Hindawi
Journal of Mathematics
Volume 2023, Article ID 2335570, 20 pages
https://doi.org/10.1155/2023/2335570

Research Article

Modeling the Effects of Algal Bloom on Dissolved Oxygen in Eutrophic Water Bodies

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Received 5 January 2023; Revised 20 February 2023; Accepted 13 April 2023; Published 8 May 2023

Academic Editor: Chang Phang

Algal blooms have a wide variety of detrimental effects on both the aquatic environment and human activities unfortunately, including a decrease in dissolved oxygen levels in water. In this study, a nonlinear mathematical model has been developed to study the effects of algal bloom on the depletion of dissolved oxygen in eutrophic water sources considering nutrient concentrations, density of algal population, detritus, and concentration of dissolved oxygen as variables. The model's existence and uniqueness, stability at the equilibrium points, and characteristics with respect to state variables have been performed as some parts of analytical solution, whereas the numerical solution has been performed using the Runge–Kutta 4^{th} order technique. The findings of this research reveal that an increase in the quantity of nutrients causes an algal bloom, which in turn lowers the equilibrium level of dissolved oxygen. However, higher levels of detritus (> 4.7 mg/litre) are hazardous to the generation of dissolved oxygen in water systems. Maximum detritus of 1.0 mg/liter is required for fast algal and dissolved oxygen growth in water bodies; beyond that, algal and dissolved oxygen growth rates are lower. Therefore, increasing public awareness is an essential step in controlling this growing issue; failing to do so will result in problems for our water supply, which will in turn pose a danger to our ecosystem.

1. Introduction

In this era of modernization, it is evident that most of the lakes around us are nutrients dense (nitrogen, phosphorus, etc.), mostly by chemical wastes from different factories, wastewater outflow from agricultural areas, as well as residential drainage. This excess amount of nutrients causes algal bloom which results in eutrophication of water bodies. Algal bloom is a dense layer of tiny green plants that occur on the surface of lakes and other bodies of water when there is an overabundance of nutrients (primarily phosphorus) on which algae depend [1]. It affects the water bodies severely and ends up in the depletion of dissolved oxygen. Nowadays, it has become a matter of great concern in both urban and rural areas. More than 40% of the world's water bodies undergo eutrophication at varying amounts, based on the fifth Global Environment Outlook (GEO-5) [2], and it is increasing year after year at an alarming rate. As a consequence of eutrophication, the water bodies become unfit for the survival of living organisms which hampers the aquatic ecosystem. Recently, the effect of budget allocation for controlling algal bloom in a lake has been discussed by Misra et al. [3]. Here, the interactions of concentration of nutrients, density of algal population, detritus, and dissolved oxygen levels are considered in this model. The objectives of our study are to analyze and find the characteristics among those variables with respect to some parameters that we will be considering.

Algal bloom is a rapid growth of microscopic algae or cyanobacteria in water, often resulting in a colored scum on the surface. These so-called algal blooms can lead to a release of toxins, taste, and odor problems and finally the depletion of dissolved oxygen in water bodies [4, 5]. Eutrophication is characterized by a significant growth of algae due to the overabundance of one or more growth factors necessary for photosynthesis, such as sunlight, carbon dioxide, and nutrients (nitrogen and phosphorus) [6, 7]. When algae are starting to grow in an unrestrained manner, an increasingly large biomass is formed, which is destined to demote. A huge quantity of organic matter collects in deep water, indicating that algae have reached the end of their life cycle. To consume all of the dead algae, microbes have to use a large amount of oxygen; in some instances, nearly all of it. Oxygen-free environment in aquatic bodies is thus created on the bottom. Without treatment, the algae will grow more every year, resulting in an unbalanced ecosystem. In order to get a clear concept of the model, the diagram in Figure 1 shows the synopsis of our work.

A research study showed that the quantity of dissolved oxygen in the water fluctuates on a regular basis, i.e., on a daily basis, since oxygen is created by photosynthesis (when there is light) and consumed by respiration (when there is darkness) (all the time) [8]. Controlling algal bloom is a much needed step for a healthy aquatic ecosystem nowadays. There were several studies about the effects of the discharge of nutrients in water bodies, causing eutrophication, conducted by some investigators. Smith presented a concept of algal deposition in his research [9]. Voinov and Tonkikh proposed a model on eutrophication in macrophyte lakes based on the assumption which is nutrient is supplied only by the detritus of algae and macrophytes [10]. They did not take the additional supply of nutrients either from household drainage or from water due to agricultural lands. Jorgenson discussed an eutrophication model on a lake using ecological concepts [11, 12]. Jayaweera and Asaeda illustrated mathematically biomanipulation in shallow eutrophic lakes related to phytoplankton, zooplankton, detritus, bacteria, and fish population, but the supply of nutrients from outside was not considered there [13]. Various ecological modeling studies included phytoplankton, zooplankton, and nutrients have also been conducted by Busenburg et al. [14] and Hallam [15]. They had not assumed the density of dissolved oxygen for the modeling process. Hulla et.al. studied about the dynamics of dissolved oxygen on coastal lagoons [16]. Shukla et al. also analyzed algal bloom and eutrophication [17]. Misra also analyzed the eutrophication caused by nutrients only and analyzed the depletion of dissolved oxygen [18]. A laboratory simulation revealed that the depletion of bloom-forming algae had an effect on the dissolved oxygen levels in coastal waters also [19]. After all of these, a new mathematical model is proposed to study the effects of increasing nutrients and algae with decreasing dissolved oxygen on the survival of fish populations using Holling type II interaction. This model is based on the assumption that nutrients are constantly released into water bodies from different sources, which makes algae grow in large numbers, depleting the dissolved oxygen, and hurting marine life, and then some mathematical analysis has been conducted [20]. But our goal is to study the effects of algal bloom on the depletion of dissolved oxygen and to find the characteristics of the considered state variables with respect to different parameters. The main purpose of our study is to construct a nonlinear mathematical model to analyze the effects of algal bloom on the depletion of dissolved oxygen in water bodies corresponding to the current situation of Bangladesh. We will be focusing on finding out the characteristics of concentration of nutrients, density of algal population, detritus, and concentration of dissolved oxygen with respect to different parameters numerically as well as graphically.

2. Formulation of the Mathematical Model

Bangladesh is a riverine country, and fish and rice are their primary food. At the same time, many people of this country are directly and indirectly dependent on the river and the fish derived from the river but most of the zooplankton cannot survive in rivers and streams [21]. The majority of people in our nation are unaware, and as a result, they constantly throw away waste referred to as detritus which poses a serious danger to the environment's water resources. So, in the context of this country, this model plays a significant role on nutrients, algae, detritus, and dissolved oxygen. In continuation of this, we have considered a water resource such as canal which is being affected due to the overgrowth of algae caused by the discharge of nutrients from domestic drainage as well as from water runoff, industrial wastage, and also from nutrients formed from detritus. The bilinear interactions of variables such as the concentration of nutrients, density of algae (phytoplankton), density of detritus, and concentration of dissolved oxygen are considered. Due to domestic drainage, industrial wastage as well as water runoff from agriculture fields, various nutrients are supplied into the water body. These nutrients may also be supplied by the death of algae. We assumed that the phytoplankton population is wholly dependent on nutrients. It is further assumed that the level of dissolved oxygen in the water body increases by other various resources like diffusion, photosynthesis, or respiration by algae. We are going to consider concentration of nutrients, density of algae, density of detritus, and concentration of dissolved oxygen in the absence of zooplankton.

Let us consider n to be the concentration of various nutrients, a be the density of algae, s be the density of detritus, and c be the concentration of dissolved oxygen (DO). We assume that the rate of discharge of nutrients into the aquatic system from outside into water bodies is q due to natural factors which is depleted with the rate $\alpha_0 n$. The growth rate of nutrients due to detritus is $\pi_0 \delta S$ and depletion of nutrients by algae proportional to both the density of algae as well as the concentration of nutrients (i.e.na).

$$\frac{\mathrm{d}n}{\mathrm{d}t} = q + \pi_0 \delta s - \alpha_0 n - \beta_1 na. \tag{1}$$

Thus, the growth rate of algae is proportional to na as it is assumed to be wholly depended on the nutrients. The natural depletion rate of algae is assumed to be proportional to its density a.

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \theta_1 \beta_1 na - \alpha_1 a. \tag{2}$$

Since some part of natural depletion of algae is converted into detritus, it is assumed to be proportional to $\alpha_1 a$, and its natural depletion rate is assumed to be proportional to s.

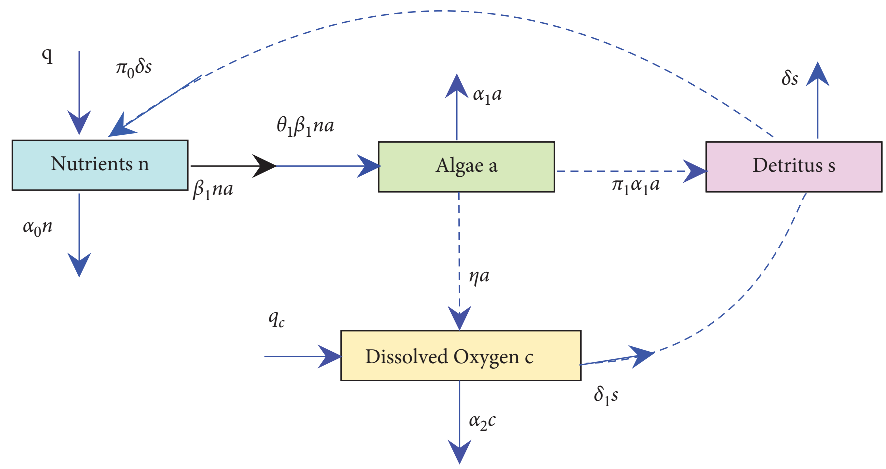


Figure 1: A diagram that shows the flow of the factors in eutrophic water bodies.

$$\frac{ds}{dt} = \pi_1\alpha_1 a - \delta s. \tag{3}$$

The rate of growth of dissolved oxygen by various sources q_c is assumed to be constant and its natural depletion rate is proportional to its concentration c. The rate of growth of dissolved oxygen by algae is proportional to a and the depletion of dissolved oxygen caused by decomposing the detritus is proportional to its concentration s.

$$\frac{dc}{dt} = q_c - \alpha_2 c + \eta a - \delta_1 s. \tag{4}$$

With the abovementioned assumptions, the algal bloom model is proposed in formulation of a nonlinear ordinary differential equation system [18] as follows:

$$\frac{dn}{dt} = q + \pi_0\delta s - \alpha_0 n - \beta_1 na = f(n,a,s,c), \tag{5}$$

$$\frac{da}{dt} = \theta_1\beta_1 na - \alpha_1 a = g(n,a,s,c), \tag{6}$$

$$\frac{ds}{dt} = \pi_1\alpha_1 a - \delta s = h(n,a,s,c), \tag{7}$$

$$\frac{dc}{dt} = q_c - \alpha_2 c + \eta a - \delta_1 s = p(n,a,s,c), \tag{8}$$

with non-negative initial conditions: $n(0)\geq 0, a(0)\geq 0, s(0)\geq 0, c(0)\geq 0$ Here, the positive co-efficients $\alpha_i; i = 0,1,2$ are depletion rate coefficients. $\beta_1, \theta_1, \delta,$ andδ_1 are proportionality constants which are positive. The positive η is the proportional constant for the growth rate of dissolved oxygen due to algae and $0 < \pi_0, \pi_1 < 1$.

3. Mathematical Analysis

The boundedness of state variables and the positivity of the system of nonlinear differential equations ((5)–(8)) have been tested in this section. Finding the equilibrium points, we have analyzed stability at the equilibrium points. Moreover, the analysis of the characteristics of state variables with respect to various parameters of the system ((5)–(8)) is presented in here.

3.1. Positivity

Theorem 1. *Considering* $n(0) > 0, a(0) > 0, s(0) > 0, c(0) > 0$, *it must be proved that* $n(t), a(t), s(t), c(t)$ *will be always positive for all* $t\epsilon[0,T]$ *in* R_4^+ *where* $T > 0$.

Proof. Taking all parameters of the system and all initial values to be positive, we have to prove that $n(t), a(t), s(t), c(t)$ will be positive for all $t\,\epsilon\,[0,T]$ in R_4^+. From the equation, we can write as follows:

$$\begin{aligned}\frac{da}{dt} &= \theta_1\beta na - \alpha_1 a,\\ \Longrightarrow \frac{da}{a} &= (\theta_1\beta n - \alpha_1)dt.\end{aligned} \tag{9}$$

Integrating both sides of the above equation, we have

$$\begin{aligned}\ln a &= \theta_1\beta\int ndt - \alpha_1\int dt,\\ \Longrightarrow a(t) &= e^{\theta_1\beta\int ndt - \alpha_1 t} > 0.\end{aligned} \tag{10}$$

Again,

$$\frac{ds}{dt} = \pi_1\alpha_1 a - \delta s. \tag{11}$$

Since $a(t) > 0$, $t\epsilon[0,T]$ where $T > 0$, we can write

$$\begin{aligned}&\Longrightarrow \frac{ds}{dt} > -\delta s,\\ &\Longrightarrow s(t) > e^{-\delta t + \xi} > 0,\end{aligned} \tag{12}$$

where ξ is an arbitrary constant.

Now,

$$\frac{\mathrm{dn}}{\mathrm{d}t} = q + \pi_0 \delta s - \alpha_0 n - \beta_1 \mathrm{na}. \tag{13}$$

As $a(t) > 0, s(t) > 0,\ t\epsilon[0, T]$ where $T > 0$, then we can write,

$$\begin{aligned} &\Longrightarrow \frac{\mathrm{d}n}{\mathrm{d}t} > (\alpha_0 - \beta_1 a)\mathrm{d}t, \\ &\Longrightarrow n(t) > e^{-\alpha_0 t - \beta_1 \int a\mathrm{d}t} > 0. \end{aligned} \tag{14}$$

Finally,

$$\begin{aligned} &\frac{\mathrm{d}c}{\mathrm{d}t} = q_c - \alpha_2 c + \eta a - \delta_1 S, \\ &\Longrightarrow c(t) > e^{-\alpha_2 t} > 0. \end{aligned} \tag{15}$$

This completes the proof. □

3.2. Boundedness of State Variables

Theorem 2. *If $0 < \pi_0, \pi_1 < 1$, then the state variables (cumulative concentration of various nutrients, the density of algae, and detritus) are bounded.*

Proof. Adding equations (5)–(7), we have

$$\begin{aligned} \frac{\mathrm{d}n}{\mathrm{d}t} + \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\mathrm{d}s}{\mathrm{d}t} &= q - (1 - \pi_0)\delta s - \alpha_0 n - (1 - \theta_1) na - (1 - \pi_1)\alpha_1 a \\ &\leq q - (1 - \pi_0)\delta s - \alpha_0 n - (1 - \pi_1)\alpha_1 a. \end{aligned} \tag{16}$$

Let $v = \min\{(1 - \pi_0)\delta, \alpha_0, (1 - \pi_1)\alpha_1\}$ for maximum of nutrients, algae, and detritus, $0 < \pi_0, \pi_1 < 1$

Taking the limit supremum, we get

$$\lim_{t \longrightarrow \infty} \mathrm{Sup}\{n(t) + a(t) + s(t)\} \leq \frac{q}{v}. \tag{17}$$

Hence, the state variables nutrients, algae, and detritus are bounded (Proved). □

3.3. Existence and Uniqueness of the Model's Solution

Theorem 3. *For all non-negative initial conditions, the solutions of the system exists, and they are also unique at the same time for all time $T \geq 0$.*

Proof. It has been decided to follow the Lipschitz criterion for the existence and uniqueness of a solution recommended by the suggested theorem [22]. For the purposes of our model, it is necessary to demonstrate that the partial derivative of the system exists and that it is continuous. Consider

$$\begin{aligned} f(n, a, s, c) &= \frac{\mathrm{d}n}{\mathrm{d}t} = q + \pi_0 \delta s - \alpha_0 n - \beta_1 na, \\ g(n, a, s, c) &= \frac{\mathrm{d}a}{\mathrm{d}t} = \theta_1 \beta na - \alpha_1 a, \\ h(n, a, s, c) &= \frac{\mathrm{d}s}{\mathrm{d}t} = \pi_1 \alpha_1 a - \delta s, \\ p(n, a, s, c) &= \frac{\mathrm{d}c}{\mathrm{d}t} = q_c - \alpha_2 c + \eta a - \delta_1 S. \end{aligned} \tag{18}$$

Using abovementioned system's equation, the partial derivative of f, g, h, p with respect to compartments n, a, s, c are obtained as follows:

$$\begin{aligned}
&\frac{\partial f}{\partial n} = -\alpha_0 - \beta_1 a, \quad \left|\frac{\partial f}{\partial n}\right| = \alpha_0 + \beta_1 a \leq \frac{q}{v} < \infty,\\
&\frac{\partial f}{\partial a} = -\beta_1 n, \quad \left|\frac{\partial f}{\partial a}\right| = \beta_1 n \leq \frac{q}{v} < \infty,\\
&\frac{\partial f}{\partial s} = \pi_0 \delta, \quad \left|\frac{\partial f}{\partial s}\right| = \pi_0 \delta \leq \frac{q}{v} < \infty,\\
&\frac{\partial f}{\partial c} = 0, \quad \left|\frac{\partial f}{\partial c}\right| = 0 \leq \frac{q}{v} < \infty.
\end{aligned} \tag{19}$$

Again,

$$\begin{aligned}
&\frac{\partial g}{\partial n} = \theta_1 \beta_1 a, \quad \left|\frac{\partial f}{\partial n}\right| = \theta_1 \beta_1 a \leq \frac{q}{v} < \infty,\\
&\frac{\partial g}{\partial a} = \theta_1 \beta_1 n - \alpha_1, \quad \left|\frac{\partial f}{\partial a}\right| = \alpha_1 - \theta_1 \beta_1 n \leq \frac{q}{v} < \infty,\\
&\frac{\partial g}{\partial s} = 0, \quad \left|\frac{\partial f}{\partial s}\right| = 0 \leq \frac{q}{v} < \infty,\\
&\frac{\partial g}{\partial c} = 0, \quad \left|\frac{\partial f}{\partial c}\right| = 0 \leq \frac{q}{v} < \infty.
\end{aligned} \tag{20}$$

Now,

$$\begin{aligned}
&\frac{\partial h}{\partial n} = 0, \quad \left|\frac{\partial h}{\partial n}\right| = 0 \leq \frac{q}{v} < \infty,\\
&\frac{\partial h}{\partial a} = \pi_1 \alpha_1, \quad \left|\frac{\partial h}{\partial a}\right| = \pi_1 \alpha_1 \leq \frac{q}{v} < \infty,\\
&\frac{\partial h}{\partial s} = -\delta, \quad \left|\frac{\partial h}{\partial s}\right| = \delta \leq \frac{q}{v} < \infty,\\
&\frac{\partial h}{\partial c} = 0, \quad \left|\frac{\partial h}{\partial c}\right| = 0 \leq \frac{q}{v} < \infty.
\end{aligned} \tag{21}$$

Finally,

$$\begin{aligned}
&\frac{\partial p}{\partial n} = 0, \quad \left|\frac{\partial p}{\partial n}\right| = 0 \leq \frac{q}{v} < \infty,\\
&\frac{\partial p}{\partial a} = \eta, \quad \left|\frac{\partial p}{\partial n}\right| = \eta \leq \frac{q}{v} < \infty,\\
&\frac{\partial p}{\partial s} = -\delta_1, \quad \left|\frac{\partial p}{\partial n}\right| = \delta_1 \leq \frac{q}{v} < \infty,\\
&\frac{\partial p}{\partial c} = -\alpha_2, \quad \left|\frac{\partial p}{\partial n}\right| = \alpha_2 \leq \frac{q}{v} < \infty.
\end{aligned} \tag{22}$$

According to the associated theorem [22], it is shown that f, g, h, p is locally continuous in R_4^+ and has a unique solution since all partial derivatives exists and are continuous. □

3.4. Equilibrium Analysis of the Model. The mathematical model is governed by differential equations ((5)–(8)). Now, for finding the equilibrium points of the model, the following relations are considered:

$$\left(\frac{\mathrm{d}n}{\mathrm{d}t}\right)_{(n^*,a^*,s^*,c^*)} = q + \pi_0 \delta s^* - \alpha_0 n^* - \beta_1 n^* a^* = 0, \tag{23}$$

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_{(n^*,a^*,s^*,c^*)} = \theta_1 \beta_1 n a^* - \alpha_1 a^* = 0, \tag{24}$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)_{(n^*,a^*,s^*,c^*)} = \pi_1 \alpha_1 a^* - \delta s^* = 0, \tag{25}$$

$$\left(\frac{\mathrm{d}c}{\mathrm{d}t}\right)_{(n^*,a^*,s^*,c^*)} = q_c - \alpha_2 c^* + \eta a^* - \delta_1 s^* = 0. \tag{26}$$

From equation (24), we have $a^* (\theta_1 \beta_1 n^* - \alpha_1) = 0$

$$\therefore a^* = 0 \text{ or } n^* = \frac{\alpha_1}{\theta_1 \beta_1}. \tag{27}$$

Substituting $a^* = 0$ in equation (25), we have

$$s^* = 0. \tag{28}$$

Substituting the values $a^* = 0$ and $s^* = 0$ in equation (23), we have

$$\begin{aligned}
&q + 0 - \alpha_0 n^* - 0 = 0\\
&\therefore n^* = \frac{q}{\alpha_0}.
\end{aligned} \tag{29}$$

Substituting the values $a^* = 0$ and $s^* = 0$ in equation (26), we have

$$\therefore c^* = \frac{q_c}{\alpha_2}. \tag{30}$$

Hence, the first equilibrium point is

$$E_1 (n^*, a^*, s^*, c^*) = \left(\frac{q}{\alpha_0}, 0, 0, \frac{q_c}{\alpha_2}\right). \tag{31}$$

Now, from equation (25), we can write

$$a^* = \frac{\delta s^*}{\pi_1 \alpha_1}. \tag{32}$$

Now, for n^* from equation (27), we have in equation (23),

$$q + \pi_0\delta s^* - \alpha_0\left(\frac{\alpha_1}{\theta_1\beta_1}\right) - \beta_1\left(\frac{\alpha_1}{\theta_1\beta_1}\right)a^* = 0,$$
$$\Longrightarrow q\theta_1\beta_1 - \alpha_0\alpha_1 - \alpha_1\beta_1 a^* + \pi_0\theta_1\beta_1\delta s^* = 0. \tag{33}$$

Substituting the value of a^* from (32) in the above equation, we have

$$q\theta_1\beta_1 - \alpha_0\alpha_1 - \alpha_1\beta_1\left(\frac{\delta s^*}{\pi_1\alpha_1}\right) + \pi_0\theta_1\beta_1\delta s^* = 0,$$
$$\therefore s^* = \frac{\pi_1\gamma}{\delta\beta_1}, \tag{34}$$

where

$$\gamma = \frac{\alpha_0\alpha_1 - q\theta_1\beta_1}{\pi_0\pi_1\theta_1 - 1}. \tag{35}$$

Putting the value of s^* from equation (34) in (32), we get

$$a^* = \frac{\delta\{\pi_1(\alpha_0\alpha_1 - q\theta_1\beta_1)/\beta_1\delta(\pi_0\pi_1\theta_1 - 1)\}}{\pi_1\alpha_1},$$
$$\therefore a^* = \frac{\gamma}{\alpha_1\beta_1}. \tag{36}$$

Putting the values a^* and s^* from equations (34) and (36), respectively, in equation (26), we have

$$q_c - \alpha_2 c^* + \eta\left\{\frac{\alpha_0\alpha_1 - q\theta_1\beta_1}{\alpha_1\beta_1(\pi_0\pi_1\theta_1 - 1)}\right\} - \delta_1\left\{\frac{\pi_1(\alpha_0\alpha_1 - q\theta_1\beta_1)}{\beta_1\delta(\pi_0\pi_1\theta_1 - 1)}\right\} = 0,$$
$$\Longrightarrow c^* = \frac{q_c}{\alpha_2} + \frac{\gamma\sigma}{\alpha_1\alpha_2\delta\beta_1}, \tag{37}$$

where $\sigma = \eta\delta - \delta_1\pi_1\alpha_1$

Hence, the other equilibrium point is

$$E_2(n^*, a^*, s^*, c^*) = \left(\frac{\alpha_1}{\theta_1\beta_1}, \frac{\gamma}{\alpha_1\beta_1}, \frac{\pi_1\gamma}{\delta\gamma}, \frac{q_c}{\alpha_2} + \frac{\gamma\sigma}{\alpha_1\alpha_2\delta\beta_1}\right). \tag{38}$$

Case 1. $E_1(q/\alpha_0, 0, 0, q_c/\alpha_2)$: This equilibrium point states that algae is absent in the water resource. As detritus is formed due to natural death of algae, which is absent, so detritus is also absent. In this case, the concentrations of nutrients and dissolved oxygen reach their respective equilibrium values.

Case 2. $E_2(n^*, a^*, s^*, c^*)$: This equilibrium point is the most fascinating equilibrium in which all the system variables are present. In this case, the concentration of nutrients is less than the concentration of nutrients in E_1 because nutrients will be utilized for the growth of algae. Here, the concentration of dissolved oxygen will be less in comparison to the abovementioned case. Here, it may also be noted that we will get the positive values of n^*, a^*, s^* and c^* when $\alpha_0\alpha_1 > q\theta_1\beta_1$, $\pi_0\pi_1\theta_1 > 1$, and $\eta\delta > \delta_1\pi_1\alpha_1$.

3.5. Future Status of Algae Populations. We shall determine the future status of algae population through the use of the next generation matrix operator R_0. This class is taken into consideration for the discussion in order to investigate the behavior and dynamics of the algae compartments.

Consequently, we take

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \theta_1\beta na - \alpha_1 a. \tag{39}$$

Therefore, two matrices M and D which represent the raise and decline of algae population are $M = \theta_1\beta_1 n^*$ and $D = \alpha_1$. Then, we get

$$R_0 = \frac{M}{D},$$
$$\Longrightarrow R_0 = \frac{\theta_1\beta_1 n^*}{\alpha_1}. \tag{40}$$

Now we can say that a decrease in the number of algae will occur if $\theta_1\beta_1 n^* < \alpha_1$, but an increase will be seen if $\theta_1\beta_1 n^* > \alpha_1$.

3.6. Stability Analysis of the Model. We can check the stability of different equilibrium points of the model by computing the eigenvalue or applying Routh–Hurwitz criterion. Here, $f_n(n^*, a^*, s^*, c^*)$ denotes the derivative of $f(n, a, s, c)$ with respect to n at the point (n^*, a^*, s^*, c^*) and the rest are similar to this one. Now, the Jacobian matrix of the model ((5)–(8)) is

$$J_{(n^*,a^*,s^*,c^*)} = \begin{pmatrix} f_n(n^*,a^*,s^*,c^*) & f_a(n^*,a^*,s^*,c^*) & f_s(n^*,a^*,s^*,c^*) & f_c(n^*,a^*,s^*,c^*) \\ g_n(n^*,a^*,s^*,c^*) & g_a(n^*,a^*,s^*,c^*) & g_s(n^*,a^*,s^*,c^*) & g_c(n^*,a^*,s^*,c^*) \\ h_n(n^*,a^*,s^*,c^*) & h_a(n^*,a^*,s^*,c^*) & h_s(n^*,a^*,s^*,c^*) & h_c(n^*,a^*,s^*,c^*) \\ p_n(n^*,a^*,s^*,c^*) & p_a(n^*,a^*,s^*,c^*) & p_s(n^*,a^*,s^*,c^*) & p_c(n^*,a^*,s^*,c^*) \end{pmatrix},$$
$$\Longrightarrow J_{(n^*,a^*,s^*,c^*)} = \begin{pmatrix} -\alpha_0-\beta_1 a^* & -\beta_1 n^* & \pi_0\delta & 0 \\ \theta_1\beta_1 a^* & \theta_1\beta_1 n^* - \alpha_1 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\delta & 0 \\ 0 & \eta & -\delta_1 & -\alpha_2 \end{pmatrix}. \tag{41}$$

At the equilibrium point $E_1(q/\alpha_0, 0, 0, q_c/\alpha_2)$, (41) becomes

$$J_{E_1\left(\frac{q}{\alpha_0},0,0,\frac{q_c}{\alpha_3}\right)} = \begin{pmatrix} -\alpha_0 & -\dfrac{\beta_1 q}{\alpha_0} & \pi_0\delta & q_c-\alpha_3 \\ 0 & \dfrac{\theta_1\beta_1 q}{\alpha_0}-\alpha_1 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\delta & 0 \\ 0 & \eta & -\delta_1 & -\alpha_2 \end{pmatrix}. \tag{42}$$

Now, the characteristics equation of the Jacobian matrix is

$$|J-\lambda I| = \left| \begin{pmatrix} -\alpha_0 & -\dfrac{\beta_1 q}{\alpha_0} & \pi_0\delta & q_c-\alpha_3 \\ 0 & \dfrac{\theta_1\beta_1 q}{\alpha_0}-\alpha_1 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\delta & 0 \\ 0 & \eta & -\delta_1 & -\alpha_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \right| = 0,$$
$$\Longrightarrow (\alpha_0+\lambda)\left(\frac{-\theta_1\beta_1 q}{\alpha_0}+\alpha_1+\lambda\right)(\delta+\lambda)(\alpha_2+\lambda) = 0. \tag{43}$$

$\therefore \lambda = -\alpha_0, \lambda = -\delta, \lambda = -\alpha_2$ and $\lambda = -(\alpha_0\alpha_1 - \theta_1\beta_1 q)/\alpha_0$

Hence, the eigenvalues $\lambda = -\alpha_0, -\delta, -\alpha_2, -(\alpha_0\alpha_1 - \theta_1\beta_1 q)/\alpha_0$

Lemma 1. *For $\alpha_0\alpha_1 > \theta_1\beta_1 q$, the equilibrium point $E_1(q/\alpha_0, 0, 0, q_c/\alpha_2)$ is asymptotically stable.*

Proof. It is known that if the eigenvalues of the characteristic equation at an equilibrium point of any system are negative, then the system is considered as asymptotically stable at that point. The eigenvalues for the equilibrium point $E_1(q/\alpha_0, 0, 0, q_c/\alpha_2)$ are $\lambda = -\alpha_0, -\delta, -\alpha_2, -(\alpha_0\alpha_1 - \theta_1\beta_1 q)/\alpha_0$. Since, all the parameters used here are positive.

Therefore, the system will be asymptotically stable if and only if $\alpha_0\alpha_1 > \theta_1\beta_1 q$ (Proved).

Now, at the equilibrium point $E_2(n^*, a^*, s^*, c^*)$, the Jacobian (41) becomes

$$J_{E_2(n^*,a^*,s^*,c^*)} == \begin{pmatrix} -\alpha_0 - \dfrac{\alpha_0\alpha_1 - q\theta_1\beta_1}{\alpha_1(\pi_0\pi_1\theta_1 - 1)} & -\dfrac{\alpha_1}{\theta_1} & \pi_0\delta & 0 \\ \dfrac{\theta_1(\alpha_0\alpha_1 - q\theta_1\beta_1)}{\alpha_1(\pi_0\pi_1\theta_1 - 1)} & 0 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\delta & 0 \\ 0 & \eta & -\delta_1 & -\alpha_2 \end{pmatrix}. \tag{44}$$

Now, the characteristics equation of the Jacobian matrix is

$$|J - \lambda I| = \left| \begin{pmatrix} -\alpha_0 - \dfrac{\gamma}{\alpha_1} & -\dfrac{\alpha_1}{\theta_1} & \pi_0\delta & 0 \\ \dfrac{\theta_1\gamma}{\alpha_1} & 0 & 0 & 0 \\ 0 & \pi_1\alpha_1 & -\delta & 0 \\ 0 & \eta & -\delta_1 & -\alpha_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \right| = 0, \tag{45}$$

$$\Longrightarrow (\lambda + \alpha_2)\left(\alpha_1\lambda^3 + (\gamma + \alpha_0\alpha_1 + \alpha_1\delta)\lambda^2 + (\alpha_1\gamma + \gamma\delta + \alpha_0\alpha_1\delta)\lambda + \alpha_1\delta\gamma - \alpha_1\delta\gamma\pi_0\pi_1\theta_1\right) = 0.$$

Therefore,

$$\begin{aligned} &\lambda = -\alpha_2, \\ &\alpha_1\lambda^3 + (\gamma + \alpha_0\alpha_1 + \alpha_1\delta)\lambda^2 + (\alpha_1\gamma + \gamma\delta + \alpha_0\alpha_1\delta)\lambda \\ &\quad + \alpha_1\delta\gamma - \alpha_1\delta\gamma\pi_0\pi_1\theta_1 = 0. \end{aligned} \tag{46}$$

Comparing the above equation with

$$A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0, \tag{47}$$

we get

$$\begin{aligned} A_1 &= \alpha_1, \\ A_2 &= \gamma + \alpha_0\alpha_1 + \alpha_1\delta, \\ A_3 &= \alpha_1\gamma + \gamma\delta + \alpha_0\alpha_1\delta, \\ A_4 &= \alpha_1\delta\gamma - \alpha_1\delta\gamma\pi_0\pi_1\theta_1, \\ A_2A_3 - A_4A_1 &= \alpha_1\gamma^2 + \delta\gamma_2 + 2\alpha_0\alpha_1\gamma(\delta + \alpha_1) + \alpha_0^2\alpha_1^2\delta + \alpha_1\delta^2\gamma + \alpha_1^2\delta\gamma\pi_0\pi_1\theta_1. \end{aligned} \tag{48}$$

□

Lemma 2. *The equilibrium point $E_2(n^*, a^*, s^*, c^*)$ will be asymptotically stable if and only if $\alpha_0\alpha_1 > q\theta_1\beta_1$ and $\pi_0\pi_1\theta_1 > 1$.*

Proof. It is known that the Routh–Hurwitz criterion for asymptotically stable of a dynamical system of the equation is $A_1 > 0, A_2 > 0$ and $A_2A_3 - A_4A_1/A_2 > 0$. These conditions are satisfied completely if and only if $\alpha_0\alpha_1 > q\theta_1\beta_1$ and $\pi_0\pi_1\theta_1 > 1$. Then, the system at the equilibrium point $E_2(n^*, a^*, s^*, c^*)$ is asymptotically stable (Proved). □

3.7. Characteristics of State Variables

3.7.1. Characteristics of the Concentration of Dissolved Oxygen (c^*) and the Density of Algae (a^*) with respect to η. Substituting the values of n^* and s^* in the model ((5)–(8)), we have

$$j(c^*, a^*, \eta) = q_c - \alpha_2 c^* + \eta a^* - \frac{\delta_1\pi_1\alpha_1 a^*}{\delta},$$

$$k(c^*, a^*, \eta) = \frac{\beta_1\theta_1 a^* q}{\alpha_0 + \beta_1 a^*} + \frac{\pi_0\pi_1\alpha_1 a^{*2}\theta_1\beta_1}{\alpha_0 + \beta_1 a^*} - \alpha_1 a^*,$$

$$\therefore \frac{\mathrm{d}c^*}{\mathrm{d}\eta} = \frac{\begin{vmatrix} (\partial j/\partial a^*) & (\partial j/\partial \eta) \\ (\partial k/\partial a^*) & (\partial k/\partial \eta) \end{vmatrix}}{\begin{vmatrix} (\partial j/\partial c^*) & (\partial j/\partial a^*) \\ (\partial k/\partial c^*) & (\partial k/\partial a^*) \end{vmatrix}} \tag{49}$$

$$= \frac{(\partial j/\partial a^*)\cdot(\partial k/\partial \eta) - (\partial j/\partial \eta)\cdot(\partial k/\partial a^*)}{(\partial j/\partial a^*)\cdot(\partial k/\partial a^*) - (\partial j/\partial a^*)\cdot(\partial k/\partial a^*)}.$$

Here, $(\partial j/\partial a^*) = \eta - \delta_1\pi_1\alpha_1/\delta$, $(\partial j/\partial c^*) = -\alpha_2$, $(\partial k/\partial c^*) = 0$, $(\partial j/\partial \eta) = a^*$, $(\partial k/\partial \eta) = 0$,

$$\frac{\partial k}{\partial a^*} = \frac{(\alpha_0 + \beta_1 a^*)\beta_1\theta_1 q - {\beta_1}^2\theta_1 q a^*}{(\alpha_0 + \beta_1 a^*)^2} + \frac{(\alpha_0 + \beta_1 a^*)\pi_0\pi_1\alpha_1 2a^*\theta_1\beta_1 - \pi_0\pi_1\alpha_1 a^{*2}\theta_1{\beta_1}^2}{(\alpha_0 + \beta_1 a^*)^2} - \alpha_1$$

$$\cdot\ \frac{\partial j}{\partial a^*}\cdot\frac{\partial k}{\partial \eta} - \frac{\partial j}{\partial \eta}\cdot\frac{\partial k}{\partial a^*} \tag{50}$$

$$= -a^*\left\{\frac{(\alpha_0 + \beta_1 a^*)\beta_1\theta_1 q - {\beta_1}^2\theta_1 q a^*}{(\alpha_0 + \beta_1 a^*)^2} + \frac{(\alpha_0 + \beta_1 a^*)\pi_0\pi_1\alpha_1 2a^*\theta_1\beta_1 - \pi_0\pi_1\alpha_1 a^{*2}\theta_1{\beta_1}^2}{(\alpha_0 + \beta_1 a^*)^2} - \alpha_1\right\},$$

and now, the denominator of $(\mathrm{d}c^*/\mathrm{d}\eta)$ is as follows:

$$\frac{\partial j}{\partial c^*}\cdot\frac{\partial k}{\partial a^*} - \frac{\partial j}{\partial a^*}\cdot\frac{\partial k}{\partial c^*}$$

$$= -\alpha_2\left\{\frac{(\alpha_0 + \beta_1 a^*)\beta_1\theta_1 q - {\beta_1}^2\theta_1 q a^*}{(\alpha_0 + \beta_1 a^*)^2} + \frac{(\alpha_0 + \beta_1 a^*)\pi_0\pi_1\alpha_1 2a^*\theta_1\beta_1 - \pi_0\pi_1\alpha_1 a^{*2}\theta_1{\beta_1}^2}{(\alpha_0 + \beta_1 a^*)^2} - \alpha_1\right\} \tag{51}$$

$$= X.$$

Therefore,

$$\frac{dc^*}{d\eta} = \frac{a^*}{\alpha_2} > 0. \quad (52)$$

Here, the change is positive which shows a proportional relation.

Therefore, when the growth rate of dissolved oxygen due to algae (η) increases, the concentration of dissolved oxygen also increases which is shown in Figure 2(b).

Again,

$$\frac{da^*}{d\eta} = \frac{\begin{vmatrix} (\partial j/\partial\eta) & (\partial j/\partial c^*) \\ (\partial k/\partial\eta) & (\partial k/\partial c^*) \end{vmatrix}}{\begin{vmatrix} (\partial j/\partial c^*) & (\partial j/\partial a^*) \\ (\partial k/\partial c^*) & (\partial k/\partial a^*) \end{vmatrix}} = \frac{(\partial j/\partial c^*)\cdot(\partial k/\partial\eta) - (\partial j/\partial\eta)\cdot(\partial k/\partial c^*)}{(\partial j/\partial c^*)\cdot(\partial k/\partial a^*) - (\partial j/\partial a^*)\cdot(\partial k/\partial c^*)}. \quad (53)$$

Now,

$$\frac{\partial j}{\partial c^*}\cdot\frac{\partial k}{\partial\eta} - \frac{\partial j}{\partial\eta}\cdot\frac{\partial k}{\partial c^*} = 0.a^* + 0.\alpha_2 = 0, \quad (54)$$

and the denominator of $(da^*/d\eta)$ has been already calculated in (52)

$$\therefore \frac{da^*}{d\eta} = 0. \quad (55)$$

There is no change of the density of algae.

Therefore, when the growth rate of dissolved oxygen due to η increases or decreases, there will be no change in the density of algae.

Now using (25), we obtain

$$\pi_1\alpha_1\frac{da^*}{d\eta} = \delta\frac{ds^*}{d\eta} \Longrightarrow \frac{ds^*}{d\eta} = 0; \left[\because \frac{da^*}{d\eta} = 0\right]. \quad (56)$$

There is no change of detritus.

Therefore, when the growth rate of dissolved oxygen due to algae (η) increases, there would be no change in detritus.

And from (23),

$$\pi_0\delta\frac{ds^*}{d\eta} = \alpha_0\frac{dn^*}{d\eta} + \beta_1 n\frac{da^*}{d\eta} + \beta_1 a\frac{dn^*}{d\eta} \Longrightarrow \frac{dn^*}{d\eta} = 0. \quad (57)$$

There is no change of nutrient.

Therefore, when the growth rate of dissolved oxygen due to algae (η) increases, there would be no change in nutrient.

3.7.2. Characteristics of the Concentration of Dissolved Oxygen (c^*) and the Density of Algae (a^*) with respect to q.

$$j(c^*, a^*, q) = q_c - \alpha_2 c^* + \eta a^* - \frac{\delta_1\pi_1\alpha_1 a^*}{\delta},$$

$$k(c^*, a^*, q) = \frac{\beta_1\theta_1 a^* q}{\alpha_0 + \beta_1 a^*} + \frac{\pi_0\pi_1\alpha_1 a^{*2}\theta_1\beta_1}{\alpha_0 + \beta_1 a^*} - \alpha_1 a^*,$$

$$\therefore \frac{dc^*}{dq} = \frac{\begin{vmatrix} (\partial j/\partial a^*) & (\partial j/\partial q) \\ (\partial k/\partial a^*) & (\partial k/\partial q) \end{vmatrix}}{\begin{vmatrix} (\partial j/\partial c^*) & (\partial j/\partial a^*) \\ (\partial k/\partial c^*) & (\partial k/\partial a^*) \end{vmatrix}} = \frac{(\partial j/\partial a^*)\cdot(\partial k/\partial q) - (\partial j/\partial q)\cdot(\partial k/\partial a^*)}{(\partial j/\partial c^*)\cdot(\partial k/\partial a^*) - (\partial j/\partial a^*)\cdot(\partial k/\partial c^*)}. \quad (58)$$

Here, $(\partial j/\partial a^*) = \eta - \delta_1\pi_1\alpha_1/\delta$, $(\partial j/\partial c^*) = -\alpha_2$, $(\partial k/\partial c^*) = 0$, $(\partial j/\partial q) = 0$, $(\partial k/\partial q) = (\beta_1\theta_1 a^*)/(\alpha_0 + \beta_1 a^*)$,

$$\frac{\partial k}{\partial a^*} = \frac{(\alpha_0 + \beta_1 a^*)\beta_1\theta_1 q - (\beta_1)^2\theta_1 q a^*}{(\alpha_0 + \beta_1 a^*)^2} + \frac{(\alpha_0 + \beta_1 a^*)\pi_0\pi_1\alpha_1 2a^*\theta_1\beta_1 - \pi_0\pi_1\alpha_1 a^{*2}\theta_1(\beta_1)^2}{(\alpha_0 + \beta_1 a^*)^2} - \alpha_1. \quad (59)$$

Now,

$$\frac{\partial j}{\partial a^*}\cdot\frac{\partial k}{\partial q} - \frac{\partial j}{\partial q}\cdot\frac{\partial k}{\partial a^*} = \left(\eta - \frac{\delta_1\pi_1\alpha_1}{\delta}\right)\left(\frac{\beta_1\theta_1 a^*}{\alpha_0 + \beta_1 a^*}\right), \quad (60)$$

and the denominator of (dc^*/dq) is same as (52).

Now,

$$\frac{dc^*}{dq} = \frac{(\eta - (\delta_1\pi_1\alpha_1/\delta))(\beta_1\theta_1 a^*/\alpha_0 + \beta_1 a^*)}{X}. \quad (61)$$

Case 1: For $\delta_1\pi_1\alpha_1 > \delta\eta$, $(dc^*/dq) > 0$.

Then, the concentration of dissolved oxygen increases with the increase of q.

Case 2: For $\delta_1\pi_1\alpha_1 < \delta\eta$, $(dc^*/dq) < 0$

Then, the concentration of dissolved oxygen decreases with the increase of q. Therefore, when the rate of discharge of nutrients from outside (q) increases, the concentration of dissolved oxygen also increases but after a certain period of time, the concentration of oxygen starts decreasing due to algal bloom which has been graphically shown in Figure 2(a).

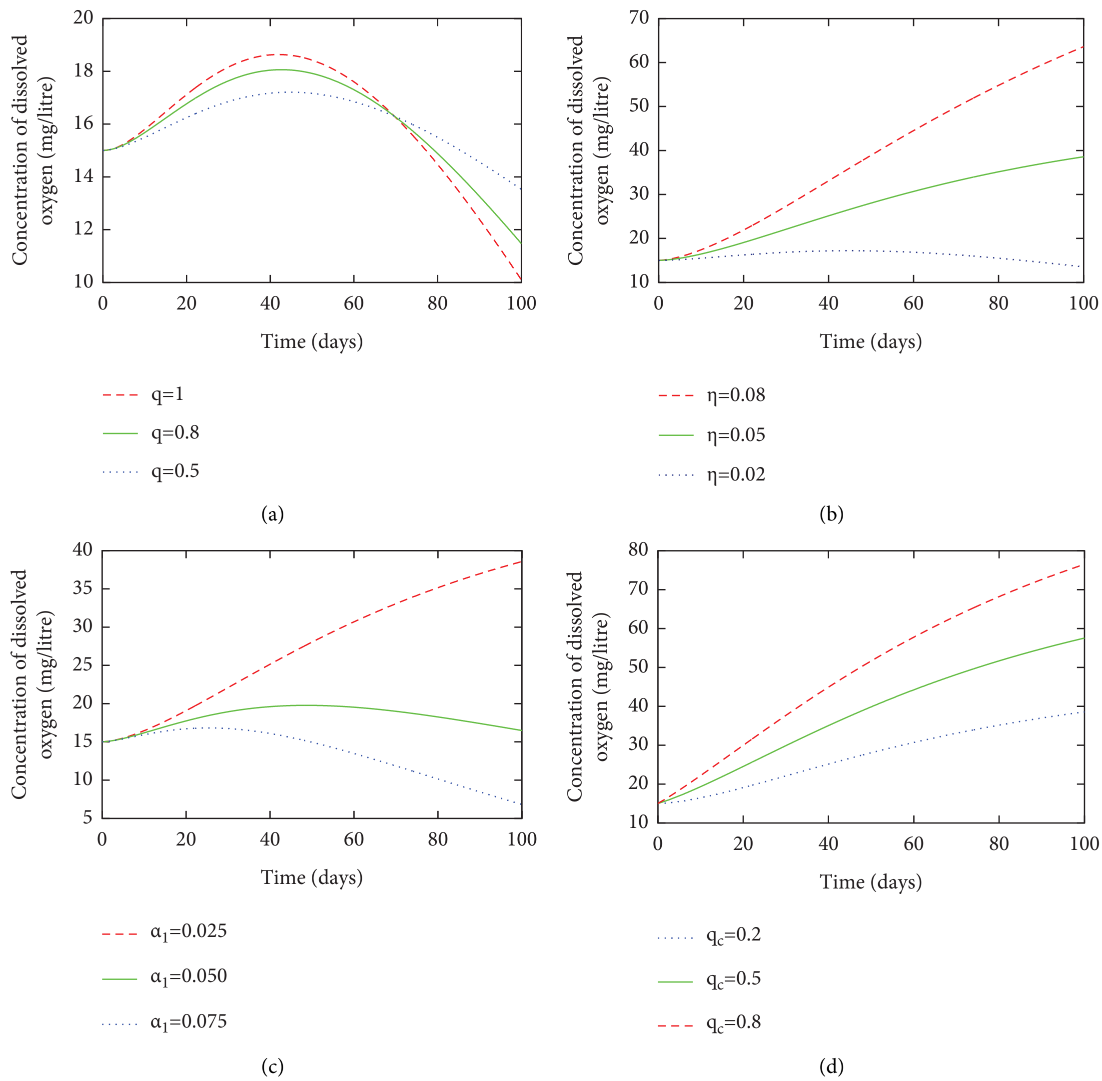


Figure 2: (a) Initially, a higher nutrient discharge rate leads to higher concentration of dissolved oxygen, but over a period of time, the reverse effect happens for water toxicity. (b) Dissolved oxygen concentration in water improves when the rate of dissolved oxygen production by algae rises. (c) The concentration of dissolved oxygen is negatively proportional to the rate at which algae deplete. (d) Monotonically raising dissolved oxygen concentrations are favorably impacted by various sources of dissolved oxygen.

Again,

$$\frac{da^*}{dq}=\frac{\begin{vmatrix}(\partial j/\partial q) & (\partial j/\partial c^*)\\ (\partial k/\partial q) & (\partial k/\partial c^*)\end{vmatrix}}{\begin{vmatrix}(\partial j/\partial c^*) & (\partial j/\partial a^*)\\ (\partial k/\partial c^*) & (\partial k/\partial a^*)\end{vmatrix}}=\frac{(\partial j/\partial c^*)\cdot(\partial k/\partial q)-(\partial j/\partial q)\cdot(\partial k/\partial c^*)}{(\partial j/\partial c^*)\cdot(\partial k/\partial a^*)-(\partial j/\partial a^*)\cdot(\partial k/\partial c^*)},$$
$$\frac{\partial j}{\partial c^*}\cdot\frac{\partial k}{\partial q}-\frac{\partial j}{\partial q}\cdot\frac{\partial k}{\partial c^*}=\alpha_2\frac{\beta_1\theta_1 a^*}{\alpha_0+\beta_1 a^*} \tag{62}$$

and the denominator of (da^*/dq) has already been calculated in (52).

Then,

$$\frac{\mathrm{d}a^*}{\mathrm{d}q}=\frac{\beta_1\theta_1 a^*(\alpha_0+\beta_1 a^*)}{\left\{2\alpha_0\alpha_1\beta_1 a^*(1-\pi_0\pi_1\theta_1)+\beta_1^2 a^{*^2}\alpha_1(1-\pi_0\pi_1\theta_1)+\alpha_0(\alpha_0\alpha_1-q\beta_1\theta_1)\right\}}. \tag{63}$$

So,

$$\frac{\mathrm{d}a^*}{\mathrm{d}q}=\frac{\beta_1\theta_1 a^*(\alpha_0+\beta_1 a^*)}{\xi}, \tag{64}$$

where $\xi=\left\{2\alpha_0\alpha_1\beta_1 a^*(1-\pi_0\pi_1\theta_1)+\beta_1^2 a^{*^2}\alpha_1\ (1-\pi_0\pi_1\theta_1)+\alpha_0(\alpha_0\alpha_1-q\beta_1\theta_1)\right\}$. Here, the change is positive for $\alpha_0(\alpha_0\alpha_1-q\beta_1\theta_1)>(\pi_0\pi_1\theta_1-1)[\beta_1 a^*\alpha_1(2\alpha_0+\beta_1 a^*)]$ which shows a proportional relation. Therefore, the rate of discharge of nutrient from outside (q) increases, and the density of algae also increases (see the Figure 3(b)).

Now using (25), we obtain

$$\pi_1\alpha_1\frac{da^*}{dq}=\delta\frac{ds^*}{dq},$$
$$\Longrightarrow\frac{ds^*}{dq}=\frac{\pi_1\alpha_1\beta_1\theta_1 a^*(\alpha_0+\beta_1 a^*)}{\xi}>0, \tag{65}$$

(using the same condition described at subsection 3.7.2).

Here, the change is positive which shows a proportional relation. Therefore, the rate of discharge of nutrient from outside (q) increases, and the density of detritus also increases which is shown in Figure 4(a). And from (23),

$$\pi_0\delta\frac{\mathrm{d}s^*}{\mathrm{d}q}=\alpha_0\frac{\mathrm{d}n^*}{\mathrm{d}q}+\beta_1 n\frac{\mathrm{d}a^*}{\mathrm{d}q}+\beta_1 a\frac{\mathrm{d}n^*}{\mathrm{d}q},$$
$$\Longrightarrow\frac{\mathrm{d}n^*}{\mathrm{d}q}=\frac{\pi_0\pi_1\alpha_1\alpha_2\beta_1\theta_1 a^*}{(\alpha_0+\beta_1 a^*)^2}+\frac{{\beta_1}^2 n\theta_1 a^*}{\xi}, \tag{66}$$
$$\therefore\frac{\mathrm{d}n^*}{\mathrm{d}q}>0,$$

(using the same condition described at subsection 3.7.2).

Here, the change is positive which shows a proportional relation. Therefore, when the rate of discharge of nutrient from outside (q) increases, and the density of nutrients also increases (see Figure 5(a)).

3.7.3. Characteristics of the Concentration of Dissolved Oxygen (c^*) and the Density of Algae (a^*) with respect to q_c.

$$j(c^*,a^*,q_c)=q_c-\alpha_2 c^*+\eta a^*-\frac{\delta_1\pi_1\alpha_1 a^*}{\delta},$$
$$k(c^*,a^*,q_c)=\frac{\beta_1\theta_1 a^* q}{\alpha_0+\beta_1 a^*}+\frac{\pi_0\pi_1\alpha_1 a^{*2}\theta_1\beta_1}{\alpha_0+\beta_1 a^*}-\alpha_1 a^*,$$
$$\frac{dc^*}{dq_c}=\frac{\begin{vmatrix}(\partial j/\partial a^*) & (\partial j/\partial q_c)\\ (\partial k/\partial a^*) & (\partial k/\partial q_c)\end{vmatrix}}{\begin{vmatrix}(\partial j/\partial c^*) & (\partial j/\partial a^*)\\ (\partial k/\partial c^*) & (\partial k/\partial a^*)\end{vmatrix}}=\frac{(\partial j/\partial a^*)\cdot(\partial k/\partial q_c)-(\partial j/\partial q_c)\cdot(\partial k/\partial a^*)}{(\partial j/\partial c^*)\cdot(\partial k/\partial a^*)-(\partial j/\partial a^*)\cdot(\partial k/\partial c^*)}. \tag{67}$$

Here, $(\partial j/\partial a^*) = \eta - \delta_1\pi_1\alpha_1/\delta$, $(\partial j/\partial c^*) = -\alpha_2$, $(\partial k/\partial c^*) = 0$, $(\partial j/\partial q_c) = 1$, $(\partial k/\partial q_c) = 0$,

$$\frac{\partial k}{\partial a^*} = \frac{(\alpha_0 + \beta_1 a^*)\beta_1\theta_1 q - \beta_1{}^2\theta_1 q a^*}{(\alpha_0 + \beta_1 a^*)^2} + \frac{(\alpha_0 + \beta_1 a^*)\pi_0\pi_1\alpha_1 2a^*\theta_1\beta_1 - \pi_0\pi_1\alpha_1 a^{*2}\theta_1\beta_1{}^2}{(\alpha_0 + \beta_1 a^*)^2} - \alpha_1. \tag{68}$$

Now,

$$\begin{aligned}
&\frac{\partial j}{\partial a^*}\cdot\frac{\partial k}{\partial q_c} - \frac{\partial j}{\partial q_c}\cdot\frac{\partial k}{\partial a^*} = -\left\{\begin{matrix} \dfrac{(\alpha_0 + \beta_1 a^*)\beta_1\theta_1 q - \beta_1{}^2\theta_1 q a^*}{(\alpha_0 + \beta_1 a^*)^2} \\ + \dfrac{(\alpha_0 + \beta_1 a^*)\pi_0\pi_1\alpha_1 2a^*\theta_1\beta_1 - \pi_0\pi_1\alpha_1 a^{*2}\theta_1\beta_1{}^2}{(\alpha_0 + \beta_1 a^*)^2} - \alpha_1 \end{matrix}\right\}, \\
&\frac{\partial j}{\partial c^*}\cdot\frac{\partial k}{\partial a^*} - \frac{\partial j}{\partial a^*}\cdot\frac{\partial k}{\partial c^*} = X.
\end{aligned} \tag{69}$$

Therefore,

$$\begin{aligned}
&\frac{dc^*}{dq_c} = \frac{1}{\alpha_2}, \\
&\therefore \frac{dc^*}{dq_c} = \frac{1}{\alpha_2} > 0.
\end{aligned} \tag{70}$$

Here, the change is positive which shows a proportional relation.

Therefore, if the growth rate of dissolved oxygen (q_c) increases, the concentration of dissolved oxygen also increases which is shown in Figure 2(d).

Again,

$$\frac{da^*}{dq_c} = \frac{\begin{vmatrix} (\partial j/\partial q_c) & (\partial j/\partial c^*) \\ (\partial k/\partial q_c) & (\partial k/\partial c^*) \end{vmatrix}}{\begin{vmatrix} (\partial j/\partial c^*) & (\partial j/\partial a^*) \\ (\partial k/\partial c^*) & (\partial k/\partial a^*) \end{vmatrix}} = \frac{(\partial j/\partial c^*)\cdot(\partial k/\partial q_c) - (\partial j/\partial q_c)\cdot(\partial k/\partial c^*)}{(\partial j/\partial c^*)\cdot(\partial k/\partial a^*) - (\partial j/\partial a^*)\cdot(\partial k/\partial c^*)}. \tag{71}$$

Here,

$$\begin{aligned}
&\frac{\partial j}{\partial c^*}\cdot\frac{\partial k}{\partial q_c} - \frac{\partial j}{\partial q_c}\cdot\frac{\partial k}{\partial c^*} = \alpha_2.0 + 1.0 = \alpha_2, \\
&\frac{\partial j}{\partial c^*}\cdot\frac{\partial k}{\partial a^*} - \frac{\partial j}{\partial a^*}\cdot\frac{\partial k}{\partial c^*} = X \therefore \frac{da^*}{dq_c} == 0.
\end{aligned} \tag{72}$$

Here, there is no change of the density of algae.

Therefore, if the growth rate of dissolved oxygen (q_c) increases or decreases, there would be no change in the density of algae.

Now using (25), we get $\pi_1\alpha_1 a^* = \delta s^* \Longrightarrow (ds^*/dq_c) = 0$; $[\because (da^*/dq_c) = 0]$

Here, there is no change of detritus.

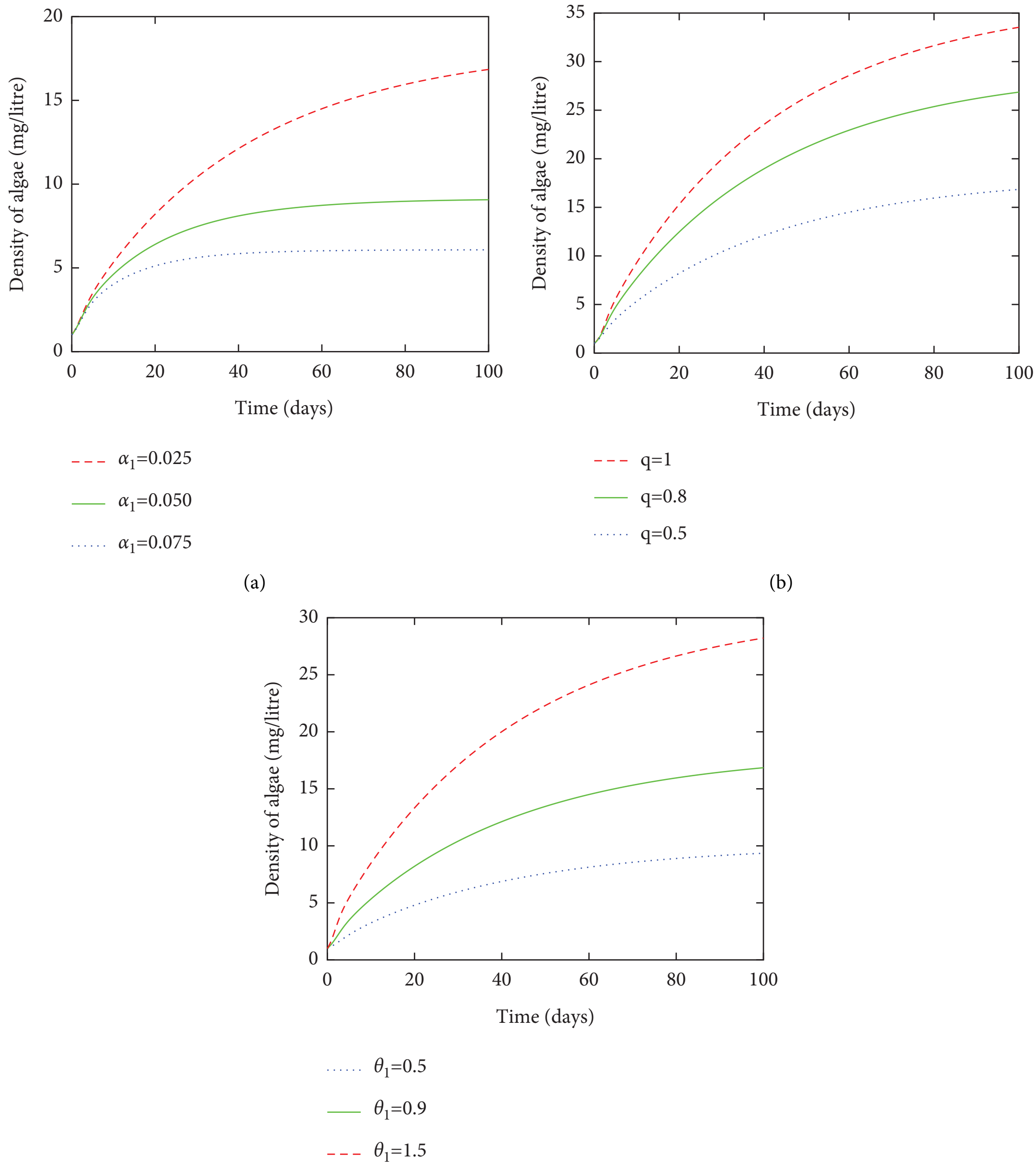
Therefore, the growth rate of dissolved oxygen (q_c) increases, and there would be no change in detritus. And from (23),

$$\pi_0\delta\frac{ds^*}{dq_c} = \alpha_0\frac{dn^*}{dq_c} + \beta_1 n\frac{da^*}{dq_c} + \beta_1 a\frac{dn^*}{dq_c}, \Longrightarrow \frac{dn^*}{dq_c} = 0. \tag{73}$$

Here, there is no change of nutrient.

Therefore, if the growth rate of dissolved oxygen (q_c) increases, there would be no change in nutrient.

3.7.4. Characteristics of the Concentration of Dissolved Oxygen (c^*) and the Density of Algae (a^*) with respect to α_1



(c)

Figure 3: (a) The more algae are depleted, the less the density of algae can rise. (b) The rate of discharged nutrients has a favorable impact on the population of algae, which is monotonously expanding. (c) Algae get denser as their growth rate due to nutrients goes up.

$$\begin{aligned}
j\left(c^*, a^*, \alpha_1\right) &= q_c - \alpha_2 c^* + \eta a^* - \frac{\delta_1 \pi_1 \alpha_1 a^*}{\delta}, \\
k\left(c^*, a^*, \alpha_1\right) &= \frac{\beta_1 \theta_1 a^* q}{\alpha_0 + \beta_1 a^*} + \frac{\pi_0 \pi_1 \alpha_1 a^{*2} \theta_1 \beta_1}{\alpha_0 + \beta_1 a^*} - \alpha_1 a^*, \\
\therefore \frac{dc^*}{d\alpha_1} &= \frac{\begin{vmatrix} \left(\partial j/\partial a^*\right) & \left(\partial j/\partial \alpha_1\right) \\ \left(\partial k/\partial a^*\right) & \left(\partial k/\partial \alpha_1\right) \end{vmatrix}}{\begin{vmatrix} \left(\partial j/\partial c^*\right) & \left(\partial j/\partial a^*\right) \\ \left(\partial k/\partial c^*\right) & \left(\partial k/\partial a^*\right) \end{vmatrix}} = \frac{\left(\partial j/\partial a^*\right) \cdot \left(\partial k/\partial \alpha_1\right) - \left(\partial j/\partial \alpha_1\right) \cdot \left(\partial k/\partial a^*\right)}{\left(\partial j/\partial c^*\right) \cdot \left(\partial k/\partial a^*\right) - \left(\partial j/\partial a^*\right) \cdot \left(\partial k/\partial c^*\right)}.
\end{aligned} \tag{74}$$

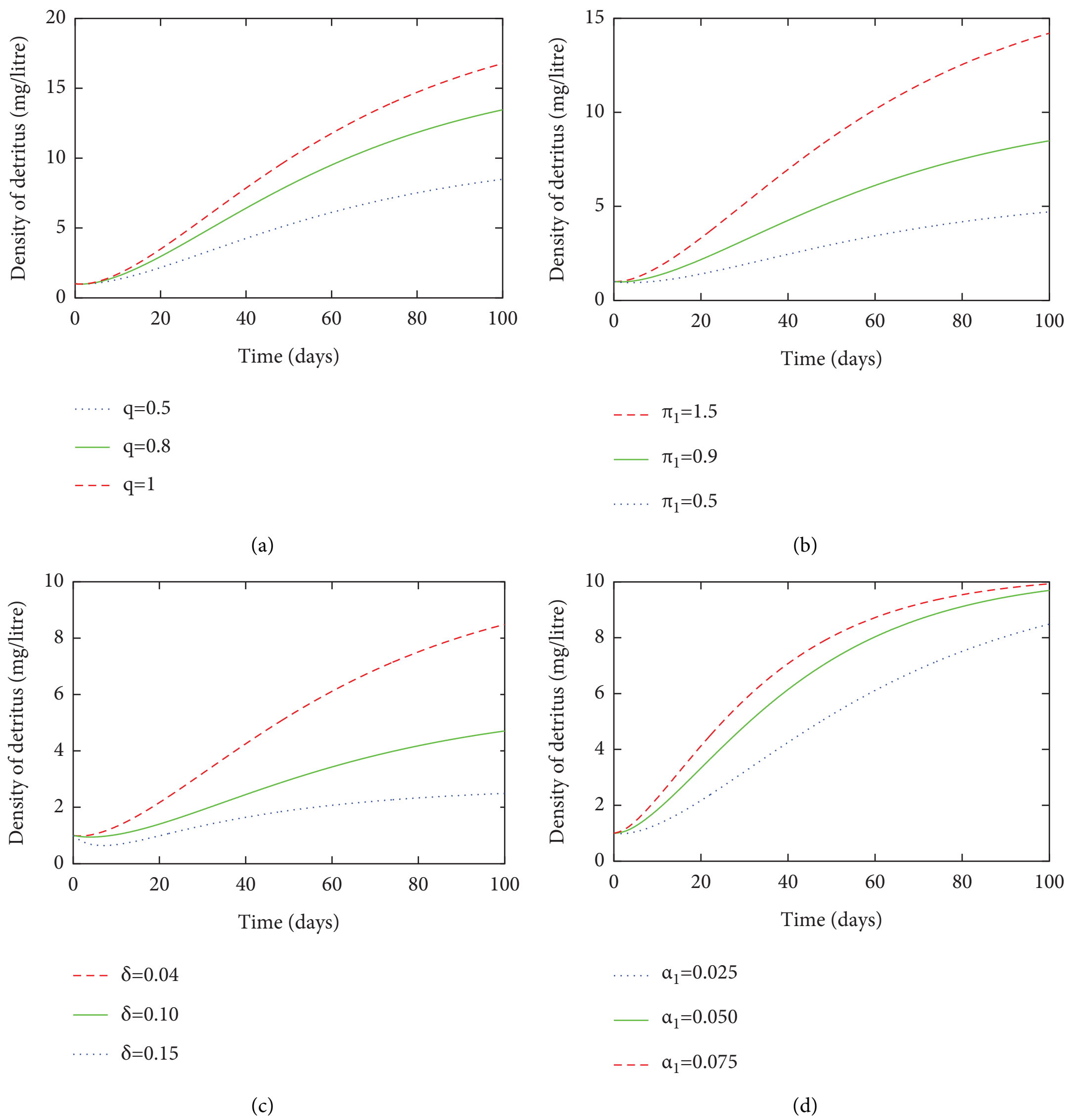


FIGURE 4: (a) When nutrient discharge rates are increased, detritus density is also enhanced. (b) Detritus is denser when algae are growing more rapidly. (c) The rate at which algae consume detritus is inversely related to the density of detritus in the environment. (d) When algae are depleted at a more natural rate, more density of detritus accumulates.

Therefore,

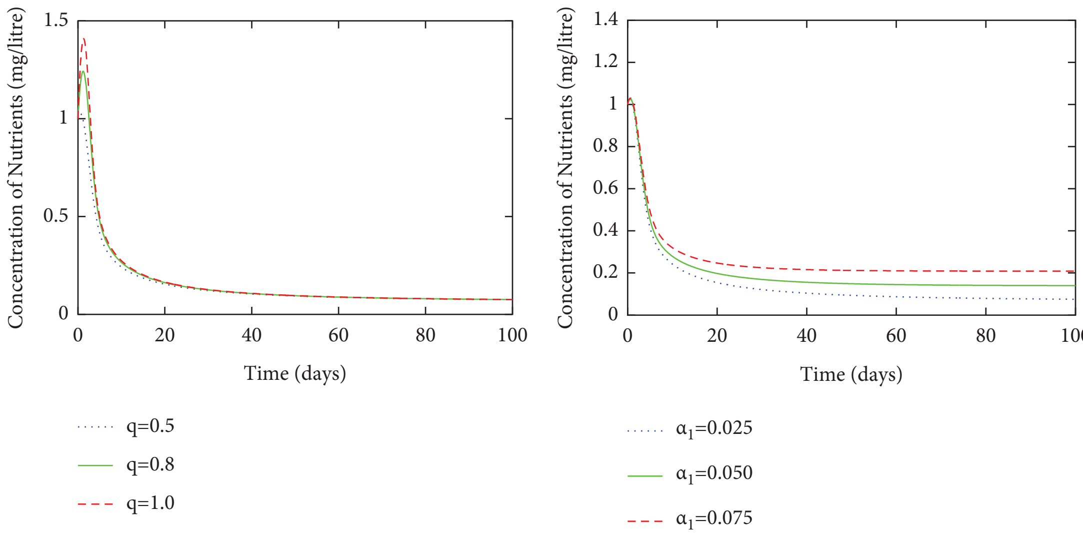
$$\frac{dc^*}{d\alpha_1} = \frac{\left(\left(\eta - \delta_1\pi_1\alpha_1/s\right)\right)\left(\left(\pi_0\pi_1 a^{*2}\theta_1\beta_1/\alpha_0 + \beta_1 a^*\right) - a^*\right)}{\alpha_2\xi + \left(\delta_1\pi_1 a^*/\delta\alpha_2\right)}. \quad (75)$$

Case: Here, $\pi_0\pi_1 a^*\theta_1\beta_1 < \alpha_0 + \beta_1 a^*$. So, using the previous condition, $(dc^*/d\alpha_1) < 0$.

Here, the change is negative which shows an inversely proportional relation.

Therefore, if the depletion rate of algae α_1 increases, the concentration of dissolved oxygen decreases (see Figure 2(c)).

Again,



(a) (b)

Figure 5: (a) More nutrient flow increases nutrient concentration; however, after a while, algal population begins to consume nutrients and lowers the concentration. (b) Concentration of nutrients decreases less with the higher depletion rate of algae.

$$\begin{aligned}
\frac{\mathrm{d}a^*}{\mathrm{d}\alpha_1} &= \frac{\begin{vmatrix} (\partial j/\partial \alpha_1) & (\partial j/\partial c^*) \\ (\partial k/\partial \alpha_2) & (\partial k/\partial c^*) \end{vmatrix}}{\begin{vmatrix} (\partial j/\partial c^*) & (\partial j/\partial a^*) \\ (\partial k/\partial c^*) & (\partial k/\partial a^*) \end{vmatrix}} = \frac{(\partial j/\partial c^*)\cdot(\partial k/\partial \alpha_1) - (\partial j/\partial \alpha_1)\cdot(\partial k/\partial c^*)}{(\partial j/\partial c^*)\cdot(\partial k/\partial c^*) - (\partial j/\partial a^*)\cdot(\partial k/\partial c^*)}, \\
\frac{\mathrm{d}a^*}{\mathrm{d}\alpha_1} &= \frac{-\left\{\pi_0\pi_1 a^{*2}\theta_1\beta_1(\alpha_0+\beta_1 a^*) - a^*(\alpha_0+\beta_1 a^*)^2\right\}}{\xi}, \\
\Longrightarrow \frac{\mathrm{d}a^*}{\alpha_1} &> 0,
\end{aligned} \tag{76}$$

(using the same condition described at subsection 3.7.2).

Here, the change is negative which shows an inversely proportional relation.

Therefore, if the depletion rate of algae (α_1) increases, the density of algae decreases (see Figure 3(a)). Now using (25), we obtain

$$\frac{\mathrm{d}s^*}{\mathrm{d}\alpha_1} = \frac{\pi_1\alpha_1}{\delta}\left[\frac{\left(\beta_1{}^2 a^{*^3} + \beta_1 a^{*^2}\right)(1-\pi_0\pi_1\theta_1) + a^*\alpha_0{}^2 + a^{*^2}\beta_1\alpha_0}{\xi}\right] > 0, \tag{77}$$

(using the same condition described at subsection 3.7.2).

Here, the change is positive which shows a proportional relation.

Therefore, if the depletion rate of algae (α_1) increases, the density of detritus also increases, which is shown in Figure 4(d).

And from (23),

$$\pi_0\delta\frac{\mathrm{d}s^*}{\mathrm{d}\alpha_1} = \alpha_0\frac{\mathrm{d}n^*}{\mathrm{d}\alpha_1} + \beta_1 n\frac{\mathrm{d}a^*}{\mathrm{d}\alpha_1} + \beta_1 a\frac{\mathrm{d}n^*}{\mathrm{d}\alpha_1} \Longrightarrow \frac{\mathrm{d}n^*}{\mathrm{d}\alpha_1} > 0. \quad (78)$$

Here, the change is positive which shows a proportional relation.

Therefore, if the depletion rate of algae (α_1) increases, the concentration of nutrient also slightly increases (see Figure 5(b)).

3.8. Numerical Results and Discussion. To analyze the effects of algal bloom on the depletion of dissolved oxygen in water bodies, the model have been numerically solved for the values of parameters which are used in this simulation from Table 1 which is related to the work of Misra [23].

The initial value of the concentration of nutrients (n) is 1 mg/litre, the density of algae (a) is 1 mg/litre, the density of detritus (s) is 1 mg/litre, and the concentration of dissolved oxygen (c) is 15 mg/litre, which are considered according to the work of Misra [23]. We have used MATLAB(R2018a) to find the result of the model by the Runge–Kutta method. The table containing the values of the parameters is given in this section. While calculating the numerical result, we have considered the time period t from 0 to 100 days because in this time interval we have found significant changes of our dynamical system. On the other hand, in some related works [17, 18, 23], the numerical results were calculated for 90 days or 120 days. However, 100 days are taken for illustrating the situation.

In Figure 2(a), the behavior of the concentration of dissolved oxygen for different values of the rate of discharge of nutrients q with respect to time (t) has been discussed. The values of q were taken 0.5, 0.8, and 1, respectively. It is clear from the graph that the concentration of dissolved oxygen has been increased gradually with respect to time. When the nutrient is discharged to the water body, it increases the amount of dissolved oxygen. The amount of dissolved oxygen is increasing for around 40 days as shown in the figure and after that it started to decrease because when algae and other plants die, a huge amount of oxygen is used to decompose the dead plant and algae. These three curves met at a point and after that it started to decrease. When $q = 0.8$, dissolved oxygen increases rapidly in comparison with $q = 0.5$, and after a certain period of time, it also decreases rapidly. Similarly, when $q = 1$, dissolved oxygen has been increased sharply in comparison with $q = 0.8$, and after a certain period of time, it has been decreased rapidly.

Hence, if the value of the rate of discharge of nutrients increases, then the concentration of dissolved oxygen decreases. After a certain period, the water bodies become toxic.

The variation of the concentration of dissolved oxygen for different values of the increasing rate of dissolved oxygen due to algae η with respect to time (t) is depicted in Figure 2(b). Three values of η have been taken in order to discuss. For $\eta = 0.08$, it is simple to understand that algae will contribute some amount of oxygen into the system which compels the curve in going upward. But after a certain period of time, it will start to decrease when increasing the rate η of dissolved oxygen due to algae is less or equal to 0.02. Because when algae will start to be produced more, then it will cause algal bloom and there will be shortage of dissolved oxygen in water bodies. A huge amount of oxygen is used to decompose the dead plant and algae. In both cases, $\eta = 0.05$ and $\eta = 0.02$, and the concentration of dissolved oxygen has been acted the same way, although in lesser quantities as the value of η is being decreased. Consequently, the concentration of dissolved oxygen grows together with the rate of dissolved oxygen owing to algae.

Figure 2(c) shows the fluctuation of the concentration of dissolved oxygen for various values of the rate of algae α_1 degradation with respect to time (t). With the increase of the depletion rate of algae, the amount of dissolved oxygen in water bodies decreases. The more the depletion rate of algae increases, the more the concentration of dissolved oxygen decreases because to decompose algae a huge amount of oxygen is used. For this reason, when the depletion rate is smaller, the dissolved oxygen increases. In short, the concentration of dissolved oxygen is inversely proportional to the depletion rate of algae.

Figure 2(d) depicts the time-dependent behavior of dissolved oxygen concentration for a variety of rising dissolved oxygen rates q_c. It is normal that if the increasing rate of dissolved oxygen by various sources increases, then the amount of dissolved oxygen in water bodies will also increase. Oxygen can be produced in many ways, including from photosynthesis by aquatic plants or it can be increased by diffusion of oxygen into the water bodies. These are the various sources that results in the increasing concentration of dissolved oxygen. As a result, the dissolved oxygen concentration rises as the rate at which it is released from different sources increases.

The variation of the density of algae is shown for different values $\alpha_1 = 0.025$, $\alpha_1 = 0.050$, and $\alpha_1 = 0.075$ with respect to time in Figure 3(a). If the depletion rate of algae is greater, the density of algae will not increase so much faster. For $\alpha_1 = 0.050$ and $\alpha_1 = 0.075$, the time for increasing the density of algae is 50 days and 25 days, respectively, and after that, it will be balanced and behaved like a straight line parallel to the horizontal axis. The curve for $\alpha_1 = 0.025$ also behaves in the same way.

Hence, if the depletion rate of algae increases, the density of algae will decrease gradually.

Table 1: Parameter values and their descriptions.

Sl. nos.	Descriptions of the parameter	Notation	Values
1	Rate of discharge of nutrients	q	$0.5\text{mgl}^{-1}\text{day}^{-1}$
2	Natural depletion rate of nutrients	α_0	$0.005\ \text{day}^{-1}$
3	Natural depletion rate of algae	α_1	$0.025\ \text{day}^{-1}$
4	Natural depletion rate of dissolved oxygen	α_2	$0.01\ \text{day}^{-1}$
5	Depletion rate of nutrients due to algae	β_1	$0.4\text{mg}^{-1}l\text{day}^{-1}$
6	Growth rate of nutrients due to detritus	π_0	0.02
7	Growth rate of detritus due to algae	π_1	0.9
8	Depletion rate of detritus due to decomposing	δ	$0.04\ \text{day}^{-1}$
9	Depletion rate of dissolved oxygen due to detritus	δ_1	$0.06\ \text{day}^{-1}$
10	Growth rate of algae due to nutrients	θ_1	0.9
11	Increasing rate of dissolved oxygen due to algae	η	$0.02\ \text{day}^{-1}$
12	Increasing rate of dissolved oxygen by various sources	q_c	$0.2\text{mgl}^{-1}\text{day}^{-1}$

At the same time, Figure 3(b) illustrates the fluctuation in the algae density with respect to time for various values of $q = 0.5$, $q = 0.8$, and $q = 1$. If the concentration of the discharge of nutrients increases, the density of algae will also increase rapidly.

So, if the concentration of discharge of nutrients increases, then the density of algae increases rapidly and causes algal bloom.

Similarly, Figure 3(c) helps us to know that the density of algae increases when the values of θ_1 improves. So, there is no doubt to say that the density of algae is directly proportional to θ_1.

Therefore, the change in Figure 3(a) is inversely proportional, where Figures 3(b) and 3(c) show completely different behavior with their corresponding parameters. A complete idea on algae has been generated in Figure 3.

In Figure 4(a), we have discussed the variation of the density of detritus s for different values of the rate of discharge of nutrients q with respect to time t. We took the values $q = 0.5$, $q = 0.8$, and $q = 1$ which makes a change in the graph. The density of detritus increases gradually with respect to time, as we can see from the figure that the amount of detritus is increasing. When algae and other plants die, the density of detritus increases. When $q = 0.8$, the density of detritus increases rapidly in comparison with $q = 0.5$. Similarly, when $q = 1$, the density of detritus increases rapidly in comparison with $q = 0.8$.

The behavior of the density of detritus for different values of the growth rate of detritus due to algae π_1 with respect to time t is shown in Figure 4(b). For increasing values of π_1, s is increasing.

Figure 4(c) shows the change of the density of detritus for different values of the depletion rate of detritus due to decomposing δ with respect to time t. For increasing values of δ, s is decreasing.

Figure 4(d) behaves in the same way as Figure 4(b) does. The variation of the density of detritus s increases with the increasing values of the natural depletion rate of algae α_1.

The density of detritus are increased for increment of the rate of discharge of nutrients, growth rate of detritus due to algae, and natural depletion rate of algae (see Figures 4(a), 4(b), 4(d)). But opposite behavior of detritus is shown for δ which is presented in Figure 4(c) because δ is the depletion rate of detritus due to decomposing.

Lastly Figure 5(a) shows the change of the concentration of nutrients n for different values of the cumulative rate of discharge of nutrients q. When the values of q increases, n starts to increase at the initial phase but after a certain period of time, it starts to decrease drastically and comes to its lowest value over the time period. It decreases because the algal population starts to increase by consuming nutrients more and more.

In Figure 5(b), variation of the concentration of nutrients n for different values of the natural depletion rate of algae α_1 has been discussed. It has been observed that the value of n goes down with respect to α_1, but when α_1 is higher, the decreased rate of n is lower.

Now, some interesting results have been presented here with the help of Figure 6. When algae start to be produced more, then it will cause algal bloom and there will be shortage of dissolved oxygen in water bodies (Figure 6(a)). Moreover, a huge amount of oxygen will be used to decompose the dead plant and algae. Observing critically Figure 6(b), the deletion of nutrients will produce algae. When the density of algae cross a level (14.5 mg/litre), the direction of dissolved oxygen is downward. For throwing of detritus (>1.0 mg/litre) in water bodies, the nutrients will be decreased (Figure 6(d)) as well as the density of algae will be produced slowly, and crossing the level (12.75 mg/litre) is a barrier of increment of dissolved oxygen (Figure 6(c)).

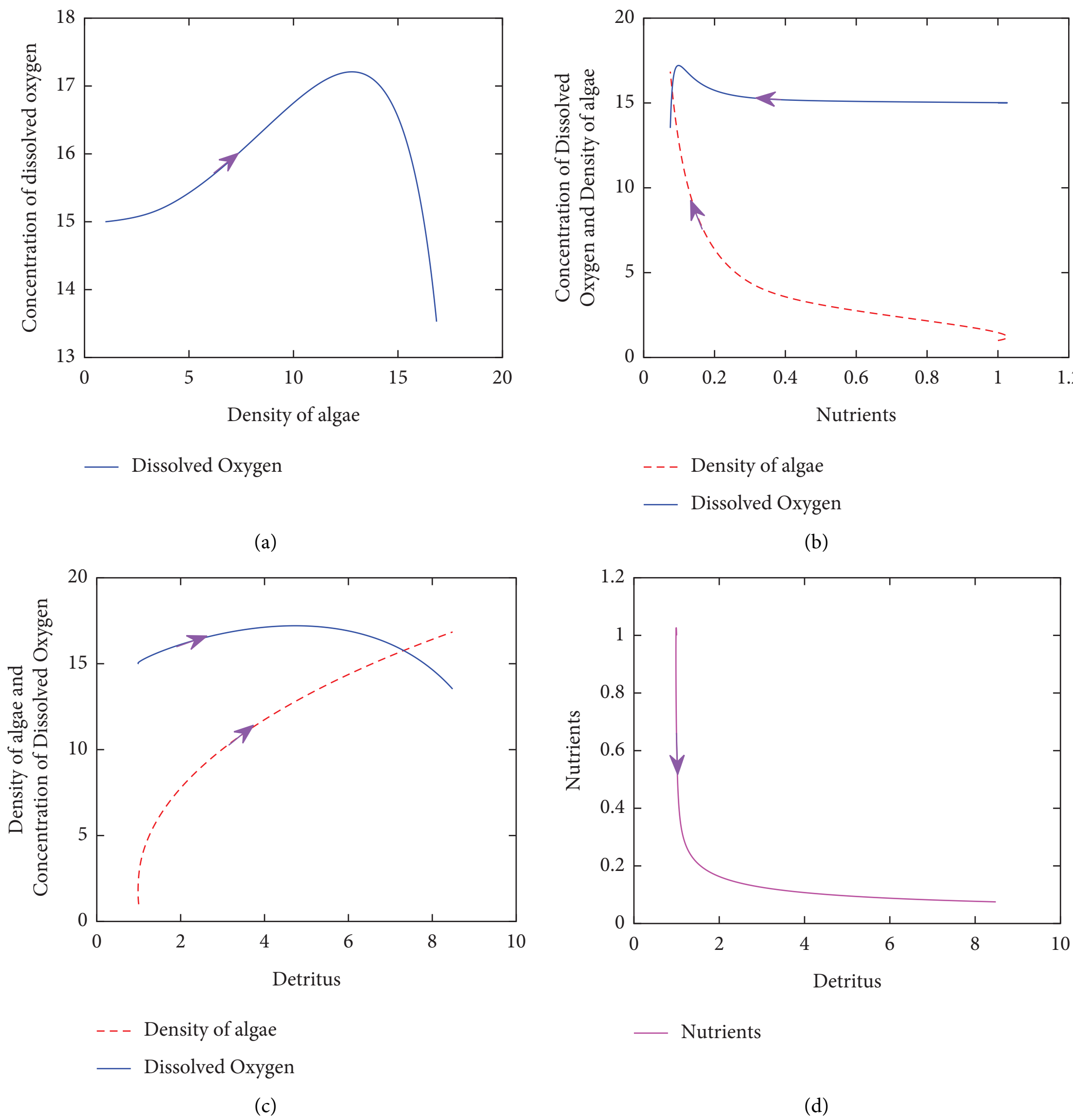


FIGURE 6: (a) Initially, the growth of density of algae increases the concentration of dissolve oxygen but after times, it decreases. (b) Density of algae grows when nutrients are depleted, yet released oxygen falls and then rises again at some point and falls again. (c) Density of oxygen and oxygen dissolved increases with detritus growth although after a period, it reduces the oxygen concentration. (d) The increase of detritus causes a diminishing of the available nutrients.

4. Conclusion

In water bodies, to get the rapid growth of algae as well as dissolved oxygen, maximum detritus is 1.0 mg/litre. However, more detritus (>4.7 mg/litre) are harmful for the production of dissolved oxygen in water bodies. Besides, analysis of the model shows that the equilibrium level of dissolved oxygen decreases if the rate of concentration of nutrients increases which results in algal bloom. It is shown both numerically and graphically in our analysis that depletion of the concentration dissolved oxygen in water bodies is caused by algal bloom. From our analysis, it is evident that a huge amount of nutrient is discharged from various mills, industries, and factories, which is alarming for the humanity as well as the environment. If this continues, there will be a huge loss economically as well as environmentally. Humans and animal life will also suffer from various water contaminated diseases. Therefore, raising public awareness is a much needed step to control this rising problem. We wish to find the optimal solution to control this problem in the extended version of our research work too. So, it is concluded that the regulatory system of our study will save water from being polluted.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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