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Research Article

Fixed Point Results of a New Family of Contractions in Metric Space Endowed with a Graph

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In this note, a general class of contractive inequality, namely, admissible hybrid $(H-\alpha-\phi)$ contraction, is proposed in metric space endowed with a graph, and new criteria for which the mapping is a Picard operator are examined. The significance of this type of contraction is connected with the possibility that its inequality can be particularized in more than one way, depending on the provided constants. Relevant examples are designed to support the assumptions of our obtained notions and to show how they are different from the known ones. A corollary which reduces our obtained result to a recently published result in the literature is pointed out and discussed.

1. Introduction and Preliminaries

The celebrated Banach contraction principle in metric space has undoubtedly laid the foundation of the modern metric fixed point theory. The importance of fixed point results via this principle runs over several fields of sciences and engineering. Examiners in this area have investigated a lot of novel ideas in metric space and have presented more than a handful of significant results.

Hereunder, every set Φ is taken nonempty, $\mathbb{N}$ depicts the set of natural numbers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{R}^+$ is the set of all non-negative real numbers.

Definition 1 (see [1]). Let Ψ be the set of all functions ϕ: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying the following:

(i) ϕ is monotone increasing, that is, $t_1 \leq t_2$ implies $\phi(t_1) \leq \phi(t_2)$

(ii) The series $\sum_{p=0}^{\infty} \phi^p(t)$ is convergent for all $t > 0$.

Then, ϕ is termed a (c)-comparison function.

Remark 1. If $\phi \in \Psi$, then $\phi(t) < t$ for any $t > 0$ and $\phi(0) = 0$, and ϕ is continuous at 0.

Definition 2 (see [2]). Let α: $\Phi \times \Phi \longrightarrow \mathbb{R}^+$ be a function. A self-mapping Γ: $\Phi \longrightarrow \Phi$ is termed α-orbital admissible if for all $r \in \Phi$,

$$\alpha(r, \Gamma r) \geq 1 \Longrightarrow \alpha\left(\Gamma r, \Gamma^2 r\right) \geq 1. \tag{1}$$

Definition 3 (see [2]). Let α: $\Phi \times \Phi \longrightarrow \mathbb{R}^+$ be a function. A self-mapping Γ: $\Phi \longrightarrow \Phi$ is termed triangular α-orbital admissible if for all $r \in \Phi$, Γ is α-orbital admissible and

$$\alpha(r, s) \geq 1 \text{ and } \alpha(s, \Gamma s) \geq 1 \Longrightarrow \alpha(r, \Gamma s) \geq 1. \tag{2}$$

Lemma 1 (see [3]). *Let $\Gamma\colon \Phi \longrightarrow \Phi$ be a triangular α-orbital admissible mapping. If we can find $r_0 \in \Phi$ such that $\alpha(r_0, \Gamma r_0) \geq 1$, then*

$$\alpha\left(r_p, r_m\right) \geq 1 \forall p, m \in \mathbb{N}, \tag{3}$$

where the sequence $\left\{r_p\right\}$ is defined by $r_{p+1} = \Gamma r_p$ and $p \in \mathbb{N}$.

Definition 4 (see [3]). Let $\alpha\colon \Phi \times \Phi \longrightarrow \mathbb{R}^+$ be a self-mapping. The set Φ is termed regular with respect to α if for a sequence $\left\{r_p\right\}$ in Φ such that $\alpha(r_p, r_{p+1}) \geq 1$, for all p and $r_p \longrightarrow r \in \Phi$ as $p \longrightarrow \infty$, we have $\alpha(r_p, r) \geq 1$ for all p.

Lately, Karapınar and Fulga [3] studied a new idea of contractive inequality which is a refinement of some known contractions in the following sense:

Definition 5 (see [3]). Let (Φ, ϖ) be a metric space. A self-mapping $\Gamma\colon \Phi \longrightarrow \Phi$ is termed an admissible hybrid contraction, if we can find $\phi \in \Psi$ and $\alpha\colon \Phi \times \Phi \longrightarrow \mathbb{R}^+$ such that

$$\alpha(r, s)\varpi(\Gamma r, \Gamma s) \leq \phi(M(r, s)), \tag{4}$$

where

$$M(r,s) = \begin{cases} \Bigg[\lambda_1 \varpi(r,s)^q + \lambda_2 \varpi(r,\Gamma r)^q + \lambda_3 \varpi(s,\Gamma s)^q, \\ +\lambda_4 \left(\dfrac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\right)^q + \lambda_5 \left(\dfrac{\varpi(s,\Gamma r)(1+\varpi(r,\Gamma s))}{1+\varpi(r,s)}\right)^q\Bigg]^{(1/q)}, \\ \text{for} \quad q > 0, r, s \in \Phi; \\ [\varpi(r,s)]^{\lambda_1} \cdot [\varpi(r,\Gamma r)]^{\lambda_2} \cdot [\varpi(s,\Gamma s)]^{\lambda_3} \cdot \left[\dfrac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\right]^{\lambda_4} \\ \cdot \left[\dfrac{\varpi(r,\Gamma s)+\varpi(s,\Gamma r)}{2}\right]^{\lambda_5} \quad \text{for } q = 0, r, s \in \Phi \backslash \mathrm{Fix}(\Gamma), \end{cases} \tag{5}$$

$\lambda_i \geq 0$ with $i = 1, 2, \ldots, 5, \sum_{i=1}^{5} \lambda_i = 1$, and $\mathrm{Fix}(\Gamma) = \{r \in \Phi\colon \Gamma r = r\}$.

For some extensions of the idea of hybrid contractions in the fixed point theory, we refer to [4–7] and the references therein.

Following Petruşel and Rus [8], a self-mapping Γ of a metric space (Φ, ϖ) is termed a Picard operator (abbr., PO) if Γ has a unique fixed point r^* and $\lim_{p \longrightarrow \infty} \Gamma^p r = r^*$ for all $r \in \Phi$ and Γ is termed a weakly Picard operator (abbr. WPO) if the sequence $\{\Gamma^p r\}_{p \in \mathbb{N}}$ converges, for all $r \in \Phi$ and the limit (which may depend on r) is a fixed point of Γ.

Jachymski [9] introduced the notion of contraction in metric space endowed with a graph H. Accordingly, let (Φ, ϖ) be a metric space and let Δ denote the diagonal of the Cartesian product $\Phi \times \Phi$. Consider a directed graph H such that the set $V(H)$ of its vertices coincides with Φ, and the set $E(H)$ of its edges contains all loops, i.e., $E(H) \supseteq \Delta$. It is assumed that H has no parallel edges, so H can be identified with the pair $(V(H), E(H))$. Moreover, H may be treated as a weighted graph (see [[10], p. 376]) by assigning to each edge the distance between its vertices. The conversion of a graph H is denoted by H^{-1}, i.e., the graph obtained from H by reversing the direction of edges. Therefore,

$$E\left(H^{-1}\right) = \{(r, s) \in \Phi \times \Phi \mid (s, r) \in E(H)\}. \tag{6}$$

The letter $\widetilde{H}$ denotes the undirected graph obtained from H by ignoring the direction of edges, or more conveniently, by treating H as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{H}) = E(H) \cup E\left(H^{-1}\right). \tag{7}$$

The pair (V', E') is said to be a subgraph of H if $V' \subseteq V(H)$ and $E' \subseteq E(H)$ and for any edge $(r, s) \in E'$ and $r, s \in V'$. If r and s are vertices in a graph H, then a path in H from r to s of length $N \in \mathbb{N}$ is a sequence $\{r_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $r_0 = r$, $r_N = s$, and $(r_{p-1}, r_p) \in E(H)$ for all $i = 1, 2, \ldots, N$. A graph H is connected if there is a path between any two vertices. H is weakly connected if $\widetilde{H}$ is connected.

Subsequently, fixed point results for Lipschitzian-type contractions in metric spaces endowed with graph have been obtained by several authors (see, e.g., [1, 11–16]). In particular, Bojor [1] obtained the following result:

Definition 6 (see [1]). Let (Φ, ϖ) be a metric space endowed with a graph H. A self-mapping $\Gamma\colon \Phi \longrightarrow \Phi$ is termed a $(H - \phi)$ contraction if

(i) $\forall r, s \in \Phi((r, s) \in E(H) \Longrightarrow (\Gamma r, \Gamma s) \in E(H))$

(ii) We can find $\phi \in \Psi$ such that

$$\varpi(\Gamma r, \Gamma s) \leq \phi(\varpi(r, s)), \tag{8}$$

for all $(r,s) \in E(H)$.

Definition 7 (see [1]). A self-mapping $\Gamma\colon \Phi \longrightarrow \Phi$ is said to be orbitally continuous if for all $r \in \Phi$ and any sequence $\{k_p\}_{p\in\mathbb{N}}$, $\Gamma^{k_p} r \longrightarrow s \in \Phi$ implies that $\Gamma(\Gamma^{k_p} r) \longrightarrow \Gamma s$ as $p \longrightarrow \infty$.

Definition 8 (see [1]). A self-mapping $\Gamma\colon \Phi \longrightarrow \Phi$ is said to be orbitally H-continuous if for all $r \in \Phi$ and a sequence $\{r_p\}_{p\in\mathbb{N}}$, $r_p \longrightarrow r$ and $(r_p, r_{p+1}) \in E(H)$ imply that $\Gamma r_p \longrightarrow \Gamma r$ as $p \longrightarrow \infty$.

Theorem 1 (see [1]). *Let (Φ,ϖ) be a complete metric space endowed with a graph H and $\Gamma\colon \Phi \longrightarrow \Phi$ be a $(H-\phi)$ contraction. Assume further that*

(i) H is weakly connected;

(ii) for any sequence $\{r_p\}_{p\in\mathbb{N}}$ in Φ with $\varpi(r_p, r_{p+1}) \longrightarrow 0$, we can find $k, p_0 \in \mathbb{N}$ such that $(r_{kp}, r_{km}) \in E(H)$ for all $p, m \in \mathbb{N}$ and $p, m \geq p_0$;

(iii) ${}_a\Gamma$ is orbitally continuous or;

(iii) ${}_b\Gamma$ is orbitally H-continuous and we can find a subsequence $\{\Gamma^{p_k} r_0\}_{k\in\mathbb{N}}$ of $\{\Gamma^{p} r_0\}_{p\in\mathbb{N}}$ such that $(\Gamma^{p_k} r_0, r^*) \in E(H)$ for each $k \in \mathbb{N}$.

Then, Γ is a PO.

As duly revealed from the available literature, we understand that hybrid fixed point concepts in metric space endowed with a graph have not been well considered. Hence, invited by the ideas in [1, 3, 9], we initiate an idea of admissible hybrid $(H-\alpha-\phi)$ contraction in metric space equipped with a graph and investigate the conditions for which this new contraction is a Picard operator. Comparative examples are constructed to demonstrate that our obtained results are valid and distinct from the existing results in the literature. In addition, a corollary is highlighted to show that the concept proposed in this manuscript complements and subsumes a well-known result in the literature.

2. Main Results

We now examine the idea of admissible hybrid $(H-\alpha-\phi)$ contraction in metric space endowed with a graph H.

Definition 9. Let (Φ,ϖ) be a metric space endowed with a graph H. A self-mapping $\Gamma\colon \Phi \longrightarrow \Phi$ is termed an admissible hybrid $(H-\alpha-\phi)$ contraction if

(i) $\forall r, s \in \Phi\,((r,s) \in E(H) \Longrightarrow (\Gamma r, \Gamma s) \in E(H))$;

(ii) we can find $\phi \in \Psi$ and $\alpha\colon \Phi \times \Phi \longrightarrow \mathbb{R}^+$ such that

$$\alpha(r,s)\varpi(\Gamma r, \Gamma s) \leq \phi(M(r,s)), \tag{9}$$

for all $(r,s) \in E(H)$, where

$$M(r,s) = \begin{cases} \Big[\lambda_1 \varpi(r,s)^q + \lambda_2 \varpi(r,\Gamma r)^q + \lambda_3 \varpi(s,\Gamma s)^q, \\ +\lambda_4 \left(\dfrac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\right)^q + \lambda_5 \left(\dfrac{\varpi(s,\Gamma r)(1+\varpi(r,\Gamma s))}{1+\varpi(r,s)}\right)^q\Big]^{(1/q)}, \\ \text{for} \quad q > 0; \\ [\varpi(r,s)]^{\lambda_1} \cdot [\varpi(r,\Gamma r)]^{\lambda_2} \cdot [\varpi(s,\Gamma s)]^{\lambda_3} \cdot \left[\dfrac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\right]^{\lambda_4}, \\ \cdot \left[\dfrac{\varpi(r,\Gamma s)+\varpi(s,\Gamma r)}{2}\right]^{\lambda_5} \quad \text{for } q = 0, r \neq s, r \neq \Gamma r, \end{cases} \tag{10}$$

$\lambda_i \geq 0$ with $i = 1, 2, \ldots, 5$ and $\sum_{i=1}^{5} \lambda_i = 1$.

Example 1. Let $\Phi = \{0,1,2,3,4\}$ with the Euclidean metric $\varpi(r,s) = |r-s| \forall r, s \in \Phi$. Define $\Gamma\colon \Phi \longrightarrow \Phi$ by

$$\Gamma r = \begin{cases} 2r, & \text{if } r \in \{0,1\}; \\ 1, & \text{if } r \in \{2,3,4\}, \end{cases} \tag{11}$$

for all $r \in \Phi$ and $\alpha\colon \Phi \times \Phi \longrightarrow \mathbb{R}^+$ by

$$\alpha(r,s) = \begin{cases} 0, & \text{if } r, s \in \{0,1\}, r \neq s; \\ 1, & \text{otherwise}. \end{cases} \tag{12}$$

Then, Γ is an admissible hybrid $(H-\alpha-\phi)$ contraction with $\phi(t) = (9t/10)$, $\lambda_1 = \lambda_5 = (2/5)$, $\lambda_2 = \lambda_4 = 0$, and $\lambda_3 = (1/5)$ for $q = 0, 2$, where the graph H is defined by $V(H) = \Phi$ and

$$E(H) = \{(0,1), (0,2), (0,3), (0,4), (2,3), (2,4), (3,4)\} \cup \Delta, \tag{13}$$

but Γ is not an admissible hybrid contraction defined in [3], since $\alpha(1,2)\varpi(\Gamma 1, \Gamma 2) = 1$ while $\phi(M(1,2)) = 0$ for $q = 0$ and $\phi(M(1,2)) = (27/50)$ for $q = 2$. See Figure 1.

The following is our main result:

Theorem 2. *Let (Φ, ϖ) be a complete metric space endowed with a graph H and $\Gamma\colon \Phi \longrightarrow \Phi$ be an admissible hybrid $(H - \alpha - \phi)$ contraction. Assume further that*

(i) Γ is triangular α-orbital admissible;

(ii) we can find $r_0 \in \Phi$ such that $\alpha(r_0, \Gamma r_0) \geq 1$;

(iii) H is weakly connected;

(iv) for any sequence $\{r_p\}_{p\in\mathbb{N}}$ in Φ with $\varpi(r_p, r_{p+1}) \longrightarrow 0$, we can find $k, p_0 \in \mathbb{N}$ such that $(r_{kp}, r_{km}) \in E(H)$ for all $p, m \in \mathbb{N}$ and $p, m \geq p_0$;

(v) ${}_a\Gamma$ is orbitally continuous or;

(v) ${}_b\Gamma$ is orbitally H-continuous, and we can find a subsequence $\{\Gamma^{p_k} r_0\}_{k\in\mathbb{N}}$ of $\{\Gamma^p r_0\}_{p\in\mathbb{N}}$ such that $(\Gamma^{p_k} r_0, r^*) \in E(H)$ for each $k \in \mathbb{N}$.

Then, Γ is a PO.

Proof. Let $r_0 \in \Phi$ be such that $(r_0, \Gamma r_0) \in E(H)$ and $\alpha(r_0, \Gamma r_0) \geq 1$ and define a sequence $\{r_p\}_{p\in\mathbb{N}}$ by $r_p = \Gamma^p r_0$ with $r_p \neq r_{p-1}$. Then, a standard induction reveals that $(\Gamma^p r_0, \Gamma^{p+1} r_0) \in E(H)$. Since Γ is an admissible hybrid $(H - \alpha - \phi)$-contraction, then by (i) and Lemma 1, inequality (9) becomes

$$\alpha(r_{p-1}, r_p)\varpi(\Gamma r_{p-1}, \Gamma r_p) \leq \phi(M(r_{p-1}, r_p)). \tag{14}$$

Considering Case 1 of (9), we have

$$\begin{aligned}
M(r_{p-1}, r_p) &= \Big[\lambda_1 \varpi(r_{p-1}, r_p)^q + \lambda_2 \varpi(r_{p-1}, \Gamma r_{p-1})^q + \lambda_3 \varpi(r_p, \Gamma r_p)^q, \\
&\quad + \lambda_4 \left(\frac{\varpi(r_p, \Gamma r_p)(1 + \varpi(r_{p-1}, \Gamma r_{p-1}))}{1 + \varpi(r_{p-1}, r_p)}\right)^q \\
&\quad + \lambda_5 \left(\frac{\varpi(r_p, \Gamma r_{p-1})(1 + \varpi(r_{p-1}, \Gamma r_p))}{1 + \varpi(r_{p-1}, r_p)}\right)^q\Big]^{(1/q)} \\
&= \Big[\lambda_1 \varpi(r_{p-1}, r_p)^q + \lambda_2 \varpi(r_{p-1}, r_p)^q + \lambda_3 \varpi(r_p, r_{p+1})^q \\
&\quad + \lambda_4 \left(\frac{\varpi(r_p, r_{p+1})(1 + \varpi(r_{p-1}, r_p))}{1 + \varpi(r_{p-1}, r_p)}\right)^q + \lambda_5 \left(\frac{\varpi(r_p, r_p)(1 + \varpi(r_{p-1}, r_{p+1}))}{1 + \varpi(r_{p-1}, r_p)}\right)^q\Big]^{(1/q)} \\
&= \left[\lambda_1 \varpi(r_{p-1}, r_p)^q + \lambda_2 \varpi(r_{p-1}, r_p)^q + \lambda_3 \varpi(r_p, r_{p+1})^q + \lambda_4 \varpi(r_p, r_{p+1})^q\right]^{(1/q)} \\
&= \left[(\lambda_1 + \lambda_2)\varpi(r_{p-1}, r_p)^q + (\lambda_3 + \lambda_4)\varpi(r_p, r_{p+1})^q\right]^{(1/q)}.
\end{aligned} \tag{15}$$

Hence, from (14), we have

$$\begin{aligned}
\varpi(r_p, r_{p+1}) = \varpi(\Gamma r_{p-1}, \Gamma r_p) &\leq \alpha(r_{p-1}, r_p)\varpi(\Gamma r_{p-1}, \Gamma r_p) \\
&\leq \phi\left(\left[(\lambda_1 + \lambda_2)\varpi(r_{p-1}, r_p)^q + (\lambda_3 + \lambda_4)\varpi(r_p, r_{p+1})^q\right]^{(1/q)}\right).
\end{aligned} \tag{16}$$

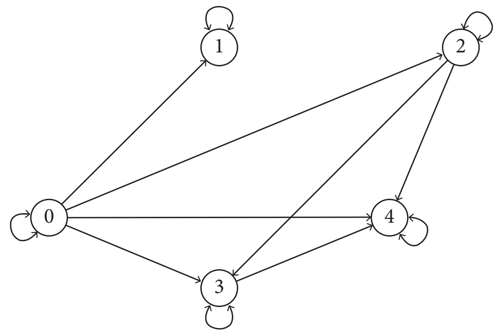

Figure 1: Graph H defined in Example 1.

Now, if $\varpi(r_{p-1}, r_p) \le \varpi(r_p, r_{p+1})$, then we have

$$\begin{aligned}\varpi(r_p, r_{p+1}) &\le \phi\Big(\big[(\lambda_1+\lambda_2)\varpi(r_{p-1}, r_p)^q + (\lambda_3+\lambda_4)\varpi(r_p, r_{p+1})^q\big]^{(1/q)}\Big)\\ &\le \phi\Big(\big[(\lambda_1+\lambda_2)\varpi(r_p, r_{p+1})^q + (\lambda_3+\lambda_4)\varpi(r_p, r_{p+1})^q\big]^{(1/q)}\Big)\\ &\le \phi\big((\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{(1/q)}\varpi(r_p, r_{p+1})\big)\\ &< (\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{(1/q)}\varpi(r_p, r_{p+1})\\ &\le \varpi(r_p, r_{p+1}),\end{aligned} \tag{17}$$

a contradiction. Therefore, $\varpi(r_p, r_{p+1}) < \varpi(r_{p-1}, r_p)$, so that (14) becomes

$$\varpi(r_p, r_{p+1}) \le \phi(\varpi(r_{p-1}, r_p)). \tag{18}$$

Continuing inductively, we obtain

$$\varpi(r_p, r_{p+1}) \le \phi^p(\varpi(r_0, \Gamma r_0)) \forall p \in \mathbb{N}. \tag{19}$$

Also by Case 2, we have

$$\begin{aligned}M(r_{p-1}, r_p) &= \varpi(r_{p-1}, r_p)^{\lambda_1} \cdot \varpi(r_{p-1}, \Gamma r_{p-1})^{\lambda_2} \cdot \varpi(r_p, \Gamma r_p)^{\lambda_3}\\ &\quad\cdot \left[\frac{\varpi(r_p, \Gamma r_p)(1+\varpi(r_{p-1}, \Gamma r_{p-1}))}{1+\varpi(r_{p-1}, r_p)}\right]^{\lambda_4}\\ &\quad\cdot \left[\frac{\varpi(r_{p-1}, \Gamma r_p)+\varpi(r_p, \Gamma r_{p-1})}{2}\right]^{\lambda_5}\\ &\le \varpi(r_{p-1}, r_p)^{\lambda_1} \cdot \varpi(r_{p-1}, r_p)^{\lambda_2} \cdot \varpi(r_p, r_{p+1})^{\lambda_3}\\ &\quad\cdot \left[\frac{\varpi(r_p, r_{p+1})(1+\varpi(r_{p-1}, r_p))}{1+\varpi(r_{p-1}, r_p)}\right]^{\lambda_4}\\ &\quad\cdot \left[\frac{\varpi(r_{p-1}, r_p)+\varpi(r_p, r_{p+1})+\varpi(r_p, r_p)}{2}\right]^{\lambda_5}\\ &\le \varpi(r_{p-1}, r_p)^{(\lambda_1+\lambda_2)} \cdot \varpi(r_p, r_{p+1})^{(\lambda_3+\lambda_4)}\\ &\quad\cdot \left[\frac{\varpi(r_{p-1}, r_p)^{\lambda_5}+\varpi(r_p, r_{p+1})^{\lambda_5}}{2}\right].\end{aligned} \tag{20}$$

Now, if $\varpi(r_{p-1}, r_p) \leq \varpi(r_p, r_{p+1})$, then

$$\begin{aligned} M\left(r_{p-1}, r_p\right) &\leq \varpi\left(r_p, r_{p+1}\right)^{(\lambda_1+\lambda_2)} \varpi\left(r_p, r_{p+1}\right)^{(\lambda_3+\lambda_4)} \varpi\left(r_p, r_{p+1}\right)^{\lambda_5} \\ &= \varpi\left(r_p, r_{p+1}\right)^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \\ &= \varpi\left(r_p, r_{p+1}\right), \end{aligned} \tag{21}$$

and so (14) becomes

$$\varpi\left(r_p, r_{p+1}\right) \leq \phi\left(\varpi\left(r_p, r_{p+1}\right)\right) < \varpi\left(r_p, r_{p+1}\right), \tag{22}$$

which is a contradiction. Therefore, we must have

$$\varpi\left(r_p, r_{p+1}\right) < \varpi\left(r_{p-1}, r_p\right), \tag{23}$$

so that (14) resolves to

$$\varpi\left(r_p, r_{p+1}\right) \leq \phi\left(\varpi\left(r_{p-1}, r_p\right)\right), \tag{24}$$

and by induction, we obtain

$$\varpi\left(r_p, r_{p+1}\right) \leq \phi^p\left(\varpi\left(r_0, \Gamma r_0\right)\right) \forall p \in \mathbb{N}. \tag{25}$$

Therefore, we see that

$$\varpi\left(\Gamma^p r_0, \Gamma^{p+1} r_0\right) \leq \phi^p\left(\varpi\left(r_0, \Gamma r_0\right)\right), \tag{26}$$

for all $p \in \mathbb{N}$ and for all $q \geq 0$. Hence, we have

$$\lim_{p \longrightarrow \infty} \varpi\left(\Gamma^p r_0, \Gamma^{p+1} r_0\right) = 0, \tag{27}$$

and by (iv), we can find $k, p_0 \in \mathbb{N}$ such that

$$\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{\mathrm{km}} r_0\right) \in E(H) \forall p, m \in \mathbb{N}, p, m \geq p_0. \tag{28}$$

Since $\varpi(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+1)} r_0) \longrightarrow 0$, then for any given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ and $N \geq p_0$ such that

$$\varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+1)} r_0\right) < \epsilon - \phi(\epsilon), \quad \text{for any } p \in \mathbb{N}. \tag{29}$$

Since $(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+1)} r_0) \in E(H)$, then for any $p \geq N$, we have

$$\begin{aligned} \varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+2)} r_0\right) &\leq \varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+1)} r_0\right) + \varpi\left(\Gamma^{k(p+1)} r_0, \Gamma^{k(p+2)} r_0\right) \\ &< \epsilon - \phi(\epsilon) + \phi^k\left(\varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+1)} r_0\right)\right) < \epsilon. \end{aligned} \tag{30}$$

Similarly, since $(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+2)} r_0) \in E(H)$, then for any $p \geq N$, we have

$$\begin{aligned} \varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+3)} r_0\right) &\leq \varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+1)} r_0\right) + \varpi\left(\Gamma^{k(p+1)} r_0, \Gamma^{k(p+3)} r_0\right) \\ &< \epsilon - \phi(\epsilon) + \phi^k\left(\varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+2)} r_0\right)\right) < \epsilon. \end{aligned} \tag{31}$$

Continuing inductively, we see that

$$\varpi\left(\Gamma^{\mathrm{kp}} r_0, \Gamma^{k(p+m)} r_0\right) \leq \epsilon, \quad \text{for any } p, m \in \mathbb{N}, p \geq N. \tag{32}$$

Therefore, $\left\{\Gamma^{\mathrm{kp}} r_0\right\}_{k \in \mathbb{N}}$ is a Cauchy sequence in (Φ, ϖ), and hence by the completeness of (Φ, ϖ), $\Gamma^{\mathrm{kp}} r_0 \longrightarrow r^*$ as $p \longrightarrow \infty$. Given that $\varpi(\Gamma^p r_0, \Gamma^{p+1} r_0) \longrightarrow 0$ as $p \longrightarrow \infty$, then we have $\Gamma^p r_0 \longrightarrow r^*$ as $p \longrightarrow \infty$.

Now, for any arbitrary $r \in \Phi$, we see that

(1) if $(r, r_0) \in E(H)$, then $(\Gamma^p r, \Gamma^p r_0) \in E(H)$ for all $p \in \mathbb{N}$. Therefore,

$$\varpi\left(\Gamma^p r, \Gamma^p r_0\right) \leq \phi^p\left(\varpi\left(r, r_0\right)\right) \forall p \in \mathbb{N}. \tag{33}$$

Letting $p \longrightarrow \infty$, we have that $\Gamma^p r \longrightarrow r^*$.

(2) if $(r, r_0) \notin E(H)$, then by (iii), we can find a path in $\widetilde{H}$ and $\{\widetilde{r}_i\}_{i=0}^N$ from r_0 to r such that $\widetilde{r}_0 = r_0$ and $\widetilde{r}_N = r$ with $(\widetilde{r}_{i-1}, \widetilde{r}_i) \in E(\widetilde{H})$ for all $i = 1, 2, \ldots, N$ such that by simple induction, we obtain

$$\begin{aligned} &\left(\Gamma^p \widetilde{r}_{i-1}, \Gamma^p \widetilde{r}_i\right) \in E(\widetilde{H}), \quad \text{for } i = 1, 2, \ldots, N, \\ &\varpi\left(\Gamma^p r_0, \Gamma^p r\right) \leq \sum_{i=1}^N \phi^p\left(\varpi\left(\widetilde{r}_{i-1}, \widetilde{r}_i\right)\right), \end{aligned} \tag{34}$$

so that $\varpi(\Gamma^p r_0, \Gamma^p r) \longrightarrow 0$, implying that $\Gamma^p r \longrightarrow r^*$.

Hence, for all $r \in \Phi$, there is a unique point $r^* \in \Phi$ such that

$$\lim_{p \longrightarrow \infty} \Gamma^p r = r^*. \tag{35}$$

We now prove that $r^* \in \text{Fix}(\Gamma)$. If $(v)_a$ holds, then obviously $r^* \in \text{Fix}(\Gamma)$. Otherwise if $(v)_b$ holds, then since $\{\Gamma^{p_k} r_0\}_{k\in\mathbb{N}} \longrightarrow r^*$ and $(\Gamma^{p_k} r_0, r^*) \in E(H)$, then by the orbital H-continuity of Γ, we have $\Gamma^{p_k+1} r_0 \longrightarrow \Gamma r^*$ as $k \longrightarrow \infty$. Hence, $\Gamma r^* = r^*$.

If we can find some $s \in \Phi$ such that $\Gamma s = s$, then from the above, we must have that $\Gamma^p s \longrightarrow r^*$, implying that $s = r^*$.

Therefore, Γ is a PO. □

Example 2. Let $\Phi = \{1, 2, 3, 4, 5, 6\}$ be endowed with the metric ϖ: $\Phi \times \Phi \longrightarrow \mathbb{R}^+$ defined by

$$\varpi(r, s) = |r - s|, \forall r, s \in \Phi. \tag{36}$$

Then, (Φ, ϖ) is a complete metric space.

Consider a mapping Γ: $\Phi \longrightarrow \Phi$ given by

$$\Gamma r = \begin{cases} \dfrac{r}{2}, & if r \in \{2,4,6\}; \\ 1, & if r \in \{1,3,5\}, \end{cases} \tag{37}$$

for all $r \in \Phi$ and α: $\Phi \times \Phi \longrightarrow \mathbb{R}^+$ by

$$\alpha(r, s) = \begin{cases} 2, & if r, s \in \{4,5\}; \\ 1, & \text{otherwise.} \end{cases} \tag{38}$$

Consider the symmetric graph $\widetilde{H}$ defined by $V(\widetilde{H}) = \Phi$ and

$$E(\widetilde{H}) = \{(1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,5), (4,6), (5,6)\} \cup \Delta. \tag{39}$$

Then, it is clear that Γ preserves edges, Γ is triangular α-orbital admissible, and H is weakly connected.

To see that Γ is an admissible hybrid $(H - \alpha - \phi)$ contraction, let $\phi(t) = (9t/10)$ for all $t \geq 0$, $\lambda_1 = \lambda_5 = (1/10)$, $\lambda_2 = \lambda_4 = (2/5)$, and $\lambda_3 = 0$ for $q = 0, 2$. We then consider the following cases:

Case 1. $r = s$ and $r, s \in \{2, 4, 6\}$;

Case 2. $r \neq s$ and $r, s \in \{2, 4, 6\}$;

Case 3. $r = s$ and $r, s \in \{1, 3, 5\}$;

Case 4. $r \neq s$ and $r, s \in \{1, 3, 5\}$;

Case 5. $r \in \{2, 4, 6\}$ and $s \in \{1, 3, 5\}$;

Case 6. $r \in \{1, 3, 5\}$ and $s \in \{2, 4, 6\}$.

We demonstrate using the following Table 1 that inequality (9) is satisfied for each of the above cases.

In Figures 2–4, we present the symmetric graph $\widetilde{H}$ defined in Example 2 and illustrate the validity of contractive inequality (9) using Example 2.

Table 1: Table of values for Cases 1–6.

Cases	r	s	$\alpha(r,s)\varpi(\Gamma r, \Gamma s)$	$\phi(M(r,s))$, $q = 0$	$\phi(M(r,s))$, $q = 2$
Case 1	2	2	0	—	1.32272
	4	4	0	—	3.88566
	6	6	0	—	7.71274
Case 2	4	6	1	2.16471	2.01067
	6	4	1	2.42870	1.93075
Case 3	1	1	0	—	0
	3	3	0	—	3.88566
	5	5	0	—	12.80912
Case 4	1	3	0	—	0.46086
	3	1	0	—	0.74215
	3	5	0	2.47339	2.65156
	5	3	0	3.03412	2.57021
Case 5	2	1	0	—	0.37107
	2	3	0	1.23669	1.32578
	2	5	0	1.45264	1.35
	4	3	1	1.97517	1.94074
	4	5	2	2.71410	3.88670
	6	5	2	3.63692	4.98261
Case 6	1	2	0	—	0.33068
	3	2	0	1.45445	1.19906
	3	4	1	1.97517	2.04242
	5	2	0	2.09573	1.66712
	5	4	2	3.32940	3.61907
	5	6	2	3.97647	4.98627

Therefore, all the hypotheses of Theorem 2 are satisfied, Γ has a unique fixed point, $r = 1$, and $\lim_{p\longrightarrow\infty} \Gamma^p r = 1$ for all $r \in \Phi$. Consequently, Γ is a PO.

Remark 2. If in Definition 9, we let $\alpha(r, s) = 1$ for all $r, s \in \Phi$, $q > 0$, $\lambda_1 = 1$, and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, then admissible hybrid $(H - \alpha - \phi)$ contraction reduces to $(H - \phi)$ contraction defined by Bojor [1] (see Definition 6). Hence, we have the following result:

Corollary 1 (see [Theorem 1]). *Let (Φ, ϖ) be a complete metric space endowed with a graph H and Γ: $\Phi \longrightarrow \Phi$ be a $(H - \phi)$ contraction. Assume further that*

(i) H is weakly connected;

(ii) for any sequence $\{r_p\}_{p\in\mathbb{N}}$ in Φ with $\varpi(r_p, r_{p+1}) \longrightarrow 0$, we can find $k, p_0 \in \mathbb{N}$ such that $(r_{kp}, r_{km}) \in E(H)$ for all $p, m \in \mathbb{N}$ and $p, m \geq p_0$;

(iii) ${}_a\Gamma$ is orbitally continuous or;

(iii) ${}_b\Gamma$ is orbitally H-continuous, and we can find a subsequence $\{\Gamma^{p_k} r_0\}_{k\in\mathbb{N}}$ of $\{\Gamma^p r_0\}_{p\in\mathbb{N}}$ such that $(\Gamma^{p_k} r_0, r^*) \in E(H)$ for each $k \in \mathbb{N}$.

Then, Γ is a PO.

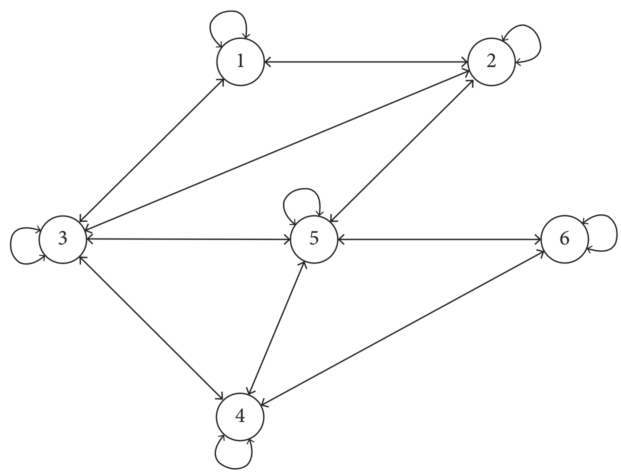

Figure 2: Symmetric graph $\tilde{H}$ defined in Example 2.

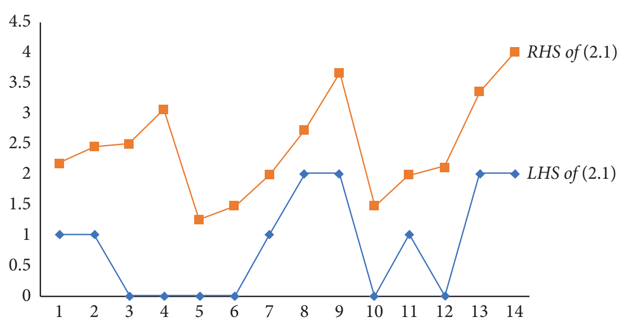

Figure 3: Illustration of contractive inequality (9) for $q = 0$.

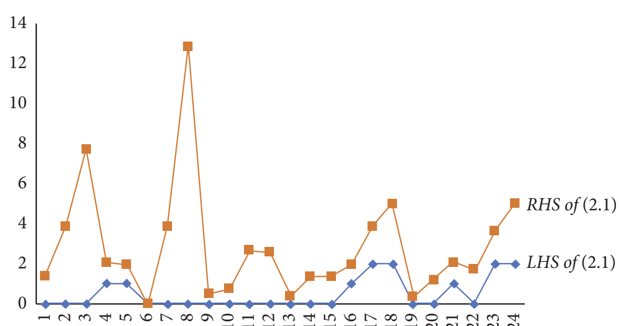

Figure 4: Illustration of contractive inequality (9) for $q = 2$.

Remark 3. It is obvious that we can obtain more consequences of our results by particularizing the values of the mappings $\alpha(r, s)$ and $\phi(t)$ and specializing the constants q and $\lambda_i\,(i = \overline{1,5})$.

3. Conclusion

In this note, the notion of admissible hybrid $(H - \alpha - \phi)$ contraction in metric space endowed with a graph is introduced (Definition 9). Sufficient conditions under which the new mapping is a Picard operator are examined (Theorem 2). To authenticate the hypotheses and indicate the generality of our new ideas, comparative examples are constructed with graphical illustrations (Examples 1 and 2). A corollary is presented to highlight that the proposed idea herein is a refinement of some known concepts in the literature (Corollary 1). In particular, the obtained results herein are inspired by and compared with [1, 3, 9].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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