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Retraction

Retracted: Novel Concepts in Rough Cayley Fuzzy Graphs with Applications

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Y. Rao, Q. Zhou, M. Akhoundi, A. A. Talebi, S. Omidbakhsh Amiri, and G. Muhiuddin, "Novel Concepts in Rough Cayley Fuzzy Graphs with Applications," *Journal of Mathematics*, vol. 2023, Article ID 2244801, 11 pages, 2023.

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Research Article

Novel Concepts in Rough Cayley Fuzzy Graphs with Applications

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Today, fuzzy graphs (FGs) have a variety of applications in other fields of study, including medicine, engineering, and psychology, and for this reason, many researchers around the world are trying to identify their properties and use them in computer sciences as well as finding the smallest problem in a network. The concept of a Cayley fuzzy graph has become a standard part of the toolkit used to investigate and describe groups. Also, Cayley fuzzy graphs are good models for interconnection networks, and they are useful in semigroup theory for establishing which elements are ℓ and R related. The previous definition limitations in the FGs have directed us to offer a new classification in terms of Cayley fuzzy graphs. So, in this paper, two new definitions of Cayley fuzzy graphs (CFGs) and pseudo-Cayley fuzzy graphs (PCFGs) are discussed and their rough approximations are studied. Also, some properties of fuzzy rough sets (FRSs) in CFGs and PCFGs have been investigated. Finally, we presented the determination of the most effective person in the Water and Sewerage Organization and the importance of using refereeing facilities in football matches between club teams, by using CFG in the presented applications.

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1. Introduction

FG is one of the most widely used topics in fuzzy theory, which has been studied by many researchers today. One of the advantages of FG is its flexibility in reducing time and costs in economic issues, which has been welcomed by all managers of institutions and companies. FGs were introduced by Rosenfeld [1], ten years after Zadeh's innovative paper (Fuzzy Sets) [2]. In 1878, the Cayley graph (CGs) was considered for finite groups by Cayley. Dehn introduced CGs under the name Gruppenbid (group diagram) from 1909 to 1910. Alshehri and Akram defined Cayley bipolar fuzzy graphs in a group [3]. Namboothiri et al. [4] discussed CFGs. Akram et al. [5] introduced the notation of Cayley's intuitionistic fuzzy graphs on groups and investigated some of their properties. Borzooei and Rashmanlou [6] defined Cayley interval-valued fuzzy graphs. The concept of RS was originally proposed by Pawlak [7] as a mathematical approach to handle uncertainty in data analysis. Skowron and Ziarko [8, 9] introduced RS, FS, and knowledge discovery. The fundamental factors in RS theory are approximations. Biswas and Nanda [10] presented the notion of rough subgroups. Kuroki and Mordeson [11] defined RSs as the structure and rough groups. He and Shi [12] introduced the RG definition and studied some operations on it. Shahzamanian et al. [13] defined the concepts of rough approximations of CGs and rough edge CGs. Chakrabarty et al. [14] introduced the LAs and UAs in FS theory to obtain FRS. Wu et al. [15] studied the concept of generalized FRSs. Mahapatra et al. [16–20] investigated several concepts in radio fuzzy graphs and planar graphs. Shi and Kosari [21] proposed some properties of domination in product vague graphs. Shao et al. [22] defined new concepts in intuitionistic fuzzy graphs. CFG is used to illustrate real-world phenomena using Cayley fuzzy models in a variety of fields, including technology, social networking, and biological sciences. It is evident that CFGs are very useful tools in theoretical computer science. CFGs are used for interconnection networks as great models. They are utilized in many applied problems in graph and group theory. In physics, CFGs seem to appear in the study of quantum walks. Therefore, in this paper, two new algebraic definitions called CFGs and PCFGs have been proposed and their rough approximations have been studied. Finally, two basic applications are presented, which we will be explained using this graph. In the first application, due to the vital importance of water in human life and better management in its consumption, we introduced the most efficient and literate person in the water organization and we expressed his mastery of technology using the CFG. Using refereeing facilities (video check camera, the use of yellow and red cards, experienced referee and assistant referee, etc.) is very important and we use CFG to express the use of these facilities in various competitions.

2. Preliminaries

All the basic notations are shown in Table 1. A fuzzy subset (FSS) of M is a function from M into $I = [0,1]$. The class of all subsets of M (fuzzy subsets) will be denoted by $\mathscr{P}(M)(\mathscr{F}(M))$. Let $\psi \in \mathscr{F}(M)$. For $\beta \in I$, ψ_β and $\psi_{\beta+}$ are defined as follows:

$$\begin{aligned}\psi_\beta &= \{r \mid r \in M, \psi(r) \geq \beta\},\\ \psi_{\beta+} &= \{r \mid r \in M, \psi(r) > \beta\},\end{aligned} \tag{1}$$

and are called the β-cut and the strong β-cut (or β-level and strong β-level) of ψ, respectively.

Let V be a nonempty set. A FG is a triple (V, ψ, φ) so that ψ is a FSS of V and φ is a symmetric FR on ψ. That is, $\psi: V \longrightarrow [0,1]$ and $\varphi: V \times V \longrightarrow [0,1]$ so that $\varphi(r,p) \leq \psi(r) \wedge \psi(p)$, $\forall r, p \in V$.

Let P and Q be two finite and nonempty universes. A FSS $T \in \mathscr{F}(P \times Q)$ is defined as a FBR from P to Q. A relation $T \in \mathscr{F}(P \times Q)$ is called reflexive if $T(r,r) = 1$, symmetric if $T(r,p) = T(p,r)$, and transitive if $T(r,k) \geq \vee_{p \in P}(T(r,p) \wedge T(p,k))$, for all $r, k \in P$.

Let G be a group. A FSS ψ of a group G is called a FSG of the group G if

(i) $\psi(rp) \geq \min\{\psi(r), \psi(p)\}$, for each $r, p \in G$

(ii) $\psi(r^{-1}) = \psi(r)$, for every $r \in G$

A FSS ψ of a group G is a FSG of the group G iff $\psi(rp^{-1}) \geq \min\{\psi(r), \psi(p)\}$, for each $r, p \in G$.

Definition 1 (see [23]). A fuzzy graph (G, ψ, φ) is complete if

$$\varphi(r,p) = \psi(r) \wedge \psi(p), \quad \forall r, p \in G. \tag{2}$$

Accordingly, some definitions are summarized which can be found in [15]. Let P and Q be two finite universes. Suppose that T be an optional relation from P to Q. We can define a set-valued function $F: P \longrightarrow \mathscr{P}(Q)$ by

Table 1: Some basic notations.

Notation	Meaning
RS	Rough set
FS	Fuzzy set
RG	Rough graph
FG	Fuzzy graph
CG	Cayley graph
CFG	Cayley fuzzy graph
CCFG	Complete Cayley fuzzy graph
PCFG	Pseudo-Cayley fuzzy graph
FSG	Fuzzy subgroup
CSS	Cayley subset
AS	Approximation space
ACFG	Approximation Cayley fuzzy graph
FR	Fuzzy relation
FSS	Fuzzy subset
CFSS	Cayley fuzzy subset
FBR	Fuzzy binary relation
FAS	Fuzzy approximation space
LA	Lower approximation
LG	Lower generalized
UA	Upper approximation
UG	Upper generalized
FAO	Fuzzy approximation operator
FRS	Fuzzy rough set
PFSG	Partial fuzzy subgraph

$$F(r) = \{p \in Q: (r,p) \in T\}, \quad r \in P. \tag{3}$$

Obviously, any set-valued function of F from P to Q defines a binary relation from P to Q by setting $T = \{(r,p) \in P \times Q: p \in F(r)\}$. The triple (P, Q, T) is referred to as a generalized AS. For any set $B \subseteq Q$, a pair of LAs and UAs, $\underline{T}(B)$ and $\overline{T}(B)$, are defined by

$$\begin{aligned}\underline{T}(B) &= \{r \in P: F(r) \subseteq C\},\\ \overline{T}(B) &= \{r \in P: F(r) \cap C \neq \varnothing\}.\end{aligned} \tag{4}$$

The pair $\underline{T}(B)$, $\overline{R}(B)$ is referred to as a generalized RS.

Let T be an optional FR from P to Q. We define the mapping of $F: P \longrightarrow \mathscr{F}(Q)$ by

$$F(r)(p) = T(r,p), (r,p) \in P \times Q. \tag{5}$$

For any $\beta \in I$, we further define $F_\beta: P \longrightarrow \mathscr{P}(Q)$ by

$$F_\beta(r) = \{p \in Q: F(r)(p) \geq \beta\}, \quad r \in U. \tag{6}$$

Also for any $M \in \mathscr{P}(Q)$, the LAs and UAs of M with respect to the AS (P, Q, H_β) are defined as follows:

$$\begin{aligned}\underline{F}_\beta(M) &= \{r \in P: F_\beta(r) \subseteq M\},\\ \overline{F}_\beta(M) &= \{r \in P: F_\beta(r) \cap M \neq \varnothing\}.\end{aligned} \tag{7}$$

Definition 2. Let T be an optional FR from P to Q and $B \in \mathscr{F}(Q)$. The triple (P, Q, T) is defined as the generalized FAS. We define the LG and UG FAOs $\underline{F}$ and $\overline{F}$ with respect to (P, Q, T) by

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$$\underline{F}(B) = \vee_{\beta\in I}\left(\beta\wedge\underline{F}_{1-\beta}\left(B_{\beta+}\right)\right),$$
$$\overline{F}(B) = \vee_{\beta\in I}\left(\beta\wedge\overline{H}_{\beta}\left(B_{\beta}\right)\right). \quad (8)$$

The pair $(\underline{F}(B), \overline{F}(B))$ is described as generalized FRS.

Theorem 1. *If T is an optional FR from P to Q, then the pair of FAOs is satisfied in the following cases.*

For all B, $D \in \mathscr{F}(Q)$ and $\beta \in I$,

(a) $\underline{F}(B) = \sim(\overline{F}(\sim B))$
(b) $\overline{F}(B) = \sim(\underline{F}(\sim B))$
(c) $\underline{F}(B\vee\widehat{\beta}) = \underline{F}(B)\vee\widehat{\beta}$
(d) $\overline{F}(B\wedge\widehat{\beta}) = \overline{F}(B)\vee\widehat{\beta}$
(e) $\underline{F}(B\wedge D) = \underline{F}(B)\wedge\underline{F}(D)$
(f) $\underline{F}(B\vee D) = \overline{F}(B)\vee\overline{F}(D)$
(g) $B\subseteq D\Rightarrow\underline{F}(B)\subseteq\underline{F}(D)$
(h) $B\subseteq D\Rightarrow\overline{F}(B)\subseteq\overline{F}(D)$
(i) $\underline{F}(B\vee D)\supseteq\underline{F}(B)\vee\underline{F}(D)$
(j) $\overline{F}(B\wedge D)\subseteq\overline{F}(B)\wedge\overline{F}(D)$

$\widehat{\beta}$ is the constant FS: $\widehat{\alpha}(r) = \alpha$, $\forall r \in P$ and $r \in Q$.

3. Cayley Fuzzy Graph and Pseudo-Cayley Fuzzy Graphs

Definition 3. Let G be a group and ψ be a FSG of G and suppose φ is a subset of ψ so that $\varphi(rp^{-1})\leq\psi(r)\wedge\psi(p)$, for all $r, p \in G$ and $\varphi(r)\neq 0$ and $\varphi(r) = \varphi(r^{-1})$, $\forall r \in G$. The FG (G, ψ, ν) so that mapping ν is defined by $\nu(r, p) = \varphi(rp^{-1})\wedge\psi(r)\wedge\psi(p)$ for all $(r, p) \in G\times G, r\neq p$ is called the CFG of ψ in G relative to φ and is denoted by $\mathrm{Cay}F(G, \psi, \varphi)$. The FSS φ with the above properties is referred to as a CFSS of ψ in G.

Example 1. Consider $G = Z_4$ is the additive group of integers modulo 4 and ψ: $Z_4 \longrightarrow [0, 1]$ is defined by $\psi(0) = 1$, $\psi(2) = 0.9$, and $\psi(1) = \psi(3) = 0.6$. Let $\varphi(0) = \varphi(2) = 0.5$ and $\varphi(1) = \varphi(3) = 0.6$. Then, the CFG $K = \mathrm{Cay}(Z_4, \psi, \varphi)$ is given in Figure 1.

Theorem 2. *A CFG, $K = \mathrm{Cay}\,F(G, \psi, \varphi)$ is a complete fuzzy graph if and only if $\psi(rp^{-1}) = \psi(r)\wedge\psi(p)$, $\forall r, p \in G$, and $\psi(r) = \varphi(r)$, $\forall r \in G$.*

Proof. Assume that $K = \mathrm{Cay}F(G, \psi, \varphi)$ is CCFG. Then,

$$\varphi\left(rp^{-1}\right) = \psi(r)\wedge\psi(p), \quad \forall r, p \in G, \quad (9)$$

Hence,

$$\varphi(r) = \varphi\left(re^{-1}\right) = \psi(r)\wedge\psi\left(e^{-1}\right)$$
$$= \psi(r), \forall, r \in G\}, \quad (10)$$

and also

$$\psi\left(rp^{-1}\right) = \varphi\left(rp^{-1}\right) = \psi(r)\wedge\psi(p), \quad \forall r, p \in G. \quad (11)$$

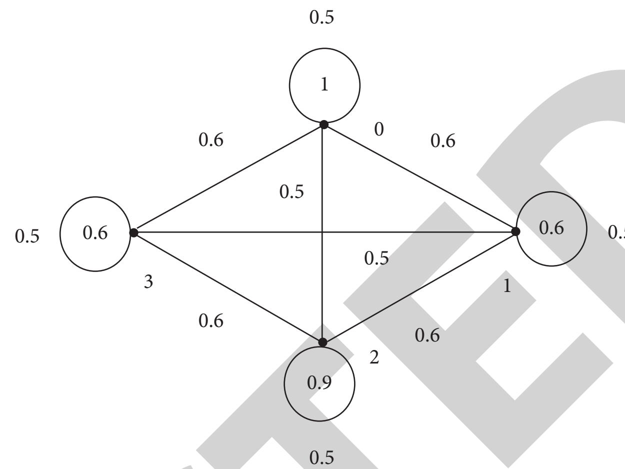

Figure 1: $\mathrm{Cay}\,F(Z_4, \psi, \varphi)$.

Conversely, let $\psi(r) = \varphi(r)$, $\forall r \in G$ and $\psi(rp^{-1}) = \psi(r)\wedge\psi(p)$, $\forall r, p \in G$. Then, we have $\psi(rp^{-1}) = \varphi(rp^{-1}) = \psi(r)\wedge\psi(p), \forall r, p \in G$; hence, k is complete. □

Definition 4. Suppose that G is a group and ψ is a FSG of G. Let τ be a subset of ψ so that $\tau(r) = \tau(r^{-1})$, $\forall r \in G$. For a CSS φ of ψ, the FG (G, τ, ν) so that mapping ν is defined by $\nu(r, p) = \varphi(rp^{-1})\wedge\tau(r)\wedge\tau(p)$, $\forall (r, p) \in G\times G$, and $r\neq p$ is named the PCFG of τ in G relative to φ and is shown by $P\mathrm{Cay}F(G, \tau, \sigma)$.

Example 2. Consider $G = Z_4$ is the additive group of integers modulo 4. ψ is defined by the condition of Example 1 and let $\varphi(x) = 0.5$, $\forall r \in Z_4$. Let τ be a FSS of ψ so that $\tau(0) = 0.5$, $\tau(1) = \tau(3) = 0.6$, and $\tau(2) = 0.8$. Then, $K = P\mathrm{Cay}\,F(Z_4, \tau, \sigma)$ is PCFG.

Theorem 3. *If $K_1 = \mathrm{Cay}\,F(G, \psi_1, \varphi_1)$ and $K_2 = \mathrm{Cay}\,F(G, \psi_2, \varphi_2)$ are CFGs, then*

(1) $K_1\cup K_2 = \mathrm{Cay}\,F(G, \psi_1\cup\psi_2, \varphi_1\cup\varphi_2)$, if $\psi_1\subseteq\psi_2$ or $\psi_2\subseteq\psi_1$
(2) $K_1\cap K_2 = \mathrm{Cay}\,F(G, \psi_1\cap\psi_2, \varphi_1\cap\varphi_2)$

Proof

(1) Let $\psi_1\subseteq\psi_2$, so $\psi_1\cup\psi_2 = \psi_2$. For $r, p \in G$, we have

$$\begin{aligned}
(\varphi_1\cup\varphi_2)\left(rp^{-1}\right) &= \varphi_1\left(rp^{-1}\right)\vee\varphi_2\left(rp^{-1}\right)\\
&\leq (\psi_1(r)\wedge\psi_1(p))\vee(\psi_2(r)\wedge\psi_2(p))\\
&= (\psi_2(r)\wedge\psi_2(p)),\\
(\varphi_1\cup\varphi_2)\left(r^{-1}\right) &= \varphi_1\left(r^{-1}\right)\vee\varphi_2\left(r^{-1}\right)\\
&= \varphi_1(r)\vee\varphi_2(r)\\
&= (\varphi_1\cup\varphi_2)(r).
\end{aligned} \quad (12)$$

(2) For $x, y \in G$, we have

$$(\varphi_1 \cap \varphi_2)(rp^{-1}) = \varphi_1(rp^{-1}) \wedge \varphi_2(rp^{-1})$$
$$\leq (\psi_1(r) \wedge \psi_1(p)) \wedge (\psi_2(r) \wedge \psi_2(p))$$
$$= (\psi_1(r) \wedge \psi_2(r)) \wedge (\psi_1(p) \wedge \psi_2(p)$$
$$= (\psi_1 \cap \psi_2)(r) \wedge (\psi_1 \cap \psi_2)(p), \quad (13)$$

also

$$(\varphi_1 \cap \varphi_2)(r^{-1}) = \varphi_1(r^{-1}) \wedge \varphi_2(r^{-1})$$
$$= \varphi_1(r) \wedge \varphi_2(r) \quad (14)$$
$$= (\varphi_1 \cap \varphi_2)(r).$$

□

Theorem 4. *If $K_1 = Cay\,F(G, \psi_1, \varphi_1)$ and $K_2 = Cay\,F(G, \psi_2, \varphi_2)$ are CFGs, then*

$$K_1 \subseteq K_2 \Leftrightarrow \psi_1 \subseteq \psi_2,$$
$$\varphi_1 \subseteq \varphi_1. \quad (15)$$

Proof. According to the definition of the fuzzy subgraph, the proof is obvious. □

Theorem 5. *Let G be a group, ψ be an FSG of G. If $K_1 = PCayF(G, \tau_1, \varphi_1)$ and $K_2 = PCayF(G, \tau_2, \varphi_2)$ are PCFG, then*

(1) $K_1 \cup K_2 = PCayF(G, \tau_1 \cup \tau_2, \varphi_1 \cup \varphi_2)$
(2) $K_1 \cap K_2 = PCayF(G, \tau_1 \cap \tau_2, \varphi_1 \cap \varphi_2)$

Proof
(1) Since τ_1 and $\tau_2 \subseteq \psi$, then $\tau_1 \cup \tau_2 \subseteq \psi$. According to the PCFGs definition,

$$(\tau_1 \cup \tau_2)(r) = \tau_1(r) \vee \tau_2(r) = \tau_1(r^{-1}) \vee \tau_2(r^{-1})$$
$$= (\tau_1 \cup \tau_2)(r^{-1}),$$
$$(\varphi_1 \cup \varphi_2)(r) = \varphi_1(r) \vee \varphi_2(r) = \varphi_1(r^{-1}) \vee \varphi_2(r^{-1}) \quad (16)$$
$$= (\varphi_1 \cup \varphi_2)(r^{-1}).$$

(2) It is similar to the proof of (1). □

4. Rough Cayley Fuzzy Graphs

Definition 5. Let ψ be a FSG of the group G so that $\psi(rp^{-1}) = \psi(r^{-1}p^{-1})$, $\forall r, p \in G$. Let R be the FR from G to G, defined by $R(r, p) = \psi(rp^{-1})$, $\forall r, p \in G$. Then, by Definition 2, we have

$$\underline{F}(\varphi)(r) = \vee\{\beta \in I; r \in \underline{F}_{1-\beta}(\varphi_{\beta+})\},$$
$$\overline{F}(\varphi)(r) = \vee\{\beta \in I; r \in \overline{F}_{\beta}(\varphi_{\beta})\}. \quad (17)$$

Lemma 1. *Let ψ be a FSG of G. Then, for any $A \in \mathscr{F}(G)$,*

(i) $\underline{F}(A) \subseteq A$
(ii) $A \subseteq \overline{F}(A)$

Proof
(i) For any $r \in G$, let $\underline{F}(A)(r) = \beta$, then $r \in \underline{F}_{1-\beta}\ (A_{\beta^+})$. So, $F_{1-\beta}(r) \subseteq A_{\beta^+}$. Now, $F(x)(x) = 1 \geq 1 - \beta$ shows that $r \in F_{1-\beta}(x)$. Hence, $r \in A_{\beta^+}$, i.e., $A(r) > \beta = \underline{F}(A)(r)$. This shows $\underline{F}(A) \subseteq A$.

(ii) For any $r \in G$, let $A(r) = \beta$, then $r \in A_\beta$. On the other hand, $F(r)(r) = \psi(rr^{-1}) = \psi(e) = 1 \geq \beta$ shows that $r \in F_\alpha(r)$.

Hence, $r \in A_\beta \cap F_\beta$, i.e., $r \in \overline{F_\beta}(A_\beta)$. Now by Definition 5, $\overline{F}(A)(r) \geq \beta = A(r)$; hence, $A \subseteq \overline{F}(A)$. □

Remark 1. If in Definition 3$\varphi(r) = 0$, for any $r \in G$, then, according to Definition 2, we have $\underline{F}(\varphi) = 0$.

Definition 6. Let ψ be a FSG of G and φ be a CFSS of ψ. For CFG $K = \text{Cay}\,F(G, \psi, \varphi)$, a pair of LAs and UAs, $\underline{K}$ and $\overline{K}$, are defined by

$$\underline{K} = \text{Cay}\,F(G, \psi, \underline{F}(\varphi)),$$
$$\overline{K} = \text{Cay}\,F(G, \psi, \overline{F}(\varphi)). \quad (18)$$

Theorem 6. *Two FGs $\underline{K}$ and $\overline{K}$ are CFGs.*

Proof. We show that $\underline{K}$ and $\overline{K}$ hold the conditions of Definition 3.

$$\underline{F}(\varphi)(r) = \vee\{\beta \in I; r \in \underline{F}_{1-\beta}(\varphi_{\beta+})\}$$
$$= \vee\{\beta \in I; F_{1-\beta}(r) \subseteq (\varphi_{\beta+})\}$$
$$= \vee\{\beta \in I; \forall p \in GR(r, p) = \psi(rp^{-1}) \geq 1 - \beta \Rightarrow \varphi(p) > \alpha\}$$
$$= \vee\{\beta \in I; \forall p \in GR(r^{-1}, p) = \psi(r^{-1}p^{-1}) \geq 1 - \beta \Rightarrow \varphi(y) > \beta\}$$
$$= \underline{F}(\varphi)(r^{-1}). \quad (19)$$

Similarly we can prove $\overline{F}(\varphi)(r) = \overline{F}(\varphi)(r^{-1})$.

Since $\varphi \leq \psi$, then $\varphi_{\beta+} \subseteq \psi_{\beta+}$. Therefore, for any $r \in G$,

$$\underline{F}(\varphi)(r) = \vee\{\beta \in I; r \in \underline{F}_{1-\beta}(\varphi_{\beta+})\}$$
$$= \vee\{\beta \in I; F_{1-\beta}(r) \subseteq \varphi_{\beta+}\}$$
$$\leq \vee\{\beta \in I; F_{1-\beta}(r) \subseteq \psi_{\beta+}\} \quad (20)$$
$$= \underline{F}(\psi)(r).$$

According to Lemma 1, since $\underline{F}(\psi) \leq \psi$, so $\underline{F}(\varphi) \leq \psi$.

Now, for any $r \in G$,

$$\overline{F}(\varphi)(r) = \vee\{\beta \in I; r \in \overline{F}_\beta(\varphi_\beta)\}$$
$$= \vee\{\beta \in I; F_\beta(r) \cap \varphi_\beta \neq \varnothing\}$$
$$= \vee\{\beta \in I; \exists p \in GR(r, p) \geq \beta \text{ and } \varphi(p) \geq \beta\}$$
$$= \vee\{\beta \in I; \exists p \in G\psi(rp^{-1}) \geq \beta \text{ and } \varphi(p) \geq \beta\}. \quad (21)$$

Suppose $\overline{F}(\varphi)(r) = \beta_0$. Then, $\exists p \in G$ so that $\psi(rp^{-1}) \geq \beta_0$ and $\varphi(p) \geq \beta_0$. Since ψ is an FSG and $\psi(p) \geq \varphi(p)$, we have

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$$\psi(r) = \psi\left(rp^{-1}p\right) \geq \psi\left(rp^{-1}\right)\wedge\psi(p)) \geq \beta_0. \tag{22}$$

Then, $\psi(r) \geq \overline{F}(\varphi)(r)$, so $\overline{F}(\varphi) \leq \psi$. Therefore, $\underline{K}$ and $\overline{K}$ are CFGs. □

Example 3. Consider $G = Z_4$ is the additive group of integers modulo 4. ψ: $Z_4 \longrightarrow [0,1]$ is defined by $\psi(0) = 1$, $\psi(1) = \psi(3) = 0.7$, and $\psi(2) = 0.9$. Let $\varphi(0) = 0.5$, $\varphi(2) = 0$, and $\varphi(1) = \varphi(3) = 0.6$. So, we have

$$\begin{aligned}&\underline{F}(\varphi)(r) = 0.5,\\&\overline{F}(\varphi)(r) = 0.6, \quad \text{for all } r \in G.\end{aligned} \tag{23}$$

Then, $K = \operatorname{Cay} F(Z_4,\psi,\varphi), \underline{K}$, and $\overline{K}$ are given in Figure 2.

Theorem 7. *Let ψ, ψ_1 and ψ_2 be FSGs of G. Let $K = Cay\,F(G,\psi,\varphi)$, $K_1 = Cay\,F(G,\psi_1,\varphi_1)$, and $K_2 = Cay\,F(G,\psi_2,\varphi_2)$ be CFGs. Then, we have*

(1) $\underline{K} \subseteq K \subseteq \overline{K}$

(2) (a) $\underline{K_1 \cap K_2} = \underline{K_1} \cap \underline{K_2}$, (b) $\underline{K_1 \cup K_2} \supseteq \underline{K_1} \cup \underline{K_2}$, if $\psi_1 \subseteq \psi_2$

(3) (a) $\overline{K_1 \cup K_2} = \overline{K_1} \cup \overline{K_2}$, if $\psi_1 \subseteq \psi_2$ or $\psi_2 \subseteq \psi_1$, (b) $\overline{K_1 \cap K_2} \subseteq \overline{K_1} \cap \overline{K_2}$

(4) $K_1 \subseteq K_2 \Rightarrow \underline{K_1} \subseteq \underline{K_2}$ and $\overline{K_1} \subseteq \overline{K_2}$

Proof

(1) It follows from Theorem 4 and Lemma 1.

(2) (a) By Theorem 1 (e) and Theorem 3 (2), we have

$$\begin{aligned}\underline{K_1} \cap \underline{K_2} &= \operatorname{Cay} F\left(G,\psi_1,\underline{F}(\varphi_1)\right) \cap \operatorname{Cay} F\left(G,\psi_2,\underline{F}(\varphi_2)\right)\\&= \operatorname{Cay} F\left(G,\psi_1 \cap \psi_2,\underline{F}(\varphi_1)\right) \cap \underline{F}(\varphi_2)),\end{aligned} \tag{24}$$

and on the other hand, we have

$$\underline{K_1 \cap K_2} = \operatorname{Cay} F\left(G,\psi_1 \cap \psi_2,\underline{F}(\varphi_1 \cap \varphi_2)\right). \tag{25}$$

Theorem 1 (e) implies $\underline{F}(\varphi_1 \cap \varphi_2) = \underline{F}(\varphi_1) \cap \underline{F}(\varphi_2)$. Therefore, $\underline{K_1 \cap K_2} = \underline{K_1} \cap \underline{K_2}$.

(b) Theorem 1 (i) implies $\underline{F}(\varphi_1 \cup \varphi_2) \supseteq \underline{F}(\varphi_1) \cup \underline{F}(\varphi_1)$. Hence, by Theorem 3 (1), we have

$$\begin{aligned}\underline{K_1 \cup K_2} &= \operatorname{Cay} F\left(G,\psi_1 \cup \psi_2,\underline{F}(\varphi_1 \cup \varphi_2)\right)\\&\supseteq \operatorname{Cay} F\left(G,\psi_1 \cup \psi_2,\underline{F}(\varphi_1) \cup \underline{F}(\varphi_2)\right)\\&= \operatorname{Cay} F\left(G,\psi_1,\underline{F}(\varphi_1)\right) \cup \operatorname{Cay} F\left(G,\psi_2,\underline{F}(\varphi_2)\right)\\&= \underline{K_1} \cup \underline{K_2}.\end{aligned} \tag{26}$$

(3) By Theorems 1 (f) and (i), the proof is similar to (2).

(4) Suppose that $K_1 \subseteq K_2$, since $\varphi_1 \subseteq \varphi_2$, then $\underline{F}(\varphi_1) \subseteq \underline{F}(\varphi_2)$. Hence, $\underline{K_1} \subseteq \underline{K_2}$.

Similarly, we can prove $\overline{K_1} \subseteq \overline{K_2}$. □

Theorem 8. *Let $K = (G,\psi,\varphi)$ be CFG. If K is CCFG, then $\overline{K}$ is complete.*

Proof. Suppose that K is a CCFG. By Theorems 2 and Lemma 1, we have

$$\begin{aligned}&\psi\left(rp^{-1}\right) = \psi(r)\wedge\psi(p), \quad \forall r,p \in G,\\&\overline{F}(\varphi)(r) \subseteq \psi(r) = \varphi(r) \subseteq \overline{F}(\varphi)(r), \quad \forall r \in G.\end{aligned} \tag{27}$$

So $\overline{F}(\varphi)(r) = \psi(r)$; hence, $\overline{K}$ is complete. □

Example 4. Consider $G = Z_4$ is the additive group of integers modulo 4 and ψ: $Z_4 \longrightarrow [0,1]$ is defined by $\psi(0) = 1$, $\psi(1) = \psi(3) = 0.7$, and $\psi(2) = 0.9$. Let $\varphi(0) = 1$, $\varphi(2) = 0.9$, and $\varphi(1) = \varphi(3) = 0.7$. By straight calculation, we have $\overline{F}(\varphi)(0) = 1$, $\overline{F}(\varphi)(1) = \overline{F}(\varphi)(3) = 0.9$, and $\overline{F}(\varphi)(2) = 0.9$. Then, $\overline{K}$ is complete.

Definition 7. Let $K = \operatorname{Cay} F(G,\psi,\varphi)$ and $\underline{K}$ and $\overline{K}$ be LA and UA of K. The complement of K is $K^c = (\underline{K}^c, \overline{K}^c)$, where

$$\begin{aligned}\underline{K}^c &= \left(G,\psi^c,\underline{F}(\varphi)^c\right),\\\overline{K}^c &= \left(G,\psi^c,\overline{F}(\varphi)^c\right),\end{aligned} \tag{28}$$

such that for all $r,p \in G$, we have

(a) $\psi^c(r) = \psi(r)$

(b) $\underline{F}(\varphi)^c(rp^{-1}) = (\psi(r)\wedge\psi(p)) - \underline{F}(\varphi)(rp^{-1})$

(c) $\overline{F}(\varphi)^c(rp^{-1}) = (\psi(r)\wedge\psi(p)) - \overline{F}(\varphi)(rp^{-1})$

Example 5. Suppose that hypothesis of Example 3 and group $G = Z_4$ are the group congruence modulo 4 integral number Z. Hence, according to Definition 7, $K^c = (G,\psi^c,\varphi^c)$, $\underline{K}^c = (G,\psi^c,\underline{F}(\varphi)^c)$, and $\overline{K}^c = (G,\psi^c,\overline{F}(\varphi)^c)$ are given in Figure 3.

5. Rough Pseudo-Cayley Graphs

Definition 8. Let ψ be a FSG of G and τ be a FSS of ψ so that $\tau(r) = \tau(r^{-1})$, $r \in G$. For the PCFG $K = PCay\,F(G,\tau,\varphi)$, a pair of LAs and UAs, $\underline{K'}$ and $\overline{K'}$, are defined by

$$\begin{aligned}\underline{K'} &= PCay\,F\left(G,\underline{F}(\tau),\varphi\right),\\\overline{K'} &= PCay\,F\left(G,\overline{F}(\tau) \cap \psi,\varphi\right).\end{aligned} \tag{29}$$

Theorem 9. *Two FGs $\underline{K'}$ and $\overline{K'}$ are PCFGs.*

Proof. According to the Definition 5, for each $r \in G$,

$$\begin{aligned}\underline{F}(\tau)(r) &= \vee\left\{\beta \in I; r \in \underline{F}_{1-\beta}(\tau_{\alpha+})\right\}\\&= \vee\left\{\beta \in I; F_{1-\beta}(r) \subseteq \left(\tau_{\beta+}\right)\right\}\\&= \vee\left\{\beta \in I; \forall p \in GR(r,p) = \psi\left(rp^{-1}\right) \geq 1-\beta \Rightarrow \tau(p) > \alpha\right\}\\&= \vee\left\{\beta \in I; \forall p \in GR\left(r^{-1},p\right) = \psi\left(r^{-1}p^{-1}\right) \geq 1-\beta \Rightarrow \tau(p) > \beta\right\}\\&= \underline{F}(\tau)\left(r^{-1}\right).\end{aligned} \tag{30}$$

Similarly, we can prove $\overline{F}(\tau)(r) = \overline{F}(\tau)(r^{-1})$, $\forall r \in G$; hence,

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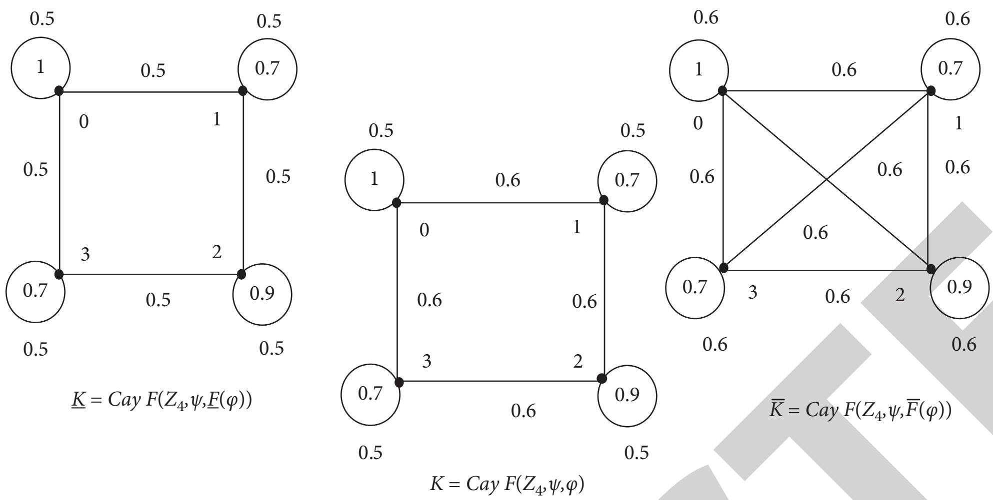

Figure 2: $K = \mathrm{Cay}\,F(Z_4,\psi,\varphi)$, $\underline{K}$ and $\overline{K}$.

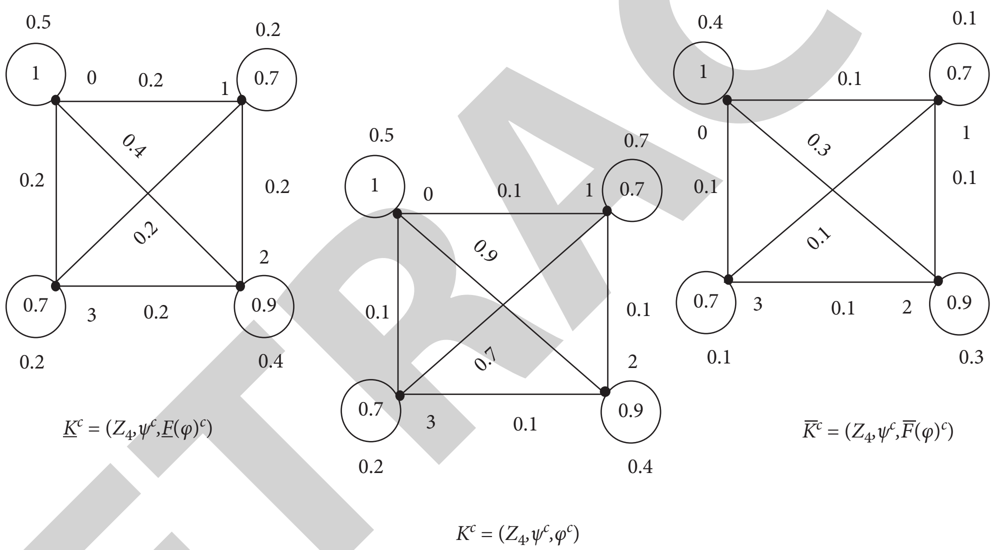

Figure 3: $K^c = (Z_4,\psi^c,\varphi^c)$, $\underline{K}^c$, and $\overline{K}^c$.

$$
\begin{aligned}
(\overline{F}(\tau)\cap\psi)(r) &= \overline{F}(\tau)(r)\wedge\psi(r) = \overline{F}(\tau)\left(r^{-1}\right)\wedge\psi\left(r^{-1}\right)\\
&= (\overline{F}(\tau)\cap\psi)\left(r^{-1}\right).
\end{aligned}
\tag{31}
$$

On the other hand, since $\overline{F}(\tau)\cap\psi\subseteq\psi$ and $\underline{F}(\tau)\subseteq\tau\subseteq\psi$, $\underline{K'}$ and $\overline{K'}$ are PCFGs. □

Theorem 10. *Let ψ be a FSG of G, and $K = PCay\,F(G,\tau,\varphi)$, $K_1 = PCay\,F(G,\tau_1,\varphi_1)$. and $K_2 = PCay\,F(G,\tau_2,\varphi_2)$ are PCFGs. Then, we have*

(1) $\underline{K}'\subseteq K\subseteq\overline{K}'$

(2) (a) $(\underline{K_1\cap K_2})' = \underline{K_1}'\cap\underline{K_2}'$, (b) $(\underline{K_1\cup K_2})'\supseteq\underline{K_1}'\cup\underline{K_2}'$, if $\tau_1\subseteq\tau_2$

(3) (a) $(\overline{K_1\cup K_2})' = \overline{K_1}'\cup\overline{K_2}'$, if $\tau_1\subseteq\tau_2$ or $\tau_2\subseteq\tau_1$, (b). $(\overline{K_1\cap K_2})'\subseteq\overline{K_1}'\cap\overline{K_2}'$

(4) $K_1\subseteq K_2\Rightarrow\underline{K_1}'\subseteq\underline{K_2}'$ and $\overline{K_1}'\subseteq\overline{K_2}'$

Proof

(1) Since $\underline{F}(\tau)\subseteq\tau$, $\tau\subseteq\overline{F}(\tau)$, and $\tau\subseteq\psi$, $\underline{F}(\tau)\subseteq\tau\subseteq\overline{F}(\tau)\cap\psi$. Now, Theorem 4 shows $\underline{K}'\subseteq K\subseteq\overline{K}'$.

(2) (a) We have

$$
\begin{aligned}
(\underline{K_1\cap K_2})' &= \mathrm{PCay}F(G,\underline{F}(\tau_1\cap\tau_2),\varphi_1\cap\varphi_2)\\
&= \mathrm{PCay}F(G,\underline{F}(\tau_1)\cap\underline{F}(\tau_2),\varphi_1\cap\varphi_2)\\
&= \mathrm{PCay}F(G,\underline{F}(\tau_1),\varphi_1)\cap\mathrm{PCay}F(G,\underline{F}(\tau_2),\varphi_2)\\
&= \underline{K_1'}\cap\underline{K_2'}.
\end{aligned}
\tag{32}
$$

(b) Theorem 1 (i) implies $\underline{F}(\tau_1\cup\tau_2)\supseteq\underline{F}(\tau_1)\cup\underline{F}(\tau_1)$. Hence,

$$
\begin{aligned}
(\underline{K_1\cup K_2})' &= \mathrm{PCay}F(G,\underline{F}(\tau_1\cup\tau_2),\varphi_1\cup\varphi_2)\\
&\supseteq \mathrm{PCay}F(G,\underline{F}(\tau_1)\cup\underline{F}(\tau_2),\varphi_1\cup\varphi_2)\\
&= \mathrm{PCay}F(G,\underline{F}(\tau_1),\varphi_1\cup\mathrm{PCay}F(G,\underline{F}(\tau_2),\varphi_2)\\
&= \underline{K_1'}\cup\underline{K_2'}.
\end{aligned}
\tag{33}
$$

(3) (a) We have

$$\begin{aligned}(\overline{K_1 \cup K_2})' &= PCayF(G, \overline{F}(\tau_1 \cup \tau_2) \cap \psi, \varphi_1 \cup \varphi_2)\\ &= PCayF(G, (\overline{F}(\tau_1) \cap \psi) \cup (\overline{F}(\tau_2) \cap \psi), \varphi_1 \cup \varphi_2)\\ &= PCayF(G, \overline{F}(\tau_1) \cap \psi, \varphi_1) \cup PCayF(G, \overline{F}(\tau_2) \cap \psi, \varphi_2)\\ &= \overline{K_1'} \cup \overline{K_2'}.\end{aligned} \tag{34}$$

(b)

$$\begin{aligned}(\overline{K_1 \cap K_2})' &= PCayF(G, \overline{F}(\tau_1 \cap \tau_2) \cap \psi, \varphi_1 \cap \varphi_2)\\ &\subseteq PCayF(G, (\overline{F}(\tau_1) \cap \psi) \cap (\overline{F}(\tau_2) \cap \psi), \varphi_1 \cap \varphi_2)\\ &= PCayF(G, \overline{F}(\tau_1) \cap \psi, \varphi_1) \cap PCayF(G, \overline{F}(\tau_2) \cap \psi, \varphi_2)\\ &= \overline{K_1'} \cap \overline{K_2'}.\end{aligned} \tag{35}$$

(4) Assume $K_1 \subseteq K_2$. Then, $\tau_1 \subseteq \tau_1$ and $\varphi_1 \subseteq \varphi_2$. So, $\underline{F}(\tau_1) \subseteq \underline{F}(\tau_2)$ and hence $\underline{K_1}' \subseteq \underline{K_2}'$. Similarly, we can prove $\overline{K_1}' \subseteq \overline{K_2}'$. □

Example 6. Consider $G = Z_5$ is the additive group of integers modulo 5 and $\psi\colon Z_5 \longrightarrow [0,1]$ is defined by $\psi(0) = 1$ and $\psi(1) = \psi(2) = \psi(3) = \psi(4) = 0.8$. Let $\varphi(r) = 0.5$, $\forall r \text{ in } Z_4$ and τ be an FSSs of ψ so that $\tau(0) = 0.7$, $\tau(1) = \tau(4) = 0.6$, and $\tau(2) = \tau(3) = 0.5$. So we have

$$\begin{aligned}&\underline{F}(\tau)(r) = 0.5,\\ &\overline{F}(\tau)(r) = 0.7, \quad \text{for } r \in G.\end{aligned} \tag{36}$$

Then, $K = PCayF(Z_5, \tau, \varphi), \underline{K'}$, and $\overline{K'}$ are given in Figure 4.

Definition 9. For the PCFG $K = PCayF(G, \tau, \varphi)$, a pair of LAs and UAs, $\underline{K''}$ and $\overline{K''}$, are defined by

$$\begin{aligned}\underline{K''} &= PCayF(G, \underline{F}(\tau), \underline{F}(\varphi)),\\ \overline{K''} &= PCayF(G, \overline{F}(\tau) \cap \psi, \overline{F}(\varphi)).\end{aligned} \tag{37}$$

Theorem 11. *Two FGs $\underline{K''}$ and $\overline{K''}$ are PCFGs.*

Proof. According to Theorem 9, $(\overline{F}(\tau) \cap \psi) \subseteq \psi$, $\underline{F}(\tau) \subseteq \tau \subseteq \psi$, $\underline{F}(\tau)(r) = \underline{F}(\tau)(r^{-1})$ and $(\overline{F}(\tau) \cap \psi)(r) = (\overline{F}(\tau) \cap \psi)(r^{-1})$, we show that $\underline{K''}$ and $\overline{K''}$ satisfy the conditions of Definition 4. For any $r \in G$, we have

$$\begin{aligned}\overline{F}(\varphi)(r) &= \vee\{\beta \in I; r \in \overline{F}_\beta(\varphi_\beta)\}\\ &= \vee\{\beta \in I; F_\beta(r) \subseteq (\varphi_\beta)\}\\ &= \vee\{\beta \in I; \exists p \in GR(r,p) = \psi(rp^{-1}) \geq \beta \Rightarrow \varphi(p) \geq \beta\}\\ &= \vee\{\beta \in I; \exists p \in GR(r^{-1},p) = \psi(r^{-1}p^{-1}) \geq \beta \Rightarrow \varphi(p) \geq \beta\}\\ &= \overline{F}(\varphi)(r^{-1}).\end{aligned} \tag{38}$$

Similarly, we can prove $\underline{F}(\varphi)(r) = \underline{F}(\varphi)(r^{-1})$, $\forall r \in G$. □

Example 7. Consider $G = Z_3$ is the additive group of integers modulo 3 and $\psi\colon Z_3 \longrightarrow [0,1]$ is defined by $\psi(0) = 1$, $\psi(1) = \psi(2) = 0.9$. Let $\varphi(0) = 0.4$ and $\varphi(1) = \varphi(2) = 0.5$. Let τ be a FSS of ψ so that $\tau(0) = 0.8$, $\tau(1) = \tau(2) = 0.6$. So we have

$$\begin{aligned}&\underline{F}(\tau)(r) = 0.6,\\ &\overline{F}(\tau)(r) = 0.8, \quad \text{for } r \in G,\\ &\underline{F}(\varphi)(r) = 0.4,\\ &\overline{F}(\varphi)(r) = 0.5, \quad \text{for } r \in G,\\ &(\overline{F}(\tau) \cap \psi)(r) = 0.8.\end{aligned} \tag{39}$$

Then, $K = PCay(Z_3, \tau, \varphi), \underline{K''}$, and $\overline{K''}$ are given in Figure 5.

Theorem 12. *Let ψ be a FSG of G. Let $K = PCayF(G, \tau, \varphi)$, $K_1 = PCayF(G, \tau_1, \varphi_1)$, and $K_2 = PCayF(G, \tau_2, \varphi_2)$ are PCFGs. Then, we have*

(1) $\underline{K''} \subseteq K \subseteq \overline{K''}$
(2) $(\underline{K_1 \cap K_2})'' = \underline{K_1}'' \cap \underline{K_2}''$
(3) $K_1 \subseteq K_2 \Rightarrow \underline{K_1}'' \subseteq \underline{K_2}''$
(4) $K_1 \subseteq K_2 \Rightarrow \overline{K_1}'' \subseteq \overline{K_2}''$
(5) $(\overline{K_1 \cap K_2})'' \subseteq \overline{K_1}'' \cap \overline{K_2}''$

Proof

(1) We have

$$\begin{aligned}\underline{K''} &= PCay\,F(G, \underline{F}(\tau), \underline{F}(\varphi))\\ &\subseteq PCay\,F(G, \tau, \varphi) = K\\ &\subseteq PCay\,F(G, \overline{F}(\tau) \cap \psi, \overline{F}(\varphi))\\ &= \overline{K''}.\end{aligned} \tag{40}$$

(2) We have

$$\begin{aligned}(\underline{K_1 \cap K_2})'' &= PCay\,F(G, \underline{F}(\tau_1 \cap \tau_2), \underline{F}(\varphi_1 \cap \varphi_2))\\ &= PCay\,F(G, \underline{F}(\tau_1) \cap \underline{F}(\tau_2), \underline{F}(\varphi_1) \cap \underline{F}(\varphi_2))\\ &= PCay\,F(G, \underline{F}(\tau_1), \underline{F}(\varphi_1)) \cap PCay\,F(G, \underline{F}(\tau_2), \underline{F}(\varphi_2))\\ &= \underline{K_1''} \cap \underline{K_2''}.\end{aligned} \tag{41}$$

(3) Assume $K_1 \subseteq K_2$. Then, $\tau_1 \subseteq \tau_1$ and $\varphi_1 \subseteq \varphi_2$. So, $\underline{F}(\tau_1) \subseteq \underline{F}(\tau_2)$ and $\underline{F}(\varphi_1) \subseteq \underline{F}(\varphi_2)$. Hence, $\underline{K_1''} \subseteq \underline{K_2''}$.

(4) By Theorem 1 (h), the proof is similar to (3).

(5) We have

$$\begin{aligned}(\overline{K_1 \cap K_2})'' &= PCay\,F(G, \overline{F}(\tau_1 \cap \tau_2) \cap \psi, \overline{F}(\varphi_1 \cap \phi_2))\\ &\subseteq PCay\,F(G, (\overline{F}(\tau_1) \cap \overline{F}(\tau_2))\\ &\quad \cap \psi, (\overline{F}(\varphi_1) \cap \overline{F}(\varphi_2))\\ &= PCay\,F(G, \overline{F}(\tau_1) \cap \psi, \overline{F}(\varphi_1))\\ &\quad \cap PCay\,F(G, \overline{F}(\tau_2) \cap \psi, \overline{F}(\varphi_2))\\ &= \overline{K_1''} \cap \overline{K_2''}.\end{aligned} \tag{42}$$

□

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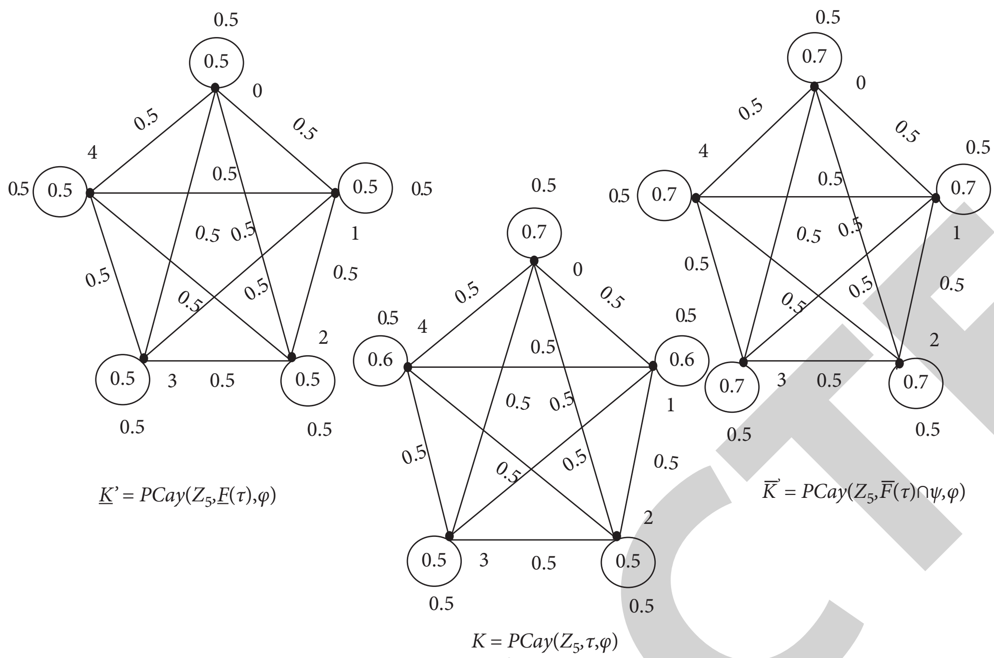

Figure 4: $K = PCayF(Z_5, \tau, \varphi)$, $\underline{K'}$, and $\overline{K'}$.

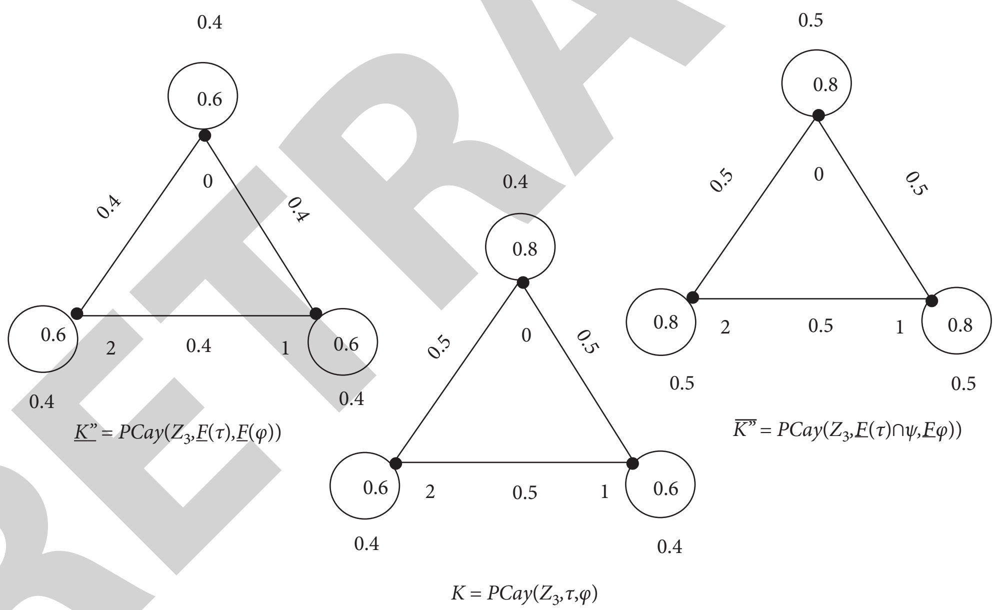

Figure 5: $K = PCay(Z_3, \tau, \varphi)$, $\underline{K''}$, and $\overline{K''}$.

6. Application

Example 8. Water and water resources are critical in maintaining adequate food resources and a productive environment for all living organisms. Demand for fresh water in the world is growing rapidly as the population grows and economies grow. The negative effects of global population growth, the effects of climate change, and lifestyle changes on our vital water resources are putting increasing pressure, which in turn is leading to widespread water tensions in many countries. As a result, there is an urgent need for water conservation. Water is the fluid of life because it has a strong impact on living standards and public health, although water is unevenly distributed around the world. Liquid water is very important and its existence is necessary to maintain vital human activities such as blood circulation, excretion, and reproduction. Water is also a place to live and one of the main constituents of the living environment. Population growth along with increasing water consumption not only leads to a sharp decline in water for all but also causes stress on biodiversity throughout the global ecosystem. Other important factors that limit the

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availability of water are rainfall rate, soil quality, and vegetation type. In addition, there are currently serious problems with the equitable allocation of freshwater resources in the world and between and within countries. Reports indicate that water consumption has increased seven times more than that of the last century. The world's drinking water resources, unsustainable urbanization, overpopulation, water pollution, water loss, rising greenhouse gases, and overindustrialization are increasingly damaging and degrading the water resources.

Therefore, considering that water is one of the most valuable elements on the planet and that we should try to use it properly and manage it better, it is necessary to have experienced and literate people who have sufficient mastery in the use of cyberspace to advance this.

Suppose K is a CFG on the group $G = Z_6 = \{0, 1, 2, 3, 4, 5\}$ is the additive group of integers modulo 6, where the nodes and edges are expressed as follows.

The nodes (Table 2) represent the employees of the Water and Sewerage Company which correspond to the members of the group Z_6. Edges (Table 3) are the degree of people mastering the authorized space (WhatsApp, Facebook, Instagram, etc.) to improve in this organization.

Figure 6 shows that financial expert has 70% awareness to guide people on how to save water. Education and the use of technology are directly related. For example, the use of technology between the manager and the deputy is 70% that is more than other edges. As we can see, the manager has the highest impact on other officials in terms of the use of technology, so the manager is the most effective person in this office.

Table 2: The staff education rate.

Employees	Label	Education rate
Manager	0	0.9
Deputy	1	0.9
Financial expert	2	0.9
Billing expert	3	0.8
Responsible for subscriber affairs	4	0.8
Services	5	0.7

Table 3: The use of technology corresponding to Figure 5.

	0	1	2	3	4	5
0	0	0.6	0.5	0.4	0.5	0.6
1	0.6	0	0.6	0.5	0.4	0.5
2	0.5	0.6	0	0.6	0.5	0.4
3	0.4	0.5	0.6	0	0.6	0.5
4	0.5	0.4	0.5	0.6	0	0.6
5	0.6	0.5	0.4	0.5	0.6	0

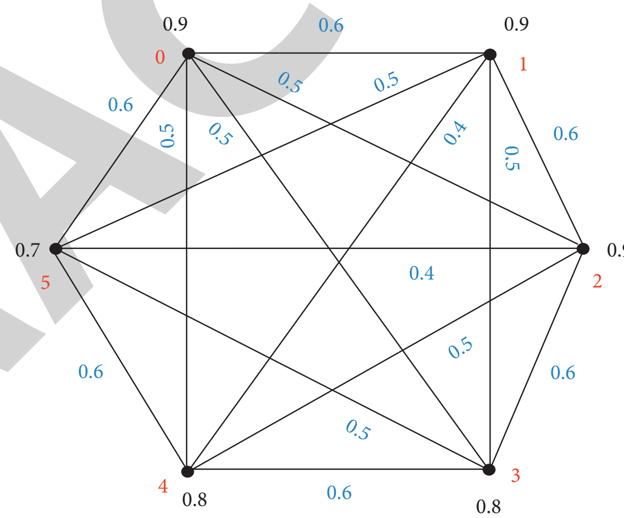

Figure 6: CFG.

Example 9. Football in Iran has become the most popular sport in this country. The Iranian football team won the championship at the highest level in Asia in several years and first qualified for the 1978 FIFA World Cup. The main infrastructure of the football industry includes clubs that play a major role in the economic development of the industry. Using their equipment and facilities, these clubs have turned skilled and specialized manpower and effective football management into a money-making industry. The better the situation of the clubs, the more developed the football of that country will be. Therefore, considering this issue, to improve the quality of matches in the national league, the use of refereeing facilities (video check cameras, etc.) in these matches is of special importance. And the stronger the teams, the more sensitive this will be. In the following, we will review 8 clubs that won the first to eighth place in 2021. Suppose K is a CFG on the group $G = Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the group congruence modulo 8 integral number$Z$, where the nodes and edges are expressed as follows:

(a) The nodes (Table 4) represent the names of the clubs and their weight. The quality of the game of these teams during the league which correspond to the members of group Z_8

Table 4: Ranking of the country's football clubs.

Team names	Label	Game quality
Esteghlal	0	0.9
Perspolis	1	0.9
Sepahan	2	0.9
Gol Gohar	3	0.8
Foolad	4	0.8
Mes Rafsanjan	5	0.8
Aluminium Arak	6	0.7
Zob Ahan	7	0.7

(b) The edges (Table 5) of the matches between these clubs and their weight are the importance of refereeing facilities about these matches

As we know, the match between the teams of the country's premier league is much more sensitive than other leagues. For example, according to Figure 7, the sensitivity of Esteghlal and Persepolis games and the sensitivity of

Table 5: Referee's sensitivity to matches according to Figure 7.

	0	1	2	3	4	5	6	7
0	0	0.6	0.5	0.4	0.4	0.4	0.5	0.6
1	0.6	0	0.6	0.5	0.4	0.4	0.4	0.5
2	0.5	0.6	0	0.6	0.5	0.4	0.4	0.4
3	0.4	0.5	0.6	0	0.6	0.5	0.4	0.4
4	0.4	0.4	0.5	0.6	0	0.6	0.5	0.4
5	0.4	0.4	0.4	0.5	0.6	0	0.6	0.5
6	0.5	0.4	0.4	0.4	0.5	0.6	0	0.6
5	0.6	0.5	0.4	0.4	0.4	0.5	0.6	0

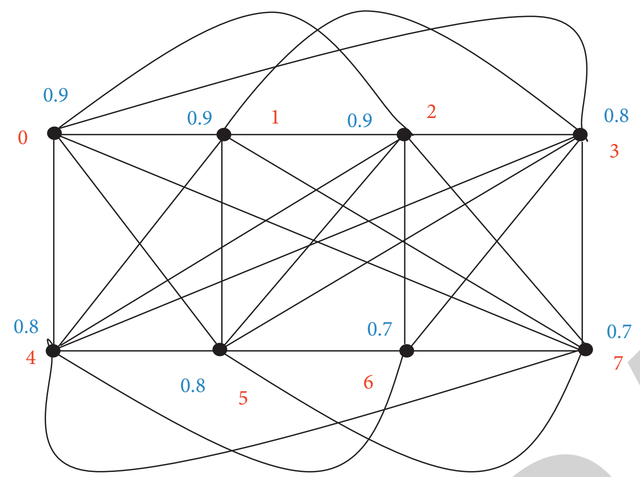

Figure 7: The CFG for the matches of 8 teams in the country's premier league.

Aluminum Arak and Zobahan are on the same level. Therefore, using a variety of refereeing facilities (video check camera, use of yellow and red cards, experienced referee and assistant referee, etc.) can change the fate of the game.

7. Conclusion

The fuzzy rough models are very important in FG problems and they give more integrity, flexibility, and suitability to the system. CFG has a wide bound of applications in the field of psychological sciences as well as the reconnoiter of individuals founded on oncological behaviors. So, in this survey, we have introduced two new algebraic definitions called CFGs and PCFGs and studied their rough approximations with several examples. Likewise, some properties of fuzzy rough sets (FRSs) in CFGs and PCFGs have been investigated. Finally, two applications are presented using CFG, in one of which the most effective and literate person in cyberspace technology in the Water and Sewerage Organization is identified, and in the other, the sensitivity of refereeing facilities in the country's Premier Football League is shown. It means we showed that the use of refereeing facilities is more among the teams that are at the top of the table than other teams.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References

[1] A. Rosenfeld, "Fuzzy graphs," in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, L. A. Zadeh, K. S. Fu, and M. Shimura, Eds., pp. 77–95, Academic Press, New York, NY, USA, 1975.
[2] L. A. Zadeh, "Fuzzy sets," *Information and ControlInformation and Control*, vol. 8, no. 3, pp. 338–353, 1965.
[3] N. O. Alshehri and M. Akram, "Cayley bipolar fuzzy graphs," *The Scientific World Journal*, vol. 2013, Article ID 156786, 8 pages, 2013.
[4] N. M. Namboothiri, V. A. Kumar, and P. T. Ramachandran, "Cayley fuzzy graphs," *Far East Journal of Mathematical Sciences*, vol. 73, pp. 1–15, 2013.
[5] M. Akram, M. G. Karunambigal, and O. K. Kaluivani, "Cayley intutionistic fuzzy graphs," *Journal of Appllied Mathematics and Informatics*, vol. 32, no. 5-6, pp. 827–842, 2014.
[6] R. A. Borzooei and H. Rashmanlou, "Cayley interval-valued fuzzy graphs," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 78, no. 3, pp. 83–94, 2016.
[7] Z. Pawlak, "Rough sets," *International Journal of Computer & Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.
[8] A. Skowron and L. Polkowski, *Rough Sets in Knowledge Discovery*, Springer-Verlag, Berlin, Germany, 1998.
[9] W. P. Ziarko, "Rough sets, fuzzy sets and knowledge discovery," in *Proceedings of the International Workshop on Rough Sets and Knowledge Discovery*, London, UK, October 1994.
[10] R. Biswas and S. Nanda, "Rough groups and rough subgroups," *Bulletin of the Polish Academy of Sciences - Mathematics*, vol. 42, pp. 251–254, 1994.
[11] N. Kuroki and J. N. Moderson, "Structures of rough sets and rough groups," *Journal of Fuzzy Mathematics*, vol. 5, no. 1, pp. 183–191, 1997.
[12] T. He and K. Shi, "Rough graph and its structure," *Journal of Shandong University*, vol. 41, no. 6, pp. 46–50, 2006.
[13] M. H. Shahzamanian, M. Shirmohammadi, and B. Davvaz, "Roughness in Cayley graphs," *Information Sciences*, vol. 180, no. 17, pp. 3362–3372, 2010.
[14] K. Chakrabarty, R. Biswas, and S. Nanda, "Fuzziness in rough sets," *Fuzzy Sets and Systems*, vol. 110, no. 2, pp. 247–251, 2000.
[15] W. Wu, M. Ju-Sheng, and W.-X. Zhang, "Generalized fuzzy rough sets," *Information Sciences*, vol. 151, pp. 263–282, 2003.
[16] R. Mahapatra, S. Samanta, M. Pal, and Q. Xin, "Link prediction in social networks by neutrosophic graph," *International Journal of Computational Intelligence Systems*, vol. 13, no. 1, pp. 1699–1713, 2020.
[17] R. Mahapatra, S. Samanta, and M. Pal, "Generalized neutrosophic planar graphs and its application," *Journal of Applied Mathematics and Computing*, vol. 65, no. 1-2, pp. 693–712, 2020.

RETRACTED

[18] R. Mahapatra, S. Samanta, M. Pal, and Q. Xin, "RSM index: a new way of link prediction in social networks," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 2, pp. 2137–2151, 2019.

[19] R. Mahapatra, S. Samanta, T. Allahviranloo, and M. Pal, "Radio fuzzy graphs and assignment of frequency in radio stations," *Computational and Applied Mathematics*, vol. 38, no. 3, 2019.

[20] R. Mahapatra, S. Samanta, and M. Pal, "Applications of edge colouring of fuzzy graphs," *Informatica*, vol. 31, no. 2, pp. 313–330, 2020.

[21] X. Shi and S. Kosari, "Certain properties of domination in product vague graphs with an application in medicine," *Frontiers in Physiology*, vol. 9, pp. 1–8, Article ID 680634, 2021.

[22] Z. Shao, S. Kosari, H. Rashmanlou, and M. Shoaib, "New concepts in intuitionistic fuzzy graph with application in water supplier systems," *Mathematics*, vol. 8, no. 8, p. 1241, 2020.

[23] A. Nagoor Gani and K. Radha, "On regular fuzzy graphs," *Journal of Physical Science*, vol. 12, pp. 33–40, 2008.

RETRACTED