} + s_{11}) + \gamma_2 = \Delta\ (r_{11}) + \Delta(s_{11}) + \gamma_3 = e\Delta(r_{11}) + e\Delta(s_{11}) + \gamma_4 = \Omega\ \ (r_{11}e) + \Omega(s_{11}e) + \gamma_5 = \Omega(r_{11}) + \Omega(s_{11}) + \gamma_5$; where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and $\gamma_5 \in Z(S)$. □

Lemma 13. *$\Omega(r_{12} + r_{21}) - \Omega(r_{12}) - \Omega(r_{21}) \in Z(S)$ for all $r_{12} \in S_{12}$ and $r_{21} \in S_{21}$.*

Proof. For any $r_{12} \in S_{12}, r_{21} \in S_{21}$ and $u_{1n} \in S_{1n}$, where $n \in \{1, 2\}$, we have

$$\begin{aligned}(\Omega(r_{12}) + \Omega(r_{21}))u_{1n} &= \Omega(r_{12})u_{1n} + \Omega(r_{21})u_{1n} = \Omega(r_{12})v_{n1}^* + \Omega(r_{21})v_{n1}^* \\ &= \Omega(v_{n1}r_{12}) - r_{12}^*\Delta(v_{n1}) + \Omega(v_{n1}r_{21}) - r_{21}^*\Delta(v_{n1}) + \gamma_1 \\ &= \Omega(v_{n1}(r_{12} + r_{21})) - r_{12}^*\Delta(v_{n1}) - r_{21}^*\Delta(v_{n1}) + \gamma_2 \\ &= \Omega(r_{12} + r_{21})v_{n1}^* + (r_{12} + r_{21})^*\Delta(v_{n1}) - r_{12}^*\Delta(v_{n1}) - r_{21}^*\Delta(v_{n1}) + \gamma_3 \\ &= \Omega(r_{12} + r_{21})u_{1n} + \gamma_3;\ \gamma_1, \gamma_2, \gamma_3 \in Z(S).\end{aligned} \tag{7}$$

Which implies $(\Omega(r_{12} + r_{21}) - \Omega(r_{12}) + \Omega(r_{21}))u_{1n} \in Z(S)$, that is,

$$(\Omega(r_{12} + r_{21}) - \Omega(r_{12}) + \Omega(r_{21}))S_{1n} \subset Z(S), \text{for all } n \in \{1, 2\}. \tag{8}$$

In a similar way, we obtained the following equation:

$$(\Omega(r_{12} + r_{21}) - \Omega(r_{12}) - \Omega(r_{21}))S_{2n} \subset Z(S), \text{for all } n \in \{1, 2\}. \tag{9}$$

Combining equations (8) and (9), we obtained $(\Omega(r_{12} + r_{21}) - \Omega(r_{12}) - \Omega(r_{21}))S \subset Z(S)$. By hypothesis (C_1), we have $\Omega(r_{12} + r_{21}) - \Omega(r_{12}) - \Omega(r_{21}) \in Z(S)$. □

Lemma 14. *$\Omega(r_{11} + r_{12}) - \Omega(r_{11}) - \Omega(r_{12}) \in Z(S)$ for all $r_{11} \in S_{11}$ and $r_{12} \in S_{12}$.*

Proof. Let $r_{11} \in S_{11}, r_{12} \in S_{12}, t_{12} \in S_{12}$ and $u_{1n} \in S_{1n}$, where $n \in \{1, 2\}$. Then, we have $(\Omega(r_{11} + r_{12}) - \Omega(r_{11}) - \Omega(r_{12}))t_{12}u_{1n} \in Z(S)$. Which implies

$$(\Omega(r_{11} + r_{12}) - \Omega(r_{11}) - \Omega(r_{12}))t_{12}S_{1n} \subset Z(S), \text{for all } n \in \{1, 2\}. \tag{10}$$

For any $u_{2n} \in S_{2n}$; $n \in \{1, 2\}$, using Lemmas 7 and 13, we find the following equation:

$$\begin{aligned}\Omega(r_{11} + r_{12})t_{12}u_{2n} &= \Omega(r_{11} + r_{12})w_{21}^*v_{n2}^* = \Omega(r_{11} + r_{12})(v_{n2}w_{21})^* \\ &= \Omega((v_{n2}w_{21})(r_{11} + r_{12})) - (r_{11} + r_{12})^*\Delta(v_{n2}w_{21}) + \nu_1 \\ &= \Omega((v_{n2}w_{21} + v_{n2})(w_{21}r_{11} + r_{12})) - (r_{11} + r_{12})^*\Delta(v_{n2}w_{21}) + \nu_1 \\ &= \Omega(w_{21}r_{11} + r_{12})(v_{n2}w_{21} + v_{n2})^* + (w_{21}r_{11} + r_{12})^*\Delta(v_{n2}w_{21} + v_{n2}) \\ &\quad - (r_{11} + r_{12})^*\Delta(v_{n2}w_{21}) + \nu_2 \\ &= \Omega(w_{21}r_{11} + r_{12})(v_{n2}w_{21} + v_{n2})^* + (w_{21}r_{11})^*\Delta(v_{n2}w_{21}) + r_{12}^*\Delta(v_{n2}w_{21}) \\ &\quad + (w_{21}r_{11})^*\Delta(v_{n2}) + r_{12}^*\Delta(v_{n2}) - r_{11}^*\Delta(v_{n2}w_{21}) - r_{12}^*\Delta(v_{n2}w_{21}) + \nu_3 \\ &= \Omega(w_{21}r_{11} + r_{12})(v_{n2}w_{21} + v_{n2})^* + (w_{21}r_{11})^*\Delta(v_{n2}w_{21}) + r_{12}^*\Delta(v_{n2}) \\ &\quad - r_{11}^*\Delta(w_{21})v_{n2}^* + \nu_4 \\ &= \Omega(w_{21}r_{11})(v_{n2}w_{21} + v_{n2})^* + \Omega(r_{12})(v_{n2}w_{21} + v_{n2})^* + (w_{21}r_{11})^*\Delta(v_{n2}w_{21})\end{aligned}$$

$$
\begin{aligned}
&+ r_{12}^*\Delta(v_{n2}) - r_{11}^*\Delta(w_{21})v_{n2}^* + v_5 \\
&= \Omega(w_{21}r_{11})(v_{n2}w_{21})^* + \Omega(w_{21}r_{11})v_{n2}^* + \Omega(r_{12})(v_{n2}w_{21})^* + \Omega(r_{12})v_{n2}^* \\
&+ (w_{21}r_{11})^*\Delta(v_{n2}w_{21}) + r_{12}^*\Delta(v_{n2}) - r_{11}^*\Delta(w_{21})v_{n2}^* + v_6 \\
&= \Omega((v_{n2}w_{21})(w_{21}r_{11})) + \Omega(w_{21}r_{11})v_{n2}^* + \Omega(r_{12})(v_{n2}w_{21})^* + \Omega(v_{n2}r_{12}) \\
&- r_{11}^*\Delta(w_{21})v_{n2}^* + v_7 \\
&= \Omega(0) + \Omega(w_{21}r_{11})v_{n2}^* + \Omega(r_{12})(v_{n2}w_{21})^* + \Omega(0) - r_{11}^*\Delta(w_{21})v_{n2}^* + v_7 \\
&= \Omega(r_{11})w_{21}^*v_{n2}^* + r_{11}^*\Delta(w_{21})v_{n2}^* + \Omega(r_{12})w_{21}^*v_{n2}^* - r_{11}^*\Delta(w_{21})v_{n2}^* + v_8 \\
&= (\Omega(r_{11}) + \Omega(r_{12}))t_{12}u_{2n} + v_8;\ v_i \in Z(S), i \in \{1,2,\ldots,8\}.
\end{aligned} \tag{11}
$$

Which implies that

$$(\Omega(r_{11} + r_{12}) - \Omega(r_{11}) - \Omega(r_{12}))t_{12}S_{2n} \subset Z(S), \text{for all } n \in \{1,2\}. \tag{12}$$

Combining equations (10) and (12), we obtained the following equation:

$$(\Omega(r_{11} + r_{12}) - \Omega(r_{11}) - \Omega(r_{12}))t_{12}S \subset Z(S). \tag{13}$$

Applying (C_1), we get $(\Omega(r_{11} + r_{12}) - \Omega(r_{11}) - \Omega(r_{12}))S_{12} \subset Z(S)$. Applying (C_2), we get $\Omega(r_{11} + r_{12}) - \Omega(r_{11}) - \Omega(r_{12}) \in Z(S)$, as desired. □

Lemma 15. $\Omega(r_{12} + r_{22}) - \Omega(r_{12}) - \Omega(r_{22}) \in Z(S)$, *for all* $r_{12} \in S_{12}, r_{22} \in S_{22}$.

Proof. Let $r_{12} \in S_{12}, r_{22} \in S_{22}$ and $u_{1n} \in S_{1n}$, where $n \in \{1,2\}$. Then, we have the following equation:

$$
\begin{aligned}
(\Omega(r_{12}) + \Omega(r_{22}))u_{1n} &= \Omega(r_{12})u_{1n} + \Omega(r_{22})u_{1n} = \Omega(r_{12})v_{n1}^* + \Omega(r_{22})v_{n1}^* \\
&= \Omega(v_{n1}r_{12}) - r_{12}^*\Delta(v_{n1}) + \Omega(v_{n1}r_{22}) - r_{22}^*\Delta(v_{n1}) + \mu_1 \\
&= \Omega(v_{n1}(r_{12} + r_{22})) - (r_{12}^* + r_{22}^*)\Delta(v_{n1}) + \mu_2 \\
&= \Omega(r_{12} + r_{22})v_{n1}^* + (r_{12} + r_{22})^*\Delta(v_{n1}) \\
-(r_{12}^* + r_{22}^*)\Delta(v_{n1}) + \mu_3 &= \Omega(r_{12} + r_{22})u_{1n} + \mu_3,
\end{aligned} \tag{14}
$$

where $\mu_i \in Z(S), i \in \{1,2,3\}$. Which implies that

$$(\Omega(r_{12} + r_{22}) - \Omega(r_{12}) - \Omega(r_{22}))S_{1n} \subset Z(S) \text{ for all } n \in \{1,2\}. \tag{15}$$

Analogously, we obtained the following equation:

$$(\Omega(r_{12} + r_{22}) - \Omega(r_{12}) - \Omega(r_{22}))S_{2n} \subset Z(S) \text{ for all } n \in \{1,2\}. \tag{16}$$

Combining equations (15) and (16), we obtained $(\Omega(r_{12}+r_{22})-\Omega(r_{12})-\Omega(r_{22}))S\subset Z(S)$. In view of (C_1), we get $\Omega(r_{12}+r_{22})-\Omega(r_{12})-\Omega(r_{22})\in Z(S)$. □

Lemma 16. *Ω is centrally-extended additive on S_{12}.*

Proof. Let $r_{12},s_{12},t_{12}\in S_{12}$ and $u_{2n}\in S_{2n}$, where $n\in\{1,2\}$. Then, we have the following equation:

$$\begin{aligned}\Omega(r_{12}+s_{12})t_{12}u_{2n} &= \Omega(r_{12}+s_{12})w_{21}^{*}v_{n2}^{*}\\ &= \Omega(r_{12}+s_{12})(v_{n2}w_{21})^{*}\\ &= \Omega((v_{n2}w_{21})(r_{12}+s_{12}))-(r_{12}+s_{12})^{*}\Delta(v_{n2}w_{21})+\mu_1\\ &= \Omega((v_{n2}w_{21}+v_{n2})(r_{12}+w_{21}s_{12}))-(r_{12}+s_{12})^{*}\Delta(v_{n2}w_{21})+\mu_1\\ &= \Omega(r_{12}+w_{21}s_{12})(v_{n2}w_{21}+v_{n2})^{*}+(r_{12}+w_{21}s_{12})^{*}\Delta(v_{n2}w_{21}+v_{n2})\\ &\quad -(r_{12}+s_{12})^{*}\Delta(v_{n2}w_{21})+\mu_2\\ &= \Omega(r_{12})(v_{n2}w_{21})^{*}+\Omega(w_{21}s_{12})v_{n2}^{*}+(w_{21}s_{12})^{*}\Delta(v_{n2})-s_{12}^{*}\Delta(v_{n2}w_{21})\\ &\quad +\mu_3;\ \text{(by Lemma 2.15)}\\ &= \Omega(r_{12})(v_{n2}w_{21})^{*}+\Omega(s_{12})w_{21}^{*}v_{n2}^{*}+\mu_4\\ &= (\Omega(r_{12})+\Omega(s_{12}))w_{21}^{*}v_{n2}^{*}+\mu_4\\ &= (\Omega(r_{12})+\Omega(s_{12}))t_{12}u_{2n}+\mu_4.\end{aligned}\tag{17}$$

Which implies that

$$(\Omega(r_{12}+s_{12})-\Omega(r_{12})-\Omega(s_{12}))t_{12}S_{2n}\subset Z(S)\ \text{for all}\ n\in\{1,2\}.\tag{18}$$

And trivially, we have the following equation:

$$(\Omega(r_{12}+s_{12})-\Omega(r_{12})-\Omega(s_{12}))t_{12}S_{1n}\subset Z(S)\ \text{for all}\ n\in\{1,2\}.\tag{19}$$

Combining equations (18) and (19), we find $(\Omega(r_{12}+s_{12})-\Omega(r_{12})-\Omega(s_{12}))t_{12}S\subset Z(S)$. By (C_1), we get $(\Omega(r_{12}+s_{12})-\Omega(r_{12})-\Omega(s_{12}))S_{12}\subset Z(S)$. Applying (C_2), we get $\Omega(r_{12}+s_{12})-\Omega(r_{12})-\Omega(s_{12})\in Z(S)$, as desired. □

Lemma 17. *Ω is centrally-extended additive on $S_{11}+S_{12}=eS$.*

Proof. Consider the arbitrary elements r_{11},s_{11} in S_{11} and r_{12}, s_{12} in S_{12}. So Lemmas 14, 16, and 12 give the following equation:

$$\begin{aligned}\Omega((r_{11}+r_{12})+(s_{11}+s_{12})) &= \Omega((r_{11}+s_{11})+(r_{12}+s_{12}))\\ &= \Omega(r_{11}+s_{11})+\Omega(r_{12}+s_{12})+\lambda_1\\ &= \Omega(r_{11})+\Omega(s_{11})+\Omega(r_{12})+\Omega(s_{12})+\lambda_2\\ &= (\Omega(r_{11})+\Omega(r_{12}))+(\Omega(s_{11})+\Omega(s_{12}))+\lambda_2\\ &= \Omega(r_{11}+r_{12})+\Omega(s_{11}+s_{12})+\lambda_3,\end{aligned}\tag{20}$$

where $\lambda_i\in Z(S)$; $i\in\{1,2,3\}$. Thus, Ω is centrally-extended additive on $S_{11}+S_{12}$, as required. □

3. Main Result

Now, we are ready to prove our main theorem.

Theorem 18. *Let S be a ring endowed with an involution $*$ containing a nontrivial symmetric idempotent e which satisfies conditions (C_1) and (C_2). If Ω is any multiplicative generalized reverse *CE-derivation of S, i.e., $\Omega(rs)=\Omega(s)r^*+s^*\Delta(r)+\rho$, for all $r,s\in S$ and $\rho\in Z(S)$ which is associated with some reverse *CE-derivation Δ of S, then Ω is centrally-extended additive.*

Proof. Let r and s be any elements of S. Consider $\Omega(r)+\Omega(s)$. Take an element k in $Se = S_{11} + S_{21}$. Thus, rk and sk are elements of Se. According to Lemma 17, we can obtain the following equation:

$$\begin{aligned}(\Omega(r)+\Omega(s))k &= (\Omega(r)+\Omega(s))t^* = \Omega(tr)+\Omega(ts)-(r+s)^*\Delta(t)+\rho_1\\ &= \Omega(tr+ts)-(r+s)^*\Delta(t)+\rho_2 = \Omega(t(r+s))-(r+s)^*\Delta(t)+\rho_2\\ &= \Omega(r+s)t^* + (r+s)^*\Delta(t)-(r+s)^*\Delta(t)+\rho_3\\ &= \Omega(r+s)k+\rho_3,\end{aligned} \tag{21}$$

where $\rho_i \in Z(S)$; $i \in \{1,2,3\}$. Thus, $\{\Omega(r+s)-\Omega(r)-\Omega(s)\}k \in Z(S)$. Since k is an arbitrary element in Se, we obtain $(\Omega(r)+\Omega(s)-\Omega(r+s))Se \subseteq Z(S)$. By condition (C_1), we get $\Omega(r+s)-\Omega(r)-\Omega(s) \in Z(S)$, which shows that the multiplicative generalized reverse *CE-derivation Ω is centrally-extended additive. □

Data Availability

No data were used to support the findings of this study.

Disclosure

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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