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Research Article

A One-Point Third-Derivative Hybrid Multistep Technique for Solving Second-Order Oscillatory and Periodic Problems

Mufutau Ajani Rufai,[1] **Ali Shokri**,[2] **and Ezekiel Olaoluwa Omole**[3,4]

[1]*Dipartimento di Matematica, Università Degli Studi di Bari Aldo Moro, Bari 70125, Italy*
[2]*Faculty of Mathematical Sciences, University of Maragheh, Maragheh, Iran*
[3]*Department of Mathematics and Statistics, Joseph Ayo Babalola University, Ikeji Arakeji, Osun State, Nigeria*
[4]*Department of Mathematics, Faculty of Science, Federal University Oye-Ekiti, Oye-Ekiti, Ekiti State, Nigeria*

Correspondence should be addressed to Ali Shokri; shokri@maragheh.ac.ir

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This paper describes a third-derivative hybrid multistep technique (TDHMT) for solving second-order initial-value problems (IVPs) with oscillatory and periodic problems in ordinary differential equations (ODEs), the coefficients of which are independent of the frequency (omega) and step size (h). This research is significant because it has numerous applications to real-life phenomena such as chaotic dynamical systems, almost periodic problems, and duffing equations. The current method is derived from the collocation of a derivative function at the equidistant grid and off-grid points. The TDHMT obtained is a continuous scheme for obtaining simultaneous approximations to the solution and its derivative at each point in the $[x0, xN]$ interval integration. The presence of high derivatives increases the order of the method, which increases the accuracy method's order and the stability property, as discussed in detail. Finally, the proposed method is compared to existing methods in the literature on some oscillatory and periodic test problems to demonstrate the technique's effectiveness and productivity.

1. Introduction

The numerical solution of general second-order IVPs of ODEs of the form (1) is the focus of this study.

$$\begin{aligned} u''(x) &= f(x, u, u'), \\ u(x_0) &= u_0, \\ u'(x_0) &= u_0', \\ u, x, f &\in \mathbb{R}^{\mathrm{d}}, \end{aligned} \tag{1}$$

whose solution $u(x)$ is assumed to have an oscillatory or periodic behaviour.

Problems with structure (1) arise much of the time in astrophysics and space, nuclear physics, celestial mechanics, molecular dynamics, circuit theory, chemical kinetics, and other various areas of applications in science and engineering. The study places emphasis on second-order IVPs, where the first-derivative does not show up explicitly, due to their applications, and many of these problems do not have known analytical solutions. Because of real-life applications of this problem, numerous scientists are propelled to examine its numerical solutions (see [1, 2] and references therein).

The authors of reference [1] developed a 3-point variable step block hybrid method (3-point VSBHM) using Lagrange polynomials as the basis function. The 3-point VSBHM was applied to difficult chemical problems such as the Belousov–Zhabotinsky reaction and Hires. It has been demonstrated that the method meets the basic requirement for convergence. Achar tenders symmetric multistep Obrechkoff methods with eight-order algebraic accuracy and zero-phase lag for periodic initial-value problems of second-order differential equations in reference [2]. The method was used to solve practical problems of the form (1), and the solution demonstrates how the problems behave when used to solve periodic initial-value problems.

In addition, Wu et al. in reference [3] presented a novel method for simulating turbulent reactive flows with stiff chemistry called efficient time-stepping techniques. The method was used to solve various types of stiff problems in chemistry, specifically turbulent reactive flows, and its applicability was demonstrated. Skwame et al. developed an equidistant one-step block hybrid method for treating second-order ordinary differential equations. The developed one-step block hybrid method was applied to a real-world physics and engineering problem, namely, the cooling of a body in reference [4]. Next, the authors of reference [5] proposed a new trigonometrically two-step hybrid method for solving second-order ODEs with one or two frequencies. The methods were examined and discussed in terms of their practical application. Meanwhile, Liu et al. [6] considered a new modified embedded 5(4) pair of explicit Runge–Kutta methods for numerically solving the Schrödinger equation. The methods were investigated and applied to the Schrodinger equation, with the solution illustrated in curves.

In references [7, 8], the effects of Soret and Dufour on heat and mass transfer of boundary layer flow over porous wedge with thermal radiation were investigated. The investigated governing equations have boundary conditions that are nondimensionalized by introducing some nondimensional variables. The following equations are solved numerically using the bivariate spectral relaxation method (BSRM). This type of ODE has a wide range of applications in engineering, including oil bed recovery, filtration, thermal insulations, heat exchangers, and geothermal analysis. Kamran et al. [9] propounded a numerical simulation of the time fractional BBM-Burger equation using cubic B-spline functions. The authors of reference [10] investigated an asymptotic reduction of the dimension of the solution space for dynamical systems. The authors of reference [11] studied the Cauchy problem for the degenerate parabolic convolution equation. In the work of reference [12], the mathematical and numerical modeling of coupled dynamic thermoelastic problems for isotropic bodies was investigated. The authors of reference [13] presented an inverse boundary value problem for the Boussinesq-love equation with a nonlocal integral condition in his paper. Later, Prasad and Isik [14] discuss numerical solutions to boundary value problems using contractions.

Furthermore, some reports on recent articles for problem (1) resolution will be explored. According to Franco and Gomez, reference [15] has developed trigonometrically fitted nonlinear two-step methods. In reference [16], Kalogiratou also presented diagonally implicit trigonometrically fitted symplectic Runge–Kutta methods. In references [17] and [18], Tsitouras and Simos have derived and implemented high phase-lag order four-step and explicit Numerov-type methods. More recently, Shokri has constructed a new eight-order symmetric two-step multiderivative technique in reference [19].

It should be noted that the majority of the methods described mentioned above do not use the first-derivative values in their derivations. Consequently, this study proposes and investigates a new third-derivative hybrid multistep method for numerically solving the problem in equation (1). The new TDHMT introduced and used first-derivative values in its formulation, giving it a broader application space and higher computational productivity than existing multistep type strategies. Lastly, the TDHMT is written in a block-by-block fashion, and approximations are provided at every grid point of the integration interval at the same time, resulting in a lower total number of function evaluations.

2. Mathematical Formulation of the TDHMT

This section's goal is to derive a TDHMT with two off-grid points in $[x_n, x_{n+2}]$. To do so, we assume that the exact solution $u(x)$ is approximated by the polynomial $\mu(x)$,

$$u(x) \simeq \mu(x) = \sum_{n=0}^{7} b_n x^n, \tag{2}$$

where $b_n \in \mathbb{R}$ are real unknown coefficients that are resolved after taking collocation and interpolation conditions into account at selected points. Let $x_{n+r} = x_n + (1/2)h, x_{n+s} = x_n + (3/2)h$ be the two off-grid points on $[x_n, x_{n+2}]$ and the approximation in equation (2), and its first derivatives applied to the point x_n, its second derivative applied to the points $x_n, x_{n+r}, x_{n+1}, x_{n+s}, x_{n+2}$, and its third derivative applied to the points x_{n+2}. Then, we get a system of 8 equations with 8 unknowns $b_n, n = 0(1)7$, given by

$$\begin{gathered} \mu(x_n) = u_n, \mu'(x_n) = u_n', \mu''(x_n) = f_n, \mu'''(x_{n+2}) = g_{n+2}, \\ \mu''(x_{n+r}) = f_{n+r}, \mu''(x_{n+1}) = f_{n+1}, \mu''(x_{n+s}) = f_{n+s}, \mu''(x_{n+2}) = f_{n+2}. \end{gathered} \tag{3}$$

After obtaining the values of the coefficients $b_n, n = 0(1)7$, substituting its values into equation (2), and changing the variable, $x = x_n + zh$, then, the polynomial in equation (2) maybe written as follows:

$$u(x_n + zh) = \alpha_0(z)u_n + h\alpha_1(z)u_n' + h^2\left(\beta_0(z)f_n + \beta_r(z)f_{n+r} + \beta_1(z)f_{n+1} + \beta_s(z)f_{n+s} + \beta_2(z)f_{n+2}\right) + h^3\left(\gamma_1(z)g_{n+2}\right), \tag{4}$$

where h is the chosen step-size and $\alpha_0(z), \alpha_1(z), \beta_0(z), \beta_r(z), \beta_1(z), \beta_s(z), \beta_2(z), \gamma_1(z)$ are continuous coefficients.

2.1. Main Formulas of the TDHMT. The main formulas are generated by substituting the values of $\alpha_j(z)$, $\beta_j(z)$, $j = 0, 1, 2$, $\beta_r(z)$, $\beta_s(z)$, and $\gamma_1(z)$ into equation (4) and evaluating $\mu(x)$ and $\mu'(x)$ at the point $x_{n+2} = x_n + 2h$ to obtain the following approximations for $u(x_{n+2})$ and $u'(x_{n+2})$.

$$\begin{aligned} u_{n+2} &= h^2\left(\frac{31f_n}{105} + \frac{8f_{n+1}}{105} + \frac{1088}{945}f_{n+r} + \frac{64}{105}f_{n+s} - \frac{25f_{n+2}}{189}\right) + \frac{2}{63}h^3 g_{n+2} + 2hu_n' + u_n, \\ u_{n+2}' &= h\left(\frac{7f_n}{45} + \frac{4f_{n+1}}{15} + \frac{7f_{n+2}}{45} + \frac{32}{45}f_{n+r} + \frac{32}{45}f_{n+s}\right) + u_n'. \end{aligned} \tag{5}$$

2.2. Additional Formulas of the TDHMT. The additional formulas are obtained by evaluating of $\mu(x)$ and $\mu'(x)$ at the points x_{n+r}, x_{n+1}, and x_{n+s}. Thus, the following additional formulas that form the proposed method, namely, TDHMT was obtained.

$$\begin{aligned} u_{n+r} &= h^2\left(\frac{3247f_n}{53760} + \frac{337f_{n+r}}{3024} + \frac{41}{560}f_{n+s} - \frac{1193f_{n+1}}{13440} - \frac{15139f_{n+2}}{483840}\right) + \frac{107h^3 g_{n+2}}{16128} + \frac{hu_n'}{2} + u_n, \\ u_{n+1} &= h^2\left(\frac{39f_n}{280} + \frac{418}{945}f_{n+r} + \frac{6}{35}f_{n+s} - \frac{5f_{n+1}}{28} - \frac{563f_{n+2}}{7560}\right) + \frac{1}{63}h^3 g_{n+2} + hu_n' + u_n, \\ u_{n+s} &= h^2\left(\frac{3891f_n}{17920} + \frac{447}{560}f_{n+r} + \frac{33}{112}f_{n+s} - \frac{297f_{n+1}}{4480} - \frac{2127f_{n+2}}{17920}\right) + \frac{45h^3 g_{n+2}}{1792} + \frac{3hu_n'}{2} + u_n, \\ u_{n+r}' &= h\left(\frac{1873f_n}{11520} + \frac{23}{45}f_{n+r} + \frac{47}{180}f_{n+s} - \frac{311f_{n+1}}{960} - \frac{1277f_{n+2}}{11520}\right) + \frac{3}{128}h^2 g_{n+2} + u_n', \\ u_{n+1}' &= h\left(\frac{37f_n}{240} + \frac{f_{n+1}}{20} + \frac{98}{135}f_{n+r} + \frac{2}{15}f_{n+s} - \frac{137f_{n+2}}{2160}\right) + \frac{1}{72}h^2 g_{n+2} + u_n', \\ u_{n+s}' &= h\left(\frac{201f_n}{1280} + \frac{99f_{n+1}}{320} + \frac{7}{10}f_{n+r} + \frac{9}{20}f_{n+s} - \frac{149f_{n+2}}{1280}\right) + \frac{3}{128}h^2 g_{n+2} + u_n'. \end{aligned} \tag{6}$$

3. Analysis of the TDHMT

This section analyzes the characteristics of the proposed TDHMT, which is based on an extension of Dahlquist barrier for multistep methods.

3.1. Accuracy and Consistency of the TDHMT. The TDHMT method given in equations (5) and (6) might be formulated as follows:

$$PV_n = hQV_n' + h^2RF_n + h^3SG_n, \tag{7}$$

where P, Q, R, S denote matrices of coefficients in equations (5) and (6), and

$$\begin{aligned} V_n &= \left(u_n, u_{n+r}, u_{n+1}, u_{n+s}, u_{n+2}\right)^T, \\ V_n' &= \left(u_n', u_{n+r}', u_{n+1}', u_{n+s}', u_{n+2}'\right)^T, \\ F_n &= \left(f_n, f_{n+r}, f_{n+1}, f_{n+s}, f_{n+2}\right)^T, \\ G_n &= \left(g_n, g_{n+r}, g_{n+1}, g_{n+s}, g_{n+2}\right)^T. \end{aligned} \tag{8}$$

Using [20], we assume that $u(x)$ is a sufficiently differentiable function, we define the operator ℓ as follows:

$$\ell[u(x);h] = \sum_{j\in J}\left[\overline{\alpha}_j u(x_n + jh) - h\overline{\beta}_j u'(x_n + jh) - h^2\overline{\gamma}_j u''(x_n + jh) - h^3\overline{\mu}_j u'''(x_n + jh)\right], \tag{9}$$

where $\overline{\alpha}_j, \overline{\beta}_j, \overline{\gamma}_j, \overline{\mu}_j$ are the vector columns of the matrices P, Q, R, S, respectively, and $J = \{0, r, 1, s, 2\}$. By using the Taylor series about x_n, we expand the formulas in equation (9) and obtain as follows:

$$\ell[u(x);h] = \overline{C}_0 u(x_n) + \overline{C}_1 hu'(x_n) + \overline{C}_2 h^2 u''(x_n) + \ldots + \overline{C}_q h^q u^q(x_n) + \ldots, \tag{10}$$

with

$$\overline{C}_q = \frac{1}{q!}\left[\sum_{j\in J}^{k} j^q\overline{\alpha}_j - q\sum_{j\in J}^{k} j^{q-1}\overline{\beta}_j - q(q-1)\sum_{j\in J}^{k} j^{q-2}\overline{\gamma}_j\right], \tag{11}$$

where $q = 0, 1, 2, \ldots$. The abovementioned operator and the associated formulas are said to be of order p, according to reference [21], if $\overline{C}_0 = \overline{C}_1 = \ldots = \overline{C}_{p+1} = 0, \overline{C}_{p+2} \neq 0$. $\overline{C}_i$ is the column vectors and $\overline{C}_{p+2}$ is the vector of local truncation errors. For the proposed TDHMT in equations (5) and (6), we have $\overline{C}_0 = 0, \overline{C}_1 = 0, \overline{C}_2 = 0, \ldots, \overline{C}_7 = 0$ and

$$\overline{C}_8 = \left(-\frac{1}{7560}, -\frac{1}{15120}, -\frac{781}{30965760}, -\frac{1}{16128}, -\frac{111}{1146880}, \frac{337}{3870720}, -\frac{1}{16128}, -\frac{11}{143360}\right)^T. \tag{12}$$

Therefore, the proposed TDHMT has six algebraic order of convergences. Since the order of the new method is not ≤ 1, then the TDHMT is also consistent.

3.2. Zero-Stability and Convergence of the TDHMT. Zero-stability of the TDHMT is concerned with the behavior of the difference scheme (7) in the limit as h tends to zero. For $h \longrightarrow 0$, (7) may be written as follows:

$$P^{(0)}\overline{V}_u - P^{(1)}\overline{V}_{u-1} = 0, \tag{13}$$

where

$$\begin{aligned} &\overline{V}_u = \left(u_{n+2}, u_{n+s}, u_{n+1}, u_{n+r}\right)^T, \overline{V}_{u-1} = \left(u_n, u_{n+s-2}, u_{n-1}, u_{n+r-2}\right)^T, \\ &P^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{14}$$

The proposed TDHMT is said to be zero-stable if the roots p_j of the first characteristic polynomial $\rho(p)$ denoted by $\rho(p) = \det[P^{(0)}p - P^{(1)}]$, satisfy $|p_j| \leq 1$, and those roots with $|p_j| = 1$ have a multiplicity which is not greater than 2 (see [22] for more details). Since $\rho(p) = (p-1)p^3$, the TDHMT is zero stable. The requirements for a multistep method to be convergent are that it must be zero-stable and consistent. Because these two conditions are met, the TDHMT is also convergent.

3.3. Linear Stability Analysis of the TDHMT. The following test equation to determine the linear stability of the newly constructed strategy is considered.

$$u''(x) = -2\nu u'(x) - \nu^2 u(x), \lambda > 0. \tag{15}$$

The TDHMT is applied to the scalar test (15), yielding the matrix difference equation shown.

$$A\begin{pmatrix} u_{n+r} \\ u_{n+1} \\ u_{n+s} \\ u_{n+2} \\ u'_{n+r} \\ u'_{n+1} \\ u'_{n+s} \\ u'_{n+2} \end{pmatrix} = B\begin{pmatrix} u_{n+r-2} \\ u_{n-1} \\ u_{n+s-2} \\ u_n \\ u'_{n+r-2} \\ u'_{n-1} \\ u'_{n+s-2} \\ \mu'_n \end{pmatrix}, \tag{16}$$

with

$$A = \begin{pmatrix} \frac{337\nu^2}{3024}+1 & -\frac{1193\nu^2}{13440} & \frac{41\nu^2}{560} & -\frac{15139\nu^2}{483840} & 0 & 0 & 0 & \frac{107\nu^2}{16128} \\ \frac{418\nu^2}{945} & 1-\frac{5\nu^2}{28} & \frac{6\nu^2}{35} & -\frac{563\nu^2}{7560} & 0 & 0 & 0 & \frac{\nu^2}{63} \\ \frac{447\nu^2}{560} & -\frac{297\nu^2}{4480} & \frac{33\nu^2}{112}+1 & -\frac{2127\nu^2}{17920} & 0 & 0 & 0 & \frac{45\nu^2}{1792} \\ \frac{1088\nu^2}{945} & \frac{8\nu^2}{105} & \frac{64\nu^2}{105} & 1-\frac{25\nu^2}{189} & 0 & 0 & 0 & \frac{2\nu^2}{63} \\ \frac{23\nu^2}{45} & -\frac{1}{960}\left(311\nu^2\right) & \frac{47\nu^2}{180} & -\frac{1277\nu^2}{11520} & 1 & 0 & 0 & \frac{3\nu^2}{128} \\ \frac{98\nu^2}{135} & \frac{\nu^2}{20} & \frac{2\nu^2}{15} & -\frac{137\nu^2}{2160} & 0 & 1 & 0 & \frac{\nu^2}{72} \\ \frac{7\nu^2}{10} & \frac{99\nu^2}{320} & \frac{9\nu^2}{20} & -\frac{149\nu^2}{1280} & 0 & 0 & 1 & \frac{3\nu^2}{128} \\ \frac{32\nu^2}{45} & \frac{4\nu^2}{15} & \frac{32\nu^2}{45} & \frac{7\nu^2}{45} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 1-\frac{3247\nu^2}{53760} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1-\frac{39\nu^2}{280} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-\frac{3891\nu^2}{17920} & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1-\frac{31\nu^2}{105} & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -\frac{1873\nu^2}{11520} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{240}\left(37\nu^2\right) & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{201\nu^2}{1280} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{45}\left(7\nu^2\right) & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (17)$$

The eigenvalues of the stability matrix $M(z) = A^{-1}B, z = \nu h$ are used to investigate the boundedness of their solutions. If $\nu \in \mathbb{R}$, the entire absolute stability region then gets smaller to a real interval known as an interval of periodicity. This interval is obtained as $(0, \infty)$. Figure 1 displays the TDHMT method stability region.

4. Implementation Details

In this part, the discussion of the step-by-step procedure for implementation of the new method tagged TDHMT is fully stated. The TDHMT is implemented in a fashion-by-fashion block method to simultaneously generate the solution at the initial point to the terminal point. The method is self-starting and does not require any separate predictors. In fact, each block integrators in equations (5) and (6) form a system of equations which are applied along with Newton's method. The derived starting values, also known as predictors, used Newton's method and are considered approximations provided by the Taylor series expansion formulas.

$$\begin{aligned} u_{n+i} &= u_n + hu_n' + \left(\frac{h^2}{2}\right) f_n + \left(\frac{h^3}{6}\right) g_n, \\ u_{n+i}' &= u_n' + hf_n + \left(\frac{h^2}{2}\right) g_n, i = 0\left(\frac{1}{2}\right)\ldots, k. \end{aligned} \quad (18)$$

$g_{n+i}, i = 0(1/2)\ldots, k$, occurring in equations (5) and (6), denotes the third derivative at $x_{n+i}, i = 0(1/2)\ldots, k$. In other to obtain a closed form solution of equations (5) and (6) which is expanded in equation (18), it is important to calculate the values of $g_{n+i}, i = 0(1/2)\ldots, k$. For more clarification,

$$\begin{aligned} u'''(x) &= \left(\frac{dw(x, u, u')}{dx}\right), i = 0\left(\frac{1}{2}\right)\ldots, k, \\ u'''(x) &= \frac{dw}{dx} + \frac{dw}{dy}y' + \frac{dw}{du'}w, i = 0\left(\frac{1}{2}\right)\ldots, k. \end{aligned} \quad (19)$$

5. Numerical Examples and Discussion of the Results

This section is concerned with the presentation of the computational results as well as the CPU time required to compute the results, and the number of function evaluation. The following notations will be used for simplicity presentation.

(i) NFE—Number of function evaluation

(ii) TDHMT—The one-pointthird-derivative hybrid multistep technique of order 6: newly proposed method

(iii) SAMTD—The multiderivative method of order 12 produced by reference [19]

(iv) SIMOSMTD—The 12th order multiderivative Obrechkoff Method developed by reference [23]

(v) VDMTD—The 12th order Obrechkoff ultiderivative method contructed by reference [24]

(vi) AMTD—The 8th order Obrechkoff multiderivative method introduced by reference [2]

(vii) WETMTD—The Obrechkoff multiderivative method of order 12 presented by reference [25]

(viii) SKMTD—The four-step multiderivative method of order 8 studied by reference [26]

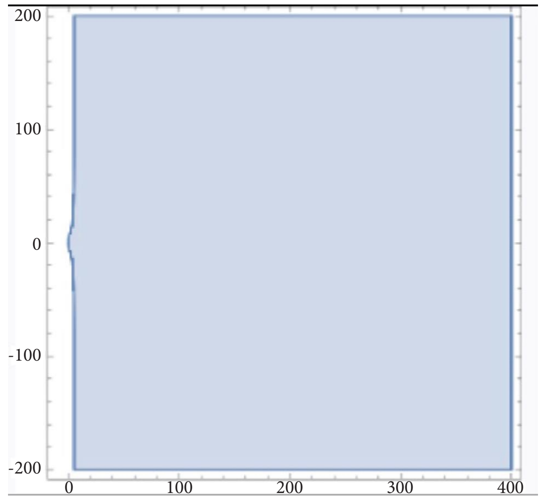

FIGURE 1: Stability region of the proposed TDHMT.

Example 1. The following is the nonlinear undamped Duffing's equation

$$y'' = -y - y^3 + B\cos(\omega x),$$
$$y(0) = 0.200426728067, \tag{20}$$
$$y'(0) = 0,$$

where $B = 0.002$, $\omega = 1.01$, and $x \in [0, 40.5\pi/1.01]$. The exact solution for equation (20) $g(x) = \sum_{i=0}^{3} K_{2i+1}\cos((2i+1)\omega x)$, where

$$\{K_1, K_3, K_5, K_7\} = \{0.200179477536, 0.246946143 \times 10^{-3}, 0.304016 \times 10^{-6}, 0.374 \times 10^{-9}\}. \tag{21}$$

The absolute errors at $x = 40.5\pi/1.01$ are given in Table 1, where $M = 40.5\pi/1.01$ and the CPU times are listed in Table 2, while Table 3 shows the comparison of the number of functions evaluation (NFE).

Example 2. The initial-value problem

$$y'' = -100y + 99\sin(x),$$
$$y(0) = 1, \tag{22}$$
$$y'(0) = 11.$$

The exact solution $y(t) = \sin(t) + \sin(10t) + \cos(10t)$. This equation has been solved numerically for $0 \le x \le 10\pi$ using exact starting values. The step lengths $h = \pi/50$, $h = \pi/100$, $h = \pi/200$, $h = \pi/300$, $h = \pi/400$, and $h = \pi/500$.

Example 3. Consider the initial-value problem

$$y'' = \frac{8y^2}{1+2x},$$
$$y(0) = 1, \tag{23}$$
$$y'(0) = -2,$$
$$x \in [0, 4.5],$$

with the exact solution

$$y(x) = \frac{1}{1+2x}. \tag{24}$$

The absolute errors at $x = 4.5$ are given in Table 4, the CPU times are listed in Table 5, and the NFE is shown in Table 6.

Example 4. Franco and Palacios investigated an almost periodic orbital problem:

Table 1: Comparison of the absolute errors at the end point of TDHMT with absolute error of other existing methods, namely, SKMTD, SAMTD, SIMOSMTD, VDMTD, AMTD, and WETMTD for Example 1.

h	TDHMT	SKMTD	SAMTD	SIMOSMTD	VDMTD	AMTD	WETMTD
$M/500$	$3.14e-10$	$8.26e-11$	$9.31e-11$	$3.15e-04$	$4.06e-05$	$4.09e-05$	$4.08e-05$
$M/1000$	$5.23e-12$	$9.52e-12$	$8.03e-12$	$1.81e-05$	$1.87e-06$	$1.27e-06$	$1.27e-06$
$M/2000$	$8.43e-14$	$3.74e-12$	$5.52e-12$	$1.07e-06$	$3.84e-08$	$3.94e-08$	$3.93e-08$
$M/3000$	$5.02e-15$	$7.16e-13$	$7.25e-12$	$2.09e-07$	$5.13e-09$	$5.18e-09$	$5.17e-09$
$M/4000$	$1.31e-15$	$2.18e-13$	$6.99e-12$	$6.55e-08$	$3.19e-09$	$1.23e-09$	$1.23e-09$
$M/5000$	$3.48e-16$	$9.65e-14$	$6.65e-12$	$2.67e-08$	$9.89e-10$	$4.09e-10$	$4.07e-10$

Table 2: Comparison of the CPU time of TDHMT with the CPU time of other existing methods, namely, SKMTD, SAMTD, SIMOSMTD, VDMTD, AMTD, and WETMTD for Example 1.

h	TDHMT	SKMTD	SAMTD	SIMOSMTD	VDMTD	AMTD	WETMTD
$M/500$	2.15	1.42	1.45	1.44	1.48	1.19	1.41
$M/1000$	3.54	2.25	2.87	2.89	2.94	2.31	2.89
$M/2000$	5.24	5.12	6.27	6.23	6.36	4.81	6.24
$M/3000$	8.14	7.61	9.86	9.86	9.72	7.55	9.55
$M/4000$	11.38	10.25	13.42	13.55	13.39	9.99	13.06
$M/5000$	15.02	14.81	16.86	16.92	16.97	12.86	16.50

Table 3: Comparison of the NFE of TDHMT with other existing methods, namely, SKMTD, SAMTD, SIMOSMTD, VDMTD, AMTD, and WETMTD for Example 1.

h	TDHMT	SKMTD	SAMTD	SIMOSMTD	VDMTD	AMTD	WETMTD
$M/500$	1468	1490	1501	1550	1500	1600	1620

Table 4: Comparison of the absolute errors at the end point of TDHMT with absolute error of other existing methods, namely, SIMOSMTD and AMTD for Example 3.

h	TDHMT	SIMOSMTD	AMTD
4.5/500	$2.73e-19$	$1.07e-09$	$7.46e-14$
4.5/1000	$4.71e-21$	$9.24e-12$	$5.36e-16$
4.5/2000	$7.46e-23$	$5.45e-14$	$2.88e-18$
4.5/3000	$6.16e-24$	$2.45e-15$	$1.25e-19$
4.5/4000	$9.03e-25$	$2.63e-16$	$1.32e-20$
4.5/5000	$2.37e-26$	$4.60e-17$	$2.29e-21$

$$\begin{aligned} y'' &= \epsilon e^{i\psi x}, \\ y(0) &= 1, \\ y'(0) &= i, \\ y &\in \mathbb{C}, \end{aligned} \tag{25}$$

or equivalently by

$$\begin{aligned} u'' + u &= \epsilon \cos(\psi x), u(0) = 1, u'(0) = 0, \\ v'' + v &= \epsilon \sin(\psi x), v(0) = 0, v'(0) = 1, \end{aligned} \tag{26}$$

$\epsilon = 0.001$, $\psi = 0.01$. The theoretical solution is as follows:

$$y(x) = u(x) + iv(x), \quad u, v \in \mathbb{R}, \tag{27}$$

where

$$\begin{aligned} u(x) &= \frac{1-\epsilon-\psi^2}{1-\psi^2}\cos(x) + \frac{\epsilon}{1-\psi^2}\cos(\psi x), \\ v(x) &= \frac{1-\epsilon\psi-\psi^2}{-\psi^2}\sin(x) + \frac{\epsilon}{1-\psi^2}\sin(\psi x). \end{aligned} \tag{28}$$

For $x in [0, 1000 pi]$, this system of equations has been solved. Table 7 shows the absolute errors at the end point,

Table 5: Comparison of the CPU time of TDHMT with the CPU time of other existing methods, namely, SIMOSMTD and WETMTD for Example 3.

h	TDHMT	SIMOSMTD	AMTD
4.5/500	0.34	0.36	0.19
4.5/1000	0.54	0.62	0.76
4.5/2000	1.08	1.23	1.20
4.5/4000	1.63	1.89	1.62
4.5/5000	2.27	2.59	2.06

Table 6: Comparison of the NFE of TDHMT with the NFE of other existing methods, namely, SIMOSMTD and WETMTD for Example 3.

h	TDHMT	SIMOSMTD	AMTD
4.5/500	1600	1750	1900

Table 7: Comparison of the absolute errors at the end point of TDHMT with absolute error of other existing methods, namely, SIMOSMTD, VDMTD, and AMTD for Example 4.

h	TDHMT	SIMOSMTD	VDMTD	AMTD
$M/500$	$2.75e-8$	$2.04e-06$	$5.19e-06$	$1.07e-04$
$M/1000$	$5.71e-10$	$2.40e-07$	$1.59e-07$	$1.52e-05$
$M/1500$	$1.06e-11$	$5.87e-08$	$9.03e-08$	$1.12e-06$
$M/2000$	$7.75e-12$	$2.64e-08$	$1.45e-08$	$4.96e-07$
$M/2500$	$2.71e-13$	$1.60e-08$	$2.43e-09$	$1.25e-07$
$M/3000$	$9.78e-14$	$1.13e-08$	$1.82e-09$	$3.45e-08$

Table 8: Comparison of the CPU time of TDHMT with the CPU time of other existing methods, namely, SIMOSMTD, VDMTD, and AMTD for Example 4.

h	TDHMT	SIMOSMTD	VDMTD	AMTD
$M/500$	0.32	0.36	0.42	0.37
$M/1000$	0.68	0.76	0.80	0.78
$M/1500$	1.09	1.17	1.19	1.20
$M/2000$	1.45	1.59	1.61	1.58
$M/2500$	1.73	1.95	2.00	1.98
$M/3000$	2.21	2.37	2.34	2.39

Table 9: Comparison of the NFE of TDHMT with the NFE of other existing methods, namely, SIMOSMTD, VDMTD, and AMTD for Example 4.

h	TDHMT	SIMOSMTD	VDMTD	AMTD
$M/500$	780	820	900	890

Table 8 shows the CPU times, and Table 9 presents the NFE where $M = 40.5\pi/1.01$.

Example 5. The almost periodic orbital problem studied by Stiefel and Bettis:

$$\begin{aligned} y'' + y &= 0.001e^{ix}, \\ y(0) &= 1, \\ y'(0) &= 0.9995i, \\ y &\in \mathbb{C}, \end{aligned} \tag{29}$$

or equivalently by

$$\begin{aligned} u'' + u &= 0.001 \cos(\psi x), u(0) = 1, u'(0) = 0, \\ v'' + v &= 0.001 \sin(\psi x), v(0) = 0, v'(0) = 0.9995. \end{aligned} \tag{30}$$

The theoretical solution: $y(x) = u(x) + iv(x)$, where $u, v \in \mathbb{R}$ and

$$\begin{aligned} u(x) &= \cos(x) + 0.0005 \sin(x), \\ v(x) &= \sin(x) - 0.0005x \cos(x). \end{aligned} \tag{31}$$

Table 10: Comparison of the absolute errors at the end point of TDHMT with absolute error of other existing methods, namely, SIMOSMTD, VDMTD, AMTD, and WETMTD for Example 2.

h	TDHMT	SIMOSMTD	VDMTD	AMTD	WETMTD
$\pi/50$	$2.32e-25$	$3.03e-06$	$3.64e-06$	$2.23e-08$	$1.76e-16$
$\pi/100$	$4.71e-27$	$1.15e-08$	$6.79e-09$	$7.98e-11$	$4.54e-20$
$\pi/200$	$8.05e-29$	$4.50e-11$	$1.07e-12$	$5.24e-14$	$1.92e-24$
$\pi/300$	$4.23e-30$	$1.76e-12$	$8.13e-15$	$6.38e-16$	$4.65e-27$
$\pi/400$	$9.83e-31$	$1.76e-13$	$2.56e-16$	$2.74e-17$	$6.34e-29$
$\pi/500$	$2.12e-32$	$2.95e-14$	$1.79e-17$	$2.38e-18$	$2.25e-30$

Table 11: Comparison of the CPU time of TDHMT with the CPU time of other existing methods, namely, SIMOSMTD, VDMTD, AMTD, and WETMTD for Example 2.

h	TDHMT	SIMOSMTD	VDMTD	AMTD	WETMTD
$\pi/50$	0.16	0.17	0.25	0.19	0.11
$\pi/100$	0.48	0.51	0.53	0.45	0.28
$\pi/200$	1.77	0.86	0.83	0.75	0.58
$\pi/300$	1.08	1.14	1.15	0.95	0.92
$\pi/400$	1.25	1.39	1.40	1.23	1.26
$\pi/500$	1.63	1.70	1.78	1.47	1.56

Table 12: Comparison of the NFE of TDHMT with the NFE of other existing methods, namely, SIMOSMTD, VDMTD, AMTD, and WETMTD for Example 2 at the end point.

h	TDHMT	SIMOSMTD	VDMTD	AMTD	WETMTD
$\pi/50$	480	790	850	800	930

Table 13: Comparison of the absolute errors at the end point of TDHMT with absolute error of other existing methods, namely, SIMOSMTD, VDMTD, and AMTD for Example 5.

h	TDHMT	SIMOSMTD	VDMTD	AMTD
$M/500$	$4.16e-05$	$1.86e-03$	$1.91e-03$	$2.55e-02$
$M/1000$	$7.71e-07$	$4.32e-04$	$7.44e-04$	$4.94e-02$
$M/1500$	$2.17e-08$	$1.08e-04$	$1.89e-05$	$8.74e-04$
$M/2000$	$8.35e-09$	$2.69e-05$	$2.61e-07$	$1.21e-05$
$M/2500$	$4.03e-10$	$6.70e-06$	$4.48e-09$	$1.96e-07$
$M/3000$	$9.78e-11$	$1.46e-06$	$8.55e-11$	$4.36e-09$

Table 14: Comparison of the CPU time of TDHMT with the CPU time of other existing methods, namely, SIMOSMTD, VDMTD, and AMTD for Example 5.

h	TDHMT	SIMOSMTD	VDMTD	AMTD
$M/500$	0.05	0.05	0.06	0.05
$M/1000$	0.11	0.11	0.12	0.11
$M/1500$	0.31	0.31	0.30	0.30
$M/2000$	0.68	0.72	0.62	0.80
$M/2500$	1.36	1.47	1.34	1.39
$M/3000$	2.65	2.71	2.64	2.66

Table 15: Comparison of the NFE of TDHMT with the NFE of other existing methods, namely, SIMOSMTD, VDMTD, and AMTD for Example 5.

h	TDHMT	SIMOSMTD	VDMTD	AMTD
$M/500$	350	400	500	450

This system has been solved for $x \in [0, 1000\pi]$.

6. Concluding Remarks

This study has successfully derived, analyzed, and implemented a third-derivative hybrid multistep technique (TDHMT) for the direct solution of second-order initial-value problems (IVPs) in ODEs whose coefficients do not depend on the frequency ($omega$) and step-size (h) in this study. The absolute error of the TDHMT, the CPU time compared with other existing methods, and the NFE compared with other existing methods in the literature, respectively, are shown in Tables 1–15. The TDHMT's numerical results show that it is more efficient, accurate, capable, and stable in solving oscillatory and periodic problems with lesser NFEs.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the study. All authors have read and approved the final manuscript.

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