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Research Article

On Degenerate Poly-Daehee Polynomials Arising from Lambda-Umbral Calculus

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In this article, we derived various identities between the degenerate poly-Daehee polynomials and some special polynomials by using λ-umbral calculus by finding the coefficients when expressing degenerate poly-Daehee polynomials as a linear combination of degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Bernoulli polynomials of the second kind, degenerate Daehee polynomials, Changhee polynomials, degenerate Bell polynomials, and degenerate Lah-Bell polynomials.

1. Introduction

The special functions or special polynomials are important and useful not only in pure or applied mathematics, but also in all fields of applied mathematics, and many interesting properties are investigated by many researchers even now (see [1]). In [2], authors introduced a new type of generating function of Appell-type Changhee-Euler polynomials and derived the differential equations arising from the generating function of the Appell-type Changhee-Euler polynomials. Boussayoud-Boughaba-Araci found explicit formulas for the k-Fibonacci numbers, k-Pell numbers, and the product of those polynomials in [3]. Simsek constructed recurrence relations for a new class of special numbers and polynomials and obtained some new and interesting identities related to the Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Stirling numbers (see [4]). In [5], authors introduced a new variant of type 2 Bernoulli polynomials and numbers by modifying a generating function and derived the explicit representations of the those polynomials in terms of the degenerate Lah-Bell polynomials and the higher-order degenerate derangement polynomials.

For a given nonzero real number λ, the degenerate exponential function is defined by the following equation:

$$e_{\lambda}^{x}(t) = (1+\lambda t)^{x/\lambda} \text{ and } e_{\lambda}(t) = (1+\lambda t)^{1/\lambda}. \tag{1}$$

By using the degenerate exponential function, Carlitz defined the degenerate Bernoulli polynomials in [6], and many degenerate versions of special functions are being actively studied by many researchers. Typically, in [7], Kim defined the degenerate Stirling numbers of the second kind, and Kim-Kim investigated the symmetric properties of degenerate Frobenius-Euler polynomials in [8]. The degenerate Bernoulli polynomials and numbers were introduced by Kim-Kim by using degenerate polylogarithm functions in [9], and in [10], authors defined degenerate polyexponential function and degenerate Bell polynomials by using these functions and derived some interesting identities. Another degenerate Bernoulli polynomials were introduced in [11] arising from degenerate polylogarithm function. The degenerate version of umbral calculus called λ-umbral calculus were introduced by authors in [12]. In addition, Kwon-Wongsason-Kim-Kim defined modified type 2 degenerate poly-Bernoulli polynomials arising from polylogarithm function in [5].

For $n, k \in \mathbb{N} \cup \{0\}$, the Stirling numbers of the first kind $S_1(n,k)$ and the Stirling numbers of the second kind $S_2(n,k)$, respectively, are defined by the generating function to be as follows:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{2}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$ are the falling factorial sequences. By equation (2), we can derive the generating function of these numbers as follows (see [13–15]):

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \text{ and } \frac{1}{k!}\left(e^t - 1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{3}$$

Let $\log_\lambda(t)$ be the compositional inverse function. Then, by the definition of compositional inverse function, we have the following equation:

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!}, \tag{4}$$

where $x_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, and $(n \geq 1)$ are degenerate falling factorial sequences.

As degenerate version of the Stirling numbers of the first and second kind in equations (2) and (3), the degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ and the degenerate Stirling numbers of the second kind $S_{2,\lambda}(n,k)$ are, respectively, introduced by Kim-Kim (see [6, 7]) as follows:

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \text{ and } \frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{5}$$

For a given integer k and a nonzero real number λ, the degenerate polylogarithm functions are defined by the following equation:

$$Li_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}x^n, \tag{6}$$

and when $k = 2$ or 3 and $\lambda = 0.9, 1, 1.9, 2, 2.9$ or 3, the graphs of $Li_{k,\lambda}(x)$ are shown in Figure 1.

By equations (4) and (6), we have the following equation:

$$Li_{1,\lambda}(x) = -\log_\lambda(1-x) \text{ and } \lim_{\lambda \longrightarrow 0} Li_{k,\lambda}(x) = Li_k(x), \tag{7}$$

where $Li_k(x)$ are the polylogarithm functions which is defined by the following equation:

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, (|x| < 1). \tag{8}$$

By using polylogarithm functions, Lim-Kwon defined the poly-Daehee polynomials by generating function to be as follows:

$$\frac{\log(1+t)}{Li_k\left(1-e^{-t}\right)}(1+t)^x = \sum_{n=0}^{\infty} D_n^{(k)}(x)\frac{t^n}{n!}. \tag{9}$$

By using equation (1), the higher-order degenerate Bernoulli polynomials are defined by the generating function to be as follows:

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{t}{e_\lambda(t) - 1}\right)^r e_\lambda^x(t). \tag{10}$$

In the special case $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ is called the higher-order degenerate Bernoulli numbers.

As a generalization of degenerate Bernoulli polynomials, Kim-Kim defined the *d*egenerate poly-Bernoulli polynomials in [9] as follows:

$$\frac{Li_{k,\lambda}(1-e_\lambda(-t))}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{11}$$

When $x = 0$, $B_{n,\lambda}^{(k)}(0) = B_{n,\lambda}^{(k)}$ is called the degenerate poly-Bernoulli numbers.

Among the many tools that can be used to study the properties of special functions, umbral calculus is one of the very useful tools. A rigorous theoretical foundations of umbral calculus are built by Rota in the 1970s based on relatively modern ideas of linear functions, linear operators (see [15, 16]).

Some different umbral calculus have been introduced over the past 30 years, and many interesting results have been found by using these powerful tools, and even now, studies using umbral calculus are being actively conducted by many researchers (see [15, 16]). In particular, the modern umbral calculus were introduced in [15–17], and authors in [18] defined q-umbral calculus and gave some interesting identities related some special functions. The λ-umbral calculus and its applications were introduced in [5, 12, 15–17, 19, 20].

2. Review of the λ-Umbral Calculus

From now on, we introduced some basic facts about the λ-umbral calculus which are introduced by Kim-Kim (see [12, 19]).

Let $\mathbb{C}$ be the complex numbers field,

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty} a_k \frac{t^n}{n!} \middle| a_k \in \mathbb{C} \right\}, \tag{12}$$

and let

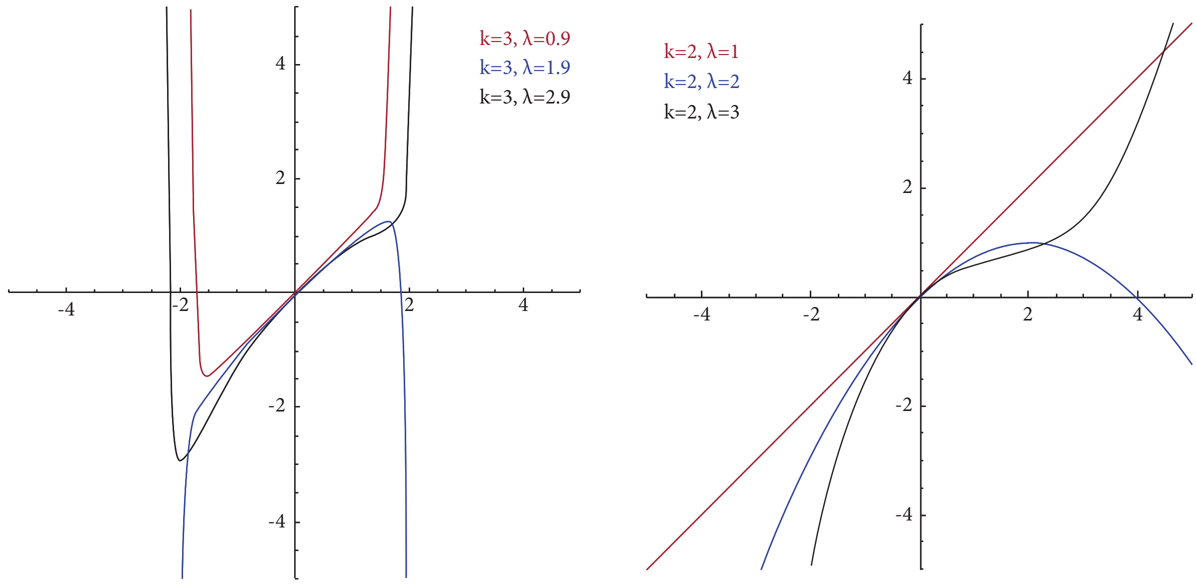

FIGURE 1: The shapes of degenerate polylogarithm functions $Li_{k,\lambda}(x)$.

$$\mathbb{P} = \mathbb{C}[x] = \left\{ \sum_{k=0}^{\infty} a_k x^k \middle| a_k \in \mathbb{C} \text{ with } a_k = 0 \text{ for all but finite number of } k \right\}. \tag{13}$$

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. Then, for each non-negative real number λ, the linear functional $\langle f(t)|\cdot\rangle_\lambda$ on $\mathbb{P}$ which are called λ-linear functional given by $f(t)$ is defined by the following equation:

$$\langle f(t)|(x)_{n,\lambda}\rangle_\lambda = a_n, \ (n \geq 0). \tag{14}$$

From equations (14), we see that

$$\langle t^k|(x)_{n,\lambda}\rangle_\lambda = n!\delta_{n,k}, \quad (n,k \geq 0), \tag{15}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Kim-Kim defined the differential operator on $\mathbb{P}$ by the following equation:

$$\left(t^k\right)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & \text{if } k \leq n, \\ 0, & \text{if } k \geq n, \end{cases} \tag{16}$$

for each nonzeor real number λ and each non-negative integer k. By equation (16), we see that for any $f(t) = \sum_{k=0}^{\infty} a_k t^k/k! \in \mathcal{F}$,

$$(f(t))_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}. \tag{17}$$

Moreover, they showed that for each $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$,

$$\begin{aligned} \langle f(t)g(t)|p(x)\rangle_\lambda &= \langle g(t)|(f(t))_\lambda p(x)\rangle_\lambda \\ &= \langle f(t)|(g(t))_\lambda p(x)\rangle_\lambda. \end{aligned} \tag{18}$$

By equations (14)–(17), we see that

$$\begin{aligned} \langle e_\lambda^y(t)|g(x)\rangle_\lambda &= g(y), \\ \langle e_\lambda^y(t) - 1|g(x)\rangle_\lambda &= g(y) - g(0), \\ \left\langle \frac{e_\lambda^y(t) - 1}{t} \middle| g(x) \right\rangle_\lambda &= \int_0^y p(u)du. \end{aligned} \tag{19}$$

For given $f(t) \in \mathcal{F} - \{0\}$, the order of $f(x)$ denoted by $o(f(t))$ is the smallest positive integer k that the coefficient of t^k does not zero. $f(x)$ is called invertible if $o(f(t)) = 0$, and delta series if $o(f(t)) = 1$. Note that, if $f(t)$ is invertible then there is a multiplication inverse $1/f(t)$, and if $f(t)$ is a delta series then $f(t)$ has the compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then, there is the unique polynomial sequence $S_{n,\lambda}(x)$ of x with $\deg S_{n,\lambda}(x) = n$ satisfying the orthogonality conditions as follows:

$$\langle g(t)(f(t))^k|S_{n,\lambda}(x)\rangle_\lambda = n!\delta_{n,k}, \quad (n,k \geq 0). \tag{20}$$

Here, $S_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t), f(t))$, and denoted by $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$. It is well-known fact that the sequence $S_{n,\lambda}(x)$ is the λ-Sheffer sequence for $(g(t), f(t))$ if and only if

$$\frac{1}{g(\overline{f}(t))} e_\lambda^y(\overline{f}(t)) = \sum_{n=0}^{\infty} S_{n,\lambda}(y) \frac{t^n}{n!}, \tag{21}$$

for all $y \in \mathbb{C}$.

The following lemma and theorem are important results in the λ-umbral calculus.

Lemma 1 ([12, 19]). *Let $S_{n,\lambda}(x)$ be the λ-Sheffer sequence of $(g(t), f(t))$ and let $h(x) = \sum_{l=0}^{n} a_l S_{l,\lambda}(x) \in \mathbb{P}$. Then,*

$$a_k = \frac{1}{k!}\langle g(t)(f(t))^k \big| h(x)\rangle_\lambda. \tag{22}$$

Proof. Let $S_{n,\lambda}(x)$ be the λ-Sheffer sequence of $(g(t), f(t))$ and let $h(x) = \sum_{l=0}^{n} a_l S_{l,\lambda}(x)$. Then,

$$\begin{aligned}\langle g(t)(f(t))^k \big| h(x)\rangle_\lambda &= \sum_{l=0}^{n} a_l \langle g(t)(f(t))^k \big| S_{l,\lambda}(x)\rangle_\lambda \\ &= k! a_k.\end{aligned} \tag{23}$$

Thus, our proof is completed. □

Theorem 1 ([12]). *Let $s_{n,\lambda}$ and $r_{n,\lambda}$ be the λ-Sheffer sequence of $(g(t), f(t))$ and $(h(t), l(t))$, respectively. Then, we have the following equation:*

$$s_{n,\lambda} = \sum_{k=0}^{n} c_{n,k} r_{k,\lambda}, \tag{24}$$

where

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))}(l(\overline{f}(t)))^k \Big| (x)_{n,\lambda}\right\rangle_\lambda. \tag{25}$$

Since

$$(x)_n \sim \left(1, e_\lambda(t) - 1\right)_\lambda \text{ and } (x)_{n,\lambda} \sim (1, t)_\lambda, \tag{26}$$

by Theorem 1, we see that

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda} \text{ and } (x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k. \tag{27}$$

In this article, we find some interesting identities among degenerate poly-Daehee polynomials, degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Bernoulli polynomials of the second kind, degenerate Daehee polynomials, Changhee polynomials, degenerate Bell polynomials, and degenerate Lah-Bell polynomials. In particular, we find the relationships among those special polynomials by finding the coefficients when expressing degenerate poly-Daehee polynomials as a linear combination of these polynomials which is motivated by the Kim-Kim methods (see [12]).

3. Degenerate Poly-Daehee Polynomials

In view point of equations (1), (9), and (11), we defined the degenerate poly-Daehee polynomials by the generating function to be as follows:

$$\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1 - e_\lambda(-t))} e_\lambda^x(\log_\lambda(1+t)) = \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{28}$$

In the special case $x = 0$, $D_{n,\lambda}^{(k)}(0) = D_{n,\lambda}^{(k)}$ is called the degenerate poly-Daehee numbers.

By equations (27) and (28), we get the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} D_{n-m,\lambda}^{(k)}(x)_m\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m} D_{n-m,\lambda}^{(k)} S_{1,\lambda}(m,l)(x)_{l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{29}$$

By comparing both sides of equation (29), we can obtain the relationship between $D_{n,\lambda}^{(k)}(x)$ and the falling factorial sequences as follows:

$$D_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m} D_{n-m,\lambda}^{(k)} S_{1,\lambda}(m,l)(x)_{l,\lambda}, \ (n \geq 0). \tag{30}$$

Note that, by equation (5), we get the following equation:

$$\begin{aligned}Li_{k,\lambda}\left(1 - e_\lambda\left(1 - e_\lambda(t)\right)\right) &= \sum_{n=1}^{\infty}\frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}\frac{1}{n!}\left(e_\lambda\left(1 - e_\lambda(t)\right) - 1\right)^n \\ &= \sum_{b=1}^{\infty}\sum_{n=1}^{b}\frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}} S_{2,\lambda}(b,n)\frac{1}{b!}\left(1 - e_\lambda(t)\right)^b \\ &= \sum_{a=1}^{\infty}\sum_{b=1}^{a}\sum_{n=1}^{b}\frac{(-1)^{b+1}\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}} S_{2,\lambda}(b,n) S_{2,\lambda}(a,b)\frac{t^a}{a!}.\end{aligned} \tag{31}$$

Theorem 2. *For each non-negative integer n, we have the following equation:*

$$D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\left(\sum_{r=l}^{n}\binom{n}{r}S_{1,\lambda}(r,l)D_{n-r,\lambda}^{(k)}\right)(x)_{l,\lambda}. \tag{32}$$

Proof. Note that, by equation (28), the λ-Sheffer sequence of degenerate poly-Daehee polynomials is as follows:

$$D_{n,\lambda}^{(k)}(x)\sim\left(\frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t},e_\lambda(t)-1\right)_\lambda. \tag{34}$$

By equation (14), (16), (21), and (34), we get the following equation:

$$\begin{aligned}D_{n,\lambda}^{(k)}(x)&=\left\langle\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}e_\lambda^x(\log_\lambda(1+t))\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{l=0}^{n}(x)_{l,\lambda}\left\langle\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\middle|\left(\frac{1}{l!}(\log_\lambda(1+t))^l\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{l=0}^{n}\sum_{r=l}^{n}\binom{n}{l}S_{1,\lambda}(r,l)(x)_{l,\lambda}\left\langle\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\middle|(x)_{n-r,\lambda}\right\rangle_\lambda\\&=\sum_{l=0}^{n}\sum_{r=l}^{n}\binom{n}{r}S_{1,\lambda}(r,l)D_{n-r,\lambda}^{(k)}(x)_{l,\lambda}.\end{aligned} \tag{35}$$

and so our proof is completed.

As the inversion formula of equation (32), we have the following equation:

$$(x)_{n,\lambda}=\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{b=1}^{n-m+1}\sum_{c=1}^{b}\binom{n}{m}\frac{(-1)^{b+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}(n-m+1)}S_{2,\lambda}(b,c)S_{2,\lambda}(n-m+1,b)S_{2,\lambda}(m,l)\right)D_{l,\lambda}^{(k)}(x). \tag{33}$$

Let $(x)_{n,\lambda}=\sum_{l=0}^{n}c_{n,l}D_{l,\lambda}^{(k)}(x)$. By Theorem 1 and equations (26) and (31), we get the following equation:

$$\begin{aligned}c_{n,l}&=\frac{1}{l!}\left\langle\frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t}(e_\lambda(t)-1)^l\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\left\langle\frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t}\middle|\left(\frac{1}{l!}(e_\lambda(t)-1)^l\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{m=l}^{n}\binom{n}{m}S_{2,\lambda}(m,l)\left\langle\frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t}\middle|(x)_{n-m,\lambda}\right\rangle_\lambda\\&=\sum_{m=l}^{n}\binom{n}{m}\frac{S_{2,\lambda}(m,l)}{n-m+1}\langle Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))|(x)_{n-m+1,\lambda}\rangle_\lambda\\&=\sum_{m=l}^{n}\sum_{b=1}^{n-m+1}\sum_{c=1}^{b}\binom{n}{m}\frac{(-1)^{b+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}(n-m+1)}S_{2,\lambda}(b,c)S_{2,\lambda}(n-m+1,b)S_{2,\lambda}(m,l).\end{aligned} \tag{36}$$

Thus our proofs are completed. □

By equation (10), we can know that the λ-Sheffer sequences of degenerate Bernoulli polynomials are

$$\beta_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, t\right)_\lambda. \tag{37}$$

Theorem 3. *For each non-negative integer n, we have the following equation:*

$$D_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r}S_{1,\lambda}(m,l)b_{r,\lambda}D_{n-m-r,\lambda}^{(k)}\right)\beta_{l,\lambda}(x). \tag{38}$$

As the inversion formula of equation (38), we have the following equation:

$$\begin{aligned}\beta_{n,\lambda}(x) = \sum_{l=0}^{n}\Bigg(&\sum_{m=l}^{n}\sum_{r=0}^{n-m}\sum_{b=1}^{n-m-r+1}\sum_{c=1}^{b}\binom{n}{m}\binom{n-m}{r}\\ &\times\frac{(-1)^{b+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}(n-m-r+1)}S_{2,\lambda}(m,l)S_{2,\lambda}(b,c)S_{2,\lambda}(n-m-r+1,b)\beta_{r,\lambda}\Bigg)D_{l,\lambda}^{(k)}(x).\end{aligned} \tag{39}$$

Proof. Let $D_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} c_{n,l}\beta_{l,\lambda}(x)$. By Theorem 1 and equations (34) and (37), we get the following equation:

$$\begin{aligned}c_{n,l} &= \frac{1}{l!}\left\langle \frac{t/\log(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))/\log_\lambda(1+t)}(\log_\lambda(1+t))^l \middle| (x)_{n,\lambda}\right\rangle_\lambda\\
&= \left\langle \frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\frac{t}{\log_\lambda(1+t)} \middle| \left(\frac{1}{l!}(\log_\lambda(1+t))^l\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda\\
&= \sum_{m=l}^{n}\binom{n}{m}S_{1,\lambda}(m,l)\left\langle \frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))} \middle| \left(\frac{t}{\log_\lambda(1+t)}\right)_\lambda (x)_{n-m,\lambda}\right\rangle_\lambda\\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r}S_{1,\lambda}(m,l)b_{r,\lambda}\left\langle \frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))} \middle| (x)_{n-m-r,\lambda}\right\rangle_\lambda\\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r}S_{1,\lambda}(m,l)b_{r,\lambda}D_{n-m-r,\lambda}^{(k)},\end{aligned} \tag{40}$$

where $b_{n,\lambda}$ are the degenerate Bernoulli numbers of the second kind defined as follows:

$$\sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!} = \frac{t}{\log_\lambda(1+t)}. \tag{41}$$

Conversely, we assume that $\beta_{n,\lambda}(x) = \sum_{l=0}^{n} c_{n,l} D_{l,\lambda}^{(k)}(x)$. Then, by Theorem 1 and equations (31), (34), and (37), we get the following equation:

$$\begin{aligned}
c_{n,l} &= \frac{1}{l!}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))/t}{e_\lambda(t)-1/t}(e_\lambda(t)-1)^l \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t}\, \frac{t}{e_\lambda(t)-1} \middle| \left(\frac{1}{l!}(e_\lambda(t)-1)^l\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\binom{n}{m} S_{2,\lambda}(m,l)\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t} \middle| \left(\frac{t}{e_\lambda(t)-1}\right)_\lambda (x)_{n-m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r} S_{2,\lambda}(m,l)\beta_{r,\lambda}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t} \middle| (x)_{n-m-r,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r} \frac{S_{2,\lambda}(m,l)\beta_{r,\lambda}}{n-m-r+1}\left\langle Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t))) \middle| (x)_{n-m-r+1,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\sum_{b=1}^{n-m-r+1}\sum_{c=1}^{b}\binom{n}{m}\binom{n-m}{r} \\
&\quad\times \frac{(-1)^{b+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}(n-m-r+1)} S_{2,\lambda}(m,l) S_{2,\lambda}(b,c) S_{2,\lambda}(n-m-r+1,b)\beta_{r,\lambda}.
\end{aligned} \tag{42}$$

□

The degenerate Daehee polynomials are defined by the generating function to be as follows:

$$\sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!} = \frac{\log_\lambda(1+t)}{t} e_\lambda^x(\log_\lambda(1+t)),\ (n \ge 0). \tag{43}$$

When $x=0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ is called the degenerate Daehee numbers. By equation (43), we see that

$$\left(\frac{e_\lambda(t)-1}{t}, e_\lambda(t)-1\right)_\lambda \sim D_{n,\lambda}(x). \tag{44}$$

Note that

$$\begin{aligned}
Li_{k,\lambda}(1-e_\lambda(-t)) &= \sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}(1-e_\lambda(-t))^n \\
&= \sum_{n=1}^{\infty}\frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}\frac{1}{n!}(e_\lambda(-t)-1)^n \\
&= \sum_{n=1}^{\infty}\frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}\sum_{l=n}^{\infty} S_{2,\lambda}(l,n)\frac{(-t)^l}{l!} \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(-1)^{n+1}\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}} S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{45}$$

Theorem 4. *For each non-negative integer n, we have the following equation:*

$$D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\left(\sum_{r=0}^{n-l}\binom{n}{l}\binom{n-l}{r}b_{r,\lambda}D_{n-l-r,\lambda}^{(k)}\right)D_{l,\lambda}(x). \tag{46}$$

As the inversion formula of equation (46), we have the following equation:

$$D_{n,\lambda}(x)=\sum_{l=0}^{n}\left(\sum_{m=1}^{n-l+1}\binom{n}{l}\frac{(-1)^{n-l}\lambda^{n-l}(1)_{m,1/\lambda}}{m^{k-1}(n-l+1)}S_{2,\lambda}(n-l+1,m)\right)D_{l,\lambda}^{(k)}(x). \tag{47}$$

Proof. Let $D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}c_{n,l}D_{l,\lambda}(x)$. Then, by Theorem 1 and equations (34) and (44), we get the following equation:

$$\begin{aligned}c_{n,l}&=\frac{1}{l!}\left\langle\frac{t/\log_{\lambda}(1+t)}{Li_{k,\lambda}(1-e_{\lambda}(-t))/\log_{\lambda}(1+t)}t^{l}\middle|(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\frac{1}{l!}\left\langle\frac{t}{\log_{\lambda}(1+t)}\frac{\log_{\lambda}(1+t)}{Li_{k,\lambda}(1-e_{\lambda}(-t))}\middle|(t^{l})_{\lambda}(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\binom{n}{l}\left\langle\frac{\log_{\lambda}(1+t)}{Li_{k,\lambda}(1-e_{\lambda}(-t))}\middle|\left(\frac{t}{\log_{\lambda}(1+t)}\right)_{\lambda}(x)_{n-l,\lambda}\right\rangle_{\lambda}\\&=\sum_{r=0}^{n-l}\binom{n}{l}\binom{n-l}{r}b_{r,\lambda}\left\langle\frac{\log_{\lambda}(1+t)}{Li_{k,\lambda}(1-e_{\lambda}(-t))}\middle|(x)_{n-l-r,\lambda}\right\rangle_{\lambda}\\&=\sum_{r=0}^{n-l}\binom{n}{l}\binom{n-l}{r}b_{r,\lambda}D_{n-l-r,\lambda}^{(k)}.\end{aligned} \tag{48}$$

Conversely, we assume that $D_{n,\lambda}(x)=\sum_{l=0}^{n}c_{n,l}D_{l,\lambda}^{(k)}(x)$. Then, by Theorem 1 and equations (31), (34), (44), and (45), we get the following equation:

$$\begin{aligned}c_{n,l}&=\frac{1}{l!}\left\langle\frac{Li_{k,\lambda}(1-e_{\lambda}(-t))/\log_{\lambda}(1+t)}{t/\log_{\lambda}(1+t)}t^{l}\middle|(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\frac{1}{l!}\left\langle\frac{Li_{k,\lambda}(1-e_{\lambda}(-t))}{t}\middle|(t^{l})_{\lambda}(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\binom{n}{l}\left\langle\frac{Li_{k,\lambda}(1-e_{\lambda}(-t))}{t}\middle|(x)_{n-l,\lambda}\right\rangle_{\lambda}\\&=\binom{n}{l}\frac{1}{n-l+1}\left\langle Li_{k,\lambda}(1-e_{\lambda}(-t))\middle|(x)_{n-l+1,\lambda}\right\rangle_{\lambda}\\&=\sum_{m=1}^{n-l+1}\binom{n}{l}\frac{(-1)^{n-l}\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}(n-l+1)}S_{2,\lambda}(n-l+1,m).\end{aligned} \tag{49}$$

□

The Changhee polynomials are defined by the generating function to be as follows:

$$\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{2}{t+2} e_\lambda^x\left(\log_\lambda(1+t)\right). \tag{50}$$

In the special case of $x=0$, $Ch_n = Ch_n(0)$ is called the Changhee numbers.

By the definition of the Changhee polynomials, we see the λ-Sheffer sequences of the Changhee polynomials are as follows:

$$Ch_n(x) \sim \left(\frac{1+e_\lambda(t)}{2}, e_\lambda(t)-1\right)_\lambda. \tag{51}$$

Note that, by equation (6), we get the following equation:

$$\begin{aligned}
\frac{Li_{k,\lambda}(1-e_\lambda(-t))}{\log_\lambda(1+t)} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}(1-e_\lambda(-t))^n \frac{1}{\log_\lambda(1+t)} \\
&= \sum_{n=1}^{\infty} \frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}} \frac{1}{n!}(e_\lambda(-t)-1)^n \frac{1}{\log_\lambda(1+t)} \\
&= \sum_{n=1}^{\infty} \frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}} \sum_{l=n}^{\infty} S_{2,\lambda}(l,n)\frac{(-t)^l}{l!} \frac{1}{\log_\lambda(1+t)} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n+1}\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}} S_{2,\lambda}(n,m)\frac{t^{n-1}}{n!} \frac{t}{\log_\lambda(1+t)} \\
&= \left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1} \frac{(-1)^{n}\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}} \frac{S_{2,\lambda}(n+1,m)}{n+1}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!}\right) \\
&= \sum_{a=0}^{\infty}\left(\sum_{r=0}^{a}\sum_{m=1}^{r+1}\binom{a}{r}\frac{(-1)^r\lambda^{m-1}(1)_{m,1/\lambda}}{m^{k-1}(r+1)} S_{2,\lambda}(r+1,m) b_{a-r,\lambda}\right)\frac{t^a}{a!}.
\end{aligned} \tag{52}$$

Theorem 5. *For each non-negative integer n, we have the following equation:*

$$D_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n-1}\left(\binom{n}{l} D_{n-l,\lambda}^{(k)} + \frac{1}{2}\binom{n}{l}(n-l) D_{n-l-1,\lambda}^{(k)}\right) Ch_l(x) + Ch_n(x). \tag{53}$$

As the inversion formula of equation (53), we have the following equation:

$$\begin{aligned} Ch_n(x) &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n-l}\sum_{r=0}^{n-l-m}\sum_{s=1}^{r+1}\binom{n}{l}\binom{n-l}{m}\binom{n-l-m}{r}\right.\\ &\quad\left.\times\frac{(-1)^r\lambda^{s-1}(1)_{s,1/\lambda}}{s^{k-1}}\frac{S_{2,\lambda}(r+1,m)}{r+1}Ch_m b_{n-l-m-r,\lambda}\right)D_{l,\lambda}^{(k)}(x)\\ &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{r=l}^{k}\sum_{b=1}^{k-r+1}\sum_{s=1}^{b}\binom{n}{m}\binom{k}{r}\frac{(-1)^{b+1}\lambda^{s-1}(1)_{s,1/\lambda}}{s^{k-1}}\right.\\ &\quad\left.\times\frac{S_{1,\lambda}(m,k)S_{2,\lambda}(r,l)S_{2,\lambda}(b,s)S_{2,\lambda}(k-r+1,b)}{k-r+1}Ch_{n-m}\right)D_{l,\lambda}^{(k)}(x). \end{aligned}\tag{54}$$

Proof. Let $D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}c_{n,l}Ch_l(x)$. By Theorem 1 and equations (34) and (51), we get the following equation:

$$\begin{aligned} c_{n,l} &= \frac{1}{l!}\left\langle\frac{t+2/2}{Li_{k,\lambda}(1-e_\lambda(-t))/\log_\lambda(1+t)}t^l\,\middle|\,(x)_{n,\lambda}\right\rangle_\lambda\\ &= \frac{1}{l!}\left\langle\frac{t+2}{2}\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\,\middle|\,\left(t^l\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda\\ &= \binom{n}{l}\left\langle\frac{t+2}{2}\,\middle|\,\left(\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\right)_\lambda(x)_{n-l,\lambda}\right\rangle_\lambda\\ &= \sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}D_{m,\lambda}^{(k)}\left\langle\frac{t+2}{2}\,\middle|\,(x)_{n-l-m,\lambda}\right\rangle_\lambda\\ &= \sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\frac{D_{m,\lambda}^{(k)}}{2}\left(\langle t|(x)_{n-l-m,\lambda}\rangle_\lambda+2\langle 1|(x)_{n-l-m,\lambda}\rangle_\lambda\right)\\ &= \sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}\frac{D_{m,\lambda}^{(k)}}{2}\left(\delta_{1,n-l-m}+2\delta_{0,n-l-m}\right). \end{aligned}\tag{55}$$

Since $D_{0,\lambda}^{(k)}=1$,

$$D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n-1}\left(\binom{n}{l}D_{n-l,\lambda}^{(k)}+\frac{1}{2}\binom{n}{l}(n-l)D_{n-l-1,\lambda}^{(k)}\right)Ch_l(x)+Ch_n(x).\tag{56}$$

Conversely, we assumed that $Ch_n(x) = \sum_{l=0}^{n} c_{n,l} D_{l,\lambda}^{(k)}(x)$. Then, by Theorem 1 and equations (34), (51), and (52), we get the following equation:

$$\begin{aligned}
c_{n,l} &= \frac{1}{l!}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(-t))/\log_\lambda(1+t)}{t+2/2} t^l \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{l!}\left\langle \frac{2}{t+2}\frac{Li_{k,\lambda}(1-e_\lambda(-t))}{\log_\lambda(1+t)} \middle| \left(t^l\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \binom{n}{l}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(-t))}{\log_\lambda(1+t)} \middle| \left(\frac{2}{t+2}\right)_\lambda (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m} Ch_m \left\langle \frac{Li_{k,\lambda}(1-e_\lambda(-t))}{\log_\lambda(1+t)} \middle| (x)_{n-l-m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n-l}\sum_{r=0}^{n-l-m}\sum_{s=1}^{r+1}\binom{n}{l}\binom{n-l}{m}\binom{n-l-m}{r}\frac{(-1)^r\lambda^{s-1}(1)_{s,1/\lambda}}{s^{k-1}}\frac{S_{2,\lambda}(r+1,m)}{r+1} Ch_m b_{n-l-m-r,\lambda}.
\end{aligned} \tag{57}$$

In addition, by Lemma 1 and equations (5) and (27), we have the following equation:

$$\begin{aligned}
c_{n,l} &= \frac{1}{l!}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t}(e_\lambda(t)-1)^l \middle| Ch_n(x)\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m} S_{1,\lambda}(m,k) Ch_{n-m}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t} \middle| \left(\frac{1}{l!}(e_\lambda(t)-1)^l\right)_\lambda (x)_{k,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{r=l}^{k}\binom{n}{m}\binom{k}{r} S_{1,\lambda}(m,k) S_{2,\lambda}(r,l) Ch_{n-m}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t} \middle| (x)_{k-r,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{r=l}^{k}\sum_{b=1}^{k-r+1}\sum_{s=1}^{b}\binom{n}{m}\binom{k}{r}\frac{(-1)^{b+1}\lambda^{s-1}(1)_{s,1/\lambda}}{s^{k-1}(k-r+1)} \\
&\quad \times S_{1,\lambda}(m,k) S_{2,\lambda}(r,l) S_{2,\lambda}(b,s) S_{2,\lambda}(k-r+1,b) Ch_{n-m}.
\end{aligned} \tag{58}$$

□

The degenerate Bell polynomials are defined by the generating function to be as follows:

$$e_\lambda^x\left((e_\lambda(t)-1)\right)=\sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}. \tag{59}$$

Note that

$$\begin{aligned}\sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}&=e_\lambda^x\left((e_\lambda(t)-1)\right)\\&=\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{1}{m!}(e_\lambda(t)-1)^m\\&=\sum_{m=0}^{\infty}(x)_{m,\lambda}\sum_{l=m}^{\infty}S_{2,\lambda}(l,m)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)(x)_{m,\lambda}\right)\frac{t^n}{n!},\end{aligned} \tag{60}$$

and thus

$$Bel_{n,\lambda}(x)=\sum_{m=0}^{n}S_{2,\lambda}(n,m)(x)_{m,\lambda}. \tag{61}$$

In addition, by the definition of Bell polynomials, the λ-Sheffer sequence of Bell polynomials is as follows

$$Bel_{n,\lambda}(x)\sim\left(1,\log_\lambda(1+t)\right)_\lambda. \tag{62}$$

Note that, in addition, we know that

$$\begin{aligned}\frac{1}{k!}\left(\log_\lambda(1+\log_\lambda(1+t))\right)^k&=\sum_{l=k}^{\infty}S_{1,\lambda}(l,k)\frac{1}{l!}\left(\log_\lambda(1+t)\right)^l\\&=\sum_{l=k}^{\infty}\sum_{m=l}^{\infty}S_{1,\lambda}(l,k)S_{1,\lambda}(m,l)\frac{t^m}{m!}\\&=\sum_{l=k}^{\infty}\sum_{m=k}^{l}S_{1,\lambda}(m,k)S_{1,\lambda}(l,m)\frac{t^l}{l!}.\end{aligned} \tag{63}$$

Theorem 6. *For each non-negative integer n, we have the following equation:*

$$D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\left(\sum_{a=l}^{n}\sum_{r=l}^{a}\binom{n}{a}S_{1,\lambda}(r,l)S_{1,\lambda}(a,r)D_{n-a,\lambda}^{(k)}\right)Bel_{l,\lambda}(x). \tag{64}$$

As the inversion formula of equation (64), we have the following equation:

$$\begin{aligned}Bel_{n,\lambda}(x)&=\sum_{l=0}^{n}\left(\sum_{a=l}^{n}\sum_{r=l}^{a}\sum_{m=0}^{n-a}\sum_{b=1}^{n-a-m+1}\sum_{s=1}^{b}\sum_{c=1}^{s}\binom{n}{a}\binom{n-a}{m}\frac{(-1)^{s+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}}\right.\\&\quad\left.\times\frac{S_{2,\lambda}(s,c)S_{2,\lambda}(b,s)S_{2,\lambda}(n-a-m+1,b)S_{2,\lambda}(r,l)S_{2,\lambda}(a,r)}{n-a-m+1}B_{m,\lambda}\right)D_{l,\lambda}^{(k)}(x)\\&=\sum_{l=0}^{n}\left(\sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=1}^{m-a+1}\sum_{r=1}^{b}\binom{m}{a}\frac{(-1)^{b+1}\lambda^{r-1}(1)_{r,1/\lambda}}{r^{k-1}}\right.\\&\quad\left.\times\frac{S_{2,\lambda}(n,m)S_{2,\lambda}(a,l)S_{2,\lambda}(b,r)S_{2,\lambda}(m-a+1,b)}{m-a+1}\right)D_{l,\lambda}^{(k)}(x).\end{aligned} \tag{65}$$

Proof. Let $D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}c_{n,l}\mathrm{Bel}_{l,\lambda}(x)$. By Theorem 1 and equations (34), (63), and (62), we get the following equation:

$$\begin{aligned}c_{n,l}&=\frac{1}{l!}\left\langle\frac{1}{Li_{k,\lambda}(1-e_\lambda(-t))/\log_\lambda(1+t)}\left(\log_\lambda(1+\log_\lambda(1+t))\right)^l\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\left\langle\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\middle|\left(\frac{1}{l!}\left(\log_\lambda(1+\log_\lambda(1+t))\right)^l\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{a=l}^{n}\sum_{r=l}^{a}\binom{n}{a}S_{1,\lambda}(r,l)S_{1,\lambda}(a,r)\left\langle\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\middle|(x)_{n,a,\lambda}\right\rangle_\lambda\\&=\sum_{a=l}^{n}\sum_{r=l}^{a}\binom{n}{a}S_{1,\lambda}(r,l)S_{1,\lambda}(a,r)D_{n-a,\lambda}^{(k)}.\end{aligned} \tag{66}$$

Conversely, we assume that $Bel_{n,\lambda}(x) = \sum_{l=0}^{n} c_{n,l} D_{l,\lambda}^{(k)}(x)$. and Note that

$$\begin{aligned}
&\frac{1}{l!}\left(e_\lambda\left(e_\lambda(t)-1\right)-1\right)^l \\
&= \sum_{r=l}^{\infty} S_{2,\lambda}(r,l)\frac{1}{r!}\left(e_\lambda(t)-1\right)^r \\
&= \sum_{r=l}^{\infty} S_{2,\lambda}(r,l)\sum_{m=r}^{\infty} S_{2,\lambda}(m,r)\frac{t^m}{m!} \\
&= \sum_{a=l}^{\infty}\sum_{r=l}^{a} S_{2,\lambda}(r,l)S_{2,\lambda}(a,r)\frac{t^a}{a!}.
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\frac{1}{t}Li(1-e(1-e(e(t)-1))) \\
&= \frac{1}{t}\sum_{n=1}^{\infty}\frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k}(1-e(1-e(e(t)-1)))^n \\
&= \frac{1}{t}\sum_{n=1}^{\infty}\frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}\ \frac{1}{n!}(e(1-e(e(t)-1))-1)^n \\
&= \frac{1}{t}\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\frac{-\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}S_{2,\lambda}(m,n)\frac{1}{m!}(1-e(e(t)-1))^m \\
&= \frac{1}{t}\sum_{a=1}^{\infty}\sum_{n=1}^{a}\frac{(-1)^{a+1}\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}S_{2,\lambda}(a,n)\frac{1}{a!}(e(e(t)-1))^a \\
&= \frac{1}{t}\sum_{b=1}^{\infty}\sum_{a=1}^{b}\sum_{n=1}^{a}\frac{(-1)^{a+1}\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}S_{2,\lambda}(a,n)S_{2,\lambda}(b,a)\frac{1}{b!}(e(t)-1)^b \\
&= \frac{1}{t}\sum_{b=1}^{\infty}\sum_{a=1}^{b}\sum_{n=1}^{a}\sum_{s=b}^{\infty}\frac{(-1)^{a+1}\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}S_{2,\lambda}(a,n)S_{2,\lambda}(b,a)S_{2,\lambda}(s,b)\frac{t^s}{s!} \\
&= \frac{1}{t}\sum_{c=1}^{\infty}\sum_{b=1}^{c}\sum_{a=1}^{b}\sum_{n=1}^{a}\frac{(-1)^{a+1}\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}}S_{2,\lambda}(a,n)S_{2,\lambda}(b,a)S_{2,\lambda}(c,b)\frac{t^c}{c!} \\
&= \sum_{c=0}^{\infty}\sum_{b=1}^{c+1}\sum_{a=1}^{b}\sum_{n=1}^{a}\frac{(-1)^{a+1}\lambda^{n-1}(1)_{n,1/\lambda}}{n^{k-1}(c+1)}S_{2,\lambda}(a,n)S_{2,\lambda}(b,a)S_{2,\lambda}(c+1,b)\frac{t^c}{c!}.
\end{aligned} \tag{68}$$

Then, by Theorem 1 and equations (31), (34), (62), (67), and (68), we get the following equation:

$$\begin{aligned}
c_{n,l} &= \frac{1}{l!}\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda\left(e_\lambda(t)-1\right)\right)\right)}{e_\lambda(t)-1}\left(e_\lambda\left(e_\lambda(t)-1\right)-1\right)^l \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda\left(e_\lambda(t)-1\right)\right)\right)}{t}\frac{t}{e_\lambda(t)-1}\middle| \left(\frac{1}{l!}\left(e_\lambda\left(e_\lambda(t)-1\right)-1\right)^l\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{a=l}^{n}\sum_{r=l}^{a} S_{2,\lambda}(r,l)S_{2,\lambda}(a,r)\binom{n}{a}\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda\left(e_\lambda(t)-1\right)\right)\right)}{t}\middle| \left(\frac{t}{e_\lambda(t)-1}\right)_\lambda (x)_{n-a,\lambda}\right\rangle_\lambda \\
&= \sum_{a=l}^{n}\sum_{r=l}^{a}\sum_{m=0}^{n-a}\binom{n}{a}\binom{n-a}{m}S_{2,\lambda}(r,l)S_{2,\lambda}(a,r)\beta_{m,\lambda} \\
&\quad\times\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda\left(e_\lambda(t)-1\right)\right)\right)}{t}\middle| (x)_{n-a-m,\lambda}\right\rangle_\lambda \\
&= \sum_{a=l}^{n}\sum_{r=l}^{a}\sum_{m=0}^{n-a}\sum_{b=1}^{n-a-m+1}\sum_{s=1}^{b}\sum_{c=1}^{s}\binom{n}{a}\binom{n-a}{m}\frac{(-1)^{s+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}(n-a-m+1)} \\
&\quad\times S_{2,\lambda}(s,c)S_{2,\lambda}(b,s)S_{2,\lambda}(n-a-m+1,b)S_{2,\lambda}(r,l)S_{2,\lambda}(a,r)\beta_{m,\lambda}.
\end{aligned} \tag{69}$$

On the other hand, by Lemma 1 and equation (31), we have the following equation:

$$\begin{aligned}
c_{n,l} &= \frac{1}{l!}\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right)}{t}\left(e_\lambda(t)-1\right)^l \middle| Bel_{n,\lambda}(x)\right\rangle_\lambda \\
&= \sum_{m=0}^{n} S_{2,\lambda}(n,m)\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right)}{t}\middle| \left(\frac{1}{l!}\left(e_\lambda(t)-1\right)^l\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{a=l}^{m}\binom{m}{a}S_{2,\lambda}(n,m)S_{2,\lambda}(a,l)\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right)}{t}\middle| (x)_{m-a,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{a=l}^{m}\sum_{b=1}^{m-a+1}\sum_{r=1}^{b}\binom{m}{a}\frac{(-1)^{b+1}\lambda^{r-1}(1)_{r,1/\lambda}}{r^{k-1}(m-a+1)}S_{2,\lambda}(n,m)S_{2,\lambda}(a,l)S_{2,\lambda}(b,r)S_{2,\lambda}(m-a+1,b).
\end{aligned} \tag{70}$$

□

The degenerate Euler polynomials are defined by the generating function to be as follows:

$$\sum_{n=0}^{\infty}\mathscr{E}_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{e_\lambda(t)+1}e_\lambda^x(t). \tag{71}$$

In the special case of $x=0$, $\mathscr{E}_{n,\lambda}=\mathscr{E}_{n,\lambda}(0)$ is called the degenerate Euler numbers. By equation (71), we see that

$$\mathscr{E}_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)+1}{2}, t\right)_\lambda. \tag{72}$$

Theorem 7. *For each non-negative integer n, we have the following equation:*

$$D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\left(\sum_{m=l}^{n-1}\binom{n}{m}S_{1,\lambda}(m,l)\left(D_{n-m,\lambda}+\frac{n-m}{2}D_{n-m-1,\lambda}\right)+S_{1,\lambda}(n,l)\right)\mathscr{E}_{l,\lambda}(x). \tag{73}$$

As the inversion formula of equation (73), we have the following equation:

$$\begin{aligned}\mathscr{E}_{n,\lambda}(x)=\sum_{l=0}^{n}\Bigg(&\sum_{m=l}^{n}\sum_{r=0}^{n-m}\sum_{b=1}^{n-m-r+1}\sum_{c=1}^{b}\binom{n}{m}\binom{n-m}{r}\\&\times\frac{(-1)^{b+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}(n-m-r+1)}S_{2,\lambda}(m,l)S_{2,\lambda}(b,c)S_{2,\lambda}(n-m-r+1,b)\mathscr{E}_{r,\lambda}\Bigg)D_{l,\lambda}^{(k)}(x).\end{aligned} \tag{74}$$

Proof. Let $D_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}c_{n,l}\mathscr{E}_{l,\lambda}(x)$. By Theorem 1 and equations (34) and (72), we get the following equation:

$$\begin{aligned}c_{n,l}&=\frac{1}{l!}\left\langle\frac{t+2/2}{Li_{k,\lambda}(1-e_\lambda(-t))/\log_\lambda(1+t)}(\log_\lambda(1+t))^l\Big|(x)_{n,\lambda}\right\rangle_\lambda\\&=\left\langle\frac{t+2}{2}\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\Big|\left(\frac{1}{l!}(\log_\lambda(1+t))^l\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{m=l}^{n}\binom{n}{m}S_{1,\lambda}(m,l)\left\langle\frac{t+2}{2}\Big|\left(\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\right)_\lambda(x)_{n-m,\lambda}\right\rangle_\lambda\\&=\frac{1}{2}\sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r}S_{1,\lambda}(m,l)D_{r,\lambda}^{(k)}\langle t+2|(x)_{n-m-r,\lambda}\rangle_\lambda\\&=\sum_{m=l}^{n-1}\binom{n}{m}S_{1,\lambda}(m,l)\left(D_{n-m,\lambda}+\frac{n-m}{2}D_{n-m-1,\lambda}\right)+S_{1,\lambda}(n,l),\end{aligned} \tag{75}$$

because of $D_{0,\lambda}^{(k)}=1$.

Conversely, we assume that $\mathscr{E}_{n,\lambda}(x)=\sum_{l=0}^{n}c_{n,l}D_{l,\lambda}^{(k)}(x)$. Then, by Theorem 1 and (31), (34), and (72), we get the following equation:

$$\begin{aligned}
c_{n,l} &= \frac{1}{l!}\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right)/t}{e_\lambda(t)+1/2}\left(e_\lambda(t)-1\right)^l \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right)}{t}\,\frac{2}{e_\lambda(t)+1} \middle| \left(\frac{1}{l!}\left(e_\lambda(t)-1\right)^l\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\binom{n}{m} S_2\lambda(m,l)\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right)}{t} \middle| \left(\frac{2}{e_\lambda(t)+1}\right)_\lambda (x)_{n-m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r} S_{2,\lambda}(m,l)\mathscr{E}_{r,\lambda}\left\langle \frac{Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right)}{t} \middle| (x)_{n-m-r,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\binom{n}{m}\binom{n-m}{r} \frac{S_{2,\lambda}(m,l)\mathscr{E}_{r,\lambda}}{n-m-r+1}\left\langle Li_{k,\lambda}\left(1-e_\lambda\left(1-e_\lambda(t)\right)\right) \middle| (x)_{n-m-r+1,\lambda}\right\rangle_\lambda \\
&= \sum_{m=l}^{n}\sum_{r=0}^{n-m}\sum_{b=1}^{n-m-r+1}\sum_{c=1}^{b}\binom{n}{m}\binom{n-m}{r} \\
&\quad\times \frac{(-1)^{b+1}\lambda^{c-1}(1)_{c,1/\lambda}}{c^{k-1}(n-m-r+1)} S_{2,\lambda}(m,l)S_{2,\lambda}(b,c)S_{2,\lambda}(n-m-r+1,b)\mathscr{E}_{r,\lambda}.
\end{aligned} \tag{76}$$

□

The unsigned Lah number $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets and has the explicit formula as follows:

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}. \tag{77}$$

By equation (77), we can derive the generating function of $L(n,k)$ to be as follows:

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!}, \ (k \geq 0). \tag{78}$$

Recently, Kim-Kim introduced the degenerate Lah-Bell polynomials as follows:

$$e_\lambda^x\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} B_{n,\lambda}^L(x)\frac{t^n}{n!}. \tag{79}$$

In the special case of $x=1$, $B_n^L = B_n^L(1)$ is called Lah-Bell numbers.

Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}^L(x)\frac{t^n}{n!} &= e_\lambda^x\left(\frac{t}{1-t}\right) \\
&= \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{1}{n!}\left(\frac{t}{1-t}\right)^n \\
&= \sum_{n=0}^{\infty}(x)_{n,\lambda}\sum_{m=n}^{\infty}L(m,n)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}L(n,m)(x)_{m,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{80}$$

Thus, we get the following equation:

$$B_{n,\lambda}^L(x) = \sum_{m=0}^{n} L(n,m)(x)_{m,\lambda}. \tag{81}$$

In addition, by the definition of Lah-Bell polynomials, the λ-sheffer sequence of Lah-Bell polynomials is as follows:

$$B_{n,\lambda}^L(x) \sim \left(1, \frac{t}{t+1}\right)_\lambda. \tag{82}$$

Note that, for a given non-negative integer l,

$$\begin{aligned}\frac{1}{l!}\left(\frac{\log_\lambda(1+t)}{1+\log_\lambda(1+t)}\right)^l &= \sum_{r=0}^{\infty}(-1)^r <l>_r \binom{r+l}{l}\frac{1}{(r+l)!}\left(\log_\lambda(1+t)\right)^{r+l}\\ &= \sum_{a=0}^{\infty}\sum_{r=0}^{a}(-1)^r <l>_r \binom{r+l}{l} S_{1,\lambda}(a+l,r+l)\frac{t^{a+l}}{(a+l)!},\end{aligned} \tag{83}$$

where $<x>_0 = 1$, $<x>_n = x(x+1)\cdots(x-n+1)$, and $(n\geq 1)$ are rising factorial sequences.

Theorem 8. *For each non-negative integer n, we have the following equation:*

$$D_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n}\left(\sum_{a=0}^{n-l}\sum_{r=0}^{a}\binom{n}{a+l}\binom{r+l}{l}(-1)^r <l>_r S_{1,\lambda}(a+l,r+l)D_{n-a-l,\lambda}^{(k)}\right)B_{l,\lambda}^{L}(x). \tag{84}$$

As the inversion formula of equation (84), we have the following equation:

$$\begin{aligned}B_{n,\lambda}^{L}(x) = \sum_{l=0}^{n}\Bigg(&\sum_{m=0}^{n}\sum_{r=l}^{m}\sum_{b=1}^{m-r+1}\sum_{a=1}^{b}\binom{m}{r}\frac{(-1)^{b+1}\lambda^{a-1}(1)_{a,1/\lambda}}{a^{k-1}}\\ &\times\frac{L(n,m)S_{2,\lambda}(b,a)S_{2,\lambda}(r,l)S_{2,\lambda}(m-r+1,b)}{m-r+1}\Bigg)D_{l,\lambda}^{(k)}(x).\end{aligned} \tag{85}$$

Proof. Let $D_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} c_{n,l}B_{l,\lambda}^{L}(x)$. By Theorem 1 and equations (34), (82), and (83), we get the following equation:

$$\begin{aligned}c_{n,l} &= \frac{1}{l!}\left\langle\frac{1}{Li_{k,\lambda}(1-e_\lambda(-t))/\log_\lambda(1+t)}\left(\frac{\log_\lambda(1+t)}{1+\log_\lambda(1+t)}\right)^l\middle|(x)_{n,\lambda}\right\rangle_\lambda\\ &= \left\langle\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\middle|\left(\frac{1}{l!}\left(\frac{\log_\lambda(1+t)}{1+\log_\lambda(1+t)}\right)^l\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda\\ &= \sum_{a=0}^{n-l}\sum_{r=0}^{a}(-1)^r <l>_r\binom{r+l}{l}\binom{n}{a+l}S_{1,\lambda}(a+l,r+l)\\ &\quad\times\left\langle\frac{\log_\lambda(1+t)}{Li_{k,\lambda}(1-e_\lambda(-t))}\middle|(x)_{n-a-l,\lambda}\right\rangle_\lambda\\ &= \sum_{a=0}^{n-l}\sum_{r=0}^{a}\binom{n}{a+l}\binom{r+l}{l}(-1)^r <l>_r S_{1,\lambda}(a+l,r+l)D_{n-a-l,\lambda}^{(k)}.\end{aligned} \tag{86}$$

Conversely, we assume that $B_{n,\lambda}^{L}(x)=\sum_{l=0}^{n}c_{n,l}D_{l,\lambda}^{(k)}(x)$. Then, by Lemma 1 and equations (31) and (34), we get the following equation:

$$\begin{aligned}
c_{n,l} &= \frac{1}{l!}\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t}(e_\lambda(t)-1)^l \middle| B_{n,\lambda}^{L}(x)\right\rangle_\lambda \\
&= \sum_{m=0}^{n} L(n,m)\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t} \middle| \left(\frac{1}{l!}(e_\lambda(t)-1)^l\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{r=l}^{m}\binom{m}{l} L(n,m)S_{2,\lambda}(r,l)\left\langle \frac{Li_{k,\lambda}(1-e_\lambda(1-e_\lambda(t)))}{t} \middle| (x)_{m-r,\lambda}\right\rangle_\lambda \\
&= \sum_{m=0}^{n}\sum_{r=l}^{m}\sum_{b=1}^{m-r+1}\sum_{a=1}^{b}\binom{m}{r}\frac{(-1)^{b+1}\lambda^{a-1}(1)_{a,1/\lambda}}{a^{k-1}(m-r+1)} \\
&\quad\times L(n,m)S_{2,\lambda}(b,a)S_{2,\lambda}(r,l)S_{2,\lambda}(m-r+1,b).
\end{aligned} \tag{87}$$

□

4. Conclusion

Degenerate exponential function was first defined by Carlitz (see [6]), and their relationships and properties with various special polynomials are being actively studied by many researchers. In addition, its extension is also being studied a lot (see [19–22]).

In this article, we find some relationships between some special polynomials and degenerate poly-Daehee polynomials by expressing linear combinations of degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Bernoulli polynomials of the second kind, degenerate Daehee polynomials, Changhee polynomials, degenerate Bell polynomials, degenerate Lah-Bell polynomials, and vice versa.

Research on the generalization of various special functions using the polylogarithm function has been studied by many researchers. In the near future, we will continue research to obtain new and interesting identities between the special functions and other polynomials by using the tools used in this study.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

JWP conceived the framework and structured the whole article. SJY and JWP wrote the article. All authors have read and agreed to the published version of the manuscript.

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References

[1] G. E. Andrew, R. Askey, and R. Roy, *Special Functions: Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, 1999.

[2] T. Nahid, M. Saif, and S. Araci, "A new class of Appell-type Changhee-Euler polynomials and related properties," *AIMS Mathematics*, vol. 6, pp. 13566–13579, 2021.

[3] A. Boussayoud, S. Boughaba, and S. Araci, "Complete symmetric functions in several variables and k-Fibonacci numbers," *Utilitas Mathematica*, vol. 115, pp. 23–37, 2020.

[4] Y. Simsek, "New classes of recurrence relations involving hyperbolic functions, special numbers and polynomials," *Applicable Analysis and Discrete Mathematics*, vol. 15, no. 2, pp. 426–443, 2021.

[5] J. Kwon, P. Wongsason, Y. Kim, and D. Kim, "Representations of modified type 2 degenerate poly-Bernoulli polynomials," *AIMS Mathematics*, vol. 7, no. 6, pp. 11443–11463, 2022.

[6] L. Carlitz, "Degenerate stirling, bernoulli and eulerian numbers," *Utill Mathametics*, vol. 15, pp. 51–88, 1979.

[7] T. Kim, "A note on degenerate Stirling polynomials of the second kind," *Proc. Jangjeon Math. Soc.*vol. 20, no. 3, pp. 319–331, 2017.

[8] T. Kim and D. S. Kim, "An identity of symmetry for the degenerate Frobenius-Euler polynomials," *Mathematica Slovaca*, vol. 68, pp. 239–243, 2018.

[9] D. S. Kim and T. Kim, "A note on a new type of degenerate Bernoulli numbers," *Russian Journal of Mathematical Physics*, vol. 27, no. 2, pp. 227–235, 2020.

[10] T. Kim and D. S. Kim, "Degenerate polyexponential functions and degenerate Bell polynomials," *Journal of Mathematical*

Analysis and Applications, vol. 487, no. 2, Article ID 124017, 2020.

[11] T. Kim, D. S. Kim, H. Y. Kim, H. Lee, and L. C. Jang, "Degenerate poly-Bernoulli polynomials arising from degenerate polylogarithm," *Advances in Difference Equations*, vol. 2020, no. 1, p. 444, 2020.

[12] D. S. Kim and T. Kim, "Degenerate Sheffer sequences and λ-Sheffer sequences *λ-Sheffer sequences*," *Journal of Mathematical Analysis and Applications*, vol. 493, no. 1, Article ID 124521, 2021.

[13] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co, Dorrecht, Netherlands, 1974.

[14] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*, World Scientific, Singapore, 2016.

[15] S. Roman, *The Umbral Calculus*, Dover Publ. Inc, New York, NY, USA, 2005.

[16] K. S. Nisar, *Umbral Calculus*, LAP LAMBERT Academic Publishing, Saarland, Germany, 2012.

[17] R. Dere and Y. Simsek, "Applications of umbral algebra to some special polynomials," *Adv. Stud. Contemp. Math. (Kyungshang)*, vol. 22, no. 3, pp. 433–438, 2012.

[18] S. Araci, M. Acikgoz, T. Diagana, and H. M. Srivastava, "A novel approach for obtaining new identities for the lambda extension of q-Euler polynomials arising from the q-umbral calculusλextension of -qEuler polynomials arising from the *-qumbral calculus*," *The Journal of Nonlinear Science and Applications*, vol. 10, no. 04, pp. 1316–1325, 2017.

[19] H. K. Kim, "Degenerate Lah-Bell polynomials arising from degenerate Sheffer sequences," *Advances in Difference Equations*, vol. 687, p. 16, 2020.

[20] J. W. Park, B. M. Kim, and J. Kwon, "Some identities of the degenerate Bernoulli polynomials of the second kind arising from λ-Sheffer sequences," *Proceedings of the Jangjeon Mathematical Society*, vol. 24, no. 3, pp. 323–342, 2021.

[21] D. Lim and J. Kwon, "A note on poly-Daehee numbers and polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 19, pp. 291–224, 2016.

[22] N. L. Wang and H. Li, "Some identities on the higher-order daehee and changhee numbers," *Pure and Applied Mathematics Journal*, vol. 4, no. 5-1, pp. 33–37, 2015.